

SPECTRAL AND REGULARITY PROPERTIES  
OF AN OPERATOR CALCULUS RELATED TO  
LANDAU QUANTIZATION

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### **Abstract**

The theme of this work is that the theory of charged particles in a uniform magnetic field can be generalized to a large class of operators if one uses an extended class of Weyl operators which we call "Landau–Weyl pseudodifferential operators". The link between standard Weyl calculus and Landau–Weyl calculus is made explicit by the use of an infinite family of intertwining "windowed wavepacket transforms"; this makes possible the use of the theory of modulation spaces to study various regularity properties. Our techniques allow us not only to recover easily the eigenvalues and eigenfunctions of the Hamiltonian operator of a charged particle in a uniform magnetic field, but also to prove global hypoellipticity results and to study the regularity of the solutions to Schrödinger equations.

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## 1 Introduction

The aim of this Communication is to compare the properties of a partial differential operator (or, more generally, a Weyl pseudodifferential operator)

$A = a^w(x, -i\hbar\partial_x)$  with those of the operator  $\tilde{A} = a(X^{\gamma,\mu}, Y^{\gamma,\mu})$  obtained by replacing formally  $x$  and  $-i\hbar\partial_x$  by the vector fields

$$X^{\gamma,\mu} = \frac{\gamma}{2}x + \frac{i\hbar}{\mu}\partial_y, \quad Y^{\gamma,\mu} = \frac{\mu}{2}y - \frac{i\hbar}{\gamma}\partial_x \quad (1)$$

where  $\gamma$  and  $\mu$  are real scalars such that  $\gamma\mu \neq 0$ . A typical situation of physical interest is the following: choose for  $A$  the harmonic oscillator Hamiltonian

$$H_{\text{har}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2 \quad (2)$$

and  $\gamma = 1$ ,  $\mu = m\omega$ ; defining the Larmor frequency  $\omega_L = \omega/2$  we obtain  $H_{\text{har}}(X^{1,m\omega}, Y^{1,m\omega}) = H_{\text{sym}}$  where

$$H_{\text{sym}} = -\frac{\hbar^2}{2m}\Delta_{x,y} - i\hbar\omega_L \left( y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right) + \frac{m\omega_L^2}{2}(x^2 + y^2) \quad (3)$$

is the Hamiltonian operator in the symmetric gauge of a charged particle moving in the  $x, y$  plane under the influence of constant magnetic field orthogonal to the  $x, y$  plane. Another interesting situation occurs if one takes  $\gamma = 2$ ,  $\mu = 1$ . In this case

$$H_{\text{har}}(X^{2,1}, Y^{2,1})\Psi = H_{\text{har}} \star_{\hbar} \Psi$$

where  $\star_{\hbar}$  is the Moyal product familiar from deformation quantization.

Of course these observations are not of earthshaking importance unless we can find a procedure for comparing the properties of both operators  $A$  and  $\tilde{A}$ . In fact, as a rule, the initial operator  $A$  is less complicated than its counterpart  $\tilde{A}$  so one would like to deduce the properties of the second from those of the first. For this we first have to find a procedure allowing us to associate to a function  $\psi \in L^2(\mathbb{R}^n)$  a function  $\Psi \in L^2(\mathbb{R}^{2n})$ ; that correspondence should be linear, and intertwine in some way the operators  $A$  and  $\tilde{A}$ ; notice that the request for linearity excludes the choice  $\Psi = W\psi$  ( $W$  the Wigner transform). It turns out that there exist many procedures for transforming a function of, say,  $x$  into a function of twice as many variables; the Bargmann transform is an archetypical (and probably the oldest) example of such a procedure. However, the Bargmann transform (and its variants) is not sufficient to recover all the spectral properties of  $\tilde{A}$  from those of  $A$ . For example, it is well known that the ‘‘Landau levels’’ of the magnetic operator  $H_{\text{sym}}$  are infinitely degenerate, so it is illusory to attempt to recover the corresponding eigenvectors from those of  $H_{\text{har}}$  (the rescaled Hermite functions) using one single transform! This difficulty is of course related to the

fact that no isometry from  $L^2(\mathbb{R}^n)$  to  $L^2(\mathbb{R}^{2n})$  can take a basis of the first space to a basis of the other, (intuitively  $L^2(\mathbb{R}^n)$  is “much smaller” than  $L^2(\mathbb{R}^{2n})$ ). We will overcome this difficulty by constructing an *infinity* of isometries  $\mathcal{U}_\phi$  parametrized by the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$ ; these isometries are defined in terms of the cross-Wigner transform, or equivalently, in terms of the windowed Fourier transform familiar from time-frequency and Gabor analysis. We will therefore call them *windowed wavepacket transforms*. For instance, in the case  $\gamma = 2$ ,  $\mu = 1$  corresponding to the Moyal product, these isometries are (up to a normalization factor) just the mappings  $\psi \mapsto \Psi = W(\psi, \phi)$  where  $W(\psi, \phi)$  is the cross-Wigner transform.

One of the goals of this paper is to emphasize the great potential of a particular class of function spaces, namely Feichtinger’s modulation spaces, whose elements can be defined in terms of decay properties of their windowed Wigner transform; these spaces have turned out to be the proper setting for the discussion of pseudodifferential operators in the last decade, and have allowed to prove (or to recover) in an elementary way many results which would otherwise requires the use of “hard analysis”. Although modulation spaces and their usefulness is well-known in time-frequency an Gabor analysis they have not received a lot of attention in quantum mechanics (they have however found some applications in the study of Schrödinger operators).

This article is structured as follows: in Sections 2–6 we develop the theory in the case  $\lambda = \mu = 1$ , that is we work with the quantization rules

$$X = \frac{1}{2}x + i\hbar\partial_y \quad , \quad Y = \frac{1}{2}y - i\hbar\partial_x$$

and their higher-dimensional generalizations

$$X_j = \frac{1}{2}x_j + i\hbar\partial_{y_j} \quad , \quad Y_j = \frac{1}{2}y_j - i\hbar\partial_{x_j}.$$

Furthermore we introduce the class of modulation spaces and recall some of their basic properties that are of relevance in the present investigation. In Section 7 we show how the general case (1) and its multi-dimensional generalization

$$X_j^{\gamma,\mu} = \frac{\gamma}{2}x + \frac{i\hbar}{\mu}\partial_{y_j} \quad , \quad Y_j^{\gamma,\mu} = \frac{\mu}{2}y_j - \frac{i\hbar}{\gamma}\partial_{x_j}$$

can be reduced to this one. We thereafter apply the previous study to deformation quantization.

