A quantum mechanical representation in phase space

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A quantum mechanical representation suitable for studying the time evolution of quantum densities in phase space is proposed and examined in detail. This representation on \( S^2 \) (2) phase space is based on definitions of the operators \( \hat{P} \) and \( \hat{Q} \) in phase space that satisfy various correspondences for the Liouville equation in classical and quantum phase space, as well as quantum position and momentum \( S^2 \) (1) spaces. The definitions presented here, \( \hat{P} = p/2 - i\hbar \partial / \partial q \) and \( \hat{Q} = q/2 + i\hbar \partial / \partial p \), are related to definitions that have been recently proposed [J. Chem. Phys. 93, 8862 (1990)]. The resulting quantum phase space representation shares many of the mathematical properties of usual representations in coordinate and momentum spaces. Within this representation, time evolution equations for complex-valued functions (wave functions) and their square magnitudes (distribution functions) are derived, and it is shown that the coordinate and momentum space time evolution equations can be recovered by a simple Fourier projection. The phase space quantum probability conservation equation obtained is a good illustration of the quantization rule that requires one to replace the classical Poisson bracket between the Hamiltonian and the probability density with the quantum commutator between the corresponding operators. The possible classical analogs to quantum probabilities densities are also considered and some of the present results are illustrated for the dynamics of the coherent state.

I. INTRODUCTION

Since the advent of quantum mechanics as the proper theory governing the nonrelativistic physics of atoms and molecules, the understanding of the relationship between quantum and classical mechanics has remained a persistent pursuit. In the search for a definitive classical-quantum correspondence, efforts have ranged from the development of semiclassical theories,\(^{1-5}\) which provide guidelines for identifying particular classical trajectories or invariant manifolds\(^{6-8}\) as being most representative of quantum states, to the establishment of quantum phase space distributions,\(^{9-11,24,32,34}\) which provides a means for expressing quantum states directly in the province of classical mechanics, namely, phase space. One of the modern goals of these studies is a better understanding of the quantum mechanical manifestations of classical chaos\(^{35-37}\) and the implied relationship between classical and quantum statistical mechanics.\(^{36,38,39}\) Along these lines, the definition of an appropriate phase space distribution for the quantum density operator, \( \hat{\rho} \), and a suitable means for determining its time evolution appear to be crucial.

Unfortunately, due to limitations imposed by the uncertainty principle, there is no unique way to define quantum mechanics on phase space and consequently, a variety of different quantum phase space distributions have been proposed.\(^{9-11,24,32,34}\) The earliest and most intensively studied of these is the Wigner function,\(^{9-23}\) defined by

\[
\rho_W(q,p) = \rho_W(\Gamma) = \frac{1}{\pi \hbar} \int_{-\infty}^{\infty} ds (q-s|\hat{\rho}|q+s)e^{2ips/\hbar},
\]

(1.1)

where \( \hat{\rho} \) can represent either a pure or mixed state, and the extension to higher dimensions is straightforward (for simplicity, we shall generally confine our treatment to a single degree of freedom). The Wigner function has been shown to satisfy a number of desirable, classical-like conditions, including (i) it is real (it arises from a Hermitian form of the state vector); (ii) it has the properties \( \int dp \rho_W(q,p) = \langle q | \hat{\rho} | q \rangle \), \( \int dq \rho_W(q,p) = \langle p | \hat{\rho} | p \rangle \), and \( \int dq \int dp \rho_W(q,p) = \text{Tr}(\hat{\rho}) = 1 \); (iii) it is invariant to translations of the inertial frame origin (Galilei invariance); (iv) it is invariant to space and time reflections; and (v) its equation of motion reduces to the classical one for the free particle problem.\(^{10}\) Conversely, these conditions also serve to uniquely define the distribution given in Eq. (1.1).\(^{21}\)

As a consequence of these properties, the Wigner function provides a convenient means for relating averages over phase space to quantum mechanical expectation values via

\[
\langle \hat{\varphi} \rangle = \text{Tr}(\hat{\rho} \hat{\varphi}) = \int dq \int dp \rho_W(q,p)A(q,p),
\]

(1.2)

where

\[
A(q,p) = \frac{1}{\pi \hbar} \int ds (q-s|\hat{\varphi}|q+s)e^{2ips/\hbar}
\]

(1.3)

and all integrations are from \(-\infty\) to \(+\infty\). One of the drawbacks in the physical interpretation of the Wigner function, and with other related phase space distributions,\(^{18}\) is that it is not everywhere nonnegative.\(^{10,12,32,40}\) One can see this by replacing \( \hat{\varphi} \) in Eq. (1.3) with the density operator, \( \hat{\varphi}' \), of an orthogonal state so that \( \text{Tr}(\hat{\varphi} \hat{\varphi}') = 0 \) and then considering the integrand in Eq.
scribed classically, can only be projected onto the correct
time evolution of \( |\psi_1| \) for potentials whose coordinate
dependence is quadratic or lower.\(^{10,40}\)

In an effort to provide a quantum phase space distribution
which is everywhere positive, and therefore can function as a probability distribution, other workers have found ways to "coarse grain" or "smooth" phase space over cells of volume \( \tau^N \) \( (N \text{ being the number of degrees of freedom}) \) so as to incorporate the limitations of the uncertainty principle implicitly.\(^{24-30}\) The most widely studied of these alternative distribution functions is the Husimi function\(^{34}\) given by

\[
\rho_H^{(2)}(q,p) = \rho_H^{(1)}(\Gamma) = \langle \varphi(\Gamma) | \hat{\rho} | \psi \rangle \langle \psi | \varphi(\Gamma) \rangle ,
\]

where

\[
\langle q' | \varphi(\Gamma) \rangle = \left( \frac{\hbar \pi}{\lambda} \right)^{-1/4} \exp \left[ -\frac{\lambda}{2\hbar} (q' - q)^2 + \frac{i\hbar}{\lambda} (q' - q) \right]
\]

and the width parameter \( \lambda \) is arbitrary. The Husimi function is equivalent to a Wigner function which has been "smoothed" by a Gaussian and the fact that it is everywhere nonnegative can be easily seen from its definition, Eq. (1.4). By insisting on a function that is non-negative in phase space, however, we no longer have a distribution that satisfies all of the classical-like properties of the Wigner function and, in particular, it no longer satisfies Eq. (1.2).\(^{18}\) Furthermore, the Husimi function introduces an arbitrary parameter, \( \lambda \), into the definition of the distribution function which fosters an additional nonuniqueness.

The nonuniqueness of phase space distribution functions is a necessary result of any attempt to map a function on \( \mathcal{L}^2 (1) \), \( \psi(q) \), onto a function on \( \mathcal{L}^2 (2) \), i.e., \( \rho(q,p) \). An alternative approach would be to formulate quantum mechanics directly in an \( \mathcal{L}^2 (2) \) space, such as phase space. Such a formulation would necessarily involve complex-valued functions, rather than real-valued distribution functions, and requires the definition of quantum operators on the \( \mathcal{L}^2 (2) \) space.

In the present paper, we develop and study a formulation of quantum mechanics on phase space. Our purpose in constructing this formulation is to explore the possibility of constructing a viable quantum mechanical representation in this space and, as a byproduct to facilitate a direct comparison between classical and quantum mechanics in the same dynamical space. As a result, our formulation is based on definitions of \( \hat{Q} \) and \( \hat{P} \) in phase space which best satisfy the correspondence between the coordinate, momentum, and phase space representations and between the classical and quantum Liouville equations. From these operators, we can construct the Hamiltonian operator in phase space, and thus the Schrödinger equation governing the time evolution of complex-valued wave functions in phase space. The quantum Liouville equation in phase space then provides a basic time evolution equation for the square magnitude of these wave functions. Consequently, we associate this square magnitude with a probability density whose time evolution can be compared with that of a classical density on phase space.

Time evolution equations for quantum phase space distributions provide a useful basis for the direct comparison of quantum and classical dynamics and can point to the origins of purely quantum effects in the dynamics. The present effort is not the first attempt to define such an equation; however, most earlier studies of time evolving distributions have focused on the Wigner function and its relationship to the time evolution of a classical distribution in phase space.\(^{10,17,19,20,44}\) Similar studies have also been performed for the Husimi function,\(^{27}\) including the recent use of a hydrodynamic-like conservation equation to propagate the Husimi density.\(^{55}\) In addition, the correspondence between classical and quantum mechanics has been investigated semiclassically through the definition of a "dynamical characteristic function" (a phase space distribution function)\(^{34}\) and through an analysis of the classical and quantum Liouville equations.\(^{55}\) Finally, numerous studies have investigated the appearance of phase space structure in quantum states by generating a quantum state, either stationary or nonstationary, within a coordinate representation, then transforming it into a phase space distribution via either Eq. (1.1) or Eq. (1.4).\(^{32,33,38,46,47}\)

Our approach differs from earlier approaches in several ways: (i) We base our conditions for establishing the forms of \( \hat{P} \) and \( \hat{Q} \) on those appropriate for a quantum representation and not on an explicit set of expectations for the classical-quantum correspondence of phase space distributions; (ii) our definition of phase space versions of \( \hat{P} \) and \( \hat{Q} \) will not follow from the definition of a specific phase space distribution function\(^{42}\) (as, e.g., the Weyl correspondence rule for operators\(^{45}\) is associated with the definition of the Wigner function\(^{14,16}\) and, therefore, may be more generally applicable to a wide range of such distribution functions; (iii) we shall avoid consideration of any particular operator correspondence rule by considering operators which depend only on \( \hat{P} \) or only on \( \hat{Q} \), although symmetrization of the operators will naturally arise in our treatment of the quantum Liouville equation; (iv) we focus on the correspondence between quantum operator equations and classical dynamical functions in the establishment of suitable time evolution equations; and (v) our time evolution equation will apply to a quantum density which is the square magnitude of a complex-valued wave function in phase space—a feature more consistent with a quantum phase space "representation" than with the statistical distribution functions used, for example, by Wigner.

The remainder of the paper is organized as follows. In Sec. II, we introduce general forms for the quantum operators \( \hat{P} \) and \( \hat{Q} \) in phase space and show that the properties associated with these general forms is consistent with the usual formulation of quantum mechanics in an \( \mathcal{L}^2 (1) \) space.
Section III is concerned with the derivation of the diagonal matrix elements of the quantum Liouville equation and their classical analogs and probability conservation equations in the coordinate and momentum $L^2$ (1) spaces and in the $L^2$ (2) phase space for the generalized operators introduced in Sec. II. To our knowledge, the equations obtained in the momentum representation and in phase space have not been reported before. These derivations also provide us with guidelines for specifying the forms of $\hat{P}$ and $\hat{Q}$ in phase space which are examined in detail in Sec. IV. In Sec. V, we illustrate the use of the time evolution equations found in the previous sections with an analysis of the dynamics of coherent states and speculate on the possible classical analogs of harmonic oscillator eigenstates in phase space. Finally, we summarize our findings in Sec. VI.

II. THE PHASE SPACE REPRESENTATION

In an analysis of the quantum dynamics of a system, one ordinarily chooses a particular representation of the abstract Hilbert space, and works with dynamical quantities and their time evolution equations in that representation. Here we attempt a similar process that will be appropriate for phase space, thereby defining a phase space representation. The basis vectors of this representation, $|\Gamma\rangle = |p,g\rangle$, are formally the eigenvectors of some Hermitian phase space operator, such that $\hat{f}|\Gamma\rangle = f|\Gamma\rangle$; however, we do not know the identity of $\hat{f}$ or its eigenvalues, nor do we know the form of the eigenvectors. A tempting assumption is that any basis vectors for a phase space representation must be simultaneous eigenvectors of both $\hat{P}$ and $\hat{Q}$, thus violating the uncertainty principle. Since $\hat{f}$ and its eigenkets $|\Gamma\rangle$ are not known, it is possible that they are not eigenkets of either $\hat{P}$ or $\hat{Q}$ individually. We then simply postulate the existence of the eigenvalue equation $\hat{f}|\Gamma\rangle = f|\Gamma\rangle$ for an operator $\hat{f}$ which incorporates the limitations of the uncertainty principle, and analyze the consequences of doing this. As we demonstrate below, it is possible to develop a self-consistent framework in which the simultaneous measurement of $\hat{P}$ and $\hat{Q}$ is impossible.

