Spreading function representation of operators and Gabor multiplier approximation

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Abstract

Modification of signals in the time-frequency domain are used in many applications. However, the modification is often restricted to be purely multiplicative. In this paper, it is shown that, in the continuous case, a quite general class of operators can be represented by a twisted convolution in the short-time Fourier transform domain. The discrete case of Gabor transforms turns out to be more intricate. A similar representation will however be derived by means of a special form for the operator's spreading function (twisted spline type function). The connection between STFT- and Gabor-multipliers, their spreading function and the twisted convolution representation will be investigated. A precise characterization of the best approximation and its existence is given for both cases. Finally, the concept of Gabor multipliers is generalized to better approximate "overspread" operators.

Key words and phrases : Operator approximation, spreading function, twisted convolution, Gabor multiplier, optimal multiplier

2000 AMS Mathematics Subject Classification —

1 Introduction

Time-frequency expansions have been shown to provide extremely efficient representations for functions and distributions, and found many relevant applications, notably in signal processing. Efficient representation of linear operators in the time-frequency domain still remains a difficult challenge. Many theoretical results have been obtained, but these have only had little practical impact so far. We here address this problem from a slightly different point of view, emphasizing the geometrical aspects of short time Fourier and Gabor transforms.

It is well known that twisted convolution plays a central role as soon as functions and operators are studied directly on the phase space. We first show that the spreading function representation of Hilbert-Schmidt operators is equivalent to a representation by left twisted convolution in the STFT domain. Unfortunately, such a remark does not hold true any more when the STFT is discretized, i.e. replaced with a Gabor transform.

Gabor multipliers are often proposed as an efficient alternative for time-frequency operator representation. It is well known that for "nice operators", i.e. operators that do not involve time-frequency shifts of large magnitude, Gabor multipliers can provide accurate approximations. We show that for more general operators suitable generalizations of Gabor multipliers may still provide good approximation. We mainly propose two possible extensions. In the first one, the analyzed operator is approximated by a linear combination of adapted Gabor multipliers, and when the latter are given on a suitably chosen lattice in the time-frequency plane, the question of optimal approximation (in Hilbert-Schmidt sense) may be well formulated.

In the second extension, an adaptation of the synthesis window to the analyzed operator is proposed, while the time-frequency transfer function is fixed. In both situations, the final result reveals interesting connections to twisted convolution.

2 The Weyl-Heisenberg group, Time-frequency representations

Classical time-frequency analysis is based on translation and modulation operators, which generate the Weyl-Heisenberg group. Even though the group theoretical language may be completely avoided in most applications and theoretical developments, it is nevertheless interesting to trace back some tools which play a central role in Gabor theory (including covariance issues, and twisted convolution) to their algebraic origins.

2.1 Preliminaries: the Weyl-Heisenberg group and its unitary representations

We start by briefly reviewing the main results concerning the Weyl-Heisenberg group and its representation theory that will be of interest here. We refer the interested reader to [6, 12] for a more thorough analysis.

Classical time-frequency analysis rests on translation operators T_b and modulation operators M_ν, defined by

$$T_b f(t) = f(t-b) \ , \quad M_\nu f(t) = e^{2i\pi\nu t} f(t) \ , \quad f \in \mathbf{L}^2(\mathbb{R}) \ ,$$

and the corresponding time-frequency shifts $\pi(b,\nu) = M_\nu T_b$, which satisfy the canonical commutation relations

$$M_\nu T_b = e^{2i\pi\nu b} T_b M_\nu \ .$$

These operators generate a three-parameter Lie group, called the (reduced) Weyl-Heisenberg group[1] $\mathbb{H}$.

$$\mathbb{H} = \{(b,\nu,\varphi) \in \mathbb{R}\times\mathbb{R}\times[0,1]\} \ , \tag{1}$$

with group multiplication

$$(b,\nu,\varphi)(b',\nu',\varphi') = (b+b', \nu+\nu', \varphi+\varphi'-\nu' b) \ . \tag{2}$$

The zero element is clearly $e=(0,0,0)$, and the inverse operation is $(b,\nu,\varphi)^{-1} = (-b,-\nu,-\varphi-b\nu)$. The group $\mathbb{H}$ is unimodular, i.e. its two Haar measures coincide, and read

$$d\mu(b,\nu,\varphi) = db\, d\nu\, d\varphi \ . \tag{3}$$

The left regular representation of $\mathbb{H}$ on $\mathbf{L}^2(\mathbb{H})$ takes the following form:

$$L(b',\nu',\varphi')F(b,\nu,\varphi) = F\left((b',\nu',\varphi')^{-1}(b,\nu,\varphi)\right) = F(b-b',\nu-\nu',\varphi-\varphi'+b'(\nu-\nu')) \ . \tag{4}$$

The unitary dual $\widehat{\mathbb{H}}$ of $\mathbb{H}$ (i.e. the set of unitary equivalence classes of irreducible unitary representations of $\mathbb{H}$) has the following form

$$\widehat{\mathbb{H}} \simeq \mathbb{R}^2 \cup \mathbb{Z}^* \ .$$

with $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$. The $\mathbb{R}^2$ part consists of the unitary characters $\{\chi_{\alpha,\beta},\ \alpha,\beta\in\mathbb{R}\}$ defined by $\chi_{\alpha,\beta}(b,\nu,\varphi) = e^{2i\pi\alpha b}e^{2i\pi\beta\nu}$, while the $\mathbb{Z}^*$ part consists in a series of representations π_λ on $\mathcal{H}_\lambda \simeq \mathbf{L}^2(\mathbb{R}^d)$:

$$\pi_\lambda(b,\nu,\varphi)f(t) = e^{2i\pi\lambda[\varphi+\nu t]} f(t-b) \ . \tag{5}$$

In other terms, $\pi_\lambda(b,\nu,\varphi) = e^{2i\pi\lambda\varphi} M_{\lambda\nu} T_b$.