## Notation

Functions (or distributions) on  $\mathbb{R}^n$  will usually be denoted by lower-case Greek letters  $\psi, \phi, \dots$  while functions (or distributions) on  $\mathbb{R}^{2n}$  will be denoted by upper-case Greek letters  $\Psi, \Phi, \dots$ . We denote the inner product on  $L^2(\mathbb{R}^n)$  by  $(\psi|\phi)$  and the inner product on  $L^2(\mathbb{R}^{2n})$  by  $(\Psi|\Phi)$ ; the associated norms are denoted by  $\|\psi\|$  and  $\|\Psi\|$ , respectively. Distributional brackets are denoted  $\langle \cdot, \cdot \rangle$  in every dimension.

The standard symplectic form on the vector space  $\mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{R}^{2n}$  is denoted by  $\sigma$ ; it is given, for  $z = (x, y), z' = (x', y')$ , by the formula  $\sigma(z, z') = Jz \cdot z'$  where  $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ . The symplectic group of  $(\mathbb{R}^{2n}, \sigma)$  is denoted by  $\text{Sp}(2n, \mathbb{R})$ .

## 2 Modulation Spaces: A Short Review

For a rather complete treatment of the theory of modulation spaces we refer to Gröchenig's book [26]. The main point we want to do in this section is that, in contrast to the standard treatment of Weyl calculus, one can use with profit Feichtinger's algebra  $M^1(\mathbb{R}^n)$  and its weighted variants  $M_{v_s}^1(\mathbb{R}^n)$  as spaces of test functions instead of the Schwartz class  $\mathcal{S}(\mathbb{R}^n)$ . We also remark that a good class of pseudodifferential symbols is provided by the modulation spaces  $M_{v_s}^{1,\infty}(\mathbb{R}^{2n})$ ; it has recently been proved that they coincide with the so-called *Sjöstrand classes*. In particular they contain the Hörmander class  $S_{0,0}^0(\mathbb{R}^{2n})$ .

### 2.1 Main definitions

Roughly speaking, modulation spaces are characterized by the matrix coefficients of the Schrödinger representation of the Heisenberg group. Recall that the *Heisenberg–Weyl operators*  $T(z_0)$  are defined, for  $z_0 = (x_0, p_0) \in \mathbb{R}^{2n}$  by:

$$T(z_0)\psi(x) = e^{\frac{i}{\hbar}(y_0 \cdot x - \frac{1}{2}y_0 \cdot x_0)}\psi(x - x_0). \quad (4)$$

We have

$$T(z_1 + z_2) = e^{-\frac{i}{2\hbar}\sigma(z_1, z_2)}T(z_1)T(z_2) \quad z_1, z_2 \in \mathbb{R}^{2n} \quad (5)$$

hence  $z \mapsto T(z)$  is a projective representation of  $\mathbb{R}^{2n}$ , it is called the Schrödinger representation of the Heisenberg group. Let  $\psi, g$  be in  $L^2(\mathbb{R}^n)$ . Then the matrix coefficient of the Schrödinger representation is given by

$$V_\phi^\hbar \psi(z_0) = \langle \psi, T(z_0)\phi \rangle = e^{\frac{i}{2\hbar}y_0 \cdot x_0} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}y_0 \cdot x} \psi(x) \overline{\phi(x - x_0)} dx. \quad (6)$$

When  $\hbar = 1/2\pi$  it is, up to the exponential in front of the integral, the *short-time Fourier transform* (STFT)  $V_\phi\psi$  familiar from time-frequency analysis:

$$V_\phi\psi(z_0) = \int_{\mathbb{R}^n} e^{2\pi iy_0 \cdot x} \psi(x) \overline{\phi(x - x_0)} dx \quad (7)$$

( $V_\phi\psi$  is also called the “voice transform” or “sliding transform” in signal theory).

In what follows  $p$  is a non-negative real number and  $m$  a real weight function on  $\mathbb{R}^{2n}$ . By definition a distribution  $\psi \in \mathcal{S}'(\mathbb{R}^n)$  is in the modulation space  $M_m^p(\mathbb{R}^n)$  if there exists  $\phi \in \mathcal{S}(\mathbb{R}^n)$ ,  $\phi \neq 0$ , such that  $V_\phi\psi \in L_m^p(\mathbb{R}^{2n})$ , that is if

$$\|\psi\|_{M_m^p} = \left( \int |V_\phi\psi(z)|^p m(z) dz \right)^{1/p} < \infty. \quad (8)$$

The essential point is that that the definition above is independent of the choice of the window  $\phi$ : if it holds for one such window, it holds for all. Formula (8) defines a norm on  $M_m^p(\mathbb{R}^n)$  and the replacement of  $\phi$  by another window leads to an equivalent norm. Modulation spaces are Banach spaces for the topology defined by (8). (See [13, 26] for the general theory).

## 2.2 Two examples

In what follows  $\langle \cdot \rangle^s$  denotes the weight function defined by

$$\langle z \rangle^s = (1 + |z|^2)^{s/2}. \quad (9)$$

- **The modulation spaces  $M_{v_s}^1(\mathbb{R}^n)$ .**

They are defined as follows: for  $s \geq 0$  we consider the weighted  $L^1(\mathbb{R}^{2n})$  space

$$L_{v_s}^1(\mathbb{R}^{2n}) = \{\Psi \in \mathcal{S}'(\mathbb{R}^{2n}) : \|\Psi\|_{L_{v_s}^1} < \infty\} \quad (10)$$

where  $\|\cdot\|_{L_{v_s}^1}$  is the norm given by

$$\|\Psi\|_{L_{v_s}^1}^2 = \int_{\mathbb{R}^{2n}} |\Psi(z)| \langle z \rangle^s dz. \quad (11)$$

The corresponding modulation space is

$$M_{v_s}^1(\mathbb{R}^n) = \{\psi \in \mathcal{S}'(\mathbb{R}^n) : V_\phi\psi \in L_{v_s}^1(\mathbb{R}^{2n})\} \quad (12)$$

where  $V_\phi$  is the STFT (7) and  $\phi$  is an *arbitrary* element of the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$ ; the formula

$$\|\psi\|_{M_{v_s}^1}^\phi = \|V_\phi\psi\|_{L_{v_s}^1} \quad (13)$$

defines a norm on  $M_{v_s}^1(\mathbb{R}^n)$  and if we change  $\phi$  into another element of  $\mathcal{S}(\mathbb{R}^n)$  we obtain an equivalent norm (Proposition 12.1.2 (p.246) in [26]). The space  $M_{v_s}^1(\mathbb{R}^n)$  can also be very simply described in terms of the Wigner transform: we have

$$M_{v_s}^1(\mathbb{R}^n) = \{\psi \in \mathcal{S}'(\mathbb{R}^n) : W\psi \in L_{v_s}^1(\mathbb{R}^{2n})\}. \quad (14)$$