The phase space basis vectors are postulated to be orthonormal, i.e.

$$\langle \Gamma'|\Gamma \rangle = \delta(\Gamma' - \Gamma), \quad (2.1)$$

in a manner similar to the orthonormalization of the coordinate and momentum representation basis vectors. The projection $\psi(\Gamma) = \langle \Gamma|\psi \rangle$, of the abstract ket $|\psi\rangle$ onto this representation, gives us a complex-valued wave function in phase space, i.e., a wave function with independent variables $p$ and $q$, and the quantity $\psi^* (\Gamma) \psi(\Gamma)$, where $\psi^* (\Gamma) = \langle \psi|\Gamma \rangle = \langle \Gamma|\psi \rangle^*$, represents a kind of probability density corresponding to $|\Gamma\rangle$. This definition ensures that the quantum density $|\psi(\Gamma)|^2$ is a nonnegative quantity in phase space.

We note that the quantum density, $|\psi(\Gamma)|^2$, fulfills the same requirements in phase space as does the probability density, $|\psi(q)|^2$, in the coordinate representation. A given wave function in a coordinate or momentum representation can be transformed into a phase space representation in many different ways, giving rise to a number of different phase space densities [cf. the Husimi parameter $\lambda$ in Eqs. (1.4) and (1.5)]. Thus the quantum density, $|\psi(\Gamma)|^2$, does not have a one-to-one correspondence with either $|\psi(q)|^2$ or $|\psi(p)|^2$ so we cannot assign a unique phase space probability density to a probability density in either coordinate or momentum space. We would face the same kind of problems if we were to generate a classical density $\rho(p,q)$ from a reduced density $\rho(p)$ or $\rho(q)$ and vice versa: It is possible to reduce $\rho(p,q)$ in several different ways. However, $|\psi(\Gamma)|^2$ does satisfy all of the requirements of a probability density, namely, it is greater than zero throughout phase space, $\int d\Gamma |\psi(\Gamma)|^2 = 1$, and the probability densities corresponding to different systems or disjoint regions of phase space are additive. Finally, we note that one cannot recover the coordinate (or momentum) space density by simply integrating $|\psi(\Gamma)|^2$ over $p$ (or $q$); in our scheme, this recovery corresponds to a projection onto a reduced representation of the wave function itself and thus requires the use of a projection function in conjunction with partial integrations over phase space [see Eqs. (4.16)-(4.23) below].

The closure relation for the basis vectors is

$$\hat{f} = \int |\Gamma\rangle d\Gamma \langle \Gamma |,$$

(2.2)

where the integration is carried over the whole phase space. The closure relation can be used in the calculation of the scalar product $\langle \psi_a|\psi_b \rangle$,

$$\langle \psi_a|\psi_b \rangle = \langle \psi_a| \int |\Gamma\rangle d\Gamma \langle \Gamma | \psi_b \rangle$$

$$= \int d\Gamma \psi^*_a(\Gamma) \psi_b(\Gamma). \quad (2.3)$$

As usual, this definition ensures that the length $\langle \psi|\psi \rangle$, of the vector $|\psi\rangle$, is greater or equal to zero, and it would be zero only for the null vector. Equation (2.3) also requires that the wave function be square integrable in phase space or, more precisely, that $|\Gamma\rangle \psi \in L^2 (2)$. This distinguishes the present formulation from the usual form of quantum mechanics in which $|q\psi \rangle \in L^2 (1)$.

The phase space representation contains elements of both the coordinate and momentum representations. For generality, we choose the following forms for $\hat{P}$ and $\hat{Q}$ acting on an arbitrary ket $|\psi\rangle$,

$$\langle \Gamma | \hat{P} | \psi \rangle \equiv \left( \alpha p + i\beta \delta \frac{\partial}{\partial p} \right) \psi(\Gamma), \quad (2.4)$$

and

$$\langle \Gamma | \hat{Q} | \psi \rangle \equiv \left( \alpha q + i\beta \delta \frac{\partial}{\partial q} \right) \psi(\Gamma), \quad (2.5)$$

where $\alpha$, $\beta$, $\gamma$, and $\delta$ are real constants to be determined. As in the Wigner, Husimi and coordinate and momentum representations, the basis vectors $|\Gamma\rangle$ are not uniquely defined and that will be reflected in the nonuniqueness of $\langle \Gamma | \hat{P} | \psi \rangle$ and $\langle \Gamma | \hat{Q} | \psi \rangle$. These operators, and any well-behaved real function of them (by "well-behaved" we
mean continuous with continuous derivatives), are Hermitian in phase space. In particular, the Hamiltonian operator, whose action on \( \langle \Gamma | \psi \rangle \) is given by
\[
\langle \Gamma | \hat{H} | \psi \rangle = \frac{1}{2m} \left( \gamma p + i \hbar \delta \frac{\partial}{\partial q} + \frac{\partial}{\partial q} \right)^2 \psi(\Gamma) + V(\alpha q + i \hbar \beta) \psi(\Gamma),
\]
(2.6)
is also Hermitian, and therefore its eigenvalues are real and its eigenfunctions are orthogonal.

If we perform a time-reversal transformation in which \( t \to -t, p \to -p, q \to -q, \) and the complex conjugate is taken, we find that the time-reversal properties of \( \hat{P} \) and \( \hat{Q} \) are
\[
\left( \gamma p + i \hbar \delta \frac{\partial}{\partial q} \right) \to - \left( \gamma p + i \hbar \delta \frac{\partial}{\partial q} \right),
\]
(2.7)
and that the Schrödinger equation
\[
i \hbar \frac{\partial}{\partial t} \langle p, q | \psi \rangle
\]
(2.8)
transforms to
\[
i \hbar \frac{\partial}{\partial t} \langle -p, q | \psi_{-}\rangle^*
\]
(2.9)
transforms to
\[
i \hbar \frac{\partial}{\partial t} \langle p, q | \psi \rangle
\]
(2.10)
transforms to
\[
i \hbar \frac{\partial}{\partial t} \langle -p, q | \psi_{-}\rangle^*
\]
(2.11)
Hence, as we require, the operators \( \hat{P}, \hat{Q}, \) and \( \hat{H} \) have the correct time-reversal symmetry, and if \( \langle p, q | \psi_i \rangle \) is a solution of the Schrödinger equation, then \( \langle -p, q | \psi_{-i} \rangle \) is also a solution.

Among the transformation properties of the operators appearing in Eqs. (2.4) and (2.5) are
\[
e^{i \gamma q \hbar / \hbar} (\gamma p + i \hbar \delta \frac{\partial}{\partial q}) e^{-i \gamma q \hbar / \hbar} = \alpha q + i \hbar \beta \frac{\partial}{\partial p},
\]
(2.12)
and
\[
e^{i \gamma q \hbar / \hbar} (i \hbar \delta \frac{\partial}{\partial q}) e^{-i \gamma q \hbar / \hbar} = \gamma p + i \hbar \delta \frac{\partial}{\partial q}.
\]
(2.13)
We can make use of these properties in examine the effect of the operators \( e^{i \gamma q \hbar / \hbar} \) and \( e^{i \gamma q \hbar / \hbar} \) on the basis vector \( | \Gamma \rangle \). For any ket \( | a \rangle \), we find that
\[
e^{i \gamma q \hbar / \hbar} | a \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i}{\hbar} \right)^n | \Gamma \rangle d\Gamma \left( \alpha q + i \hbar \beta \frac{\partial}{\partial p} \right)^n | a \rangle
\]
(2.14)
where the last equality follows from the use of Eq. (2.11) above. However, the sum that appears on the right-hand side of Eq. (2.13) is just the Taylor series in powers of \( (-\xi \beta) \) of the function \( \exp[-i \gamma q (p - \xi \beta) / \hbar] \langle p - \xi \beta, q | a \rangle \), then
\[
e^{i \gamma q \hbar / \hbar} | a \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i}{\hbar} \right)^n e^{-i \gamma q (p - \xi \beta) / \hbar} | p - \xi \beta, q | a \rangle
\]
(2.15)
whereas, \( e^{i \gamma q \hbar / \hbar} | q \rangle = | q - \eta \rangle \) in the coordinate representation, and \( e^{i \gamma q \hbar / \hbar} | p \rangle = e^{i \alpha q / \hbar} | p \rangle \) in the momentum representation. These results reinforce the idea that the phase space representation contains elements of both the coordinate and momentum representations.

We can also solve for the eigenkets of the \( \hat{P} \) and \( \hat{Q} \) operators using the eigenvalue equations
\[
\hat{P} | u_p \rangle = p | u_p \rangle,
\]
(2.16)
and
\[
\hat{Q} | u_q \rangle = q | u_q \rangle,
\]
(2.17)
projected onto the \( | \Gamma \rangle \) basis. For the first eigenvalue equation, we have
\[
\langle \Gamma | u_p \rangle = \frac{1}{\sqrt{2\pi \hbar}} e^{-i q (p' - \gamma p') / \hbar},
\]
(2.18)
which has normalized solutions of the form
\[
| u_p \rangle = \frac{1}{\sqrt{2\pi \hbar}} e^{-i q (p' - \gamma p') / \hbar}.
\]
(2.19)
Similarly, the eigenvalue equation for \( \hat{Q} \),
\[
\langle \Gamma | u_q \rangle = q | u_q \rangle,
\]
(2.20)
has normalized solutions
\[
| u_q \rangle = \frac{1}{\sqrt{2\pi \hbar}} e^{-i q (q' - \alpha q) / \hbar}.
\]
(2.21)
Since we have not yet placed any restrictions on the values of the free parameters \( \alpha, \beta, \gamma, \) and \( \delta \), it is clear that we can choose them so that there are no simultaneous eigenkets of the coordinate and momentum operators. Later, we will
incorporate the commutation relation \[ \{\hat{Q}, \hat{P}\} = i\hbar \] in defining a suitable choice of these parameters, thereby enforcing the uncertainty principle on phase space operators constructed from \( \hat{P} \) and \( \hat{Q} \).

We note that, although we can write down expressions for the eigenkets of \( \hat{P} \) and \( \hat{Q} \) in the \(| \Gamma \rangle \) basis, these matrix elements do not represent transition matrix elements for effecting a change in representation between \(| \Gamma \rangle \) and \(| \rho \rangle \) or \(| q \rangle \). In this case, \(| u_\rho \rangle \) and \(| u_q \rangle \) are not strictly equivalent to \(| \rho \rangle \) and \(| q \rangle \), respectively, because the former two kets are applicable on an \( \mathcal{L}^2 \) space of vectors and the latter two are defined for \( \mathcal{L}^2 \) (1). Actually, since none of these kets is square integrable in the usual sense, they are not themselves elements of \( \mathcal{L}^2 \) (1) or \( \mathcal{L}^2 \) (2), but can act as basis vectors on these function spaces. Thus quantities such as \( \langle q | u_q \rangle \), \( \langle p | u_\rho \rangle \), etc., are ill-defined because they represent inner products of vectors with different dimensions.

Now, we can use the Hamiltonian operator \( \hat{H} \) to construct a Schrödinger equation in phase space that governs the time evolution of complex wave functions in that space, as in Eq. (2.9). Another way of writing Eq. (2.9) is obtained by making use of the identities

\[
\begin{align*}
g(\alpha q + i\hbar \beta \frac{\partial}{\partial p}) f(p, q) &= g(\alpha q) \exp \left[ i\hbar \delta \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} + i\hbar \beta \frac{\partial}{\partial (\alpha q)} \frac{\partial}{\partial p} \right] f(p, q), \\
h(\gamma p + i\hbar \delta \frac{\partial}{\partial q}) f(p, q) &= h(\gamma p) \exp \left[ i\hbar \delta \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} + i\hbar \beta \frac{\partial}{\partial (\alpha q)} \frac{\partial}{\partial p} \right] f(p, q).
\end{align*}
\]

where \( g \) and \( h \) are functions that can be written as a power series in their arguments (see Appendix A), and the definition

\[
A(p, q) \exp \left[ i\hbar \delta \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} + i\hbar \beta \frac{\partial}{\partial (\alpha q)} \frac{\partial}{\partial p} \right] B(p, q) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \left(i\hbar \delta\right)^m (i\hbar \beta)^n A(p, q) \frac{\partial^{m+n}}{\partial (\gamma p)^m \partial (\alpha q)^n} B(p, q),
\]

for any functions \( A(p, q) \) and \( B(p, q) \). Hence, the Schrödinger equation in phase space becomes

\[
\begin{align*}
\frac{\partial}{\partial t} \langle \Gamma | \psi \rangle &= H(\gamma p, \alpha q) \exp \left[ i\hbar \delta \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} + i\hbar \beta \frac{\partial}{\partial (\alpha q)} \frac{\partial}{\partial p} \right] \langle \Gamma | \psi \rangle, \\
&= \left[i\hbar (\partial / \partial q) \sum_{n=0}^{\infty} \frac{1}{n!} \left( i\hbar \delta \right)^n \frac{\partial}{\partial (\gamma p)} \right] \langle \psi | H | \psi \rangle.
\end{align*}
\]

This last form will prove to be convenient for comparing the quantum time evolution of densities on phase space with that of classical mechanics in a later section.