The representations π_λ, which form the so-called *discrete series*, may be shown to be square-integrable, i.e. such that there exist $g \in \mathcal{H}_\lambda$ such that

$$0 < \int_{\mathbb{H}} |\langle \pi_\lambda(h)g, g\rangle|^2 \, d\mu(h) < \infty \ .$$

It follows (see [8, 9] and references therein) that the integral transform $\mathscr{V}_g$, defined by

$$f \in \mathcal{H}_\lambda \longmapsto \mathscr{V}_g f = \frac{1}{\|g\|} \langle f, \pi_\lambda(\cdot) g\rangle \in \mathbf{L}^2(\mathbb{H}) \tag{6}$$

maps isometrically $\mathcal{H}_\lambda$ to $\mathbf{L}^2(\mathbb{H})$.

[1] Notice that there exist other versions of the Weyl-Heisenberg group, including a more symmetric one generated by time-frequency shifts $M_{\nu/2} T_b M_{\nu/2}$, and another one often used in signal processing, generated by time-frequency shifts $T_b M_\nu$.

2.2 More details on $\widehat{\mathbb{H}}$ and $\mathscr{V}_g$

The dual $\widehat{\mathbb{H}}$ of the Weyl-Heisenberg group $\mathbb{H}$ and the integral transform $\mathscr{V}_g$ turn out to be closely connected, as we outline next. Classical $\mathbf{L}^2$ Fourier series theory yields the following

Lemma 1. *For $F \in \mathbf{L}^2(\mathbb{H})$, the mapping $F \to \{F_\ell,\ \ell \in \mathbb{Z}\}$ defined by*

$$F_\ell(b,\nu,\varphi) = e^{2i\pi\ell\varphi} F_\ell(b,\nu,0) = e^{2i\pi\ell\varphi} \int_0^1 F(b,\nu,\alpha) e^{-2i\pi\ell\alpha}\, d\alpha \tag{7}$$

establishes a bijective isometry $\mathbf{L}^2(\mathbb{H}) \simeq \bigoplus_{\ell\in\mathbb{Z}^+} \mathcal{E}_\ell$, where

$$\mathcal{E}_\ell = \left\{F \in \mathbf{L}^2(\mathbb{H}),\ F(b,\nu,\varphi) = e^{2i\pi\ell\varphi} F_\ell(b,\nu,0)\right\} \simeq \mathbf{L}^2(\mathbb{R}^2)\ . \tag{8}$$

Hence, we have for all $F \in \mathbf{L}^2(\mathbb{H})$

$$F(b,\nu,\varphi) = \sum_{\ell=-\infty}^{\infty} F_\ell(b,\nu,\varphi) = \sum_{\ell=-\infty}^{\infty} F_\ell(b,\nu,0) e^{2i\pi\ell\varphi} \tag{9}$$

and

$$\int_{\mathbf{G}} |F(b,\nu,\varphi)|^2 d\mu_L(b,\nu,\varphi) = \sum_{\ell=-\infty}^{\infty} \|F_\ell\|^2_{\mathbf{L}^2(\mathbb{R}^2)}\ .$$

In addition, we immediately get

Lemma 2. *Let π_λ be an irreducible unitary representation in the discrete series of $\widehat{\mathbb{H}}$, $g \in \mathbb{H}$ and let $\mathscr{V}_g$ be defined as above in (6). Then $\mathscr{V}_g x \in \mathcal{E}_{-\lambda}$.*

From now on, we shall limit ourselves to the case $\lambda = 1$, i.e. $\mathscr{V}_g x \in \mathcal{E}_{-1}$ for all $x \in \mathbf{L}^2(\mathbb{R})$. We denote by $\pi = \pi_1$ the corresponding irreducible unitary representation of $\mathbb{H}$ on $\mathcal{H}_{-1}$, and identify the latter with $\mathbf{L}^2(\mathbb{R})$.

2.3 Twisted convolution and Plancherel transform

The left convolution on the Weyl-Heisenberg group induces a twisted convolution product on $\mathcal{E}_{-1}$, as follows. Let $F, G \in \mathcal{E}_{-1}$. Then

$$\begin{aligned} F * G(b,\nu,\varphi) &= \int F(b',\nu',\varphi') G(b-b',\nu-\nu',\varphi-\varphi'+b'(\nu-\nu')) db'd\nu'd\varphi' \\ &= e^{-2i\pi[\varphi'+b'(\nu-\nu')]} \int_{\mathbb{R}^2} F(b',\nu',0) G(b-b',\nu-\nu',0) e^{-2i\pi b'(\nu-\nu')}\, db'd\nu' \\ &= e^{-2i\pi\varphi'} F * G(b,\nu,0)\ . \end{aligned}$$

Therefore, $F*G \in \mathcal{E}_{-1}$. In addition, identifying $\mathcal{E}_{-1}$ with $\mathbf{L}^2(\mathbb{R}^2)$ yields the following *twisted convolution* product on $\mathbf{L}^2(\mathbb{R}^2)$ (see [5] for more details)

$$(F \natural G)(b,\nu) = \int_{\mathbb{R}^2} F(b',\nu') G(b-b',\nu-\nu') e^{-2i\pi b'(\nu-\nu')}\, db'd\nu'\ . \tag{10}$$

The Plancherel transform may be seen as a generalization of usual Fourier transform to more general groups. In the case of the Heisenberg group, the Plancherel transform associates with any function $F \in \mathbf{L}^2(\mathbb{H})$ the operator valued function on $\widehat{\mathbb{H}}$ defined by

$$\hat{F} : \sigma \in \widehat{\mathbb{H}} \longmapsto \hat{F}(\sigma) = \int_{\mathbb{H}} f(h)\sigma(h)\, d\mu(h)\ , \tag{11}$$

the equality being understood in the weak operator sense. In the particular case $\sigma = \pi$, identifying again $\mathcal{E}_{-1}$ with $\mathbf{L}^2(\mathbb{R}^2)$ one may associate with any $F \in \mathbf{L}^2(\mathbb{R}^2)$ the operator valued function $\hat{F}(\pi)$ defined as

$$\hat{F}(\pi) f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(b,\nu) M_\nu T_b f\, dbd\nu\ , \quad f \in \mathbf{L}^2(\mathbb{R}) \tag{12}$$