In the case  $s = 0$  we obtain the *Feichtinger algebra*  $M^1(\mathbb{R}^n) = M_0^1(\mathbb{R}^n)$ . We have the inclusions

$$\mathcal{S}(\mathbb{R}^n) \subset M^1(\mathbb{R}^n) \subset C^0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \quad (15)$$

The Feichtinger algebra can be used with profit as a space of test functions; its dual Banach space

$$M^\infty(\mathbb{R}^n) = \{\psi \in \mathcal{S}'(\mathbb{R}^n) : \sup_{z \in \mathbb{R}^{2n}} \left( \langle z \rangle^N |V_\phi \psi| \right) < \infty \text{ for all } N\} \quad (16)$$

contains the Dirac distribution  $\delta$ . Furthermore  $M^1(\mathbb{R}^n)$  is the smallest Banach space invariant under Heisenberg–Weyl operators (it is the space of integrable vectors of the Heisenberg–Weyl representation). Note that step functions are not in  $M^1(\mathbb{R}^n)$  but triangle functions (which are the convolutions of two step functions) are. For these reasons the modulation spaces  $M_{v_s}^1(\mathbb{R}^n)$  are considerably larger classes of test functions than the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$ .

The spaces  $M_{v_s}^1(\mathbb{R}^n)$ , besides other properties, are invariant under Fourier transform and more generally, under the action of the metaplectic group. They are also preserved by rescalings:

**Lemma 1** *Let  $\lambda$  be a real number different from zero and set  $\psi_\lambda(x) = \psi(\lambda x)$ . We have  $\psi_\lambda \in M_{v_s}^p(\mathbb{R}^n)$  if and only if  $\psi \in M_{v_s}^p(\mathbb{R}^n)$ .*

**Proof.** It immediately follows from definition (7) of the STFT that we have

$$V_\phi \psi_\lambda(z) = \lambda^{-n} V_{\phi_{1/\lambda}} \psi(z/\lambda) \quad (17)$$

hence, performing a simple change of variable,

$$\int_{\mathbb{R}^{2n}} |V_\phi \psi_\lambda(z)|^p \langle z \rangle^s dz = \lambda^n \int_{\mathbb{R}^{2n}} |V_{\phi_{1/\lambda}} \psi(z)|^p \langle \lambda z \rangle^s dz.$$

Since  $\langle \lambda z \rangle^s \leq (1 + \lambda^2)^{s/2} \langle z \rangle^s$  it follows that there exists a constant  $C_\lambda > 0$  such that

$$\int_{\mathbb{R}^{2n}} |V_\phi \psi_\lambda(z)|^p \langle z \rangle^s dz \leq C_\lambda \int_{\mathbb{R}^{2n}} |V_{\phi_{1/\lambda}} \psi(z)|^p \langle z \rangle^s dz;$$

the integral in the right hand side is convergent if and only if  $\psi \in M_{v_s}^p(\mathbb{R}^n)$ , proving the necessity of the condition. That the condition is sufficient follows replacing  $\psi_\lambda$  by  $\psi$  in the argument above. ■

- **The modulation spaces**  $M_{v_s}^{1,\infty}(\mathbb{R}^{2n})$ .

They are defined by the condition  $a \in M_{v_s}^{1,\infty}(\mathbb{R}^{2n})$  if and only if we have

$$\|a\|_{M_{v_s}^{1,\infty}}^\Phi = \int_{\mathbb{R}^{2n}} \sup_{z \in \mathbb{R}^{2n}} |V_\Phi a(z, \zeta)| \langle \zeta \rangle^{-s} d\zeta < \infty$$

for one (and hence all)  $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$ ; here  $V_\Phi$  is the STFT of functions on  $\mathbb{R}^{2n}$ . The use of the letter  $a$  for the elements of  $M_{v_s}^{1,\infty}(\mathbb{R}^{2n})$  suggests that this space could be used as a symbol class. This is indeed the case; we mention that when  $s = 0$  one recovers the so-called ‘‘Sjöstrand classes’’ [44, 45] which have been studied and developed by Gröchenig [27, 28] from the point of view of modulation space theory. As a symbol class is rather large; for instance

$$C^{2n+1}(\mathbb{R}^{2n}) \subset M^{1,\infty}(\mathbb{R}^{2n}) = M_0^{1,\infty}(\mathbb{R}^{2n}). \quad (18)$$

The following result, follows from Theorem 4.1 and its Corollary 4.2 in [28]; it clearly demonstrates the usefulness of the spaces  $M_{v_s}^{1,\infty}(\mathbb{R}^{2n})$  in the theory of pseudodifferential operators:

**Proposition 2** (i) *An operator  $A$  with Weyl symbol  $a \in M_{v_s}^{1,\infty}(\mathbb{R}^{2n})$  is bounded on every modulation space  $M_{v_s}^1(\mathbb{R}^n)$ ; (ii) If  $a \in M_0^{1,\infty}(\mathbb{R}^{2n})$  then  $A$  maps  $L^1(\mathbb{R}^n)$  into Feichtinger’s algebra  $M_0^1(\mathbb{R}^n)$ .*

(It is actually proven in [28] that  $A$  is bounded on any modulation space  $M_m^{p,q}(\mathbb{R}^n)$  when  $m$  is an arbitrary moderate weight).

### 2.3 Duality and kernel theorems

The dual of  $M_{v_s}^1(\mathbb{R}^n)$  is the Banach space  $M_{1/v_s}^\infty(\mathbb{R}^n)$  with the norm

$$\|\psi\|_{M_{1/v_s}^\infty} = \sup_{z \in \mathbb{R}^{2n}} (V_\phi \psi(z) | \langle z \rangle^s) < \infty. \quad (19)$$

Note that the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  is the projective limit of the spaces  $\{M_{v_s}^1(\mathbb{R}^n) : s \geq 0\}$  and consequently  $\mathcal{S}'(\mathbb{R}^n)$  has a description as inductive limit of the spaces  $\{M_{1/v_s}^\infty(\mathbb{R}^n) : s \geq 0\}$ , i.e.