### III. QUANTUM LIOUVILLE EQUATION

In this section, we derive the diagonal matrix elements of the abstract quantum Liouville equation

\[
\frac{\partial}{\partial t} \rho_i = \frac{i}{\hbar} [\rho_i, \hat{H}],
\]

in each of the coordinate, momentum and phase space representations and discuss their classical analogs. Although the probability conservation equation in the coordinate representation is well known, along with its derivation, some of the ideas that will be needed later in the construction of the phase space version of Eq. (3.1) can be introduced. To our knowledge, the momentum and phase space versions of the probability conservation equation have not been previously reported.

#### A. Coordinate representation

The diagonal matrix elements of Eq. (3.1), in the coordinate representation are given by

\[
\frac{\partial}{\partial t} \langle q | \rho_i | q \rangle = \frac{i}{\hbar} \langle q | [\rho_i, \hat{H}] | q \rangle
\]

\[
= \frac{i}{2\hbar} \left( \langle q | \hat{P}^2 \rho_i | q \rangle - \langle q | \hat{P} \rho_i \hat{P}^\dagger | q \rangle \right)
\]

\[
+ \frac{i}{\hbar} \left( \langle q | \hat{P} \rho_i \hat{P}^\dagger | q \rangle - \langle q | \hat{P}^\dagger \hat{P} \rho_i | q \rangle \right).
\]

We can rewrite the matrix element

\[
\langle q | \rho_i \hat{P}^2 | q \rangle - \langle q | \hat{P}^2 \rho_i | q \rangle
\]

by making use of the following results. For any two kets \(| \chi \rangle \) and \(| \Psi \rangle \), we have that

\[
\langle q | \chi \rangle \langle \Psi | \hat{P} | q \rangle = i\hbar \langle q | \chi \rangle \frac{\partial}{\partial q} \langle \Psi | q \rangle
\]

\[
= -i\hbar \frac{\partial \langle q | \chi \rangle}{\partial q} \langle \Psi | q \rangle + \hbar \frac{\partial}{\partial q} \langle q | \chi \rangle \langle \Psi | q \rangle
\]

\[
= \langle q | \hat{P} \chi \rangle \langle \Psi | q \rangle + \hbar \frac{\partial}{\partial q} \langle q | \chi \rangle \langle \Psi | q \rangle.
\]

(3.3)

Now, one can make use of Eq. (3.3) as many times as needed in order to obtain

\[
\langle q | \chi \rangle \langle \Psi | \hat{P}^n | q \rangle - \langle q | \hat{P}^n \chi \rangle \langle \Psi | q \rangle
\]

\[
= \left[ i\hbar \frac{\partial}{\partial q} \sum_{n=0}^{\infty} \frac{1}{n!} \left( i\hbar \delta \right)^n \frac{\partial}{\partial (\gamma p)} \right] \langle \psi | H | \psi \rangle.
\]

(3.4)

Since the density operator \( \rho_n \) in general, can be written as the sum of terms of the form \(| \chi \rangle \langle \Psi | \), Eq. (3.4) implies that
\[ \langle q | [ \hat{\rho}_n \hat{P} ] | q \rangle \]
\[ = \begin{cases} i \hbar \frac{\partial}{\partial q} \sum_{n=0}^{\infty} \langle q | \hat{P}_n \hat{P}^{n-1} | q \rangle, & \text{if } n > 0 \\ 0, & \text{if } n = 0. \end{cases} \quad (3.5) \]

Then, substituting Eq. (3.5) into Eq. (3.2), using \( n = 2 \), and recalling that \( \langle q | \hat{\rho}_n \hat{V}(Q) | q \rangle - \langle q | \hat{V}(Q) \hat{\rho}_n | q \rangle = 0 \), one obtains the following diagonal matrix element of the quantum Liouville equation
\[ \frac{\partial}{\partial t} \langle q | \hat{\rho}_n | q \rangle = - \frac{1}{m} \frac{\partial}{\partial q} \left[ \langle q | \hat{\rho}_n \hat{P} | q \rangle + \langle q | \hat{P} \hat{\rho}_n | q \rangle \right]. \quad (3.6) \]

This equation shows the way in which the momentum in the system affects the time evolution of the coordinate probability density, and is reminiscent of the coordinate part of the classical Liouville equation. If the classical Liouville equation,
\[ \frac{\partial}{\partial t} \rho(r; t) = - \frac{m}{\hbar} \frac{\partial}{\partial p} \left[ \rho(r; t) + F(q) \rho(r; t) \right], \quad (3.7) \]
is integrated over \( p \), and one assumes that the probability density \( \rho(r; t) \) vanishes at \( p = \pm \infty \), then
\[ \frac{\partial}{\partial t} \int dp \rho(\Gamma; t) = - \frac{1}{m} \frac{\partial}{\partial q} \int dp \rho(\Gamma; t). \quad (3.8) \]

Hence, terms like the ones appearing on the right-hand side of Eq. (3.6) are expected since all the quantum analogs to \( PP(r; t) \) have to be taken into account [this is basically a symmetrization of the product \( PP(\Gamma; t) \)]. As was noted in Ref. 43 for the Schrödinger equation, Eqs. (3.6) and (3.8) suggest that the coordinate representation involves quantities whose momentum dependence has been averaged out.

A more explicit form of Eq. (3.6) can be obtained for density operators of the form \( \hat{\rho}_n = \sum_n \langle \psi_{n,t} | \psi_{n,t} \rangle \). In this case,
\[ \langle q | \hat{\rho}_n \hat{P} | q \rangle + \langle q | \hat{P} \hat{\rho}_n | q \rangle = \sum_n \left( \langle q | \psi_{n,t} | \psi_{n,t} \rangle \langle q | \hat{P} | q \rangle + \langle q | \hat{P} | \psi_{n,t} \rangle \langle q | \psi_{n,t} | q \rangle \right) \]
\[ = \sum_n \left( \langle q | \psi_{n,t} | \psi_{n,t} \rangle \left( i \hbar \frac{\partial}{\partial q} \langle q | \psi_{n,t} | q \rangle + \left( -i \hbar \frac{\partial}{\partial q} \langle q | \psi_{n,t} | q \rangle \right) \right) \right. \]
\[ = \sum_n \left( \langle q | \psi_{n,t} | \psi_{n,t} \rangle \left( i \hbar \frac{\partial}{\partial q} \langle q | \psi_{n,t} | q \rangle - \langle q | \psi_{n,t} | q \rangle \right) \right). \quad (3.9) \]

Now, Eq. (3.6) takes the form
\[ \frac{\partial}{\partial t} \langle q | \hat{\rho}_n | q \rangle = - \frac{m}{\hbar} \frac{\partial}{\partial q} \left[ \langle q | \hat{P} \hat{\rho}_n + \hat{\rho}_n \hat{P} \rangle \right] \]
\[ = \frac{\partial}{\partial q} \left( \frac{\partial}{\partial q} \right) \left( \frac{\partial}{\partial q} \langle q | \phi_n | q \rangle \right) \quad . \quad (3.10) \]

B. Momentum representation

The diagonal matrix elements of Eq. (3.1), in the momentum representation, can be written as
\[ \frac{\partial}{\partial t} \langle p | \hat{\rho}_n | p \rangle = \frac{i}{\hbar} \langle p | \left[ \hat{\rho}_n, \hat{H} \right] | p \rangle \]
\[ = \frac{i}{2\hbar m} \left[ \langle p | \hat{P} \hat{P}^2 | p \rangle - \langle p | \hat{P}^2 \hat{P} | p \rangle \right] \]
\[ + \frac{i}{\hbar} \left[ \langle p | \hat{\rho}_n \hat{V}(Q) | p \rangle - \langle p | \hat{V}(Q) \hat{\rho}_n | p \rangle \right]. \quad (3.11) \]

This time, the matrix elements \( \langle p | \hat{\rho}_n \hat{V}(Q) | p \rangle - \langle p | \hat{V}(Q) \hat{\rho}_n | p \rangle \) can be recast by making use of some results involving the arbitrary kets \( | \chi \rangle \) and \( | \Psi \rangle \);
\[ \langle p | \chi \rangle \langle \Psi | \hat{V}^* | p \rangle = -i \hbar \langle p | \chi \rangle \frac{\partial}{\partial p} \langle \Psi | p \rangle \]
\[ = i \hbar \frac{\partial}{\partial p} \langle \Psi | p \rangle \langle p | \chi \rangle \langle \Psi | p \rangle \]
\[ = \langle p | \hat{V}(Q) \langle \Psi | p \rangle - i \hbar \frac{\partial}{\partial p} \langle p | \chi \rangle \langle \Psi | p \rangle. \quad (3.12) \]

Using Eq. (3.12) as many times as needed, one finds
\[ \langle p | \chi \rangle \langle \Psi | \hat{V}^* | p \rangle - \langle p | \hat{V}^* \hat{\rho}_n | p \rangle \]
\[ = \begin{cases} -i \hbar \frac{\partial}{\partial p} \sum_{n=0}^{n-1} \langle p | \hat{V}(Q) \langle \Psi | \hat{V}^* | p \rangle, & \text{if } n > 0 \\ 0, & \text{if } n = 0. \end{cases} \quad (3.13) \]

Thus if the potential function \( V(q) \) can be written as a power series in \( q \), \( V(q) = \sum_{n=0}^{\infty} V_n q^n \), then
\[
\langle p|\hat{Q}|\hat{P}|p\rangle - \langle p|\hat{P}|\hat{Q}|p\rangle = -\hat{H} \frac{\partial}{\partial p} \sum_{n=1}^{\infty} V_n \sum_{l=0}^{n-1} \langle p|\hat{Q}|\chi\rangle \langle \chi|\hat{P}^{n-l-1}|p\rangle,
\]

which implies that

\[
\langle p|\hat{Q}|\hat{P}|p\rangle - \langle p|\hat{P}|\hat{Q}|p\rangle = -i\hbar \frac{\partial}{\partial p} \sum_{n=1}^{\infty} V_n \sum_{l=0}^{n-1} \langle p|\hat{Q}|\chi\rangle \langle \chi|\hat{P}^{n-l-1}|p\rangle.
\]

Note that the summation over \(n\) starts at \(n = 1\). Upon substitution of Eq. (3.15) into Eq. (3.11), and noting that

\[
(\hat{P}\hat{P}|p) - (\hat{P}|\hat{P}p)|p\rangle = 0,
\]

the diagonal matrix element of the quantum Liouville equation becomes

\[
\frac{\partial}{\partial t} \langle p|\hat{P}|p\rangle = \frac{\partial}{\partial p} \sum_{n=1}^{\infty} V_n \sum_{l=0}^{n-1} \langle p|\hat{Q}|\chi\rangle \langle \chi|\hat{P}^{n-l-1}|p\rangle.
\]

This equation demonstrates the effect of the potential of the system on the time evolution of the momentum probability density, and is reminiscent of the momentum part of the classical Liouville equation. If we integrate the classical Liouville equation, Eq. (3.7), over \(q\) and assume that the probability density \(p(q;\tau)\) vanishes at \(q = \pm \infty\), then

\[
\frac{\partial}{\partial t} \int dq \rho(\Gamma;\tau) = -\frac{\partial}{\partial p} \int dq F(q) \rho(\Gamma;\tau).
\]

Again, terms such as those appearing on the right-hand side of Eq. (3.16) arise due to the symmetrization of the classical product \(q^n\rho(\Gamma;\tau)\). In the present development, Eqs. (3.16) and (3.17) suggest that the momentum representation of quantum mechanics involves quantities whose coordinate dependence has somehow been averaged out.

