

The Multi-Dimensional Hardy Uncertainty Principle and its Interpretation in Terms of the Wigner Distribution; Relation With the Notion of Symplectic Capacity

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Abstract

We extend Hardy's uncertainty principle for a square integrable function ψ and its Fourier transform to the n -dimensional case using a symplectic diagonalization. We use this extension to show that Hardy's uncertainty principle is equivalent to a statement on the Wigner distribution $W\psi$ of ψ . We give a geometric interpretation of our results in terms of the notion of symplectic capacity of an ellipsoid.

1 Introduction

A folk metatheorem is that a function ψ and its Fourier transform $F\psi$ cannot be simultaneously sharply localized. An obvious manifestation of this “principle” is when ψ is of compact support: in this case the Fourier transform $F\psi$ can be extended into an entire function, and is hence never of compact support. A less trivial way to express this kind of trade-off between

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ψ and $F\psi$ was discovered in 1933 by G.H. Hardy [12]. Hardy showed, using methods from complex analysis (the Phragmén–Lindelöf principle), that if $\psi \in L^2(\mathbb{R})$ and its Fourier transform

$$F\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}px} \psi(x) dx$$

satisfy, for $|x| + |p| \rightarrow \infty$, estimates

$$\psi(x) = \mathcal{O}(e^{-\frac{a}{2\hbar}x^2}) \quad , \quad F\psi(p) = \mathcal{O}(e^{-\frac{b}{2\hbar}p^2}) \quad (1)$$

with $a, b > 0$, then the following holds true:

- (1) If $ab > 1$ then $\psi = 0$;
- (2) If $ab = 1$ we have $\psi(x) = Ce^{-\frac{a}{2\hbar}x^2}$ for some complex constant C ;
- (3) If $ab < 1$ we have $\psi(x) = Q(x)e^{-\frac{a}{2\hbar}x^2}$ where Q is a polynomial function.

The principal aim of this paper is to reformulate Hardy’s theorem in terms of phase-space objects. We will actually give a non-trivial restatement of Hardy’s theorem for functions $\psi \in L^2(\mathbb{R}^n)$ satisfying estimates

$$\psi(x) = \mathcal{O}(e^{-\frac{1}{2\hbar}x^T Ax}) \quad , \quad F\psi(p) = \mathcal{O}(e^{-\frac{1}{2\hbar}x^T Bx}) \quad (2)$$

where A, B are positive-definite symmetric matrices, and show that the estimates

$$\psi(x) = \mathcal{O}(e^{-\frac{1}{2\hbar}Ax^2}) \quad , \quad F\psi(p) = \mathcal{O}(e^{-\frac{1}{2\hbar}Bx^2}) \quad (3)$$

are equivalent to a single estimate

$$W\psi(x, p) = \mathcal{O}(e^{-\frac{1}{\hbar}(x^T Ax + p^T Bp)}) \quad \text{for } |x| + |p| \rightarrow \infty$$

for the Wigner transform of ψ . We will see that the geometric interpretation of the conditions on the matrices A, B is that the symplectic capacity of the “Wigner ellipsoid”

$$\mathcal{W} : x^T Ax + p^T Bp \leq \hbar$$

is at least $\frac{1}{2}\hbar$, the half of the quantum of action. This property is related to the fact that the notion of symplectic capacity is a natural tool for expressing the uncertainty principle of quantum mechanics in a symplectically covariant and intrinsic form as discussed in de Gosson [4, 5, 6]; also see de Gosson

and Luef [7, 8].); it turns out that, more generally, if a function $\psi \in L^2(\mathbb{R}^n)$ satisfies an estimate

$$W\psi(z) = \mathcal{O}(e^{-\frac{1}{\hbar}z^T M z}) \quad \text{for } |z| \rightarrow \infty$$

where $z = (x, p)$ then the symplectic capacity of $\mathcal{W} : z^T M z \leq \hbar$ must be $\geq \frac{1}{2}\hbar$.

The present article is structured as follows:

- In Section 2 we prove a multi-dimensional variant of Hardy's theorem, as a property of the symplectic spectrum of the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ extracted from the conditions (3) where A and B are positive-definite symmetric matrices; here F denotes the \hbar -dependent n -dimensional Fourier transform. In Lemma 1 (Subsection 2.1) we show that it is possible to perform a *symplectic* diagonalization of a positive-definite block-diagonal matrix using symplectic block-diagonal matrices.
- In Section 3 we give a purely geometric interpretation of Hardy's uncertainty principle in terms of the notion of the symplectic capacity, which is closely related to Gromov's non-squeezing theorem; we take the opportunity to quickly review the main definitions and properties concerning these objects. In particular, we point out that all symplectic capacities agree on phase-space ellipsoids;
- In Section 4 we restate the results above in terms of the Wigner transform, by showing that the conditions (3) are equivalent to

$$W\psi(x, p) \leq C e^{-\frac{1}{\hbar}(x^T A x + p^T B p)} \quad (4)$$

for some constant $C \geq 0$. In Subsection 4.3 we give an equivalent geometric statement of the results above in terms of the topological notion of symplectic capacity. We will in fact prove that if (1) holds for $\psi \neq 0$ then the symplectic capacity of the phase space ellipsoid $\mathcal{W} : \frac{1}{2}z^T \Sigma^{-1} z \leq 1$ is at least $\frac{1}{2}\hbar$; here Σ is the covariance matrix defined by

$$\Sigma = \begin{pmatrix} (\Delta x)^2 & 0 \\ 0 & (\Delta p)^2 \end{pmatrix}$$

where $\Delta x = \sqrt{\hbar/2a}$ and $\Delta p = \sqrt{\hbar/2b}$. The condition $ab \leq 1$ is thus equivalent to the Heisenberg inequality $\Delta x \Delta p \geq \frac{1}{2}\hbar$.

- In Section 5 we give two non-obvious extensions of the results obtained in the previous sections. The first extension (Subsection 5.1) consists in replacing the x, p coordinate system by an arbitrary “Lagrangian frame” (ℓ, ℓ') and to use the transitivity of the action of the symplectic group on the set of all such frames. In the second extension (Subsection 5.2) we consider estimates of the type $W\psi(z) \leq C e^{-\frac{1}{\hbar}Q(z)}$ where Q is a twice continuously differentiable function which is uniformly convex. We express the necessary condition on that function in terms of the symplectic capacity of the convex set $Q(z) \leq \hbar$.

Notation. We will use the shorthand notation $z = (x, p)$ for points of the phase space $\mathbb{R}^{2n} \equiv \mathbb{R}^n \times \mathbb{R}^n$. The symplectic product of two vectors $z = (x, p), z' = (x', p')$ in \mathbb{R}^{2n} is

$$\sigma(z, z') = p \cdot x - p' \cdot x$$

where the dot \cdot stands for the usual (Euclidean) scalar product; alternatively $\sigma(z, z') = Jz \cdot z'$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ (0 (resp. I) is the zero (resp. identity) matrix of order n). The symplectic group is denoted by $\text{Sp}(n)$: we have $S \in \text{Sp}(n)$ if and only if S is a real matrix of order $2n$ such that $\sigma(Sz, Sz') = \sigma(z, z')$; equivalently $S^T J S = S J S^T = J$.

When M is a symmetric matrix we will often write Mx^2, Mp^2, Mz^2 instead of $Mx \cdot x$ (or $x^T M x$), $Mp \cdot p$, $Mz \cdot z$. To express that M is symmetric and positive-definite we will use the notation $M > 0$.