2.4 Connection to short time Fourier transform

Given the representation of the Weyl-Heisenberg group $\pi = \pi_1$ (see Eq. (5)), and $g \in \mathbf{L}^2(\mathbb{R})$, $\|g\| = 1$, define the short time Fourier transform (STFT, also called continuous Gabor transform, CGT) $\mathscr{V}_g x \in \mathbf{L}^2(\mathbb{R}^2)$ of $x \in \mathbf{L}^2(\mathbb{R})$ by

$$\mathscr{V}_g x(b,\nu) = \langle x, \pi(b,\nu,0)g\rangle = \langle x, M_\nu T_b g\rangle \ , \tag{13}$$

where by an abuse of notation we have identified $\mathcal{E}_{-1}$ with $\mathbf{L}^2(\mathbb{R}^2)$.[2] Then it is a well known fact that the image of $\mathbf{L}^2(\mathbb{R}^2)$ by $\mathscr{V}_g$ is a proper subspace $\mathcal{K}_g$ of $\mathbf{L}^2(\mathbb{R}^2)$, consisting of solutions of the kernel equation

$$\mathcal{K}_g = \left\{ F \in \mathbf{L}^2(\mathbb{R}^2),\ F(b,\nu) = \int F(b',\nu')\langle \pi(b',\nu')\psi, \pi(b,\nu)\psi\rangle\, db' d\nu' \ , \forall b,\nu \in \mathbb{R} \right\} \ .$$

The following result is proved by direct calculations. We denote by $\mathscr{H}$ the set of Hilbert-Schmidt operators on $\mathbf{L}^2(\mathbb{R})$.

Proposition 1. *The short time Fourier transform intertwines twisted convolution and action of Hilbert-Schmidt operators on $\mathbf{L}^2(\mathbb{R})$. More precisely, let $F \in \mathbf{L}^2(\mathbb{R}^2)$.*

1. *There exists a linear operator $H_F = \widehat{F}(\pi_1) \in \mathscr{H}$ such that*

$$F \natural \mathscr{V}_g x = \mathscr{V}_g H_F x \ . \tag{14}$$

   Therefore, $F \natural \mathscr{V}_g x \in \mathcal{K}_g$.

2. *The linear operator $H_F \in \mathscr{H}$ also satisfies*

$$\mathscr{V}_g x \natural F = \mathscr{V}_{H_F^* g} x \ .$$

Remark 1. The consequence of this result is the fact that the image of the STFT of a signal $x \in \mathbf{L}^2(\mathbb{R})$ by a twisted convolution with some function of two variables is itself the STFT of some signal $H_F x$: $\mathcal{K}_g$ is invariant under left twisted convolution. Similarly, right twisted convolution of $\mathscr{V}_g x$ with some function yields another STFT of x with respect to a different window: right twisted convolution maps the reproducing kernel space $\mathcal{K}_g$ to the reproducing kernel space $\mathcal{K}_{H_F^* g}$.

The Gabor transform (see for example [7] for a review) is defined as a sampled version of the STFT. Given two constants $a_0, \nu_0 \in \mathbb{R}^+$, the corresponding Gabor transform associates with any $f \in \mathbf{L}^2(\mathbb{R})$ the sequence of Gabor coefficients

$$\mathscr{V}_g f(mb_0, n\nu_0) = \langle f, M_{n\nu_0} T_{mb_0} g\rangle = \langle f, g_{mn}\rangle \ , \tag{15}$$

the functions $g_{mn} = M_{n\nu_0} T_{mb_0} g$ being the Gabor atoms associated with g and the lattice constants b_0, ν_0. Under suitable assumptions, the Gabor transform is left invertible, and there exists $h \in \mathbf{L}^2(\mathbb{R})$ such that any f may be expanded as

$$f = \sum_{m,n} \mathscr{V}_g f(mb_0, n\nu_0) h_{mn} \ . \tag{16}$$

Proposition 1 does not have any simple analogue if STFT is replaced with Gabor transform. However, connections between Gabor representations of Hilbert-Schmidt operators and twisted convolutions will appear below.

[2] Notice that this transform is only modulation covariant, i.e. $\mathscr{V}_\psi M_{\nu_0} x(b,\nu) = \mathscr{V}_\psi x(b, \nu+\nu_0)$, but not translation-covariant, i.e. $\mathscr{V}_\psi T_{b_0} x(b,\nu) \neq \mathscr{V}_\psi (b+b_0,\nu)$. The other non-symmetric version of the Weyl-Heisenberg group yields a waveform transform that is shift-covariant, but not modulation covariant.

2.5 The finite Weyl-Heisenberg group and associated tools

Let us stress that similar developments may be done for Weyl-Heisenberg groups on arbitrary locally compact abelian groups. The case of practical interest is the finite case, i.e. the case of the Weyl-Heisenberg group on $\mathbb{Z}_N$, which we briefly review here. The group is $\mathbb{H}_N = \mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N$, with group law

$$(m,n,\varphi)(m',n',\varphi') = (m+m', n+n', \varphi+\varphi' - n'b) \ ,$$

all operations being understood modulo N.