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_{s \geq 0} M_{v_s}^1(\mathbb{R}^n) \quad \text{and} \quad \mathcal{S}'(\mathbb{R}^n) = \bigcup_{s \geq 0} M_{1/v_s}^\infty(\mathbb{R}^n) \quad (20)$$

with  $(\|\cdot\|_{M_{v_s}^1})_{s \geq 0}$  and  $(\|\cdot\|_{M_{1/v_s}^\infty})_{s \geq 0}$  as family of seminorms for  $\mathcal{S}(\mathbb{R}^n)$  and  $\mathcal{S}'(\mathbb{R}^n)$ , respectively.

Therefore results about the class of modulation spaces  $M_{v_s}^1(\mathbb{R}^n)$  and its dual  $M_{1/v_s}^\infty(\mathbb{R}^n)$  translate into corresponding results about the Schwartz class  $\mathcal{S}(\mathbb{R}^n)$  and the tempered distributions  $\mathcal{S}'(\mathbb{R}^n)$ . The great relevance of the pair  $\mathcal{S}(\mathbb{R}^n)$  and  $\mathcal{S}'(\mathbb{R}^n)$  comes from the kernel theorem and Feichtinger showed that for the pair of modulation spaces  $M_{v_s}^1(\mathbb{R}^n)$  and  $M_{1/v_s}^\infty(\mathbb{R}^n)$  there also exists a kernel theorem (see [26], §14.4, for a short proof):

**Theorem 3 (Kernel theorem)** *Let  $A$  be a continuous operator  $M_{v_s}^1(\mathbb{R}^n) \longrightarrow M_{1/v_s}^\infty(\mathbb{R}^n)$ . There exists a distribution  $\mathcal{K}_A \in M_{1/v_s}^\infty(\mathbb{R}^{2n})$  such that*

$$\langle A\psi, \phi \rangle = \langle \mathcal{K}_A, \psi \otimes \bar{\phi} \rangle \quad \text{for } \phi, \psi \in M_{v_s}^1(\mathbb{R}^n). \quad (21)$$

As a consequence, using the intersections (20) we get the following version of the Schwartz kernel theorem.

**Theorem 4** *Let  $A$  be a continuous linear operator from  $\mathcal{S}(\mathbb{R}^n)$  to  $\mathcal{S}'(\mathbb{R}^n)$ . Then  $A$  extends to a bounded operator from  $M_{v_s}^1(\mathbb{R}^n)$  to  $M_{1/v_s}^\infty(\mathbb{R}^n)$  for some  $s \geq 0$ .*

Therefore our framework covers the traditional setting of pseudo-differential calculus.

### 3 Landau–Weyl Calculus

For the reader's convenience we begin by quickly reviewing the basics of standard Weyl calculus. See for instance [36, 46] for details and proofs.

#### 3.1 Review of standard Weyl calculus

In view of the kernel theorem above, there exists, for every linear continuous operator  $A : M_{v_s}^1(\mathbb{R}^n) \longrightarrow M_{1/v_s}^\infty(\mathbb{R}^n)$ , a distribution  $\mathcal{K}_A \in M_{1/v_s}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  such that  $A\psi(x) = \langle \mathcal{K}_A(x, \cdot), \psi \rangle$  for every  $\psi \in M_{v_s}^1(\mathbb{R}^n)$ .

By definition the contravariant symbol of  $A$  is the distribution  $a$  defined by the Fourier transform

$$a(x, y) = \int_{\mathbb{R}^n} e^{-\frac{i}{h}y \cdot \eta} \mathcal{K}_A(x + \frac{1}{2}\eta, x - \frac{1}{2}\eta) d\eta \quad (22)$$

(the integral is interpreted in the distributional sense) and we can thus write formally

$$A\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{\frac{i}{\hbar}y(x-x')} a\left(\frac{1}{2}(x+x'), y\right) \psi(x') dx' dy; \quad (23)$$

strictly speaking this formula only makes sense when  $a \in M_{v_s}^1(\mathbb{R}^{2n})$ ; when this is not the case the double integral has to be reinterpreted in some way, for instance as a repeated or ‘‘oscillatory’’ integral; see for instance Trèves’ book [50] for an exposition of various techniques which are useful in this context.

By definition the covariant symbol  $a_\sigma$  of  $A$  is the symplectic Fourier transform

$$a_\sigma(z) = F_\sigma a(z) = \left(\frac{1}{2\pi\hbar}\right)^n \left\langle e^{-\frac{i}{\hbar}\sigma(z,\cdot)}, a(\cdot) \right\rangle. \quad (24)$$

Using the covariant symbol we can rewrite (23) as an operator-valued (Bochner) integral

$$A = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z) T(z) dz \quad (25)$$

where  $T(z)$  is the Heisenberg–Weyl operator (4).

Note that Weyl operators are composed in the following way. Assume that  $A$  and  $B$  are mappings on  $M_{v_s}^1(\mathbb{R}^n)$ ; then  $C = BA$  is defined and its contravariant and covariant symbols are given by the formulae

$$c(z) = \left(\frac{1}{4\pi\hbar}\right)^{2n} \iint_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} e^{\frac{i}{2\hbar}\sigma(u,v)} a\left(z + \frac{1}{2}u\right) b\left(z - \frac{1}{2}v\right) dudv \quad (26)$$

$$c_\sigma(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{\frac{i}{2\hbar}\sigma(z,z')} a_\sigma(z - z') b_\sigma(z') dz'. \quad (27)$$

The last two equations have a natural interpretation in terms of involutive representations of the twisted group algebra  $L^1(\mathbb{R}^{2n}, \chi)$  for the 2-cocycle  $\chi(z, z') = e^{\frac{i}{2\hbar}\sigma(z, z')}$ . Namely, the unitary representation of the Heisenberg group by the Heisenberg–Weyl operators  $T(z)$  gives an involutive faithful representation of  $L^1(\mathbb{R}^{2n}, \chi)$  via

$$\pi_{\text{int}}(a) = \int_{\mathbb{R}^{2n}} a(z) T(z) dz \quad (28)$$

for  $a \in L^1(\mathbb{R}^{2n})$ . In representation theory  $\pi_{\text{int}}$  is called the *integrated representation* of the representation  $T(z)$ . The product of  $\pi_{\text{int}}(a)$  and  $\pi_{\text{int}}(b)$

for  $a, b \in L^1(\mathbb{R}^{2n})$  yields another element  $\pi_{\text{int}}(c)$  of  $L^1(\mathbb{R}^{2n}, \chi)$ , where  $c$  is obtained from  $a$  and  $b$  by *twisted convolution*:

$$c(z) = a \natural b(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{\frac{i}{2\hbar}\sigma(z, z')} a_\sigma(z - z') b_\sigma(z') dz' \quad (29)$$

(see for instance [51]). Consequently, the composition of two operators in the Weyl calculus is actually the twisted convolution of their covariant symbols in the twisted group algebra  $L^1(\mathbb{R}^{2n}, \chi)$ .