F denotes the n -dimensional \hbar -dependent Fourier transform. It is the unitary operator $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ defined for $\psi \in \mathcal{S}(\mathbb{R}^n)$ by

$$F\psi(p) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} \int e^{-\frac{i}{\hbar}p \cdot x} \psi(x) d^n x. \quad (5)$$

2 Hardy’s Theorem in Dimension n

Using classical results on the simultaneous diagonalization of a pair of symmetric matrices it is possible to extend Hardy’s theorem to the case of \mathbb{R}^n (see for instance Sitaram *et al.* [19]). We are going to prove a variant of this result using a *symplectic* diagonalization; this will allow us to relate our statements to the notion of symplectic capacity later on in this work.

2.1 A symplectic diagonalization result

The following result, although being of an elementary nature is very useful. We will see that it is a refined version of Williamson's diagonalization theorem [21] in the block-diagonal case.

We make the preliminary observation that if A and B are positive definite matrices then the eigenvalues of AB are real because AB has the same eigenvalues as the symmetric matrix $A^{1/2}BA^{1/2}$.

Lemma 1 *Let $A, B > 0$. There exists $L \in GL(n, \mathbb{R})$ such that*

$$L^T AL = L^{-1}B(L^T)^{-1} = \Lambda \quad (6)$$

where $\Lambda = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$ is the diagonal matrix whose eigenvalues are the square roots of the eigenvalues $\lambda_1, \dots, \lambda_n$ of AB .

Proof. We claim that there exists $R \in GL(n, \mathbb{R})$ such that

$$R^T AR = I \text{ and } R^{-1}B(R^T)^{-1} = D \quad (7)$$

where $D = \text{diag}(\lambda_1, \dots, \lambda_n)$. In fact, first choose $P \in GL(n, \mathbb{R})$ such that $P^T AP = I$ and set $B_1^{-1} = P^T B^{-1} P$. Since B_1^{-1} is symmetric, there exists $H \in O(n, \mathbb{R})$ such that $B_1^{-1} = H^T D^{-1} H$ where D^{-1} is diagonal. Set now $R = PH^T$; we have $R^T AR = I$ and also

$$R^{-1}B(R^T)^{-1} = HP^{-1}B(P^T)^{-1}H^T = HB_1H^T = D$$

hence the equalities (7). Let $\Lambda = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$. Since

$$R^T AB(R^T)^{-1} = R^T AR(R^{-1}B(R^T)^{-1}) = D$$

the diagonal elements of D are indeed the eigenvalues of AB hence $D = \Lambda^2$. Setting $L = R\Lambda^{1/2}$ we have

$$\begin{aligned} L^T AL &= \Lambda^{1/2} R^T A R \Lambda^{1/2} = \Lambda \\ L^{-1} B (L^{-1})^T &= \Lambda^{-1/2} R^{-1} B (R^T)^{-1} \Lambda^{-1/2} = \Lambda \end{aligned}$$

hence our claim. ■

The result above is a precise statement of a classical theorem of Williamson [21] in the block-diagonal case. That theorem says that every positive-definite symmetric matrix can be diagonalized using symplectic matrices. More precisely: let M be a positive definite real $2n \times 2n$ matrix; the eigenvalues of JM are those of the antisymmetric matrix $M^{1/2}JM^{1/2}$ and are thus of the type $\pm i\lambda_j^\sigma$ with $\lambda_j^\sigma > 0$. We have:

Theorem 2 (Williamson) (i) There exists $S \in \text{Sp}(n)$ such that $S^T M S = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$ where $\Lambda = \text{diag}(\lambda_1^\sigma, \dots, \lambda_n^\sigma)$.

(ii) The symplectic matrix S is unique up to a unitary factor: if S' is another Williamson diagonalizing symplectic matrix then $S(S')^{-1} \in U(n)$.

Proof. (i) See for instance [5, 13] for “modern” proofs. (ii) See de Gosson [5]. ■

We will always arrange the λ_j^σ in decreasing order: $\lambda_1^\sigma \geq \lambda_2^\sigma \geq \dots \geq \lambda_n^\sigma$ and call $(\lambda_1^\sigma, \dots, \lambda_n^\sigma)$ the *symplectic spectrum* of the positive definite matrix M . The positive numbers λ_j^σ (which only depend on M , and not on S) are the *Williamson invariants* of M . Writing the diagonalizing symplectic matrix as $S = (X_1, \dots, X_n; Y_1, \dots, Y_n)$ where the X_j and Y_k are column vectors, the set $\mathcal{B} = \{X_1, \dots, X_n; Y_1, \dots, Y_n\}$ is called a *Williamson basis* for M (it is of course not uniquely defined in general). A Williamson basis is a symplectic basis of $(\mathbb{R}^{2n}, \sigma)$, that is $\sigma(X_j, X_k) = \sigma(Y_j, Y_k) = 0$ and $\sigma(X_j, Y_k) = \delta_{jk}$ for $1 \leq j, k \leq n$.

The following result relates Lemma 1 to Williamson’s theorem:

Lemma 3 Let $A, B > 0$. The symplectic spectrum $(\lambda_1^\sigma, \dots, \lambda_n^\sigma)$ of $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ consists of the decreasing sequence $\sqrt{\lambda_1} \geq \dots \geq \sqrt{\lambda_n}$ of square roots of the eigenvalues λ_j of AB .

Proof. Let $(\lambda_1^\sigma, \dots, \lambda_n^\sigma)$ be the symplectic spectrum of M . The λ_j^σ are the eigenvalues of

$$JM = \begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix};$$

they are thus the moduli of the zeroes of the polynomial

$$P(t) = \det(t^2 I + AB) = \det(t^2 I + D)$$

where $D = \text{diag}(\lambda_1, \dots, \lambda_n)$; these zeroes are the numbers $\pm i\sqrt{\lambda_j}$, $j = 1, \dots, n$; the result follows. ■

For L invertible set

$$M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}. \quad (8)$$

Obviously $M_L \in \text{Sp}(n)$; Lemma 1 can be restated by saying that if (A, B) is a pair of symmetric positive definite then there exists L such that

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = M_{L^T} \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} M_L. \quad (9)$$

Lemma 1 is thus a precise version of Williamson's theorem for block-diagonal positive matrices —it is not at all obvious from the statement of this theorem that such a matrix can be diagonalized using only a block-diagonal symplectic matrix!

2.2 Application to Hardy's theorem

Lemma 1 allows us to give a simple proof of a multi-dimensional version of this theorem. The following elementary remark will be useful:

Lemma 4 *Let $n > 1$. For $1 \leq j \leq n$ let f_j be a function of $(x_1, \dots, \tilde{x}_j, \dots, x_n) \in \mathbb{R}^{n-1}$ (the tilde suppressing the term it covers), and g_j a function of $x_j \in \mathbb{R}$. If*

$$h = f_1 \otimes g_1 = \dots = f_n \otimes g_n$$

then there exists a constant C such that $h = C(g_1 \otimes \dots \otimes g_n)$.