For a density operator of the form \(\hat{\rho}_\Gamma = \sum_n |\chi_{n,\tau}\rangle \langle \psi_{n,\tau}|\), one finds that

\[
\langle p|\hat{Q}|\hat{P}|p\rangle = \sum_n \left\{ \left( i\hbar \frac{\partial}{\partial p} \right)^n \langle p|\chi_{n,\tau}\rangle \right\} \left\{ \left( -i\hbar \frac{\partial}{\partial p} \right)^{n-1} \langle \psi_{n,\tau}|p\rangle \right\}
\]

so Eq. (3.16) becomes

\[
\frac{\partial}{\partial t} \langle p|\hat{P}|p\rangle = \frac{\partial}{\partial p} \sum_{n=1}^{\infty} V_n \sum_{l=0}^{n-1} \langle p|\hat{Q}|\chi\rangle \langle \chi|\hat{P}^{n-l-1}|p\rangle.
\]

This equation indicates the effect of both the momentum and the potential of the system on the time evolution of the quantum density in phase space. Note that this equation is very similar to the classical Liouville equation [see Eq. (3.7) and Sec. IV], and provides a novel description of quantum systems in terms of \(p\) and \(q\).

We can make use of the Hermitian properties of \(\hat{P}, \hat{Q}\), and \(\hat{\rho}_\Gamma\) to show that the probability current terms in Eq. (3.23) are real. We find that

\[
\langle \Gamma|\hat{P}|\hat{P}_\Gamma\rangle + \langle \Gamma|\hat{P}_\Gamma|\hat{P}\rangle = 2 \Re \langle \Gamma|\hat{P}_\Gamma|\hat{P}\rangle
\]

and

\[
\text{C. Phase space representation}
\]

Finally, in the phase space representation, the diagonal matrix elements of Eq. (3.1) are given by

\[
\frac{\partial}{\partial t} \langle \Gamma|\hat{P}_\Gamma|\Gamma\rangle = \frac{i}{\hbar} \left[ \langle \Gamma|\hat{P}_\Gamma|\hat{P}\rangle - \langle \Gamma|\hat{P}^2|\Gamma\rangle \right]
\]

and

\[
\langle \Gamma|\hat{P}_\Gamma|\hat{P}\rangle - \langle \Gamma|\hat{P}|\hat{P}_\Gamma|\Gamma\rangle = \frac{i}{2\hbar m} \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \langle \Gamma|\hat{P}_\Gamma|\hat{P}^{n-l-1}|\Gamma\rangle.
\]

Following a similar procedure to the one that led to Eqs. (3.6) and (3.16), one can show that [see Eqs. (2.4) and (2.5) and Appendix B]

\[
\langle \Gamma|\hat{P}_\Gamma|\hat{P}\rangle - \langle \Gamma|\hat{P}^2|\Gamma\rangle
\]

and

\[
\langle \Gamma|\hat{P}_\Gamma|\hat{P}\rangle - \langle \Gamma|\hat{P}|\hat{P}_\Gamma|\Gamma\rangle
\]

Substituting Eqs. (3.21) (with \(n = 2\)) and Eq. (3.22) into Eq. (3.20), the diagonal matrix element of the quantum Liouville equation becomes

\[
\frac{\partial}{\partial t} \langle \Gamma|\hat{P}_\Gamma|\Gamma\rangle = \frac{\delta}{\partial q} \sum_{n=1}^{\infty} V_n \sum_{l=0}^{n-1} \langle \Gamma|\hat{Q}_\Gamma|\hat{P}^{n-l-1}|\Gamma\rangle.
\]

This equation indicates the effect of both the momentum and the potential of the system on the time evolution of the quantum density in phase space. Note that this equation is very similar to the classical Liouville equation [see Eq. (3.7) and Sec. IV], and provides a novel description of quantum systems in terms of \(p\) and \(q\).
For a density operator of the form \( \rho_i = \sum_n |\psi_{n,i}\rangle \langle \psi_{n,i}| \) we have

\[
\langle \Gamma | \hat{P}^m \rho_i \hat{P}^i | \Gamma \rangle = \left[ \left( \chi' + i \delta \frac{\partial}{\partial q'} \right)^m \left( \chi - i \delta \frac{\partial}{\partial p} \right)^i \langle \Gamma' | \rho_i | \Gamma \rangle \right]_{\Gamma' = \Gamma},
\]

and

\[
\langle \Gamma | \hat{Q}^m \rho_i \hat{Q}^i | \Gamma \rangle = \left[ \left( \alpha q' + i \epsilon \frac{\partial}{\partial p'} \right)^m \left( \alpha p - i \epsilon \frac{\partial}{\partial p} \right)^i \langle \Gamma' | \rho_i | \Gamma \rangle \right]_{\Gamma' = \Gamma},
\]

which allows us to rewrite Eq. (3.23) in a more explicit form

\[
\frac{\partial}{\partial t} \langle \Gamma | \rho_i | \Gamma \rangle = \frac{\partial}{\partial q m} \Re \left[ \left( \chi + i \delta \frac{\partial}{\partial q} \right) \langle \Gamma | \rho_i | \Gamma' \rangle \right]_{\Gamma' = \Gamma}
\]

\[
+ \frac{\partial}{\partial p} \beta \sum_{n=1}^{\infty} V_n \sum_{l=0}^{n-1} \left[ \left( \alpha q' + i \epsilon \frac{\partial}{\partial p'} \right)^l \left( \alpha p - i \epsilon \frac{\partial}{\partial p} \right)^{n-l-1} \langle \Gamma' | \rho_i | \Gamma \rangle \right]_{\Gamma' = \Gamma}.
\]

This form of the quantum Liouville equation allows us to investigate its time-reversal properties. The time-reversal transformation is carried out by making the replacements \( p \rightarrow -p, \ q \rightarrow q, \ t \rightarrow -t \), and by taking the complex conjugate. When this is done, Eq. (3.28) is transformed to an equation with exactly the same form but with \( \langle -p,q | \hat{P}_{-i} | -p,q \rangle \) replacing \( \langle p,q | \hat{P}_i | p,q \rangle \). Therefore, if \( \langle p,q | \hat{P}_i | p,q \rangle \) is a solution to the quantum probability conservation equation, \( \langle -p,q | \hat{P}_{-i} | -p,q \rangle \) is also a solution.

Equation (3.28) may be expressed in an equivalent form by using Eqs. (2.22)–(2.24) with the result

\[
\frac{\partial}{\partial t} \langle \Gamma | \rho_i | \Gamma \rangle = \frac{1}{\hbar} H(\gamma p, \alpha q) \left[ \exp \left[ i \hbar \delta \frac{\partial}{\partial q} \frac{\partial}{\partial q} + i \epsilon \frac{\partial}{\partial p} \frac{\partial}{\partial p} \right] \langle \Gamma | \rho_i | \Gamma' \rangle \right. \\
- \exp \left[ -i \hbar \delta \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} - i \epsilon \frac{\partial}{\partial (\alpha q)} \frac{\partial}{\partial p} \right] \langle \Gamma' | \rho_i | \Gamma \rangle \left. \right]_{\Gamma' = \Gamma}
\]

\[
= \frac{2}{\hbar} H(\gamma p, \alpha q) \Im \left[ \exp \left[ i \hbar \delta \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} + i \epsilon \frac{\partial}{\partial (\alpha q)} \frac{\partial}{\partial p} \right] \langle \Gamma | \rho_i | \Gamma' \rangle \right. \\
- \left[ 1 - i \hbar \delta \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} - i \epsilon \frac{\partial}{\partial (\alpha q)} \frac{\partial}{\partial p} \right] \langle \Gamma' | \rho_i | \Gamma \rangle \left. \right]_{\Gamma' = \Gamma}
\]

We note that if the evaluation \( \Gamma' = \Gamma \) is made before applying any operator in the above equation, a time evolution equation similar to that for the Wigner density is obtained. Furthermore, if we expand the exponential and keep terms only up to first order in \( \hbar \), we can write down an approximate form of Eq. (3.29)

\[
\frac{\partial}{\partial t} \langle \Gamma | \rho_i | \Gamma \rangle \approx \frac{1}{\hbar} H(\gamma p, \alpha q) \left[ \left[ 1 + i \hbar \delta \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} + i \epsilon \frac{\partial}{\partial (\alpha q)} \frac{\partial}{\partial p} \right] \langle \Gamma | \rho_i | \Gamma' \rangle \right. \\
- \left[ 1 - i \hbar \delta \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} - i \epsilon \frac{\partial}{\partial (\alpha q)} \frac{\partial}{\partial p} \right] \langle \Gamma' | \rho_i | \Gamma \rangle \left. \right]_{\Gamma' = \Gamma}
\]

\[
= \left[ \frac{\partial H(\gamma p, \alpha q)}{\partial p} \frac{\partial}{\partial p} + \frac{\partial H(\gamma p, \alpha q)}{\partial q} \frac{\partial}{\partial q} \right] 2 \Re \langle \Gamma | \rho_i | \Gamma' \rangle \right]_{\Gamma' = \Gamma}
\]
which can be compared with the classical Liouville equation

$$\frac{\partial}{\partial t} \rho_{\text{cl}}(\Gamma,t) = -\left[ \frac{\partial H(p,q)}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H(p,q)}{\partial q} \frac{\partial}{\partial p} \right] \rho_{\text{cl}}(\Gamma,t).$$  \hfill (3.31)

Of course, we should be careful not to draw extensive conclusions using the above approximation since the dependence of $\langle \Gamma | \hat{p}_i | \Gamma \rangle$ upon $\hbar$ has not been taken into account. As a result, the classical Liouville equation cannot legitimately be recovered in this way, as was shown for the coherent state in the harmonic oscillator potential in Ref. 43.

IV. MOMENTUM AND POSITION OPERATORS IN PHASE SPACE

The general form of the quantum Liouville equation in phase space, Eq. (3.23), now provides the first criterion to be used in determining appropriate values for the parameters $\alpha$, $\beta$, $\gamma$, and $\delta$. In particular, $\beta$ and $\delta$ are coefficients for the flux terms in Eq. (3.23). Comparing this equation with the classical Liouville equation, Eq. (3.7), as well as the quantum Liouville equations in the coordinate and momentum $L^2(1)$ representations, Eqs. (3.6) and (3.16), respectively, we find that the best correspondence is obtained for the values $\beta=1$ and $\delta=-1$. Interestingly, these same values are implied by coordinate and momentum representations [see Eqs. (2.15) and (2.16) and the discussion following]. Thus these values for $\beta$ and $\delta$ provide internal consistency for correspondences between quantum phase space and quantum coordinate and momentum spaces, as well as classical phase space.

We can establish another criterion for determining values for $\alpha$, $\beta$, $\gamma$, and $\delta$ by requiring that the quantum commutator $[Q, \hat{P}] = i\hbar \hat{F}$ in the phase space representation. For an arbitrary ket $|\psi\rangle$ we have

$${\langle \Gamma \mid [\hat{Q}, \hat{P}] \mid \psi \rangle} = \left( \frac{p}{2} - i\hbar \frac{\partial}{\partial q} \right) {\langle \Gamma \mid \psi \rangle}$$

and

$${\langle \Gamma \mid \hat{Q} \mid \psi \rangle} = \left( \frac{q}{2} + i\hbar \frac{\partial}{\partial p} \right) {\langle \Gamma \mid \psi \rangle}.$$  \hfill (4.1)

This proposed form for the operators $\hat{P}$ and $\hat{Q}$ provides the foundation for a “standard” or “canonical” phase space representation. The time evolution equations for complex wave functions (Schrödinger) and real-valued densities (Liouville) now are written

$$i\hbar \frac{\partial}{\partial t} \langle \Gamma | \psi \rangle = H \left[ \left( \frac{p}{2} - i\hbar \frac{\partial}{\partial q} \right) \left( \frac{q}{2} + i\hbar \frac{\partial}{\partial p} \right) \right] \langle \Gamma | \psi \rangle$$

\quad \times \langle \Gamma | \psi \rangle.$$  \hfill (4.6)
and
\[ \frac{\partial}{\partial t} \langle \Gamma | \hat{\rho}_t | \Gamma \rangle = \frac{\partial}{\partial q} \frac{1}{2m} \left[ \langle \Gamma | \hat{\rho}_t | \Gamma \rangle + \langle \Gamma | \hat{\rho} \hat{\rho}_t | \Gamma \rangle \right] + \frac{\partial}{\partial p} \sum_{n=1}^{\infty} V_n \sum_{l=0}^{n-1} \langle \Gamma | \hat{Q}_l \hat{Q}^{n-l-1} | \Gamma \rangle, \tag{4.7} \]

respectively. In Table I, we summarize our proposal for this standard phase space representation.