Two particularly nice features of the Weyl pseudodifferential calculus are the following:

- Assume that  $A : M_{v_s}^1(\mathbb{R}^n) \longrightarrow M_{1/v_s}^\infty(\mathbb{R}^n)$ . Then the contravariant symbol of  $A^*$  is complex conjugate to that of  $A$ . In particular, the Weyl operator  $A$  is self-adjoint if and only if its contravariant symbol  $a$  is real;
- For every  $s \in \text{Sp}(2n, \mathbb{R})$  there exists a unitary operator  $S$ , uniquely defined up to a complex factor, such that the Weyl operator  $B$  with contravariant symbol  $b = a \circ s^{-1}$  is given by the formula

$$B = SAS^{-1}; \quad (30)$$

$S$  can be chosen as a multiple of any of the two metaplectic operators covering  $s$ .

We recall that the metaplectic group  $\text{Mp}(2n, \mathbb{R})$  is a faithful unitary representation of the twofold connected covering of  $\text{Sp}(2n, \mathbb{R})$ ; for a detailed account of the well-known (and less well-known) properties of  $\text{Mp}(2n, \mathbb{R})$  see [20], Chapter 7.

### 3.2 Landau–Weyl operators

In our discussion of the Weyl calculus we stressed its interpretation in terms of the integrated representation of the Heisenberg–Weyl operators. A basic result about the representation theory of groups says that there is a one-to-one correspondence between projective representations of a group and integrated representations of the twisted group algebra of the group. Therefore a new representation of the Heisenberg group yields a new kind of calculus for pseudo-differential operators. In a series of papers on phase space Schrödinger equations the first author has implicitly made use of this fact. In the following we want to present these results in terms of integrated representation of a representation of the Heisenberg group on  $L^2(\mathbb{R}^{2n})$ .

We define unitary operators  $\tilde{T}(z)$  on  $L^2(\mathbb{R}^{2n})$  by the formula

$$\tilde{T}(z_0)\Psi(z) = e^{-\frac{i}{2\hbar}\sigma(z, z_0)}\Psi(z - z_0). \quad (31)$$

We point out that these operators satisfy the relation

$$\tilde{T}(z_0 + z_1) = e^{-\frac{i}{2\hbar}\sigma(z, z_0)}\tilde{T}(z_0)\tilde{T}(z_1)$$

which is formally similar to the relation (5) for the Heisenberg–Weyl operators (4). In fact, in [20], Chapter 10, one of us has shown that these operators can be used to construct an unitary irreducible representation of the Heisenberg group  $H_n$  on (infinitely many) closed subspace(s) of  $L^2(\mathbb{R}^{2n})$ . This is achieved using the wavepacket transforms we define in Section 4.

We now *define* the operator  $\tilde{A} : \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$  by replacing  $T(z)$  with  $\tilde{T}(z)$  in formula (25):

$$\tilde{A} = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z)\tilde{T}(z)dz, \quad (32)$$

i.e. as integrated representation of  $\tilde{T}(z)$ .

**Definition 5** *We will call the operator  $\tilde{A}$  defined by (32) the Landau–Weyl (for short: LW) operator with symbol  $a$  (or: associated with the Weyl operator  $A$ ).*

This terminology is motivated by the fact that the magnetic operator (3) appears as a particular case of these operators, choosing for  $a$  the harmonic oscillator Hamiltonian. Let us in fact determine the contravariant symbol of  $\tilde{A}$ , viewed as Weyl operator  $M_{v_s}^1(\mathbb{R}^{2n}) \longrightarrow M_{1/v_s}^\infty(\mathbb{R}^{2n})$ . Because of the importance of this result for the rest of this paper we give it the status of a theorem:

**Theorem 6** *Let  $a$  be the contravariant symbol of the Weyl operator  $A$ . Let  $(z, \zeta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$  and  $z = (x, y)$ ,  $\zeta = (p_x, p_y)$ . The contravariant symbol of  $\tilde{A}$ , viewed as a Weyl operator  $M_{v_s}^1(\mathbb{R}^{2n}) \longrightarrow M_{1/v_s}^\infty(\mathbb{R}^{2n})$  is given by the formula*

$$\tilde{a}(z, \zeta) = a\left(\frac{1}{2}z - J\zeta\right) = a\left(\frac{1}{2}x - p_y, \frac{1}{2}y + p_x\right). \quad (33)$$

**Proof.** The kernel of  $\tilde{A}$  is given by the formula

$$\mathcal{K}_{\tilde{A}}(z, u) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} e^{\frac{i}{2\hbar}\sigma(z, u)} a_\sigma(z - u) \quad (34)$$

as is easily seen by performing the change of variables  $u = z - z_0$  in definition (32) and noting that  $\sigma(z, z - u) = -\sigma(z, u)$ . We have (cf. formula (22))

$$\tilde{a}(z, \zeta) = \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\zeta \cdot \eta} \mathcal{K}_{\tilde{A}}(z + \frac{1}{2}\eta, z - \frac{1}{2}\eta) d\eta$$

hence, using the identity  $\sigma(z + \frac{1}{2}\eta, z - \frac{1}{2}\eta) = -\sigma(z, \eta)$ ,

$$\tilde{a}(z, \zeta) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\zeta \cdot \eta} e^{-\frac{i}{2\hbar}\sigma(z, \eta)} a_\sigma(\eta) d\eta \quad (35)$$

where  $a_\sigma$  is the covariant symbol of  $A$ . By definition (24) of the symplectic Fourier transform we have

$$e^{-\frac{i}{\hbar}\zeta \cdot \eta} a_\sigma(\eta) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\zeta \cdot \eta} e^{-\frac{i}{\hbar}\sigma(\eta, z)} a(z) dz$$

hence, observing that  $\sigma(\eta, z) + \zeta \cdot \eta = \sigma(\eta, z + J\zeta)$ ,

$$e^{-\frac{i}{\hbar}\zeta \cdot \eta} a_\sigma(\eta) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\sigma(\eta, z)} T_{J\zeta} a(z) dz$$

where  $T_{J\zeta} a(z) = a(z - J\zeta)$ , that is

$$e^{-\frac{i}{\hbar}\zeta \cdot \eta} a_\sigma(\eta) = F_\sigma(T_{J\zeta} a)(\eta).$$

Formula (35) can thus be rewritten as

$$\tilde{a}(2z, \zeta) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\sigma(z, \eta)} F_\sigma(T_{J\zeta} a)(\eta) d\eta$$

hence  $\tilde{a}(2z, \zeta) = T_{J\zeta} a(z)$  (the symplectic Fourier transform is involutive); formula (33) follows. ■

Here are two immediate consequences of the result above. The first says that  $\tilde{A}$  is self-adjoint if and only  $A$  is; the second says that the LW operators compose as the usual Weyl operators.