Proof. Assume that $n = 2$; then

$$h(x_1, x_2) = f_1(x_2)g_1(x_1) = f_2(x_1)g_2(x_2).$$

If $g_1(x_1)g_2(x_2) \neq 0$ then

$$f_1(x_2)/g_2(x_2) = f_2(x_1)/g_1(x_1) = C$$

hence $f_1(x_2) = Cg_2(x_2)$ and $h(x_1, x_2) = Cg_1(x_1)g_2(x_2)$. If $g_1(x_1)g_2(x_2) = 0$ then $h(x_1, x_2) = 0$ hence $h(x_1, x_2) = Cg_1(x_1)g_2(x_2)$ in all cases. The general case follows by induction on the dimension n : suppose that

$$h = f_1 \otimes g_1 = \dots = f_n \otimes g_n = f_{n+1} \otimes g_{n+1};$$

for fixed x_{n+1} the function $k = f_1 \otimes g_1 = \dots = f_n \otimes g_n$ is given by

$$k(x, x_{n+1}) = C(x_{n+1})g_1(x_1) \dots g_n(x_n).$$

Since we also have

$$k(x, x_{n+1}) = f_{n+1}(x_1, \dots, x_n)g_{n+1}(x_{n+1})$$

it follows that $C(x_{n+1}) = C$. ■

Theorem 5 Let A and B be two real positive definite matrices and $\psi \in L^2(\mathbb{R}^n)$, $\psi \neq 0$. Assume that

$$|\psi(x)| \leq C_A e^{-\frac{1}{2\hbar}Ax^2} \quad \text{and} \quad |F\psi(p)| \leq C_B e^{-\frac{1}{2\hbar}Bp^2} \quad (10)$$

for some constants $C_A, C_B > 0$. Then:

- (i) The eigenvalues λ_j , $j = 1, \dots, n$, of AB are ≤ 1 ;
- (ii) If $\lambda_j = 1$ for all j , then $\psi(x) = C e^{-\frac{1}{2\hbar}Ax^2}$ for some some complex constant C .
- (iii) If $\lambda_j < 1$ for some j then $\psi(x) = Q(x) e^{-\frac{1}{2\hbar}Ax^2}$ for some polynomial function $Q : \mathbb{R}^n \rightarrow \mathbb{C}$.

Proof. *Proof of (i).* It is of course no restriction to assume that $C_A = C_B = C$. Let L be as in Lemma 1 and order the eigenvalues of AB decreasingly: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. It suffices to show that $\lambda_1 \leq 1$. Setting $\psi_L(x) = \psi(Lx)$ we have

$$F\psi_L(p) = F\psi((L^T)^{-1}p);$$

in view of (6) in Lemma 1 condition (10) is equivalent to

$$|\psi_L(x)| \leq C e^{-\frac{1}{2\hbar}\Lambda x^2} \quad \text{and} \quad |F\psi_L(p)| \leq C e^{-\frac{1}{2\hbar}\Lambda p^2} \quad (11)$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Setting $\psi_{L,1}(x_1) = \psi_L(x_1, 0, \dots, 0)$ we have

$$|\psi_{L,1}(x_1)| \leq C e^{-\frac{1}{2\hbar}\lambda_1 x_1^2}. \quad (12)$$

On the other hand, by the Fourier inversion formula,

$$\begin{aligned} \int F\psi_L(p) dp_2 \cdots dp_n &= (2\pi\hbar)^{n/2} \iint e^{-\frac{i}{\hbar}p \cdot x} \psi_L(x) dx dp_2 \cdots dp_n \\ &= (2\pi\hbar)^{(n-1)/2} F\psi_{L,1}(p_1) \end{aligned}$$

and hence

$$|F\psi_{L,1}(p_1)| \leq C_{L,1} e^{-\frac{1}{2\hbar}\lambda_1 p_1^2} \quad (13)$$

for some constant $C_{L,1} > 0$. Applying Hardy's theorem to the inequalities (12) and (13) we must have $\lambda_1^2 \leq 1$ hence the assertion (i). *Proof of (ii).* The condition $\lambda_j = 1$ for all j means that

$$|\psi_L(x)| \leq C e^{-\frac{1}{2\hbar}x^2} \quad \text{and} \quad |F\psi_L(p)| \leq C e^{-\frac{1}{2\hbar}p^2} \quad (14)$$

for some $C > 0$. Let us keep $x' = (x_2, \dots, x_n)$ constant; the partial Fourier transform of ψ_L in the x_1 variable is $F_1\psi_L = (F')^{-1}F\psi_L$ where $(F')^{-1}$ is the inverse Fourier transform in the x' variables, hence there exists $C' > 0$ such that

$$|F_1\psi_L(x_1, x')| \leq \left(\frac{1}{2\pi\hbar}\right)^{\frac{n-1}{2}} \int |F\psi_L(p)| dp_2 \cdots dp_n \leq C' e^{-\frac{1}{2\hbar}p_1^2}.$$

Since $|\psi_L(x)| \leq C(x')e^{-\frac{1}{2\hbar}x_1^2}$ with $C(x') \leq e^{-\frac{1}{2\hbar}x'^2}$ it follows from Hardy's theorem that we can write

$$\psi_L(x) = f_1(x')e^{-\frac{1}{2\hbar}x_1^2}$$

for some real C^∞ function f_1 on \mathbb{R}^{n-1} . Applying the same argument to the remaining variables x_2, \dots, x_n we conclude that there exist C^∞ functions f_j for $j = 2, \dots, n$, such that

$$\psi_L(x) = f_j(x_1, \dots, \tilde{x}_j, \dots, x_n)e^{-\frac{1}{2\hbar}x_1^2}. \quad (15)$$

In view of Lemma 4 above we have $\psi_L(x) = C_L e^{-\frac{1}{2\hbar}x^2}$ for some constant C_L ; since $\Lambda = I = L^T A L$ we thus have $\psi(x) = C_L e^{-Ax^2/2\hbar}$ as claimed. *Proof of (iii).* Assume that $\lambda_1 < 1$ for $j \in \mathcal{J}$, \mathcal{J} a subset of $\{1, \dots, n\}$. By the same argument as in the proof of part (ii) establishing formula (15), we infer, using Hardy's theorem in the case $ab < 1$, that

$$\psi_L(x) = f_j(x_1, \dots, \tilde{x}_j, \dots, x_n) Q_j(x_j) e^{-\frac{1}{2\hbar}x_j^2}$$

where Q_j is a polynomial with degree 0 if $j \notin \mathcal{J}$. One concludes the proof using one again Lemma 4. ■

3 Geometric interpretation

Let us give a geometric interpretation of Theorem 5. We begin by making an obvious observation: Hardy's uncertainty principle can be restated by saying that if $\psi \neq 0$ then the conditions $\psi(x) = \mathcal{O}(e^{-\frac{1}{2\hbar}ax^2})$ and $F\psi(p) = \mathcal{O}(e^{-\frac{1}{2\hbar}bp^2})$ imply that the ellipse $\mathcal{W} : ax^2 + bp^2 \leq \hbar$ has area $\pi\hbar/\sqrt{ab} \geq \pi\hbar = \frac{1}{2}h$:

$$\text{Area}(\mathcal{W}) \geq \frac{1}{2}h.$$

More precisely:

If the area of the ellipse \mathcal{W} is smaller than $\frac{1}{2}h$ then $\psi = 0$; if this area equals $\frac{1}{2}h$ then $\psi(x) = Ce^{-\frac{1}{2h}ax^2}$ and if it is larger than $\frac{1}{2}h$ then $\psi(x) = Q(x)e^{-\frac{1}{2h}ax^2}$ where Q is a polynomial function.

When trying to generalize this observation to higher dimensions, one should resist the pitfall of copying the statement above *mutatis mutandis* and replacing everywhere the word “area” by “volume”. As we will see, volume is not the right answer; one has instead to use the more subtle notion of *symplectic capacity*, introduced by Ekeland and Hofer [2] following Gromov’s work [11] on pseudoholomorphic curves.

3.1 Symplectic capacities and symplectic spectrum

A symplectic capacity on the symplectic space $(\mathbb{R}^{2n}, \sigma)$ assigns to every subset Ω of \mathbb{R}^{2n} a number $c(\Omega) \geq 0$ or $+\infty$; this assignment has the four properties listed below. (We denote by $B(R)$ the ball $|z| \leq R$ and by $Z_j(R)$ the cylinder $x_j^2 + p_j^2 \leq R^2$.)

SC1 Monotonicity: $c(\Omega) \leq c(\Omega')$ if $\Omega \subset \Omega'$;

SC2 Symplectic invariance: $c(f(\Omega)) = c(\Omega)$ for every symplectomorphism f defined near Ω ;

SC3 Conformality: $c(\lambda\Omega) = \lambda^2 c(\Omega)$ if $\lambda \in \mathbb{R}$;

SC4 Nontriviality: We have $c(B(R)) = c(Z_j(R)) = \pi R^2$.