The unitary dual $\widehat{\mathbb{H}_N}$ has a structure similar to $\widehat{\mathbb{H}}$: $\widehat{\mathbb{H}_N} = \mathbb{Z}_N^2 \cup \mathbb{Z}_N^*$, and the $\mathbb{Z}_N^*$ component of the dual consists in representations π_λ on $\mathcal{H}_\lambda = \mathbb{C}^N$, given by

$$\pi_\lambda(m,n,\varphi) = e^{2i\pi\lambda\varphi/N} M_n T_m \ ,$$

with the following definition for periodic discrete translation and modulation operators

$$T_m x[k] = x[(k-m)[\text{mod } N]] \ , \quad M_n x[k] = e^{2i\pi kn/N} x[k] \ .$$

Focusing on the case $\lambda = 1$, and following the lines developed in the continuous case, one is naturally led to the following discrete version of the twisted convolution: for $F, G \in \mathbb{C}^N$,

$$F \natural G[m,n] = \sum_{m',n'=0}^{N-1} F[m',n'] G[m-m', n-n'] e^{-2i\pi m(n-n')/N} \ , \tag{17}$$

all operations being understood modulo N.

From now on, we limit ourselves to the case $\lambda = 1$. Given $g \in \mathbb{C}^N$, the corresponding short time Fourier transform maps any $x \in \mathbb{C}^N$ to $\mathscr{V}_g x \in \mathbb{C}^{2N}$, defined by

$$\mathscr{V}_g x[m,n] = \langle x, \pi_\lambda[m,n,0] g\rangle \ .$$

The Gabor case is treated similarly. Given b_0, ν_0, two positive divisors of N, set $N_b = N/b_0$ and $N_\nu = N/\nu_0$, and introduce the Gabor functions

$$g_{mn}[t] = e^{2i\pi n\nu_0 t/N} g[t - mb_0] \ .$$

Under suitable assumptions, these form a frame in $\mathbb{C}^N$.

3 Spreading function and twisted convolution operator representation

3.1 The continuous case

Let $H \in \mathscr{H}$, the class of Hilbert-Schmidt operator on $\mathbf{L}^2(\mathbb{R})$. Then there exists a function $\eta = \eta_H \in \mathbf{L}^2(\mathbb{R}^2)$, called the *spreading function* (see for example [7] and references therein), such that [3]

$$H = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta(b,\nu)\pi(b,\nu)\, dbd\nu \ . \tag{18}$$

Such a representation turns out to be closely related to the *twisted convolution*, defined in (10) above.

Our first remark is the fact that the spreading function representation of the operator $H \in \mathscr{H}$ actually takes the form of a twisted convolution in the (continuous) time-frequency domain. η be its spreading function. Indeed, the following result [2] is a direct consequence of Proposition 1, that follows from the comparison of the spreading function representation with the Plancherel transform (12).

[3] Here, this representation must be interpreted in the weak sense. More general, the spreading representation can be given an interpretation for $\eta \in \mathcal{S}'$, i.e. , if the spreading function is a tempered distribution. For more regular spreading function, e.g., $\eta \in \mathbf{L}^1$, the integral is absolutely convergent.

Figure 1: *Transposition performed by twisted convolution in the TF-domain*

Figure 2: *Spreading functions and resulting operators applied to random signal*

Theorem 1. *Let $H \in \mathscr{H}$, with spreading function η_H. Let $g, h \in \mathbf{L}^2(\mathbb{R})$ be such that $\langle g, h\rangle = 1$. Then H may be realized as a left twisted convolution in the time-frequency domain: for all $f \in \mathbf{L}^2(\mathbb{R})$,*

$$Hf = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\eta_H \natural \mathscr{V}_g f)(b,\nu) M_\nu T_b h \, dbd\nu \ . \tag{19}$$

An immediate consequence is the fact, that the range of $\mathscr{V}_g$ is invariant under left twisted convolution.

Remark 2. Again, let us notice that no simple analogue of this expression is available when the STFT is replaced with the Gabor transform. Replacing the integrals with discrete sums does not yield simple operators.

3.2 The finite Heisenberg group

The spreading function representation of operators takes here a particularly simple form: the family of operators $\{M_n T_m,\ m, n \in \mathbb{Z}_N\}$ is an orthonormal basis of $\mathrm{Hom}(\mathbb{C}^N)$. Every $H \in \mathrm{Hom}(\mathbb{C}^N)$ is characterized by a spreading function $\eta_H \in \mathbb{C}^{N^2}$, such that

$$H = \sum_{m,n=0}^{N-1} \eta_H[m,n] M_n T_m \ .$$

The consequence of this result is the same as in the continuous theory: let $H \in \mathrm{Hom}(\mathbb{C}^N)$, and let η_H be its spreading function. Let $g \in \mathbb{C}^N$ be a unit norm vector. Then H may be realized as a twisted convolution in the time-frequency domain: for all $x \in \mathbb{C}^N$,

$$Hx = \sum_{m,n} (\eta_H \natural \mathscr{V}_g x)[m,n] M_n T_n g \ .$$

Actually, a similar theory may be developed for Weyl-Heisenberg groups associated with any locally compact abelian group.

3.3 Examples

As a first example, a simple transposition has been realized by a twisted convolution with a "Dirac pulse" in the TF-domain. See Figure 1. The reconstruction is performed with the dual of the original Gabor frame [4] and yields a perfect result. This twisted convolution corresponds, of course, to a pure time- or frequency shift. However, adding several dirac pulses, i.e. setting the spreading function to 1 in several places, will yield an amalgam of the signal components present in the original signal.

Figure 2 shows a nicely concentrated spreading function and the images of a random signal under the corresponding operator. The second pair of images shows the spreading function of a convolution operator and again the result of this operator being applied to a random signal. Both operators have a strong smoothing character, however, the convolution performs infinite time-shifts.

4 STFT and Gabor multipliers

Twisted convolution operator representations such as the ones described above are not suitable for practical implementations, mainly because of the difficulty of sampling such representations, as stressed in Remark 2.