**Corollary 7** (i) *The operator  $\tilde{A}$  is self-adjoint if and only  $a$  is real;* (ii) *The contravariant symbol of  $\tilde{C} = \tilde{A}\tilde{B}$  is given by  $\tilde{c}(z) = c(\frac{1}{2}z - J\zeta)$  where  $c$  is the contravariant symbol of  $C = AB$ .*

**Proof.** To prove property (i) it suffices to note that  $\tilde{a}$  is real if and only  $a$  is. Property (ii) immediately follows from (26) and (33). ■

Another consequence of these results is a statement about Landau–Weyl operators on  $M_{v_s}^1(\mathbb{R}^n)$ . A well-known result due to Feichtinger (see Gröchenig [24] for a proof) asserts that a Weyl pseudodifferential operator bounded on  $M_{v_s}^1(\mathbb{R}^n)$  is of trace-class. Consequently the same results holds for operators in the Landau–Weyl calculus. Therefore we can compute the trace of these operators by integrating their kernel along the diagonal

**Corollary 8** *Let  $\tilde{A}$  be a bounded selfadjoint operator on  $M_{v_s}^1(\mathbb{R}^n)$  with kernel  $\mathcal{K}_{\tilde{A}}$ . The trace of  $\tilde{A}$  is given by*

$$\mathrm{Tr}(\tilde{A}) = \int_{\mathbb{R}^{2n}} \mathcal{K}_{\tilde{A}}(z, z) dz. \quad (36)$$

**Remark 9** *We emphasize that trace formulas of the type (36) are usually not true for arbitrary trace-class operators (see the very relevant discussion of “trace formulas” in Reed and Simon [40].)*

### 3.3 Symplectic covariance and metaplectic operators

The symplectic covariance property (30) carries over to the LW calculus: for every  $\tilde{S} \in \mathrm{Mp}(4n, \mathbb{R})$  with projection  $s$  we have

$$\tilde{S}\tilde{T}(z)\tilde{S}^{-1} = \tilde{T}(sz) \quad , \quad \tilde{S}\tilde{A}\tilde{S}^{-1} = \tilde{B} \quad (37)$$

where  $\tilde{B}$  corresponds to  $b = a \circ s^{-1}$ .

Metaplectic operators are Weyl operators in their own right (see [19, 20]). Let us determine the corresponding LW operators.

**Proposition 10** *Let  $S \in \mathrm{Mp}(2n, \mathbb{R})$  have projection  $s \in \mathrm{Sp}(2n, \mathbb{R})$ . If  $\det(s - I) \neq 0$  then*

$$\tilde{S} = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma^S(z) \tilde{T}(z) dz \quad (38)$$

where the function  $a_\sigma^S$  is given by

$$a_\sigma^S(z) = \frac{i^{\nu(S)}}{\sqrt{|\det(s - I)|}} \exp\left(\frac{i}{2\hbar} M_s z \cdot z\right) \quad (39)$$

with  $M_s = \frac{1}{2}J(s + I)(s - I)^{-1}M_s^T$ . The integer  $\nu(S)$  is the class modulo 4 of the Conley–Zehnder index [21, 22] of a path joining the identity to  $S$  in  $\mathrm{Sp}(2n, \mathbb{R})$ .

**Proof.** In [19] one of us showed that every  $S \in \text{Mp}(2n, \mathbb{R})$  with  $\det(s-I) \neq 0$  can be written in the form

$$S = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma^S(z) T(z) dz$$

where  $a_\sigma^S$  is given by (39). Property (i) follows. ■

The operators  $\tilde{S}$  are metaplectic operators belonging to  $\text{Mp}(4n, \mathbb{R})$ ; we will not prove this fact here, but rather focus on a class of elementary operators which will be very useful for defining the general parameter dependent LW calculus in Section 7:

**Lemma 11** For  $(\gamma, \mu) \in \mathbb{R}^2$ ,  $\gamma\mu \neq 0$ , let  $\tilde{S}^{\gamma,\mu}$  be the unitary operator on  $L^2(\mathbb{R}^{2n})$  defined by

$$\tilde{S}^{\gamma,\mu} \Psi(x, y) = |\gamma\mu|^n \Psi(\gamma x, \mu y). \quad (40)$$

We have  $\tilde{S}^{\gamma,\mu} \in \text{Mp}(4n, \mathbb{R})$ ,  $(\tilde{S}^{\gamma,\mu})^{-1} = \tilde{S}^{\frac{1}{\gamma}, \frac{1}{\mu}}$ , and the projection of  $\tilde{S}^{\gamma,\mu}$  onto  $\text{Sp}(4n, \mathbb{R})$  is the diagonal matrix

$$S^{\gamma,\mu} = \text{diag}(\gamma^{-1}I, \mu^{-1}I, \gamma I, \mu I) \quad (41)$$

( $I$  the  $n \times n$  identity).