A fundamental example of symplectic capacity is provided by the “Gromov width”, defined by

$$c_{\text{Gr}}(\Omega) = \sup_{f \in \text{Symp}(n)} \{\pi r^2 : f(B(r)) \subset \Omega\} \quad (16)$$

where $\text{Symp}(n)$ is the group of all symplectomorphisms of $(\mathbb{R}^{2n}, \sigma)$. Properties (i)–(iii) and $c_{\text{Gr}}(B(R)) = \pi R^2$ are trivially verified; that we also have $c_{\text{Gr}}(Z_j(R)) = \pi R^2$ is just Gromov’s non-squeezing theorem [11] which asserts that a phase-space ball cannot be squeezed inside a symplectic cylinder with smaller radius using symplectomorphisms (but such a squeezing can, of course, be performed using general volume-preserving diffeomorphisms).

We will also use the “linear symplectic capacity” c_{lin} defined by

$$c_{\text{lin}}(\Omega) = \sup_{f \in \text{ISp}(n)} \{\pi r^2 : f(B(r)) \subset \Omega\} \quad (17)$$

where f this time ranges over the group $\text{ISp}(n)$ of all affine symplectic automorphisms of $(\mathbb{R}^{2n}, \sigma)$ (the “inhomogeneous symplectic group”). The capacity c_{lin} has the same properties as general symplectic capacities, except that it is only invariant under linear or affine symplectomorphisms.

We have the following result, which allows us to talk about *the* symplectic capacity of a phase-space ellipsoid:

Lemma 6 *For $M > 0$ let $\Omega_M = \{z \in \mathbb{R}^{2n} : Mz^2 \leq 1\}$. For any symplectic capacity c on $(\mathbb{R}^{2n}, \sigma)$ we have*

$$c(\Omega_M) = c_{\text{lin}}(\Omega_M) = \frac{\pi}{\lambda_1^\sigma} \quad (18)$$

where $\lambda_1^\sigma \geq \dots \geq \lambda_n^\sigma$ is the symplectic spectrum of M .

Proof. See for instance Hofer and Zehnder [13], Proposition 2, §2.1, p. 54 or de Gosson [5], Proposition 8.25, p. 251 (where λ_n^σ should be replaced by λ_1^σ). ■

3.2 Application to the Wigner ellipsoid

We can restate Hardy’s theorem in a very simple geometric way in terms of the symplectic capacity of the “Wigner ellipsoid” (the terminology seems to be due to Littlejohn [17]):

Proposition 7 *Let $\psi \in L^2(\mathbb{R}^n)$, $\psi \neq 0$. Assume that*

$$|\psi(x)| \leq C_A e^{-\frac{1}{2\hbar}Ax^2} \quad \text{and} \quad |F\psi(p)| \leq C_B e^{-\frac{1}{2\hbar}Bp^2}. \quad (19)$$

Then the symplectic capacity of the Wigner ellipsoid

$$\mathcal{W} : Ax^2 + Bp^2 \leq \hbar$$

satisfies $c(\mathcal{W}) \geq \frac{1}{2}h$.

Proof. Setting $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ the equation of \mathcal{W} is $Mz^2 \leq \hbar$. Consider the ellipsoid $\Omega_M : Mz^2 \leq 1$. Let $(\lambda_1^\sigma, \lambda_2^\sigma, \dots, \lambda_n^\sigma)$ be the symplectic spectrum of M ; by formula (18) in Lemma 6 we have $c(\Omega_M) = \pi/\lambda_1^\sigma$. In view of Lemma 3 $\lambda_j^\sigma = \sqrt{\lambda_j}$ where the λ_j are the eigenvalues of AB , and by Theorem 5 we must have $\lambda_j \leq 1$, hence $c(\Omega_M) \geq \pi$. Since $\mathcal{W} = \sqrt{\hbar}\Omega_M$ we have $c(\mathcal{W}) = \hbar c(\Omega_M)$ in view of the conformality property (CZ3); the result follows. ■

The result above will be extended to the Wigner distribution in next section.

4 Hardy's Theorem and Wigner's distribution

It turns out that Hardy's theorem – which involves two conditions, one about a function and the other about the Fourier transform of that function – is equivalent to a *single condition* on the *Wigner transform* of ψ . This condition will be made explicit in Theorem 9 below; let us first prove some preliminary results about Wigner transforms.

4.1 Wigner Distributions

The Wigner transform of a function was introduced by Wigner in [20], following joint work with Szilard. It is defined, for $\psi \in L^2(\mathbb{R}^n)$, by the formula

$$W\psi(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}p \cdot y} \psi\left(x + \frac{1}{2}y\right) \overline{\psi\left(x - \frac{1}{2}y\right)} d^n y. \quad (20)$$

The function $W\psi$ has a simple interpretation in terms on the theory of Weyl pseudodifferential operators. For $\psi \in L^2(\mathbb{R}^n)$ with $\|\psi\| = 1$ consider the orthogonal projection P_ψ on the ray $\{\lambda\psi : \lambda \in \mathbb{C}\}$; we have $P_\psi\phi(x) = (\phi|\psi)\psi$ for $\phi \in L^2(\mathbb{R}^n)$ hence the operator kernel of P_ψ is $K_\psi = \psi \otimes \bar{\psi}$. Writing P_ψ in Weyl operator form we have

$$P_\psi\phi(x) = \iint e^{\frac{i}{\hbar}p \cdot (x-y)} \rho_\psi\left(\frac{1}{2}(x+y), p\right) \phi(y) d^n p d^n y$$

where the symbol ρ_ψ is given by

$$\rho_\psi(x, p) = \left(\frac{1}{2\pi\hbar}\right)^n K_\psi\left(x + \frac{1}{2}y, x - \frac{1}{2}y\right) = W\psi(z).$$

More generally, one might want to consider the cross-Wigner transform (also called Wigner–Moyal transform) which associates to a pair $(\psi, \phi) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ the function

$$W(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}p \cdot y} \psi\left(x + \frac{1}{2}y\right) \overline{\phi\left(x - \frac{1}{2}y\right)} d^n y;$$

(it is the Weyl symbol of the operator defined by the kernel $\psi \otimes \bar{\phi}$); of course $W(\psi, \psi) = W\psi$. The following properties of the (cross-) Wigner transform are well-known

W1 $W(\psi, \phi) = \overline{W(\phi, \psi)}$ (hence $W\psi$ is real);

W2 If $\psi, F\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then

$$\int W\psi(z) d^n p = |\psi(x)|^2, \quad \int W\psi(z) d^n x = |F\psi(p)|^2. \quad (21)$$

Recall that the metaplectic group $\text{Mp}(n)$ is generated by the following unitary operators on $L^2(\mathbb{R}^{2n})$ ([5, 3]): the scaling operators $\widehat{M}_{L,m}$ ($L \in GL(n, \mathbb{R})$), the “chirps” \widehat{V}_P ($P = P^T$), and the modified Fourier transform $\widehat{J} = i^{-n/2}F$; by definition

$$\widehat{M}_{L,m}\psi(x) = i^m \sqrt{|\det L|} \psi(Lx) \quad , \quad \widehat{V}_P\psi(x) = e^{\frac{i}{2\hbar}Px^2} \psi(x) \quad (22)$$

(m corresponds to a choice of argument for $\det L$). $\text{Mp}(n)$ is a faithful representation of the double covering group of $\text{Sp}(n)$; the projection $\pi : \text{Mp}(n) \longrightarrow \text{Sp}(n)$ is determined by its action on the generators:

$$\pi(\widehat{M}_{L,m}) = M_L \quad , \quad \pi(\widehat{V}_P) = \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \quad , \quad \pi(\widehat{J}) = J \quad (23)$$

(M_L defined by formula (8)).