[4] If synthesis is desired, we need to assume that the Gabor system is a *frame*, meaning that there exist constants $0 < A, B < \infty$, such that

$$A\|f\|^2 \leq \sum_m \sum_n |\langle f, g_{mn}\rangle|^2 \leq B\|f\|^2. \tag{20}$$

In addition, even if such twisted convolutions could be treated practically, they would hardly lead to efficient algorithms, as a twisted convolution involves quite a large number of terms.

STFT and Gabor multipliers have been proposed as a valuable alternative for the approximation of operators in the time-frequency domain. In the present section we analyse the relationship between such multipliers and the spreading function representations, referring to [4] for a survey.

4.1 Time-frequency multipliers

Let $g, h \in \mathbf{L}^2(\mathbb{R}^2)$ be such that $\langle g, h\rangle = 1$, and $\mathbf{m} \in \mathbf{L}^\infty(\mathbb{R}^2)$, and define the STFT multiplier $\mathbb{M}_{\mathbf{m};g,h}$ by

$$\mathbb{M}_{\mathbf{m};g,h} f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{m}(b,\nu)\mathscr{V}_g f(b,\nu)\,\pi(b,\nu)h\,dbd\nu, \quad f \in \mathbf{L}^2(\mathbb{R})\ . \tag{21}$$

This clearly defines a bounded operator on $\mathbf{L}^2(\mathbb{R}^d)$. We shall call $\mathbf{m}$ the *time-frequency transfer function*, in analogy with signal processing notation, where time-invariant linear filters (convolution operator) are defined by multiplication in the Fourier domain with a bounded function called transfer function (which is the Fourier transform of the filter's impulse response). We refer to [10] and references therein for a presentation in a signal processing context. In the mathematical literature, $\mathbf{m}$ is also called the upper symbol of the multiplier, but we prefer to reserve this terminology for the Kohn-Nirenberg symbol or the Weyl symbol of operators (see [5, 7] for example). $\mathbf{m}$ is sometimes also called a *mask*.

STFT multipliers may also be extended to various functional settings. For example, suitable assumptions on the transfer function $\mathbf{m}$ ensure that the corresponding multiplier is compact. We refer to [4] for a thorough review of these results.

In a similar way, given lattice constants $b_0, \nu_0 \in \mathbb{R}^+$, set $\pi_{mn} = \pi(mb_0, n\nu_0) = M_{n\nu_0}T_{mb_0}$. Then, for $\mathbf{m} \in \ell^\infty(\mathbb{Z}^2)$, the corresponding Gabor multiplier is defined as

$$\mathbb{M}^G_{\mathbf{m};g,h} f = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} \mathbf{m}(m,n)\mathscr{V}_g f(mb_0, n\nu_0)\,\pi_{mn}h\ . \tag{22}$$

In both cases, the spreading function may be computed explicitely, and yields the following result [2].

Lemma 3. *1. The spreading function of the STFT multiplier $\mathbb{M}_{\mathbf{m};g,h}$ is given by*

$$\eta_{\mathbb{M}_\mathbf{m}}(b,\nu) = \mathscr{M}(b,\nu)\mathscr{V}_g h(b,\nu)\ , \tag{23}$$

where $\mathscr{M}$ is the symplectic Fourier transform of the transfer function $\mathbf{m}$

$$\mathscr{M}(t,\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{m}(b,\nu)e^{2i\pi(\nu t - \xi b)}\,dbd\nu\ .$$

2. The spreading function of the Gabor multiplier $\mathbb{M}^G_{\mathbf{m};g,h}$ is given by

$$\eta^G_{\mathbb{M}_\mathbf{m}}(b,\nu) = \mathscr{M}^{(d)}(b,\nu)\mathscr{V}_g h(b,\nu)\ , \tag{24}$$

where the $({\nu_0}^{-1}, {b_0}^{-1})$-periodic function $\mathscr{M}^{(d)}$ is the symplectic Fourier transform of the transfer function $\mathbf{m}$

$$\mathscr{M}^{(d)}(t,\xi) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \mathbf{m}(m,n)e^{2i\pi(n\nu_0 t - mb_0\xi)}\ .$$

Therefore, we can see that the spreading function of a STFT or a Gabor multiplier has a very specific form. In the STFT case, the following result is easily proved.

Lemma 4. *Let $H \in \mathscr{H}$, with spreading function η. Assume there exists a non-negative function $\mu \in \mathbf{L}^2(\mathbb{R}^2)$ such that*

$$|\eta(b,\nu)| \le \mu(b,\nu)\,|\mathscr{V}_g h(b,\nu)|\ ,$$

then H admits a STFT multiplier representation.

In a nutshell, when the spreading function of H decays faster than the ambiguity function, H admits a STFT multiplier representation. However, when the decay is not fast, the transfer function $\mathbf{m}$ may lack smoothness. A simple way of controlling such a property is to assume compact support for the spreading function [10, 11]. However, compact support is not a necessary condition. For example, let us consider the case of a Gaussian window $g(t) = e^{-t^2/2}$. The corresponding ambiguity function is given by $\sqrt{\pi}e^{-3b^2/4}e^{-\pi^2\nu^2}e^{-i\pi\nu b}$, which never vanishes. Therefore, whenever the spreading function has sufficiently fast decay in the (b, ν) plane, the corresponding operator may be realized as a STFT multiplier.

In the case of Gabor multiplier approximations, the periodicity of the function $\mathscr{M}^{(d)}$ imposes stronger constraints, which we analyze in more detail below. Let us simply stress at this point that if the lattice constant become smaller and smaller, the dual lattice becomes coarser, and the periodicity assumption becomes less and less constraining.

4.2 Further Examples

We now give some examples for illustration of the relation between Gabor Multipliers as well as more general operators and their spreading functions.