**Proof.** That  $\tilde{S}^{\gamma,\mu} \in \text{Mp}(4n, \mathbb{R})$  and formula (41) are standard results from the theory of metaplectic operators [20, 32, 33]. ■

## 4 Windowed Wavepacket Transforms

In this section we treat *resolutions of identity* from a representation theoretic point of view. This approach has been of great relevance in various works in mathematics and physics (see e.g. [15, 43] for a very general discussion of the topic). In the terminology of [43] we investigate in the present section coherent vectors and coherent projections generated by the square integrable representations  $T(z)$  and  $\tilde{T}(z)$ . The square-integrability of the representation  $T(z)$  of the Heisenberg group on  $L^2(\mathbb{R}^n)$  is the *Moyal identity*:

$$((V_{\phi_1} \psi_1 | V_{\phi_2} \psi_2)) = (\psi_1 | \psi_2) \overline{(\phi_1 | \phi_2)} \quad (42)$$

for  $\phi_1, \phi_2, \psi_1, \psi_2$  in  $L^2(\mathbb{R}^n)$ . Note that Moyal's identity is equivalent to the equality:

$$(\psi_1 | \psi_2) \overline{(\phi_1 | \phi_2)} = \left( \iint_{\mathbb{R}^{2n}} \langle f, T(z) \psi_1 \rangle T(z) \psi_2 dz | \psi_2 \right). \quad (43)$$

Setting  $\psi_1 = \psi$  and assuming that  $(\phi_1|\phi_2) \neq 0$  this equality becomes a resolution of the identity:

$$\psi = (\phi_1|\phi_2)^{-1} \iint_{\mathbb{R}^{2n}} (\psi|T(z)\phi_1)T(z)\phi_2 dz. \quad (44)$$

In the language of frames in Hilbert spaces, this resolution of the identity amounts to the statement that the set  $\{T(z)\psi : z \in \mathbb{R}^{2n}\}$  is a tight frame for  $L^2(\mathbb{R}^n)$  (see [26] for a thorough discussion of frames in time-frequency analysis). In the present setting we can always find  $\phi_1$  in  $M_{v_s}^1(\mathbb{R}^n)$  or in  $\mathcal{S}(\mathbb{R}^n)$ , i.e. there exist tight frames  $\{T(z)\psi_1 : z \in \mathbb{R}^{2n}\}$  for  $L^2(\mathbb{R}^n)$  with good phase space localization. The main purpose of this section is to discuss the consequences of the square-integrability of  $\tilde{T}(z)$  on  $L^2(\mathbb{R}^{2n})$ .

Unless otherwise specified  $\phi$  will denote a function in  $\mathcal{S}(\mathbb{R}^n)$  such that  $\|\phi\| = 1$ ; we will call  $\phi$  a ‘‘window’’.

#### 4.1 Definition and functional properties

By definition the wavepacket transform  $\mathcal{U}_\phi$  on  $L^2(\mathbb{R}^n)$  with window  $\phi$  is defined by

$$\mathcal{U}_\phi\psi(z) = \left(\frac{\pi\hbar}{2}\right)^{n/2} W(\psi, \phi)\left(\frac{1}{2}z\right), \quad \psi \in \mathcal{S}'(\mathbb{R}^n); \quad (45)$$

here  $W(\psi, \phi)$  is the cross-Wigner distribution, defined for  $\psi, \phi \in L^2(\mathbb{R}^n)$  by

$$W(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}y \cdot \eta} \psi\left(x + \frac{1}{2}\eta\right) \overline{\phi\left(x - \frac{1}{2}\eta\right)} d\eta. \quad (46)$$

We observe for further use that  $\mathcal{U}_\phi\psi$  can be written

$$\mathcal{U}_\phi\psi(z) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} (\widehat{\Pi}\left(\frac{1}{2}z\right)\psi|\phi) \quad (47)$$

where  $\widehat{\Pi}(z_0)$  is the Grossmann–Royer operator [31, 41] defined by

$$\widehat{\Pi}(z_0)\psi(x) = e^{\frac{2i}{\hbar}y_0(x-x_0)}\psi(2x_0 - x). \quad (48)$$

It is useful to have a result showing how the windowed wavepacket transform behaves under the action of symplectic linear automorphisms.

**Proposition 12** *Let  $s \in \text{Sp}(2n, \mathbb{R})$  and  $\psi \in L^2(\mathbb{R}^n)$ . We have*

$$\mathcal{U}_\phi\psi(s^{-1}z) = \mathcal{U}_{S\phi}(S\psi)(z) \quad (49)$$

where  $S$  is any of the two operators in the metaplectic group  $\text{Mp}(2n, \mathbb{R})$  covering  $s$ .

**Proof.** It immediately follows from the well-known covariance formula

$$W(\psi, \phi) \circ s^{-1} = W(S\psi, S\phi)$$

satisfied by the cross-Wigner distribution (see for instance de Gosson [20], Proposition 7.14, p.207). ■

The following theorem, part of which was proven in [20, 22], summarizes the main functional analytical properties of the wavepacket transform.

**Theorem 13** *The wavepacket transform  $\mathcal{U}_\phi$  is a partial isometry from  $L^2(\mathbb{R}^n)$  into  $L^2(\mathbb{R}^{2n})$ . More explicitly, the wavepacket transform has the following properties: (i)  $\mathcal{U}_\phi$  is a linear isometry of  $L^2(\mathbb{R}^n)$  onto a closed subspace  $\mathcal{H}_\phi$  of  $L^2(\mathbb{R}^{2n})$ ; (ii) Let  $\mathcal{U}_\phi^* : L^2(\mathbb{R}^{2n}) \rightarrow L^2(\mathbb{R}^n)$  be the adjoint of  $\mathcal{U}_\phi$ . We have  $\mathcal{U}_\phi^* \mathcal{U}_\phi = I$  on  $L^2(\mathbb{R}^n)$  and the operator  $P_\phi = \mathcal{U}_\phi \mathcal{U}_\phi^*$  is the orthogonal projection in  $L^2(\mathbb{R}^{2n})$  onto the space  $\mathcal{H}_\phi$ ; (iii) The inverse  $\mathcal{U}_\phi^{-1} : \mathcal{H}_\phi \rightarrow L^2(\mathbb{R}^n)$  is given by the formula*

$$\psi(x) = \frac{(2\pi\hbar)^{n/2}}{(\gamma|\phi)} \int_{\mathbb{R}^n} \mathcal{U}_\phi \psi(z_0) \widehat{\Pi}(\frac{1}{2}z_0) \gamma(x) dz_0 \quad (50)$$

where  $\gamma \in L^2(\mathbb{R}^n)$  is such that  $(\gamma|\phi) \neq 0$ ; (iv) The adjoint  $\mathcal{U}_\phi^*$  of  $\mathcal{U}_\phi$  is given by

$$\mathcal{U}_\phi^* \Psi(z) = \left(\frac{2}{\pi\hbar}\right)^{n/2} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{\frac{2i}{\hbar} p \cdot (x-y)} \phi(2y-x) \Psi(y,p) dp dy. \quad (51)$$