W3 Let \widehat{S} be any of the two metaplectic operators associated with $S \in \text{Sp}(n)$. The following *metaplectic covariance* formula holds:

$$W(\widehat{S}\psi)(z) = W\psi(S^{-1}z). \quad (24)$$

As a particular case of (24) we have

$$W(F\psi)(z) = W(\widehat{J}\psi)(z) = W\psi(-Jz). \quad (25)$$

Recall that the Heisenberg operator $\widehat{T}(z_0)$ is defined, for $z_0 = (x_0, p_0) \in \mathbb{R}^{2n}$, by

$$\widehat{T}(z_0)\psi(x) = e^{\frac{i}{\hbar}(p_0 \cdot x - \frac{1}{2}p_0 \cdot x_0)} \psi(x - x_0). \quad (26)$$

W4 We have

$$W(\widehat{T}(z_0)\psi)(z) = W\psi(z - z_0). \quad (27)$$

The Wigner transform behaves well under tensor products: if $x = (x', x'')$ with $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{n-k}$ and $\psi' \in L^2(\mathbb{R}^k)$, $\psi'' \in L^2(\mathbb{R}^{n-k})$, then

$$W(\psi' \otimes \psi'') = W'\psi' \otimes W''\psi'' \quad (28)$$

where W' and W'' are the Wigner transforms on $L^2(\mathbb{R}^k)$ and $L^2(\mathbb{R}^{n-k})$, respectively. More generally, if W, W', W'' now denote cross-Wigner distributions:

$$W(\psi' \otimes \psi'', \phi' \otimes \phi'') = W'(\psi', \phi') \otimes W''(\psi'', \phi''). \quad (29)$$

4.2 Wigner transform and Hermite functions

The k -th state of the quantum harmonic oscillator with classical Hamiltonian $H(x, p) = \frac{1}{2}(x^2 + p^2)$ is the Hermite function

$$\psi_k(x) = h_k\left(\frac{1}{\sqrt{\hbar}}x\right)e^{-\frac{1}{2\hbar}x^2}$$

(h_k the k -th Hermite polynomial). One shows that

$$W(\psi_k, \psi_\ell)(z) = e^{-\frac{1}{\hbar}|z|^2} \sum_{j=0}^{\min(k, \ell)} C_j(k, \ell) z^{\ell-j} \bar{z}^{k-j} \quad (30)$$

where the $C_j(k, \ell)$ are real constants and z is identified with $x + ip \in \mathbb{C}^n$ in the right-hand side (see *e.g.* [3], p. 66–67). Notice that in particular

$$|W(\psi_k, \psi_\ell)(z)| \leq e^{-\frac{1}{\hbar}|z|^2} P_{k\ell}(|z|) \quad (31)$$

where $P_{k\ell}$ is a real polynomial of degree $k + \ell$.

We will need the following Lemma which says that the Wigner transform of a Hermite function is the product of an exponential and of a polynomial with positive leading coefficient. (For related results see [15]).

Lemma 8 *Let Q be a (complex) polynomial function on \mathbb{R}^n and $\psi(x) = Q(x)e^{-\frac{1}{2\hbar}Ax^2}$, $A > 0$. Then:*

(i) *The Wigner transform of ψ is given by*

$$W\psi(x, p) = R(z_A, \bar{z}_A)e^{-\frac{1}{\hbar}|z_A|^2} \quad (32)$$

where R is a polynomial function and $z_A = A^{1/2}x + iA^{-1/2}p$ ($A^{1/2}$ the positive square root of A);

(ii) *In particular*

$$|W\psi(x, p)| \leq T(|z_A|)e^{-\frac{1}{\hbar}|z_A|^2} \quad (33)$$

where T is a polynomial with real coefficients.

Proof. (i) Let us set $\varphi = \widehat{M}_{A^{-1/2}, 0}\psi$ where $\widehat{M}_{A^{-1/2}, 0} \in \text{Mp}(n)$ is defined by (22). Thus

$$\varphi(x) = P(x)e^{-\frac{1}{2\hbar}|x|^2} \quad \text{with} \quad P(x) = \sqrt{\det A^{-1}}Q(A^{-1/2}x)$$

and we have, by property (24) of the Wigner transform and the first formula (23),

$$W\psi(z) = W\varphi(A^{1/2}x, A^{-1/2}p). \quad (34)$$

Writing $P(x) = \sum_{\alpha} a_{\alpha} x^{\alpha}$ (we are using multi-index notation $\alpha = (\alpha_1, \dots, \alpha_n)$, $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$) we have

$$\varphi(x) = \sum_{\alpha} a_{\alpha} \varphi^{\alpha}(x) \quad , \quad \varphi^{\alpha} = \varphi_1^{\alpha_1} \otimes \cdots \otimes \varphi_n^{\alpha_n}$$

with $\varphi_j^{\alpha_j}(x_j) = x_j^{\alpha_j} e^{-x_j^2/2}$. By the sesquilinearity of the cross-Wigner transform we get

$$W\varphi = \sum_{\alpha, \beta} a_{\alpha} \bar{a}_{\beta} W(\varphi^{\alpha}, \varphi^{\beta}) \quad (35)$$

and by the tensor product property (29)

$$\begin{aligned} W(\varphi^{\alpha}, \varphi^{\beta}) &= W(\varphi_1^{\alpha_1} \otimes \cdots \otimes \varphi_n^{\alpha_n}, \varphi_1^{\beta_1} \otimes \cdots \otimes \varphi_n^{\beta_n}) \\ &= W(\varphi_1^{\alpha_1}, \varphi_1^{\beta_1}) \otimes \cdots \otimes W(\varphi_n^{\alpha_n}, \varphi_n^{\beta_n}). \end{aligned}$$

The Hermite functions ψ_k forming an orthonormal basis of $L^2(\mathbb{R})$ each $\varphi_j^{\alpha_j}$ is a finite linear combination of these functions; using again sesquilinearity and applying formula (30) there exist polynomials P_{α_j, β_j} such that

$$W(\varphi_j^{\alpha_j}, \varphi_j^{\beta_j})(x_j, p_j) = P_{\alpha_j, \beta_j}(z_j, \bar{z}_j) e^{-\frac{1}{\hbar}|z_j|^2}$$

with $z_j = x_j + ip_j$ and hence

$$W(\varphi^{\alpha}, \varphi^{\beta})(z) = P_{\alpha\beta}(z, \bar{z}) e^{-\frac{1}{\hbar}|z|^2}$$

where $P_{\alpha\beta} = P_{\alpha_1, \beta_1} \otimes \cdots \otimes P_{\alpha_n, \beta_n}$ is a polynomial function in $2n$ variables. It follows from (35) that

$$W\varphi(z) = \sum_{\alpha, \beta} a_{\alpha} \bar{a}_{\beta} P_{\alpha\beta}(z, \bar{z}) e^{-\frac{1}{\hbar}|z|^2} = R(z, \bar{z}) e^{-\frac{1}{\hbar}|z|^2} \quad (36)$$

and hence, in view of (34),

$$W\psi(z) = \sum_{\alpha, \beta} a_{\alpha} \bar{a}_{\beta} P_{\alpha\beta}(A^{1/2}x, A^{-1/2}p) e^{-\frac{1}{\hbar}(A^{-1}x^2 + Ap^2)}$$

as claimed. (ii) Since $W\varphi$ is a real function we have $W\varphi(z) \leq |W\varphi|$ and hence, taking (36) into account,