Other forms for \( \hat{P} \) and \( \hat{Q} \) in phase space have previously been proposed.\textsuperscript{10,17,35,42} In one definition of the Wigner equivalents of quantum operators due to Bopp,\textsuperscript{10,17,42} the operators \([p - (i\hbar/2) \partial/\partial q] \text{ and } [q + (i\hbar/2) \partial/\partial p]\) are used.\textsuperscript{42} This choice leads to an extra factor of 1/2 on the right-hand side of Eq. (4.7), which is in disagreement with Eqs. (3.6), (3.7), and (3.16), as well as the translation generator equations in Eqs. (2.15) and (2.16). In addition, the set of values \( \alpha = 1/2 \) and \( \beta = -1 \) with \( \gamma \) undefined, was used by Skodje and coworkers\textsuperscript{35} in a hydrodynamiclike probability conservation equation in the coherent state representation though it is difficult and perhaps inappropriate to compare their results with the present formulation. Recently, in Ref. 43, the values \( \beta = 1, \delta = -1 \) and other values (than 1/2) for \( \alpha \) and \( \gamma \) were used. The only difference between that choice and the proposed standard form for these operators is the implicit definition of a phase factor for the wave function. This can be easily seen if we consider the transformation properties
\[ e^{-i \alpha \rho / \hbar} \left[ \left( \frac{1}{2} - \lambda \right) p - i \hbar \frac{\partial}{\partial q} \right] e^{i \alpha \rho / \hbar} = \frac{p - i \hbar \frac{\partial}{\partial q}}{2} \tag{4.8} \]
and
\[ e^{-i \alpha \rho / \hbar} \left[ \left( \frac{1}{2} + \lambda \right) q + i \hbar \frac{\partial}{\partial p} \right] e^{i \alpha \rho / \hbar} = \frac{q + i \hbar \frac{\partial}{\partial p}}{2}. \tag{4.9} \]

Since the quantity of interest is the quantum density which is given by the square magnitude of the wave function, all the equations
\[ i \hbar \frac{\partial}{\partial t} \psi(\Gamma; t) = \frac{1}{2m} \left[ \left( \frac{1}{2} - \lambda \right) p - i \hbar \frac{\partial}{\partial q} \right]^2 \psi(\Gamma; t) + V \left[ \left( \frac{1}{2} + \lambda \right) q + i \hbar \frac{\partial}{\partial p} \right] \psi(\Gamma; t), \tag{4.10} \]
are dynamically equivalent for \(-1/2 < \lambda < 1/2\). In the case of the standard forms for \( \hat{P} \) and \( \hat{Q} \), we have simply chosen \( \lambda = 0 \). Similar changes in the implicit phase factor of the wave function have the effect of changing the forms of \( \hat{P} \) and \( \hat{Q} \) in the standard coordinate and momentum \( \mathcal{L}^2 \) (1) representation, also.\textsuperscript{49}

An interesting point to note is that the set of values \( \alpha = 1 = -\delta, \beta = \gamma = 0 \) gives the standard forms of \( \hat{P} \) and \( \hat{Q} \) in the coordinate representation while the set \( \alpha = \delta = 0, \beta = \gamma = 1 \) does the same for the momentum representation. Nevertheless, one should be careful not to confuse the general forms given for \( \hat{P} \) and \( \hat{Q} \) in Eqs. (2.4) and (2.5) which are meant to be used with \( \mathcal{L}^2 \) (2) representations. The result that both \( \beta \) and \( \delta \) are nonzero for the \( \mathcal{L}^2 \) (2) phase space representation is significant in that it causes both \( \hat{P} \) and \( \hat{Q} \) to be nonlocal operators in that representation. This means that the measurement of \( \hat{P} \) or \( \hat{Q} \) cannot be determined by knowledge of the wave function at a single point in phase space which, in turn, implies that one cannot have simultaneous knowledge of both the position and momentum of the system.

This point becomes clearer when we reconsider the eigenkets of \( \hat{P} \) and \( \hat{Q} \) using the standard forms of those operators. These eigenkets become [see Eqs. (2.19) and (2.21)]
\[ \langle \Gamma | u_p \rangle = \frac{1}{2\pi \hbar} e^{i \alpha \rho (p' - p)/\hbar} \tag{4.11} \]
and
\[ \langle \Gamma | u_q \rangle = \frac{1}{2\pi \hbar} e^{-i \alpha \rho (q' - q)/\hbar} \tag{4.12} \]
respectively, when \( \alpha = \gamma = \frac{1}{2} \) and \( \beta = \delta = -1 \), and one can easily confirm that there are no simultaneous eigenstates of \( \hat{P} \) and \( \hat{Q} \) for these definitions. In quantum mechanics, the measurement of a dynamical quantity is represented mathematically by the action of the associated operator on the ket describing the state of the system, \( \langle \psi | \psi \rangle \). The result of such a measurement is one of the eigenvalues of that operator and after the measurement, the system is described by its corresponding eigenket.

For example, if we measure the position of the system, \( \langle \Gamma | \psi \rangle \) reduces to the eigenstate \( \langle \Gamma | u_q \rangle \) of \( \hat{Q} \) with probability given by \( | \langle u_q | \psi \rangle |^2 \). If we then measure the momentum of the system, we need to express the new state \( \langle \Gamma | u_p \rangle \) as an expansion in the eigenstates of \( \hat{P} \),
\[ \langle \Gamma | u_p \rangle = \int dp' \langle \Gamma | u_{p'} \rangle \langle u_{p'} | u_p \rangle, \tag{4.13} \]
where
\[ \langle u_{p'} | u_{p'} \rangle = \int d\Gamma \langle u_{p'} | \Gamma \rangle \langle \Gamma | u_{p'} \rangle = \int d\Gamma \left( \frac{1}{2\pi \hbar} \right)^2 e^{-i \alpha \rho (p' - p)/\hbar} e^{-i \alpha \rho (q' - q)/2/\hbar} = \frac{1}{2\pi \hbar} e^{-i \alpha \rho (q' - q)/2/\hbar}. \tag{4.14} \]

Consequently,
\[ \langle \Gamma | u_q \rangle = \frac{1}{2\pi \hbar} \int dp' e^{-i \alpha \rho (q' - q)/\hbar} \langle \Gamma | u_{p'} \rangle, \tag{4.15} \]
and it is apparent that once the position has been measured, one can find the system with any value for the momentum. This result is further evidence that a wave function dependent on \( p \) and \( q \) need not violate the uncertainty principle as long as the operators \( \hat{P} \) and \( \hat{Q} \) are nonlocal and satisfy \([\hat{Q}, \hat{P}] = i\hbar\). One should also note that Eq. (4.15) implies a Fourier transform relationship between the \( | u_{p'} \rangle \) and \( | u_p \rangle \) basis sets (albeit with a normalization appropri-
ate for vectors on an $L^2$ (2) space, just as one finds for the $L^2$ (1) basis vectors $|q\rangle$ and $|p\rangle$. We reiterate that the quantities $\langle \Gamma | q \rangle$ and $\langle \Gamma | p \rangle$ do not exist because the right and left sides of the inner product have different dimensionality, but $\langle \Gamma | u_p \rangle$ and $\langle \Gamma | u_p \rangle$ do exist.

Although there is no direct way to change from the $| \Gamma \rangle L^2$ (2) representation to either the $|q\rangle$ or $|p\rangle L^2$ (1) representations, one can define projections from the phase space representation which serve to recover the appropriate $L^2$ (1) representations (however, this process is not, in principle, reversible). Consider the following Fourier transform identities involving the kernel $(4\pi\hbar)^{-1/2} e^{ipq/2\hbar}$ with integration over $p$:

$$\int_{-\infty}^{+\infty} dp \frac{e^{ipq/2\hbar}}{\sqrt{4\pi\hbar}} \left( \frac{p}{2} - i\frac{\partial}{\partial q} \right)^n f(\Gamma) = \left( -i\hbar \frac{\partial}{\partial q} \right)^n \int_{-\infty}^{+\infty} dp \frac{e^{ipq/2\hbar}}{\sqrt{4\pi\hbar}} f(\Gamma),$$

(4.16)

and

$$\int_{-\infty}^{+\infty} dp \frac{e^{ipq/2\hbar}}{\sqrt{4\pi\hbar}} \left( \frac{q}{2} + i\frac{\partial}{\partial p} \right)^n f(\Gamma) = q^n \int_{-\infty}^{+\infty} dp \frac{e^{ipq/2\hbar}}{\sqrt{4\pi\hbar}} f(\Gamma),$$

(4.17)

where $f(\Gamma)$ is an arbitrary analytic complex function of $p$ and $q$ and the usual boundary conditions are obeyed. If we apply this identity to the Schrödinger equation in the $\hat{Q}$ space, then

$$i\hbar \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dp \frac{e^{ipq/2\hbar}}{\sqrt{4\pi\hbar}} \langle \Gamma | \psi \rangle = \frac{1}{2m} \int_{-\infty}^{+\infty} dp \frac{e^{ipq/2\hbar}}{\sqrt{4\pi\hbar}} \left( \frac{p}{2} - i\frac{\partial}{\partial q} \right)^2 \langle \Gamma | \psi \rangle + \sum_n V_n \int_{-\infty}^{+\infty} dp \frac{e^{ipq/2\hbar}}{\sqrt{4\pi\hbar}} \left( \frac{q}{2} + i\frac{\partial}{\partial p} \right)^n \langle \Gamma | \psi \rangle = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + \sum_n V_n q^n \right] \int_{-\infty}^{+\infty} dp \frac{e^{ipq/2\hbar}}{\sqrt{4\pi\hbar}} \langle \Gamma | \psi \rangle,$$

(4.18)

which is essentially the coordinate representation of the Schrödinger equation for a wave function defined by

$$\langle q | \psi \rangle = \int_{-\infty}^{+\infty} dp \frac{e^{ipq/2\hbar}}{\sqrt{4\pi\hbar}} \langle \Gamma | \psi \rangle.$$

(4.19)

This last equation describes a Fourier projection of $\langle \Gamma | \psi \rangle = \psi(p,q) \in L^2$ (2) onto the coordinate space wave function $\langle q | \psi \rangle \in L^2$ (1).

A similar projection can be devised to recover the momentum $L^2$ (1) representation from the phase space representation using the Fourier kernel $(4\pi\hbar)^{-1/2} e^{-ipq/2\hbar}$ and integration over all $q$. In this case, the identities

$$\int_{-\infty}^{+\infty} dq \frac{e^{-iq\hbar/2\hbar}}{\sqrt{4\pi\hbar}} \left( \frac{p}{2} - i\frac{\partial}{\partial q} \right)^n f(\Gamma) = p^n \int_{-\infty}^{+\infty} dq \frac{e^{-iq\hbar/2\hbar}}{\sqrt{4\pi\hbar}} f(\Gamma),$$

(4.20)

and

$$\int_{-\infty}^{+\infty} dq \frac{e^{-iq\hbar/2\hbar}}{\sqrt{4\pi\hbar}} \left( \frac{q}{2} + i\frac{\partial}{\partial p} \right)^n f(\Gamma) = \left( i\hbar \frac{\partial}{\partial p} \right)^n \int_{-\infty}^{+\infty} dq \frac{e^{-iq\hbar/2\hbar}}{\sqrt{4\pi\hbar}} f(\Gamma),$$

(4.21)

are obtained for arbitrary $f(\Gamma)$ with appropriate boundary conditions. Performing the Fourier projection on the phase space Schrödinger equation, one finds

$$i\hbar \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dq \frac{e^{-ipq/2\hbar}}{\sqrt{4\pi\hbar}} \langle \Gamma | \psi \rangle = \left[ \frac{p^2}{2m} + \sum_n V_n \left( i\hbar \frac{\partial}{\partial p} \right)^n \right] \int_{-\infty}^{+\infty} dq \frac{e^{-ipq/2\hbar}}{\sqrt{4\pi\hbar}} \langle \Gamma | \psi \rangle,$$

(4.22)

which is simply the Schrödinger equation in the $L^2$ (1) momentum representation for the state defined by $\langle p | \psi \rangle = \int_{-\infty}^{+\infty} dq \frac{e^{-ipq/2\hbar}}{\sqrt{4\pi\hbar}} \langle \Gamma | \psi \rangle.$

(4.23)

Consequently, one can recover the Schrödinger equation in either the coordinate or momentum $L^2$ (1) representations by means of the appropriate Fourier projection. This is significant because it provides a direct connection between the dynamics generated by the phase space Schrödinger equation and the dynamics of $L^2$ (1) functions expressed in either the coordinate or momentum representations.