Figure 3: *Spreading function of a Gabor multiplier*

Figure 3 shows the STFT of a synthezised signal with two components, the spreading function of a Gabor multiplier, which removes one of the components, and the STFT of the resulting signal. Figure 4 (left) shows the spreading function of an operator that again filters out one of the components, but simultaneously moves it in the TF-plane, as well as, again, the STFT of the resulting signal. Note that, while still being well-concentrated, this operator's spreading function is no longer centered about the origin. Finally, Figure 4 (right) shows the spreading function of an operator that filters the same component as before, but "duplicates" it: once, it is shifted in time and once in frequency. Quite obviously, the spreading function has two peaks in this case. The two latter operators will not be well-approximated by a simple Gabor multiplier and require a more general treatment, as suggested in Section 5.1 below.

4.3 Approximation by Gabor multipliers

It follows from (24) that not any $\eta \in \mathbf{L}^2(\mathbb{R}^2)$ can be the spreading function of a STFT or Gabor multiplier. In particular, for Gabor multipliers, the function $\eta/\mathscr{V}_g h$ has to be periodic, which is generally not true. In such cases, the optimal (in Hilbert-Schmidt sense) approximation may be seeked, provided that the family of rank one operators

$$P_{mb_0,n\nu_0} : f \in \mathbf{L}^2(\mathbb{R}) \to P_{mb_0,n\nu_0} f = \langle f, g_{mb_0,n\nu_0}\rangle h_{mb_0,n\nu_0} \ , \quad m, n \in \mathbb{Z}$$

is a Riesz sequence. This question has been addressed by several authors (see [4] and references therein). The answer turns out to be positive [2] if and only if there exist real constants $0 < A \leq B < \infty$ such that for all (t, ξ)

$$0 < A \leq \sum_{k,\ell=-\infty}^{\infty} \left| \mathscr{V}_g h\left(t + \frac{k}{\nu_0}, \xi + \frac{\ell}{b_0}\right)\right|^2 \leq B < \infty \ . \tag{25}$$

The latter condition is obtained from the more classical one found in [4] by applying the Poisson summation formula.

Now, if the above condition is fulfilled, the best Gabor multiplier approximation (in Hilbert-Schmidt sense) of $H \in \mathscr{H}$ with spreading function η_H is defined by the time-frequency transfer function $\mathbf{m}$ whose discrete symplectic Fourier transform reads

$$\mathscr{M}(b, \nu) = \frac{\sum_{k,\ell=-\infty}^{\infty} \overline{\mathscr{V}_g h}\left(b + k/\nu_0, \nu + \ell/b_0\right) \eta_H\left(b + k/\nu_0, \nu + \ell/b_0\right)}{\sum_{k,\ell=-\infty}^{\infty} \left|\mathscr{V}_g h\left(b + k/\nu_0, \nu + \ell/b_0\right)\right|^2} \tag{26}$$

Figure 4: *Spreading function of two different modifying Gabor multipliers*

Therefore, assuming that the windows g, h and the lattice constants b_0, ν_0 have been chosen so that condition (25) holds, the above expression may be used to compute the best Gabor multiplier approximation for any linear operator $H \in \mathscr{H}$. For numerical realisation of approximation by Gabor multipliers, see, for example, [3].

5 Generalized Gabor Multipliers

Gabor multipliers have the advantage of having a diagonal matrix in a suitable Gabor representation. Unfortunately, approximating a Hilbert-Schmidt operator by a Gabor multiplier may turn out to result in poor approximations when the spreading function of the operator is not "concentrated" enough, i.e. when the operators are not *underspread* [10, 11].

Two strategies may be developped in such a situation. A first strategy amounts to "split" the operator into a finite sum of operators, whose spreading function is much more concentrated in the time-frequency domain, and may thus be conveniently approximated by a Gabor multiplier. When the construction preserves the action of the Weyl-Heisenberg group, the corresponding optimal time-frequency transfer functions may be given a closed form. This leads to multiple Gabor multipliers, whose contruction is briefly outlined below.

The second possible strategy rests on a spline type approximation of the spreading function, expressed as a sum of building blocks with smaller support. Again, when this is done in a $\mathbb{H}$-covariant way, and if the building blocks are spreading functions of Gabor multipliers, the resulting approximation turns out to take an extremely simple form. A simple instance of this construction is described below, and we refer again to [2] for a thorough analysis of the general case.

5.1 Multiple Gabor multipliers

The strategy to restrict the modification of the Gabor transform to be multiplicative (i.e. the operator is represented by a diagonal matrix in the Gabor system) may be generalized to allow for *banded matrices* in this modification. This is the equivalent to allowing generalized Gabor multipliers, in the following sense.

Definition 1 (Multiple Gabor Multiplier)**.** *Let a lattice* $\Lambda = b_0\mathbb{Z} \times \nu_0\mathbb{Z}$, $g, h_k \in \mathbf{L}^2(\mathbb{R}^2)$ *and bounded sequences* $\mathbf{m}_k = (\mathbf{m}_k(\lambda))_{\lambda\in\Lambda}$ *, with* $k \in \mathbb{Z}$*, be given. Then the* multiple Gabor multiplier *(MGM)* $\mathbb{G}_{g,h_k,\Lambda,\mathbf{m_k}}$ *is defined by*

$$\mathbb{G}_{g,h_k,\Lambda,\mathbf{m_k}}(f) = \sum_k \sum_{\lambda\in\Lambda} \mathbf{m}_k(\lambda)\langle f, \pi(\lambda)g\rangle \pi(\lambda)h_k. \tag{27}$$

Remark 3. For $h_k = \pi(\mu_k)h$ with $\mu_k \in \Lambda$, $|\mu_k| \leq K$ this extends Gabor multiplier by replacing their diagonal matrix by a banded one. This is a generalization of ample relevance in applications, for example as prominent ones as mobile communication, see [13] for an overview and justification of banded matrices in mobile communication.

The operator described in the last example of Section 4.2 would actually be a (simple) example of a Multiple Gabor Multiplier.

Given such a scheme, the same questions as before may be asked. In particular, whether for a given Hilbert-Schmidt operator $H \in \mathscr{H}$, it is possible to find time-frequency transfer functions such that the corresponding multiple Gabor multiplier minimizes the quadratic approximation error. To our knowledge, this question hasn't received an answer so far.