$$W\varphi(z) \leq |R(z, \bar{z})| e^{-\frac{1}{\hbar}|z|^2} \leq T(|z|) e^{-\frac{1}{\hbar}|z|^2}$$

which concludes the proof in view of (34). ■

4.3 Phase-space formulation of Hardy's theorem

When dealing with Gaussian functions related to “squeezed coherent states” we obtain Gaussian estimates where the quadratic form in the exponent no longer is block-diagonal. For instance, the Wigner transform of a Gaussian of the type

$$\psi_{X,Y}(x) = e^{-\frac{1}{2\hbar}(X+iY)x^2}$$

(X and Y real symmetric, $X > 0$) is given by the formula

$$W\psi_{X,Y}(z) = (\pi\hbar)^{-n/2}(\det X)^{-1/2}e^{-\frac{1}{2\hbar}Gz^2} \quad (37)$$

where the matrix G is given by

$$G = \begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix} \quad (38)$$

(see Proposition 8.4, p. 263, in de Gosson [5]); the result seems to go back to Bastiaans according to Littlejohn [17]). An important observation is that G is a positive-definite symplectic matrix as follows from the obvious factorization

$$G = S^T S \quad \text{with} \quad S = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{pmatrix} \in \text{Sp}(n). \quad (39)$$

Setting $\Sigma = \frac{\hbar}{2}G^{-1}$ the ellipsoid $\mathcal{W} : \frac{1}{2}\Sigma^{-1}z^2 \leq 1$ is the set $\{z : S^T S z^2 \leq \hbar\}$; \mathcal{W} is thus the image of the ball $B(\sqrt{\hbar})$ by a linear symplectic transformation, and thus has symplectic capacity $\frac{1}{2}h$.

Let us now show, as claimed in the introduction, that Hardy's uncertainty principle for a function ψ is equivalent to a condition on its Wigner transform $W\psi$.

Theorem 9 *Let $\psi \in L^2(\mathbb{R}^n)$ and A, B two positive real $n \times n$ matrices. Let $C_A, C_B > 0$. The condition*

$$|\psi(x)| \leq C_A e^{-\frac{1}{2\hbar}Ax^2} \quad \text{and} \quad |F\psi(p)| \leq C_B e^{-\frac{1}{2\hbar}Bp^2} \quad (40)$$

is equivalent to the existence of a constant $C_{AB} > 0$ such that

$$W\psi(z) \leq C_{AB} e^{-\frac{1}{\hbar}(Ax^2 + Bp^2)}. \quad (41)$$

Proof. In view of properties (21) of $W\psi$, condition (41) implies that there exist constants $C_A, C_B \geq 0$ such that

$$|\psi(x)|^2 \leq C_A^2 e^{-\frac{1}{\hbar}Ax^2}, \quad |F\psi(p)|^2 \leq C_B^2 e^{-\frac{1}{\hbar}Bp^2}$$

hence (41) \implies (40). Let us prove that conversely (40) \implies (41). Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of AB . If there exists j such that $\lambda_j > 1$ then $\psi = 0$ by Theorem 5 and (41) is trivially verified. We may thus assume from now on that $\lambda_j \leq 1$ for $j = 1, \dots, n$. Let $L \in GL(n, \mathbb{R})$ be as in Lemma 1, that is $L^T A L = L^{-1} B (L^T)^{-1} = \Lambda$ where Λ is the diagonal matrix whose eigenvalues are the $\sqrt{\lambda_j}$. We have, setting $\psi_L(x) = \psi(Lx)$ as in the proof of Theorem 5,

$$|\psi_L(x)| \leq C_A e^{-\frac{1}{2\hbar}\Lambda x^2} \quad \text{and} \quad |F\psi_L(p)| \leq C_B e^{-\frac{1}{2\hbar}\Lambda p^2}.$$

Since $\lambda_j \leq 1$ for all $j = 1, \dots, n$ Theorem 5 implies that we have

$$\psi_L(x) = Q_L(x) e^{-\frac{1}{\hbar}\Lambda x^2}$$

where Q_L is a polynomial function which is constant when all the λ_j are equal to one. It follows, by Lemma 8 that

$$W\psi_L(z) \leq R_L(y_1, \dots, y_n) e^{-\frac{1}{\hbar}(\Lambda x^2 + \Lambda^{-1}p^2)}$$

where R_L is a polynomial function with positive leading coefficient and $y_j = \lambda_j x_j^2 + \lambda_j^{-1} p_j^2$. Let $C_L > 0$ be a constant such that

$$W\psi_L(z) \leq C_L \prod_{j=1}^n y_j^{m_j} e^{-\frac{1}{\hbar}(\Lambda x^2 + \Lambda^{-1}p^2)};$$

for every $\varepsilon > 0$ there exists $C_{L,\varepsilon} > 0$ such that

$$\prod_{j=1}^n y_j^{m_j} e^{-\frac{1}{\hbar}(\Lambda x^2 + \Lambda^{-1}p^2)} \leq C_{L,\varepsilon} e^{-\frac{1}{\hbar}((\Lambda - \varepsilon)x^2 + \Lambda^{-1}p^2)}$$

(we are writing $\Lambda - \varepsilon$ for $\Lambda - I\varepsilon$) and hence

$$W\psi_L(z) \leq C_{L,\varepsilon} e^{-\frac{1}{\hbar}((\Lambda - \varepsilon)x^2 + \Lambda^{-1}p^2)}. \quad (42)$$

Applying the same argument to $W(F\psi_L)(x, p) = W\psi_L(-p, x)$ we also have

$$W\psi_L(z) \leq C_{L,\varepsilon} e^{-\frac{1}{\hbar}(\Lambda x^2 + (\Lambda^{-1} - \varepsilon)p^2)}. \quad (43)$$

Since

$$\sup[(\Lambda - \varepsilon)x^2 + \Lambda^{-1}p^2, \Lambda x^2 + (\Lambda^{-1} - \varepsilon)p^2] = \Lambda x^2 + \Lambda^{-1}p^2$$

the inequalities (42)–(43) imply that we have

$$W\psi_L(z) \leq C_{L,\varepsilon} e^{-\frac{1}{\hbar}(\Lambda x^2 + \Lambda^{-1}p^2)},$$

since $\psi_L(x) = \psi(Lx)$ this is just condition (41). ■

Theorem 9 has the following consequence which contains Hardy's theorem as a particular case (we have proved a particular case of that result, using different methods, in [7, 8]).

Corollary 10 *Let $\psi \in L^2(\mathbb{R}^n)$, $\psi \neq 0$. Assume that there exists a positive-definite real matrix M , a vector $a \in \mathbb{R}^{2n}$ and $C > 0$ such that*

$$W\psi(z) \leq C e^{-\frac{1}{\hbar}(Mz^2 + 2a \cdot z)}. \quad (44)$$

Then the ellipsoid $\mathcal{W} = \{z : Mz^2 \leq \hbar\}$ has symplectic capacity $c(\mathcal{W}) \geq \frac{1}{2}\hbar$ (equivalently $\lambda_1^\sigma \leq \hbar$).