V. CLASSICAL CORRESPONDENCE

The standard form of the quantum Liouville equation in phase space, Eq. (4.7) (see Table I), provides a good illustration of the correspondence rule that requires replacing the classical Poisson bracket $\{H(\Gamma), p(\Gamma; t)\}$ with the quantum commutator $i\hbar [\hat{H}, \hat{p}]$. A classical evaluation of the matrix elements in Eq. (4.7), i.e., one in which $\langle \Gamma | \hat{p} \hat{p} \hat{p}^{n-1} | \Gamma \rangle = p^n \langle \Gamma | \hat{p} \hat{p}^{n-1} | \Gamma \rangle$, $\langle \Gamma | \hat{Q}^n \hat{Q}^{n-1} | \Gamma \rangle = q^n \langle \Gamma | \hat{Q}^n \hat{Q}^{n-1} | \Gamma \rangle$, and $\langle \Gamma | \hat{Q} \hat{p} | \Gamma \rangle = \langle \Gamma | \hat{p} \rangle$, gives the equation

$$\frac{\partial}{\partial t} \rho(\Gamma; t) = -\frac{1}{2m} \frac{\partial}{\partial q} \left[ 2p \rho(\Gamma; t) \right] + \frac{\partial}{\partial p} \sum_{n=1}^{\infty} V_n \sum_{l=0}^{n-1} q^l \rho(\Gamma; t) q^{n-1-l},$$

(5.1)

where $F(\Gamma) = -\sum_{n=1}^{\infty} nV_n q^n$ is the force corresponding to the potential $V(q) = \sum_{n=0}^{\infty} V_n q^n$. Thus Eq. (5.1) is simply the classical Liouville equation of motion. The appearance of terms like those on the right-hand side of Eq. (4.7) arise...
due to the noncommutivity of quantum operators. As a result, the quantum expressions must be symmetrized and all the possible orderings of the classical products $pp(\Gamma;t)$ and $qq(\Gamma;t)$ must be included.

It is now possible to take a density in phase space and propagate it using either quantum or classical mechanics and look for quantum effects. One obvious choice for selecting an initial classical density is to set it equal to the quantum density. For example, consider the dynamics of a coherent state in a harmonic potential. In Ref. 43 an expression for the coherent state $a(\Gamma;t)$ in phase space, that can be compared with its classical analog, is derived with its form in the standard phase space representation (see Table 1) given by

$$a(\Gamma;t) = (2\pi\hbar)^{-1/2} \exp \left[ -\frac{m\omega}{2\hbar} \left[ q - q_c(t) \right]^2 \right]$$

$$- \frac{1}{4m\hbar\omega} \left[ p - p_c(t) \right]^2 + i \frac{\hbar}{2\hbar} \left[ q_p(t) - p_q(t) \right]$$

$$- p_q(t) \left[ \frac{i}{2} \omega t \right],$$

(5.2)

where

$$q_c(t) = q_c(0) \cos(\omega t) + \frac{1}{m\omega} p_c(0) \sin(\omega t),$$

$$p_c(t) = p_c(0) \cos(\omega t) - m\omega q_c(0) \sin(\omega t)$$

(5.3)

and $[p_c(0),q_c(0)]$ is the initial position of the center of the coherent state. The time dependence of the probability density follows from taking the square magnitude of Eq. (5.2) and the resulting expression

$$|a(\Gamma;t)| = \left[ \frac{1}{(2\pi\hbar)^{1/2}} \exp \left[ -\frac{m\omega}{2\hbar} \left[ q - q_c(t) \right]^2 \right] \right]$$

$$\left. \right. - \frac{1}{2\hbar\omega} \left[ p - p_c(t) \right]^2 \right]$$

(5.4)

is identical to the classical time evolution of the same density. The fluxes arising from the probability conservation equation for the quantum state, moving in a harmonic oscillator potential, are given by

$$- \frac{1}{2m\hbar^2} \left[ \Gamma \left\{ \hat{p}\hat{q}_\Gamma + \Gamma \left\{ \hat{p}\hat{\rho}_\Gamma \right\} \right\} \right.$$  

$$\left. \Gamma \left\{ \hat{p}\hat{\rho}_\Gamma + \Gamma \left\{ \hat{p}\hat{\rho}_\Gamma \right\} \right\} \right.$$  

$$\left. \frac{1}{2m} \left[ \langle \Gamma | \hat{p}\hat{\rho}_\Gamma | \Gamma \rangle + \langle \Gamma | \hat{ \rho}_\Gamma \hat{p} | \Gamma \rangle \right] \right.$$  

$$\left. \frac{1}{2m} \left[ \langle \Gamma | \hat{p}\hat{\rho}_\Gamma | \Gamma \rangle + \langle \Gamma | \hat{ \rho}_\Gamma \hat{p} | \Gamma \rangle \right] \right.$$

(5.5)

and

$$\left. \frac{1}{2m} \left[ \langle \Gamma | \hat{p}\hat{\rho}_\Gamma | \Gamma \rangle + \langle \Gamma | \hat{ \rho}_\Gamma \hat{p} | \Gamma \rangle \right] \right.$$

$$\left. \frac{1}{2m} \left[ \langle \Gamma | \hat{p}\hat{\rho}_\Gamma | \Gamma \rangle + \langle \Gamma | \hat{ \rho}_\Gamma \hat{p} | \Gamma \rangle \right] \right.$$

(5.6)

As we can see, the quantum fluxes (5.5) and (5.6) qualitatively resemble the classical fluxes, given by $-m\omega^2 q\rho(\Gamma;t)$ and $(p/m)\rho(\Gamma;t)$, respectively. In fact, they are the same when $\rho(q) = [p_c(t),q_c(t)]$, i.e., at the center of the coherent state. Therefore, only the center of the coherent state behaves in a classical way and the rest of the wave packet exhibits different behavior which can be attributed to quantum effects. One might expect this as the quantum dynamics is uncertainty limited and global—the behavior of the width of a nonstationary state reflects spreading and interference effects arising from the wave packet as a whole and which are clearly quantum mechanical in origin. These results are consistent with Heller's semiclassical wave packet analysis in which the center of the wave packet follows a classical path but additional equations of motion are introduced to account for spreading in a local approximation of the quantum dynamics.  

The interesting point here is that the apparent dynamics of the coherent state is the same for both the classical and quantum time evolutions, but the fluxes associated with the two cases are still quite different! This fundamental difference between the quantum and classical cases may be at the heart of such quantum mechanical phenomena as tunneling.

In the classical flux, the factor in front of the probability density is related to $(\hat{p}, \hat{q})$. We can make use of this association in Eqs. (5.5) and (5.6) to generate "quantum trajectories" for the coherent state moving in a harmonic oscillator potential. Then, the equations of motion for the quantum trajectories are

$$\dot{p} = -\frac{1}{2m\hbar^2} \left[ q + q_c(t) \right],$$

$$\dot{q} = \frac{1}{2m} \left[ p + p_c(t) \right],$$

(5.7)

where the time dependence of the center of the coherent state, $[p_c(t),q_c(t)]$, is still given by Eqs. (5.3). The solution to Eqs. (5.7) is given by

$$p(t) = [p(0) - p_c(0)] \cos(\frac{\omega}{2} t)$$

$$+ \frac{m\omega}{2\hbar^2} \left[ q_c(0) \sin(\frac{\omega}{2} t) \right]$$

$$- \frac{1}{2m} \left[ (p(0) - p_c(0)) \sin(\frac{\omega}{2} t) \right]$$

$$+ \frac{1}{2m} \left[ q_c(0) \sin(\frac{\omega}{2} t) \right]$$

(5.8)

This trajectory consists of a harmonic oscillator within a harmonic oscillator: harmonic motion of frequency $\omega/2$, and energy $m\omega^2 [q_c(0) - q_c(0)]^2/2 + [p(0) - p_c(0)]^2/2m$, revolves about a center, $[p_c(t),q_c(t)]$, which itself is orbiting the origin in phase space with frequency $\omega$ and energy $m\omega^2 q_c^2/2 + p_c^2/2m$. Here if the initial point on the trajectory is the initial position of the center, $[p_c(t),q_c(t)] = [p_c(0),q_c(0)]$, then the dynamics reduces to the classical motion of the center of the wave packet. Otherwise, the quantum trajectory of a point not at the center of the wave packet is similar to the motion of a satellite about an orbiting object in a planetary system.

Some of the quantum trajectories for the coherent state are shown in a dimensionless phase space in Fig. 1. We can see how an initial condition, different from that of the center of the coherent state, is exploring regions in phase space.
with different classical energies, oscillating around the center of the wave packet. These quantum trajectories are different from classical trajectories in three ways: (i) the quantum trajectory can cross itself in phase space, as well as other trajectories; (ii) the quantum trajectory is constantly changing its classical energy; (iii) if we take as the initial condition another point on the trajectory that is different from the one that generated that trajectory, we will obtain another, completely different trajectory. This last point follows from the fact that there are time-dependent terms, \( q_n(t) \) and \( p_n(t) \), on the right side of Eq. (5.7), so it matters not only where in phase space the trajectory starts, but also when it starts relative to the motion of the center of the coherent state. The dynamical behavior inferred from the quantum flux is clearly nonclassical in nature and is indicative of the differences between classical and quantum mechanics in phase space. We note that similar results for the harmonic oscillator have been obtained by Skodje et al. in the context of the coherent state representation.35

One can also investigate classical-quantum correspondence by examining classical distributions on phase space that share certain properties with their quantum mechanical counterparts. For example, one possible definition for the classical density corresponding to a quantum energy eigenstate in phase space is to define it so that it is both a classical and quantum mechanics in phase space. We note, however, that classical and quantum averaging processes on phase space (according to the results of this work) are different (the quantum average is an expectation value of a nonlocal Hermitian operator), so it is not necessarily appropriate to require that the classical average energy be the same as the quantum energy expectation value. Nevertheless, one can use this average energy criterion to help determine classical analogs of the quantum eigenstates of a given system in phase space.