However, such a very general formulation turns out to assume a much simpler form when the synthesis windows h_k are constructed through the action of the Weyl-Heisenberg group on the time-frequency plane. Indeed, it may be shown [2] that when the synthesis windows are taken from a suitable lattice in the time-frequency plane, i.e.

$$h_k(t) = \pi(b_k, \nu_k)h(t) = e^{2i\pi\nu_k t}h(t - b_k) \ ,$$

the existence of optimal transfer functions $\mathbf{m}_k$ is actually equivalent to the invertibility of some (right) discrete twisted convolution operator. When this twisted convolution operator is invertible, then a closed form may be obtained for the time-frequency transfer functions $\mathbf{m}_k$.

In a finite discrete setting, the problem can be given a matrix formulation, which is easily realized numerically. Let N be the length of a signal under consideration, in other words, the function space of interest is $\mathbb{C}^N$. The number of lattice points in a given Gabor frame with lattice constants b_0 and ν_0 is then given by $k = \frac{N}{b_0}\frac{N}{\nu_0}$. If we allow for J different synthesis windows h^j in the Multiple Gabor multiplier approximation, the number of projection operators

$$P^j_{mb_0,n\nu_0} : P^j_{mb_0,n\nu_0} f = \langle f, g_{mb_0,n\nu_0}\rangle h^j_{mb_0,n\nu_0} \ , \quad m = 1,\dots,\frac{N}{b_0},\ n = 1,\dots,\frac{N}{\nu_0},\ j = 1,\dots,J$$

amounts to $J \cdot k$. Each of these projection operators can be written as a matrix of dimension $N \times N$. By writing theM as row-vectors of size $1 \times N^2$, we can build the matrix $\mathcal{A}$ of size $Jk \times N^2$ representing the system of projection operators $(P^j_{mb_0,n\nu_0})$. Let us now denote the pseudo-inverse of a matrix $\mathcal{A}$ by $\mathcal{A}^+$. We consider the best approximation $\mathbf{K}_{app}$ of a given operator $\mathbf{K}$ in the system described by $\mathcal{A}$. Clearly, $\mathbf{K}$ and $\mathbf{K}_{app}$ are $N \times N$-matrices and may be written as $1 \times N^2$-vectors, so that $\mathbf{K}_{app}$ may be written as follows:

$$\mathbf{K}_{app} = \mathbf{K} \cdot \mathcal{A}^+ \cdot \mathcal{A}.$$

Recall now, that $\mathcal{A}^+ = \mathcal{A}^* \cdot (\mathcal{A} \cdot \mathcal{A}^*)^+$. Now, the system $P^j_{mb_0,n\nu_0}, m = 1,\dots,\frac{N}{b_0}, n = 1,\dots,\frac{N}{\nu_0}, j = 1,\dots,J$ is a Riesz basis if and only if $\mathcal{A}$ has full rank, which means, that $(\mathcal{A} \cdot \mathcal{A}^*)$ is invertible such that $\mathcal{A}^+ = \mathcal{A}^* \cdot (\mathcal{A} \cdot \mathcal{A}^*)^{-1}$. In this case we obtain

$$\mathbf{K}_{app} = \mathbf{K} \cdot \mathcal{A}^* \cdot (\mathcal{A} \cdot \mathcal{A}^*)^{-1} \cdot \mathcal{A}$$

and $[\mathbf{K} \cdot \mathcal{A}^* \cdot (\mathcal{A} \cdot \mathcal{A}^*)^{-1}]$ is a row vector of size $1 \times Jk$ containing the coefficients $\mathbf{m}^j_{m,n}$ of the best-approximation of $\mathbf{K}$ in the expansion

$$\mathbf{K}_{app} = \sum_j \sum_{m,n} \mathbf{m}^j_{m,n} P^j_{mb_0,n\nu_0}.$$

The construction just given has been realized for synthetic examples and allows interesting insight in the quality of approximation by Multiple Gabor multipliers as compared to regular Gabor multipliers. For "overspread" operators, i.e. operators with a spreading function of larger support, it seems more appropriate to go for an approximation by a Multiple Gabor multiplier (MGM) than to try a Gabor multiplier with higher redundancy. In experiments examples show that an appropriately chosen MGM with an overall number of $K = Jk$ projection operators may yield better results than the approximation by a Gabor multiplier with the according redundancy of K. Here, a second remark is due. For regular Gabor multiplier, the structure of the (Gramian) matrix $\mathcal{A}$ leads to an extremely simple and efficient inversion of $(\mathcal{A} \cdot \mathcal{A}^*)$, which is a circulant matrix and may be inverted via the FFT of its convolution kernel. See [1, Section 5.4] for more details. Now, as suggested above, if the various synthesis windows h^j are in fact chosen to be time-frequency shifted versions of one particular synthesis window, on the given lattice or its dual, a similarly structured Gramian matrix of the over-all system may be expected. The investigation of these properties is the topic of current research and will lead to more efficient methods in the search of optimal approximation by Multiple Gabor multipliers.