Proof. Assume first that $a = 0$. Let $S \in \text{Sp}(n)$ be such that $S^T M S$ is in Williamson diagonal form

$$D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$$

with $\Lambda = \text{diag}(\lambda_1^\sigma, \dots, \lambda_n^\sigma)$, $\lambda_1^\sigma \geq \dots \geq \lambda_n^\sigma$. Choose $\widehat{S} \in \text{Mp}(n)$ with projection S . It follows from the metaplectic covariance property (24) of the Wigner transform that

$$W(\widehat{S}^{-1}\psi)(z) \leq C e^{-\frac{1}{\hbar}(\Lambda x^2 + \Lambda p^2)}.$$

Applying Theorem (9) there exist constants $C_1, C_2 > 0$ such that

$$|\widehat{S}^{-1}\psi(x)| \leq C_1 e^{-\frac{1}{2\hbar}\Lambda x^2} \quad \text{and} \quad |F\widehat{S}^{-1}\psi(x)| \leq C_2 e^{-\frac{1}{2\hbar}\Lambda p^2}.$$

In view of the multidimensional Hardy theorem 5 we must have $\lambda_j^\sigma \leq 1$ for $j = 1, \dots, n$ hence (Lemma 6) $c(\mathcal{W}) = \pi\hbar/\lambda_1^\sigma \geq \pi\hbar$ which concludes the proof in the case $a = 0$. Assume now a is arbitrary, and set $Q(z) = Mz^2 + 2a \cdot z$; choosing $S \in \text{Sp}(n)$ and $\widehat{S} \in \text{Mp}(n)$ as above we have $Q(Sz) = Dz^2 + 2b \cdot z$ where $b = S^T a$; completing squares we get

$$Q(Sz) = D(z + D^{-1}b)^2 - D^{-1}b^2.$$

It follows that for a new constant C' we have

$$W(\widehat{S}^{-1}\psi)(z) \leq C e^{-\frac{1}{\hbar}Q(Sz)} \leq C' e^{-\frac{1}{\hbar}D(z+D^{-1}b)^2}.$$

We next observe that

$$W(\widehat{S}^{-1}\psi)(z - D^{-1}b) = W(\widehat{T}(b)\widehat{S}^{-1}\psi)(z)$$

where $\widehat{T}(D^{-1}b)$ is a Heisenberg operator; we thus have, using (27),

$$W(\widehat{T}(D^{-1}b)\widehat{S}^{-1}\psi)(z) \leq C' e^{-\frac{1}{\hbar}Dz^2}$$

and it now suffices to apply the case $a = 0$ to $\psi' = \widehat{T}(D^{-1}b)\widehat{S}^{-1}\psi$. ■

It is instructive to see how the sub-Gaussian estimate (44) is related to the uncertainty principle of Quantum Mechanics. Setting $\Sigma = \frac{\hbar}{2}M^{-1}$ we can define the multivariate Gaussian probability density

$$\rho(z) = \left(\frac{1}{2\pi}\right)^n (\det \Sigma)^{-1/2} e^{-\frac{1}{2}\Sigma^{-1}z^2}$$

and view Σ as a statistical covariance matrix. The Wigner ellipsoid $\mathcal{W} : Mz^2 \leq \hbar$ is identical with the set $\mathcal{W}_\Sigma = \{z : \frac{1}{2}\Sigma^{-1}z^2 \leq 1\}$.

One of us has proven in [5] the following result:

Proposition 11 *The two following conditions are equivalent:*

- (i) $c(\mathcal{W}) = c(\mathcal{W}_\Sigma) \geq \frac{1}{2}\hbar$
- (ii) *The Hermitian matrix $\Sigma + i\frac{\hbar}{2}J$ is positive semidefinite.*

Write now Σ in block-matrix form

$$\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XP} \\ \Sigma_{PX} & \Sigma_{PP} \end{pmatrix}$$

where Σ_{XX} , $\Sigma_{XP} = \Sigma_{PX}^T$, and Σ_{PP} are the $n \times n$ partial covariance matrices $\Sigma_{XX} = (\text{Cov}(x_j, x_k))_{j,k}$, $\Sigma_{XP} = \Sigma_{PX}^T = (\text{Cov}(x_j, p_k))_{j,k}$, and $\Sigma_{PP} = (\text{Cov}(p_j, p_k))_{j,k}$; the covariances are defined with respect to the probability density ρ_Σ : setting $z_j = x_j$ for $1 \leq j \leq n$ and $z_j = p_j$ for $n+1 \leq j \leq 2n$ we have

$$\text{Cov}(z_j, z_k) = \int z_j z_k \rho(z) d^{2n}z - \int z_j \rho(z) d^{2n}z \int z_k \rho(z) d^{2n}z$$

The conditions (i) and (ii) in Proposition 11 above are equivalent to

$$(\Delta x_j)^2 (\Delta p_j)^2 \geq (\text{Cov}(x_j, p_j))^2 + \frac{1}{4}\hbar^2 \quad (45)$$

for $j = 1, \dots, n$ (see Narcowich [18] and de Gosson [5], and the references therein). The inequalities (45) (known in the quantum-mechanical literature as the Schrödinger–Robertson uncertainty relations) are a precise form of the usual text-book Heisenberg inequalities $\Delta X_j \Delta P_j \geq \frac{1}{2} \hbar$ to which they reduce if one neglects correlations.

5 Two Extensions of Hardy’s Theorem

5.1 Restatement in an arbitrary Lagrangian frame

The statements of Hardy’s uncertainty principle we have been considering correspond to a particular choice of coordinates namely the positions x and the momenta p . These statements thus correspond to the choice of frame (ℓ_X, ℓ_P) where ℓ_X is the horizontal Lagrangian plane $\mathbb{R}^n \times \{0\}$ and ℓ_P the vertical Lagrangian plane $\{0\} \times \mathbb{R}^n$. This choice is of course to a great extent arbitrary, and we are going to extend our results to arbitrary Lagrangian frames.

Recall that a subspace ℓ of the phase space $(\mathbb{R}^{2n}, \sigma)$ is called *isotropic*, if the symplectic form σ vanishes identically on ℓ . If ℓ has maximal dimension n , then ℓ is called a *Lagrangian plane*. The set of all Lagrangian planes in $(\mathbb{R}^{2n}, \sigma)$ is called the *Lagrangian Grassmannian* of $(\mathbb{R}^{2n}, \sigma)$ and denoted by $\text{Lag}(n)$.

The subgroup of all symplectic matrices S such that $S\ell = \ell$ is called the *stabilizer* of ℓ and denoted by $St(\ell)$. Note that $S \in St(\ell)$ if and only if $S^T \in St(J\ell)$.

If ℓ and ℓ' are Lagrangian planes in $(\mathbb{R}^{2n}, \sigma)$ satisfying $\ell \cap \ell' = 0$, then (ℓ, ℓ') are called *transversal*; equivalently $\ell \oplus \ell' = \mathbb{R}^{2n}$. We will call a pair (ℓ, ℓ') of transversal Lagrangian planes a *Lagrangian frame*. An important property is that the symplectic group $\text{Sp}(n)$ acts transitively not only on the Lagrangian Grassmannian $\text{Lag}(n)$, but also on the set of Lagrangian frames: if (ℓ_1, ℓ'_1) and (ℓ_2, ℓ'_2) are pairs of Lagrangian planes satisfying $\ell_1 \cap \ell'_1 = \ell_2 \cap \ell'_2 = 0$, then there exists $S \in \text{Sp}(n)$ such that $(\ell_2, \ell'_2) = (S\ell_1, S\ell'_1)$ (see de Gosson [5]).

We interpret the marginal properties (21) of Wigner’s distribution in terms of the horizontal and vertical Lagrangian plane ℓ_X and ℓ_P : if $\psi, F\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then we can rewrite (21) as

$$\int_{\ell_X} W\psi(z) dz = |F\psi(p)|^2 \quad , \quad \int_{\ell_P} W\psi(z) dz = |\psi(x)|^2.$$

Recall that the metaplectic covariance property (W5) of the Wigner distribution tells us that if $\widehat{S} \in \text{Mp}(n)$ has projection S on $\text{Sp}(n)$ then $W(\widehat{S}\psi)(z) = W\psi(S^{-1}z)$. Therefore, we get

$$\int_{\ell_X} W(\widehat{S}\psi)(z)dz = \int_{\ell_X} W\psi(S^{-1}z)dz,$$

or, equivalently,

$$\int_{S\ell_X} W\psi(z)dz = |\widehat{S}\psi(z)|^2 \text{ for } z \in S\ell_X.$$

Since $J\ell_X = \ell_P$ and $\widehat{J} = F$ we get the analogous results for the Lagrangian plane SJ . Note that $(S\ell_X, SJ\ell_X)$ are a transversal pair of Lagrangian planes. In other words the transitivity of the symplectic group on $\text{Lag}(n)$ allows to translate a statement about ℓ_X and ℓ_P into a statement of another pair of Lagrangian planes (ℓ, ℓ') . One just has to choose the correct S to go from (ℓ_X, ℓ_P) to (ℓ, ℓ') . Then the statements about ψ and $F\psi$ translate into statements about $\widehat{S}\psi$ and $\widehat{S} \circ \widehat{J}$.