As an example, we again turn to the harmonic oscillator system. We have previously shown that the quantum density corresponding to the \( n \)th harmonic oscillator eigenstate in the standard phase space representation is given by

\[
|\psi_n(\Gamma)|^2 = \frac{1}{2\pi\hbar} \left[ \frac{1}{\hbar \omega} \left( \frac{1}{2} m \omega^2 q^2 + \frac{1}{2} p^2 \right) \right]^n \times \exp \left[ -\frac{1}{\hbar \omega} \left( \frac{1}{2} m \omega^2 q^2 + \frac{p^2}{2m} \right) \right],
\]

for \( n = 0, 1, 2, \ldots \), and where \( m \) is the mass of the quantum particle, and \( \omega \) is the angular frequency of the harmonic oscillator. The quantum average energy, \( \langle \psi_n | \hat{H} | \psi_n \rangle \), for this stationary solution is \( (n+1/2) \hbar \omega \) in agreement with the standard \( \mathcal{L}^2 \) (1) space treatment. Now, we require that the classical density, \( \rho_n(\Gamma) \), be similar in form to the quantum density [Eq. (5.9)] and stationary with respect to the classical Liouville operator, just as the quantum density is stationary with respect to the quantum Liouville operator. Furthermore, we set the classical average energy, given by \( \langle \hat{H} \rangle_n = \int d\Gamma H(\Gamma) \rho_n(\Gamma) \), where \( H(\Gamma) = p^2/2m + m \omega^2 q^2/2 \) is the classical Hamiltonian, equal to the quantum value. A classical probability density that satisfies these requirements is

\[
\rho_n(\Gamma) = \frac{\sigma_n}{2\pi\hbar} \left[ \frac{\sigma_n}{\hbar \omega} H(\Gamma) \right]^n \times \exp \left[ -\frac{\sigma_n}{\hbar \omega} H(\Gamma) \right],
\]

where \( \sigma_n = (n+1)/(n+1/2) \) and \( H(\Gamma) \) is given in Eq. (5.9). If we let \( \bar{E} \equiv (1/\hbar \omega) (m \omega^2 q^2/2 + p^2/2m) \), the maximum of the quantum density for the \( n \)th eigenstate is located at \( \bar{E}_{n,\text{max}} = n \), and its height at this point is \((1/2\pi\hbar) n^{n} \exp(-n)\). The maximum of the quantum density for the \( n \)th eigenstate, located in Eq. (5.10), is located at \( \bar{E}_{n,\text{max}} = n/\sigma_n \) and its height at this point is \((1/2\pi\hbar) n^{n-1} \sigma_n e^{-n} \). Thus as a consequence of the uncertainty relationship \( \Delta q \Delta p > \hbar /2 \) and of the different ways in which quantum and classical averages are calculated, the quantum density is wider and shorter than the classical density for small \( n \). This means that the quantum model will appear more diffuse than its classical analog as shown in Fig. 2. In this figure, the density is plotted as a function of the quantum \( \bar{E} \) which is effectively the distance from the origin in phase space. For large \( n \), however, both densities approach the same limit.

In quantum mechanics, the wave function \( \psi(q;p) \) in coordinate space can be obtained from the wave function \( \psi(\Gamma;t) \) in phase space by taking the Fourier projection over \( p \) [see Eq. (4.19)], but the classical analog for the quantum density \( \langle \langle q | \psi \rangle \rangle^2 \), in the coordinate representation is not


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FIG. 2. A plot of quantum, Eq. (4.8), and classical, Eq. (4.9), probability densities for the harmonic oscillator for various states \( n \). The quantum probability density (dashed line) is more diffuse and shorter than the classical density (solid line). Note that as \( n \) increases, the apparent differences between the two decrease.

well defined. A procedure commensurate with the treatment of the Wigner function would be to take the average of the classical density \( \rho(\Gamma; t) \) over \( \rho \), that is \( \int dp \, \rho(\Gamma; t) \). Thus by integrating the density in Eq. (5.10) over \( \rho \), one should obtain the classical analog of the quantum density for the harmonic oscillator eigenstate in the coordinate representation

\[
\rho_n(q) = \int dp \, \rho_n(\Gamma) = \frac{1}{2\pi\hbar} \sqrt{\frac{\sigma \mu_0}{2\pi\hbar}} \exp^{-\frac{\pi m_0 q^2}{\hbar^2}} \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) \left( \frac{\hbar}{2} \right)^l \left( \frac{2n-2l-1}{2} \right)!! \left( \frac{\hbar}{2} \right)^l \left( \frac{2n-2l-1}{2} \right)!! \left( 2n \right)!
\]

(5.11)

In Fig. 3, a plot of the quantum and classical densities for the states \( n=0,2,4,6,8, \) and 10 is shown. The quantum and classical densities for the ground state are the same, and for other states, both densities extend approximately over the same region in coordinate space. In these states, however, the quantum density appears more “diffuse” since the distance between the maxima of the density is larger than for the classical density. This comparison differs with the usual way of comparing quantum and classical densities, however, in that one should compare the quantum probability density with the classical probability density for a single particle. In the present case, it is more appropriate to associate the classical analog, \( \rho_n(q) \) with a statistical ensemble of identical noninteracting systems having energy \( E_n \).

VI. SUMMARY

In this paper, we have introduced a phase space representation for quantum mechanics of \( L^2(2) \) functions which permits one to write down time evolution equations for both complex-valued wave functions \( \langle \Gamma | \psi \rangle \) (Schrödinger equation) and quantum densities \( \langle \Gamma | \rho(\Gamma) \rangle \) (Liouville equation) in phase space. This representation has been characterized by defining generalized forms for the operators \( \hat{P} \) and \( \hat{Q} \) in phase space [Eqs. (2.4) and (2.5)] and then employing various conditions, including the commutation relation \([\hat{Q}, \hat{P}]=i\hbar\) and correspondences between the classical and quantum Liouville equations to specify their form. From these arguments, we define

\[
\langle \Gamma | \hat{P} | \psi \rangle = \left( \frac{\rho}{2} - i\hbar \frac{\partial}{\partial \rho} \right) \langle \Gamma | \psi \rangle
\]

(6.1)

and

\[
\langle \Gamma | \hat{Q} | \psi \rangle = \left( \frac{q}{2} + i\hbar \frac{\partial}{\partial q} \right) \langle \Gamma | \psi \rangle
\]

(6.2)

to be the standard or canonical phase space representations of \( \hat{P} \) and \( \hat{Q} \). In deriving this standard representation, we have obtained explicit expressions for the quantum Liouville equation in both the \( L^2(1) \) momentum representation, Eq. (3.16), and the \( L^2(2) \) phase space representation, Eqs. (3.23) and (4.7). Our results for the phase space representation are summarized in Table I.

The phase space representation that has been constructed here is distinctly different and in some ways simpler than the more common ways of investigating quantum
mechanics in phase space. Those investigations have typically defined distribution functions on phase space by utilizing integral transformations of coordinate representation wave functions. In the present work, we have sought to establish a quantum theory appropriate to an $L^2(2)$ function space such as phase space by incorporating the mathematical properties postulated for $L^2(1)$ representations of quantum mechanics. As a result, the phase space representation is built on a foundation of complex-valued wave functions. In the present work, we have sought to establish a quantum theory appropriate to an $L^2(2)$ function space such as phase space by incorporating the mathematical properties postulated for $L^2(1)$ representations of quantum mechanics. As a result, the phase space representation is built on a foundation of complex-valued wave functions, rather than real-valued distribution functions.

Because the kets in phase space are elements of $L^2(2)$, that is $\langle \Gamma | \psi \rangle \in L^2(2)$, while those in the usual coordinate or momentum representations belong to $L^2(1)$, the transformation matrix elements $\langle \Gamma | \hat{q} \rangle$ and $\langle \Gamma | \hat{p} \rangle$ do not exist. In other words, the phase space representation of $|\psi\rangle$ contains more information than is available from the coordinate or momentum $L^2(1)$ representations of $|\psi\rangle$ and it is not possible to make a change of representation between $|\Gamma\rangle$ and either $|\hat{q}\rangle$ or $|\hat{p}\rangle$. We have shown that a simple Fourier projection of the phase space Schrödinger equation, Eq. (4.6), allows one to recover the Schrödinger equation in either the coordinate or momentum $L^2(1)$ representation. Consequently, the Fourier projection provides a useful means for verifying the physical viability of the postulated $L^2(2)$ phase space representation by casting the coordinate or momentum $L^2(1)$ representations as “subsets” of the phase space representation.

By defining quantum time evolution equations appropriate to phase space, the phase space representation also enables us to better elucidate the dynamics of a system in both the quantum and classical limits, thereby providing a more complete picture of the correspondence between the two. Since the dynamics of quantum and classical systems are now formulated in the same space, among other things, it is possible to propagate an initial probability density in a quantum or classical way and make comparisons between the two types of behavior.

As an example, we have examined the quantum dynamics of the coherent state of the harmonic oscillator in phase space. Interestingly, although the global dynamics of the coherent state is apparently the same for both classical and quantum mechanics, the fluxes are quite different in each case. Classically, all parts of the distribution follow classical paths, remaining on the same phase cycles throughout the dynamics. Quantum mechanically, however, only the center of the coherent state follows a classical path while other pieces of the wave packet circulate through regions of higher and lower energy during a single oscillation around the phase cycle.

In some cases, it may be desirable to define the classical density corresponding to some quantum density so that it mimics some of the average properties of the quantum density. As a result, we have also examined the classical density in phase space corresponding to a quantum eigenstate of the harmonic oscillator and found that, if the average energies of the two distributions are the same, the quantum distribution is more diffuse than the corresponding classical distribution. In the large $n$ limit, where $n$ is the quantum number, the two densities become identical.

In this paper, we have focused on simple examples involving single degree of freedom systems. The generalization of the present work to systems of higher dimension...
is straightforward. However, the quantum phase space dynamics of a state evolving in more complicated systems,\textsuperscript{52} for example, those involving purely quantum effects such as tunneling, should provide additional insight into the correspondence between classical and quantum mechanics.\textsuperscript{51} Ongoing work is involved with exploring those aspects of quantum dynamics in phase space.

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### APPENDIX A

In this Appendix, we present a proof for Eqs. (2.22), (2.23), (2.25) and (3.29). The equality

\[ q^n = \frac{(n-1)!}{n!} d^n q^n, \]  

(A1)

and the fact that \( aq \) and \( \hbar \beta (\partial / \partial p) \) commute with each other, allows us to write

\[
(aq)!' \exp \left[ i \hbar \beta \frac{\partial}{\partial (aq)} \frac{\partial}{\partial p} + i \hbar \delta \frac{\partial}{\partial (yp)} \frac{\partial}{\partial q} \right] f(p,q) = (aq)!' \sum_{m=0}^{\infty} \frac{1}{m!} \left( i \hbar \delta \right)^m \left( i \hbar \beta \right)^n \frac{\partial^m}{\partial (yp)^m} \frac{\partial^n}{\partial (aq)^n} f(p,q)
\]

\[
= (aq)!' \exp \left[ i \hbar \beta \frac{\partial}{\partial (aq)} \frac{\partial}{\partial p} \right] f(p,q),
\]

(A4)

Therefore, we can see that if \( g(q) = \sum \delta g \cdot q^n \), then

\[
g(aq + i \hbar \beta \frac{\partial}{\partial p}) f(p,q) = g(aq) \exp \left[ i \hbar \beta \frac{\partial}{\partial (aq)} \frac{\partial}{\partial p} + i \hbar \delta \frac{\partial}{\partial (yp)} \frac{\partial}{\partial q} \right] f(p,q),
\]

(A5)

which is Eq. (2.22) in the text. With the above results at hand, it is easy to verify that

\[
h(\gamma p + i \hbar \delta \frac{\partial}{\partial q}) f(p,q) = h(\gamma p) \exp \left[ i \hbar \delta \frac{\partial}{\partial (yp)} \frac{\partial}{\partial q} + i \hbar \beta \frac{\partial}{\partial (aq)} \frac{\partial}{\partial p} \right] f(p,q),
\]

(A6)

for any function \( h(p) \) of the form \( h(p) = \sum \delta h \cdot p^n \).