5.2 Twisted spline type spreading functions

Let us finally discuss a different form of operators, which allow for a discrete twisted convolution representation in the Gabor transform domain. Let ϕ be a square-integrable function, and assume that the spreading function η of $H \in \mathscr{H}$ may be written as a *twisted spline type* (TST) function

$$\eta(b,\nu) = \sum_{k,\ell} \alpha_{k\ell}\phi(b - kb_0, \nu - \ell\nu_0)e^{-2i\pi(\nu-\ell\nu_0)kb_0} = \sum_{k,\ell} \alpha_{k\ell}\phi_{k\ell}(b\nu) \ , \tag{28}$$

i.e. as a sum of time-frequency translates of ϕ, using the $\mathbb{H}$-covariant time-frequency translations. Then the following result expresses that covariance at the operator level

Lemma 5. *Let H be a Hilbert-Schmidt operator associated with a TST spreading function. Let $H_\phi \in \mathscr{H}$ denote the operator with spreading function ϕ. Then*

$$H = \sum_{k,\ell} \alpha_{k\ell}\pi(kb_0, \ell\nu_0)H_\phi \ . \tag{29}$$

Proof: Simply compute

$$\begin{aligned} H &= \sum_{k,\ell} \alpha_{k\ell} \int \phi(b',\nu') e^{-2i\pi\nu' k b_0} M_{\nu'+\ell\nu_0} T_{b'+kb_0}\, db' d\nu' \\ &= \sum_{k,\ell} \alpha_{k\ell} \int \phi(b',\nu') e^{-2i\pi\nu' k b_0} M_{\ell\nu_0} M_{\nu'} T_{kb_0} T_{b'}\, db' d\nu' \\ &= \sum_{k,\ell} \alpha_{k\ell} \pi(kb_0,\ell\nu_0) \int \phi(b',\nu') M_{\nu'} T_{b'}\, db' d\nu'\ , \end{aligned}$$

which proves the result. ♠

Now, suppose that H_ϕ admits a Gabor multiplier representation, with time-frequency transfer function $\mathbf{m}$; then[5] for $f = \sum_{m,n} \mathscr{V}_g f(mb_0, n\nu_0) h_{mn}$, we have

$$H_\phi f = \sum_{m,n} \mathbf{m}(m,n) \mathscr{V}_g f(mb_0, n\nu_0) h_{mn}\ ,$$

for some $h \in \mathbf{L}^2(\mathbb{R}^d)$, with $h_{mn} = \pi_{mn} h$, and some bounded sequence of weights $\mathbf{m}(m,n)$. Plugging this expression into the above lemma, we obtain

$$\begin{aligned} Hf &= \sum_{k,\ell} \alpha_{k\ell} \pi_{k\ell} \sum_{m,n} \mathbf{m}(m,n) \mathscr{V}_g f(mb_0, n\nu_0) h_{mn} \\ &= \sum_{m,n} \mathbf{m}(m,n) \mathscr{V}_g f(mb_0, n\nu_0) \sum_{k,\ell} \alpha_{k\ell} \pi(kb_0,\ell\nu_0) \pi_{mn} h\ . \end{aligned}$$

This yields the following result

Theorem 2. *Assume that the spreading function η of $H \in \mathcal{H}$ is of the form (28), with ϕ the spreading function of a Gabor multiplier $\mathbb{M}_{\mathbf{m};g,h}$. Then H may be realized as follows*

$$Hf = \sum_{p,q} \widetilde{\mathscr{V}}_g f(p,q) h_{pq}\ , \quad f \in \mathbf{L}^2(\mathbb{R}^d)\ , \tag{30}$$

where

$$\widetilde{\mathscr{V}}_g f(p,q) = (\alpha \natural \mathscr{W})_{pq} \tag{31}$$

$$\mathscr{W}_{mn} = \mathbf{m}(m,n) \mathscr{V}_g f(mb_0, n\nu_0)\ . \tag{32}$$

In addition, if the right twisted convolution operator $\alpha \to \alpha \natural G$ with the sequence G defined by

$$G_{mn} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(b,\nu) \overline{\phi}(b - mb_0, \nu - n\nu_0) e^{-2i\pi m b_0 (\nu - n\nu_0)}\, db d\nu$$

is invertible, then the coefficients α in (28) may be recovered from the scalar products $\langle \eta, \phi_{k\ell} \rangle$.

Remark 4. Notice that the expression provided in the theorem above is **not** a Gabor multiplier. It indeed expresses Hf as a sum of elementary atoms h_{pq}, but the latter are not time-frequency shifts of a unique window function.

From a practical point of view, Gabor coefficients of Hf may be obtained by the following two-steps procedure: given $f \in \mathbf{L}^2(\mathbb{R}^d)$, and a (discrete) Gabor transform $\mathscr{V}_g f$

1. Weight coefficients $\mathscr{V}_g f(mb_0, n\nu_0)$ using the time-frequency transfer function $\mathbf{m}$ of the Gabor multiplier $\mathbb{M}_{\mathbf{m};g,h}$.

[5] Here, once more, we implicitly assume, that the system of time-frequency translated functions g_{mn} constitutes a frame and thus allows for reconstruction by means of a dual frame generated by a dual window h, see [7] for a review of Gabor frames.

2. Evaluate the twisted convolution of the so-obtained weighted coefficients with the coefficients α of the TST expansion of the spreading function η of H.

This results in a fairly simple algorithm. Corresponding numerical results will be reported elsewhere.

Let us point out that this scheme may be modified in several respects. In particular, it may be shown that using a different sampling lattice for the TST functions, one ends up with approximations involving true Gabor multipliers. These results are described in [2].

6 Conclusions and perspectives

In this paper, we have mainly focused on the approximation of 1D continuous time Hilbert Schmidt operators by Gabor multipliers and generalizations. Generalizing Gabor multipliers may be performed in various ways. We have described mainly two approaches, in which the synthesis windows or the time-frequency transfer function are varied. More general results are described in [2].

Let us also point out that this approach generalizes *mutatis mutandis* to more complex situations, for example Gabor transforms in higher dimensions, or Gabor transforms on non-rectangular (regular) lattices. It may also be formulated in a completely discrete setting, which makes it suitable for numerical applications.

This was actually one of our main motivations for developing such methods in a purely group theoretical setting. The latter opens interesting perspectives for generalizations, including wavelet multiplier approximations, or even more general situations.

Acknowledgements

We wish to thank P. Balazs, H.G. Feichtinger, Y.B. Kouagou and D. Marelli for extremely stimulating discussions.

This work received support from the European Commission funded Research Training Network HASSIP (HPRN-CT-2002-00285).

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