5.2 The case of convex exponents

We are going to extend the Corollary 10 of Theorem 9 to the case where the inequality (44) is replaced by

$$W\psi(z) \leq C e^{-\frac{1}{\hbar}Q(z)} \quad (46)$$

where Q is a uniformly convex function on \mathbb{R}^{2n} ; we will assume that $Q(0) = 0$ (the case $Q(0) \neq 0$ is trivially reduced to this case by changing the constant C). Using the same trick as in the proof of Corollary 10 we may moreover assume, replacing ψ by $\widehat{T}(z_0)\psi$ for a suitably chosen $z_0 \in \mathbb{R}^{2n}$, that

$$Q'(0) = \nabla_z Q(0) = 0.$$

Let us briefly recall a few basic facts on convex functions (see Andrei [1] for a concise review of the topic). A function $Q : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is strictly convex if we have

$$Q(\alpha z + (1 - \alpha)z') < \alpha Q(z) + (1 - \alpha)Q(z')$$

for $0 < \alpha < 1$ and $z \neq z'$. If the function Q is of class C^2 this condition is equivalent to $Q''(z) > 0$ for all $z \in \mathbb{R}^{2n}$ ($Q''(z)$ is the Hessian matrix of

Q calculated at z). In what follows we will make in addition the following uniformity assumption:

$$\text{There exists } c > 0 \text{ such that } Q''(z_0)z^2 \geq c|z|^2 \text{ for all } z, z_0 \in \mathbb{R}^{2n}. \quad (47)$$

Diagonalizing $Q''(z_0)$ by an orthogonal matrix $H(z_0)$ this condition is easily seen to be equivalent to the condition $\lambda_Q > 0$ where

$$\lambda_Q = \inf_{z_0 \in \mathbb{R}^{2n}} \{\lambda(z) : \lambda(z) \text{ is an eigenvalue of } Q''(z)\}; \quad (48)$$

the smallest constant c for which (47) holds is then precisely λ_Q .

Proposition 12 *Under the same assumptions on Q as above let $\psi \in L^2(\mathbb{R}^{2n})$, $\psi \neq 0$ be such that $W\psi(z) \leq Ce^{-\frac{1}{\hbar}Q(z)}$ for some $C > 0$. Then the convex set*

$$\mathcal{C} = \{z \in \mathbb{R}^{2n} : Q(z) \leq \hbar\} \quad (49)$$

satisfies $c(\mathcal{C}) \geq \frac{1}{2}\hbar$ for every symplectic capacity c on $(\mathbb{R}^{2n}, \sigma)$.

Proof. Let us begin by showing that we have $0 < \lambda_Q \leq 2$. Since the uniformity assumption is equivalent to $\lambda_Q > 0$ it suffices to show that $\lambda_Q \leq 2$. In view of the mean value theorem we have

$$Q(z) = Q(0) + Q'(0) \cdot z + \frac{1}{2}Q''(z')z^2 = \frac{1}{2}Q''(z')z^2 \quad (50)$$

where z' lies on the line segment joining 0 to z ; we have $Q''(z')z^2 \geq \lambda_Q|z|^2$ hence

$$Q(z) \geq \frac{1}{2}\lambda_Q|z|^2 \quad (51)$$

so that

$$W\psi(z) \leq Ce^{-\frac{1}{\hbar}Q(z)} \leq C'e^{-\frac{1}{2\hbar}\lambda_Q|z|^2}.$$

The symplectic spectrum of $\lambda_Q I$ consists of the point λ_Q hence we must have $\lambda_Q \leq 2$ choosing $M = \frac{1}{2}\lambda_Q I$ in Corollary 10. The proposition follows: in view of (51) the condition $Q(z) \leq \hbar$ implies $\frac{1}{2}\lambda_Q|z|^2 \leq \hbar$ hence the set \mathcal{C} contains the ball $|z|^2 \leq 2\hbar/\lambda_Q \leq \hbar$; one concluded using the monotonicity of symplectic capacities (Property (SC1)). ■

Let \mathcal{C} be a compact and convex set. We recall (John [16]) that there exists a unique ellipsoid \mathcal{W} contained in \mathcal{C} having maximal volume. This ellipsoid, called the ‘‘John ellipsoid’’ [?], has the property that

$$\mathcal{W} \subset \mathcal{C} \subset z_0 + 2n(\mathcal{W} - z_0) \quad (52)$$

where z_0 is the center of \mathcal{W} . The result above has the following immediate consequence:

Corollary 13 *Let \mathcal{W} be the John ellipsoid associated to the convex and compact set (49). We have $c(\mathcal{W}) \geq \frac{1}{2}h$.*

Proof. The uniform convexity of Q implies that the convex set $\mathcal{C} = \{z : Q(z) \leq \hbar\}$ is compact (Andrei [1]); John's ellipsoid is thus well-defined. In the proof of Proposition 12 we have seen that \mathcal{C} contains the ball $|z|^2 \leq 2\hbar/\lambda_Q \leq \hbar$; this ball is contained in John's ellipsoid. The result follows again in view of the monotonicity of a symplectic capacity. ■

Proposition 12 has another interesting non-trivial consequence. In [13] Hofer and Zehnder construct a symplectic capacity c_{HZ} having the following property:

If Ω is a convex and compact subset of \mathbb{R}^{2n} with smooth boundary $\partial\Omega$ then

$$c_{\text{HZ}}(\Omega) = \inf_{\gamma} \left\{ \oint_{\gamma} p dx \right\}$$

where γ ranges over the set of all periodic Hamiltonian orbits on $\partial\Omega$.

Corollary 14 *Under the same assumptions on Q and ψ as above we have*

$$\oint_{\gamma} p dx \geq \frac{1}{2}h \tag{53}$$

for every periodic Hamiltonian orbit γ on the hypersurface defined by $Q(z) = \hbar$.

Proof. The boundary of \mathcal{C} is precisely the hypersurface defined by $Q(z) = \hbar$. In view of Proposition 12 we have $c(\mathcal{C}) \geq \frac{1}{2}h$ for every symplectic capacity c hence, choosing $c = c_{\text{HZ}}$

$$\inf_{\gamma} \left\{ \oint_{\gamma} p dx \right\} \geq \frac{1}{2}h$$

which proves (53). ■

6 Concluding Remarks

We have proved a n -dimensional version of Hardy's uncertainty principle, and showed that it is equivalent to a statement on the Wigner distribution of

a sub-Gaussian state. The extension of this result to more general estimates involving convex exponents in Subsection 5.2 opens the door to the study of non-trivial properties for the density matrix of quantum systems. Such applications are very important for the understanding of non-linear quantum optics and the theory of entangled quantum states.

We mention that Hogan and Lakey [14] have done a very interesting analysis of the interplay between Hardy’s uncertainty principle and rotations. It would certainly be useful to restate their results in our context; we leave this possibility for further work. Also, Gröchenig and Zimmermann [9] have studied Gaussian estimates from the point of view of the short-time Fourier transforms; the methods they use are very different from ours.

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