Equations (A5) and (A6) allow us to rewrite the phase space representation of the Schrödinger equation in the form of Eq. (2.25) of the text

\[
i \hbar \frac{\partial}{\partial t} \langle \Gamma | \psi_t \rangle = \hbar (\gamma p, aq) \exp \left[ i \hbar \delta \frac{\partial}{\partial (yp)} \frac{\partial}{\partial q} + i \hbar \beta \frac{\partial}{\partial (aq)} \frac{\partial}{\partial p} \right] \langle \Gamma | \psi_t \rangle.
\]

(A7)

Thus, if \( \hat{\rho}_t = \sum_n \langle \chi_n | \psi_t \rangle \langle \psi_n | \rangle \), the diagonal matrix elements of the quantum Liouville equation in the phase space representation become

\[
\frac{\partial}{\partial t} \langle \Gamma | \hat{\rho}_t | \Gamma \rangle = \sum_n \frac{1}{i \hbar} \left( \frac{\partial}{\partial t} \langle \chi_n | \langle \psi_n | \Gamma \rangle + \langle \Gamma | \chi_n \rangle \frac{\partial}{\partial t} \langle \psi_n | \Gamma \rangle \right)
\]

\[
= \sum_n \frac{1}{i \hbar} \left( \frac{\partial}{\partial t} \hbar (\gamma p, aq) \exp \left[ i \hbar \delta \frac{\partial}{\partial (yp)} \frac{\partial}{\partial q} + i \hbar \beta \frac{\partial}{\partial (aq)} \frac{\partial}{\partial p} \right] \langle \Gamma | \chi_n \rangle \langle \psi_n | \Gamma \rangle \right)
\]
\[ -\sum_n \langle \Gamma | \chi_n \rangle \left[ \frac{1}{i\hbar} H(\gamma p, aq) \exp \left[ i\hbar \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} + i\hbar \frac{\partial}{\partial (aq)} \frac{\partial}{\partial p} \right] \langle \psi_n | \Gamma \rangle \right] \]

\[ = \left[ \frac{1}{i\hbar} H(\gamma p, aq) \exp \left[ i\hbar \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} + i\hbar \frac{\partial}{\partial (aq)} \frac{\partial}{\partial p} \right] \sum_n \langle \Gamma | \chi_n \rangle \langle \psi_n | \Gamma' \rangle \right] \]

\[ - \frac{1}{i\hbar} H(\gamma p, aq) \exp \left[ -i\hbar \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} - i\hbar \frac{\partial}{\partial (aq)} \frac{\partial}{\partial p} \right] \sum_n \langle \Gamma' | \chi_n \rangle \langle \psi_n | \Gamma \rangle \right]_{\Gamma' = \Gamma} \]

\[ = \frac{2}{\hbar} H(\gamma p, aq) \text{Im} \left[ \exp \left[ i\hbar \frac{\partial}{\partial (\gamma p)} \frac{\partial}{\partial q} + i\hbar \frac{\partial}{\partial (aq)} \frac{\partial}{\partial p} \right] \langle \Gamma | \hat{\rho} \rangle \right]_{\Gamma' = \Gamma}, \tag{A8} \]

which is Eq. (3.29) in the text.

**APPENDIX B**

In this Appendix, we derive equations (3.21) and (3.22) of the text. For any two kets \(|\chi\rangle\) and \(|\Psi\rangle\), we have that

\[ \langle \Gamma | \chi \rangle \langle \Psi | \hat{\rho} | \Gamma \rangle = \langle \Gamma | \chi \rangle \left[ \left( \gamma p + i\hbar \frac{\partial}{\partial q} \right) \langle \Gamma | \Psi \rangle \right]^* \]

\[ = \langle \Gamma | \chi \rangle \gamma p \langle \Psi | \Gamma \rangle - i\hbar \delta \langle \Gamma | \chi \rangle \frac{\partial}{\partial q} \langle \Psi | \Gamma \rangle \]

\[ = \langle \Gamma | \chi \rangle \gamma p \langle \Psi | \Gamma \rangle + i\hbar \delta \frac{\partial}{\partial q} \langle \Psi | \Gamma \rangle - i\hbar \delta \frac{\partial}{\partial q} \langle \Gamma | \chi \rangle \langle \Psi | \Gamma \rangle \]

\[ = \left[ \left( \gamma p + i\hbar \delta \frac{\partial}{\partial q} \right) \langle \Gamma | \chi \rangle \right] \langle \Psi | \Gamma \rangle - i\hbar \delta \frac{\partial}{\partial q} \langle \Gamma | \chi \rangle \langle \Psi | \Gamma \rangle \]

\[ = \langle \Gamma | \hat{\rho} | \chi \rangle \langle \Psi | \Gamma \rangle - i\hbar \delta \frac{\partial}{\partial q} \langle \Gamma | \chi \rangle \langle \Psi | \Gamma \rangle. \tag{B1} \]

We can make use of Eq. (B1) as many times as needed in order to get

\[ \langle \Gamma | \chi \rangle \langle \Psi | \hat{\rho}^n | \Gamma \rangle - \langle \Gamma | \hat{\rho}^n | \chi \rangle \langle \Psi | \Gamma \rangle = \left[ -i\hbar \delta (\partial/\partial q) \sum_{n=0}^{\infty} \langle \Gamma | \hat{\rho}^{n-1} | \chi \rangle \langle \Psi | \hat{\rho} | \Gamma \rangle \langle \hat{\rho} | \Gamma \rangle \right] \]

\[ - i\hbar \delta (\partial/\partial q) \sum_{n=0}^{\infty} \langle \Gamma | \hat{\rho}^{n-1} | \chi \rangle \langle \Psi | \Gamma \rangle \langle \hat{\rho} | \Gamma \rangle \]

\[ = \left[ -i\hbar \delta (\partial/\partial q) \sum_{n=0}^{\infty} \langle \Gamma | \hat{\rho}^{n-1} | \chi \rangle \langle \Psi | \Gamma \rangle \langle \hat{\rho} | \Gamma \rangle \right] \]

\[ - i\hbar \delta (\partial/\partial q) \sum_{n=0}^{\infty} \langle \Gamma | \hat{\rho}^{n-1} | \chi \rangle \langle \Psi | \Gamma \rangle \langle \hat{\rho} | \Gamma \rangle \]

\[ = \langle \Gamma | \hat{\rho} | \chi \rangle \langle \Psi | \Gamma \rangle - i\hbar \delta \frac{\partial}{\partial q} \langle \Gamma | \chi \rangle \langle \Psi | \Gamma \rangle. \tag{B2} \]

Since the density operator \(\hat{\rho}_n\) in general, can be written as a sum of terms of the form \(|\chi\rangle \langle \Psi|\), we conclude that

\[ \langle \Gamma | \hat{\rho}_n | \Gamma \rangle - \langle \Gamma | \hat{\rho}_n | \chi \rangle \langle \Psi | \Gamma \rangle = \left[ -i\hbar \delta (\partial/\partial q) \sum_{n=0}^{\infty} \langle \Gamma | \hat{\rho}^{n-1} | \chi \rangle \langle \Psi | \hat{\rho} | \Gamma \rangle \right] \]

\[ - i\hbar \delta (\partial/\partial q) \sum_{n=0}^{\infty} \langle \Gamma | \hat{\rho}^{n-1} | \chi \rangle \langle \Psi | \Gamma \rangle \]

\[ = \left[ -i\hbar \delta (\partial/\partial q) \sum_{n=0}^{\infty} \langle \Gamma | \hat{\rho}^{n-1} | \chi \rangle \langle \Psi | \Gamma \rangle \langle \hat{\rho} | \Gamma \rangle \right] \]

\[ - i\hbar \delta (\partial/\partial q) \sum_{n=0}^{\infty} \langle \Gamma | \hat{\rho}^{n-1} | \chi \rangle \langle \Psi | \Gamma \rangle \langle \hat{\rho} | \Gamma \rangle \]

\[ = \langle \Gamma | \hat{\rho} | \chi \rangle \langle \Psi | \Gamma \rangle - i\hbar \delta \frac{\partial}{\partial q} \langle \Gamma | \chi \rangle \langle \Psi | \Gamma \rangle. \tag{B3} \]

which is Eq. (3.21) of the text.

Now, for any two kets \(|\chi\rangle\) and \(|\Psi\rangle\), we have that

\[ \langle \Gamma | \chi \rangle \langle \Psi | \hat{Q} | \Gamma \rangle = \langle \Gamma | \chi \rangle \left[ \left( a q + i\hbar \beta \frac{\partial}{\partial p} \right) \langle \Gamma | \Psi \rangle \right]^* \]

\[ = \langle \Gamma | \chi \rangle a q \langle \Psi | \Gamma \rangle - i\hbar \beta \langle \Gamma | \chi \rangle \frac{\partial}{\partial p} \langle \Psi | \Gamma \rangle \]

\[ = \langle \Gamma | \chi \rangle a q \langle \Psi | \Gamma \rangle + i\hbar \beta \frac{\partial}{\partial p} \langle \Psi | \Gamma \rangle - i\hbar \beta \frac{\partial}{\partial p} \langle \Gamma | \chi \rangle \langle \Psi | \Gamma \rangle \]

\[ = \left[ \left( a q + i\hbar \beta \frac{\partial}{\partial p} \right) \langle \Gamma | \chi \rangle \right] \langle \Psi | \Gamma \rangle - i\hbar \beta \frac{\partial}{\partial p} \langle \Gamma | \chi \rangle \langle \Psi | \Gamma \rangle \]

\[ = \langle \Gamma | \hat{Q} | \chi \rangle \langle \Psi | \Gamma \rangle - i\hbar \beta \frac{\partial}{\partial p} \langle \Gamma | \chi \rangle \langle \Psi | \Gamma \rangle. \tag{B4} \]

We can make use of the above relationship as many times as needed in order to get

\[ \langle \Gamma | \chi \rangle \langle \Psi | \hat{Q}^n | \Gamma \rangle - \langle \Gamma | \hat{Q}^n | \chi \rangle \langle \Psi | \Gamma \rangle = \left[ -i\hbar \beta (\partial/\partial p) \sum_{n=0}^{\infty} \langle \Gamma | \hat{Q}^{n-1} | \chi \rangle \langle \Psi | \hat{Q} | \Gamma \rangle \right] \]

\[ - i\hbar \beta (\partial/\partial p) \sum_{n=0}^{\infty} \langle \Gamma | \hat{Q}^{n-1} | \chi \rangle \langle \Psi | \Gamma \rangle \]

\[ = \left[ -i\hbar \beta (\partial/\partial p) \sum_{n=0}^{\infty} \langle \Gamma | \hat{Q}^{n-1} | \chi \rangle \langle \Psi | \Gamma \rangle \langle \hat{Q} | \Gamma \rangle \right] \]

\[ - i\hbar \beta (\partial/\partial p) \sum_{n=0}^{\infty} \langle \Gamma | \hat{Q}^{n-1} | \chi \rangle \langle \Psi | \Gamma \rangle \langle \hat{Q} | \Gamma \rangle \]

\[ = \langle \Gamma | \hat{Q} | \chi \rangle \langle \Psi | \Gamma \rangle - i\hbar \beta \frac{\partial}{\partial p} \langle \Gamma | \chi \rangle \langle \Psi | \Gamma \rangle. \tag{B5} \]
Hence, if the potential function $V(q)$ can be written as a power series in $q$, $V(q) = \sum_{n=0}^{\infty} V_n q^n$, we can say that

$$\langle \Gamma | \hat{\chi}(\hat{\phi}) | \Gamma \rangle = \langle \Gamma | \hat{\chi}(\hat{\phi}) | \Gamma \rangle = -i\hbar \frac{\partial}{\partial p} \sum_{n=1}^{\infty} V_n \sum_{l=0}^{n-1} \langle \Gamma | \hat{Q}^l \hat{\phi}^{n-l-1} | \Gamma \rangle.$$  

(B6)

And, since the density operator $\hat{n}_p$ in general, can be written as a sum of terms of the form $| \chi \rangle \langle \Psi |$, we conclude that

$$\langle \Gamma | \hat{\rho}(\hat{\phi}) | \Gamma \rangle = \langle \Gamma | \hat{\chi}(\hat{\phi}) \hat{n}_p | \Gamma \rangle = -i\hbar \frac{\partial}{\partial p} \sum_{n=1}^{\infty} V_n \sum_{l=0}^{n-1} \langle \Gamma | \hat{Q}^l \hat{\phi}^{n-l-1} | \Gamma \rangle.$$  

(B7)

This ends the proof of Eq. (3.22).


41. Throughout the paper, we refer to the space of square integrable functions of a single variable as $L_2 $ and the space of square integrable functions of two variables as $L_2^2 $. Inner products of states belonging to different function spaces are ill-defined and prevent the evaluation of quantities such as $\langle \Gamma | q \rangle $, for example.

42. F. Bopp, in Werner Heisenberg und die Physik unserer Zeit (Vieweg, Braunschweig, 1961).


49. In his paper of 1932 (Ref. 12), Wigner mentions that his transformation is not the only one possible and that that particular form was chosen because it seems to be the simplest. Subsequently, Cohen (Ref. 18) has investigated a family of similar transformations. Likewise, the Husimi transform (Ref. 24) is not unique thanks to the arbitrary parameter $\lambda$, in its definition [see Eqs. (1.4) and (1.5)]. Finally, the basis vectors for the coordinate and momentum representations are nonunique in that they can incorporate an arbitrary phase factor. This phase factor has no effect on the probability density, but does give rise to different expressions for $\hat{\rho}$, for example [see E. Merzbacher, Quantum Mechanics (Wiley, New York, 1970)].