**Proof.** Properties (i) and (ii) were proven in [22] and [20], Chapter 10, §2; note that the fact that  $\mathcal{U}_\phi$  is an isometry immediately follows from Moyal's identity (42):

$$((W(\psi, \phi)|W(\psi', \phi'))) = \left(\frac{1}{2\pi\hbar}\right)^n (\psi|\psi') \overline{(\phi|\phi')}. \quad (52)$$

Let us prove the inversion formula (50). Set

$$\psi'(x) = C_\gamma \int_{\mathbb{R}^n} \Psi(z_0) \widehat{\Pi}(\frac{1}{2}z_0) \gamma(x) dz_0$$

where  $C_\gamma$  is a constant. For every  $\theta \in L^2(\mathbb{R}^n)$  we have, using successively (47) and the Moyal identity (52),

$$\begin{aligned} (\psi'|\theta) &= C_\gamma \int_{\mathbb{R}^n} \mathcal{U}_\phi \psi(z_0) (\widehat{\Pi}(\frac{1}{2}z_0) \gamma|\theta) dz_0 \\ &= C_\gamma 2^{-n/2} (\pi\hbar)^{3n/2} \int_{\mathbb{R}^{2n}} W(\psi, \phi)(\frac{1}{2}z_0) W(\gamma, \theta)(\frac{1}{2}z_0) dz_0 \\ &= C_\gamma (2\pi\hbar)^{3n/2} \int_{\mathbb{R}^{2n}} W(\psi, \phi)(z_0) W(\gamma, \theta)(z_0) dz_0 \\ &= C_\gamma (2\pi\hbar)^{n/2} (\psi|\theta) \overline{(\phi|\gamma)}. \end{aligned}$$

It follows that  $\psi' = \psi$  if we choose the constant  $C_\gamma$  so that  $C_\gamma (2\pi\hbar)^{n/2} \overline{(\phi|\gamma)} = 1$ , which proves (50). Formula (51) for  $\mathcal{U}_\phi^*$  is obtained by a straightforward calculation using the identity  $(\mathcal{U}_\phi\psi|\Psi)_{L^2(\mathbb{R}^{2n})} = (\psi|\mathcal{U}_\phi^*\Psi)_{L^2(\mathbb{R}^n)}$  and the definition of  $\mathcal{U}_\phi$  in terms of the cross-Wigner distribution. ■

## 4.2 The intertwining property

Here is the key result which shows how the operators  $A$  and  $\tilde{A}$  are linked by the wavepacket transforms:

**Proposition 14** *Let  $\mathcal{U}_\phi$  be an arbitrary wavepacket transform. The following intertwining formula hold:*

$$\tilde{T}(z)\mathcal{U}_\phi = \mathcal{U}_\phi T(z) \quad \text{and} \quad \tilde{A}\mathcal{U}_\phi = \mathcal{U}_\phi A \quad (53)$$

$$\mathcal{U}_\phi^* \tilde{T}(z)\mathcal{U}_\phi = T(z) \quad \text{and} \quad \mathcal{U}_\phi^* \tilde{A}\mathcal{U}_\phi = A. \quad (54)$$

**Proof.** The proof of formulae (53) is purely computational (see [20], Theorem 10.10, p.317, where  $\tilde{T}$  and  $A$  are denoted there by  $T_{\text{ph}}$  and  $A_{\text{ph}}$ , respectively). Formulae (54) immediately follow since  $\mathcal{U}_\phi^* \mathcal{U}_\phi = I$  on  $L^2(\mathbb{R}^n)$ . ■

For instance, if

$$H_{\text{har}} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2} x^2$$

is the harmonic oscillator operator and

$$H_{\text{sym}} = -\frac{\hbar^2}{2m} \Delta_{x,y} - i\hbar\omega_L \left( y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) + \frac{m\omega_L^2}{2} (x^2 + y^2)$$

is the magnetic operator considered in the introduction we have

$$H_{\text{sym}}\mathcal{U}_\phi = \mathcal{U}_\phi H_{\text{har}} \quad (55)$$

as an immediate consequence of the second formula (53). We will use this intertwining relation in Section 5 to recover the Landau levels and the corresponding Landau eigenfunctions.

## 4.3 WPT and modulation spaces

The *cross-Wigner transform* (46) is related to the STFT by the formula

$$W(\psi, \phi)(z) = \left(\frac{2}{\pi\hbar}\right)^{n/2} e^{\frac{2i}{\hbar}y \cdot x} V_{\phi, \sqrt{2\pi\hbar}} \psi_{\sqrt{2\pi\hbar}} \left(\frac{1}{\sqrt{2\pi\hbar}}z\right) \quad (56)$$

where we set as usual  $\psi_\lambda(x) = \psi(\lambda x)$  for a non-zero  $\lambda \in \mathbb{R}$ ,  $\phi^\vee(x) = \phi(-x)$ . It follows that there is a simple relationship between the windowed wavepacket transform  $\mathcal{U}_\phi$  and the short-time Fourier transform  $V_\phi$ . In fact, using formula (56) relating  $V_\phi$  to the cross-Wigner transform together with definition (45) of  $\mathcal{U}_\phi$  we have:

$$\mathcal{U}_\phi\psi(z) = e^{\frac{i}{2\hbar}y \cdot x} V_{\phi_{\sqrt{2\pi\hbar}}^\vee} \psi_{\sqrt{2\pi\hbar}} \left( z/2\sqrt{2\pi\hbar} \right) \quad (57)$$

and hence also

$$V_\phi\psi(z) = e^{-4\pi iy \cdot x} \mathcal{U}_{\phi_{1/\sqrt{2\pi\hbar}}^\vee} \psi_{1/\sqrt{2\pi\hbar}} \left( 2\sqrt{2\pi\hbar}z \right). \quad (58)$$

**Proposition 15** (i) We have  $\psi \in M_{v_s}^1(\mathbb{R}^n)$  if and only if  $\mathcal{U}_\phi\psi \in L_{v_s}^1(\mathbb{R}^{2n})$  for one (and hence for all) window(s)  $\phi \in \mathcal{S}(\mathbb{R}^n)$ . (ii) For  $\phi \in \mathcal{S}(\mathbb{R}^n)$ ,  $\phi \neq 0$ , the formula

$$\|\psi\|_{M_{v_s}^1, \phi} = \|\mathcal{U}_\phi\psi\|_{L_{v_s}^1} \quad (59)$$

defines a family of norms on  $M_{v_s}^1(\mathbb{R}^n)$  which are equivalent to the norms  $\|\psi\|_{M_{v_s}^1}^\phi$  defined by (13). (iii) The operator  $\mathcal{U}_\phi^*$  maps  $L_{v_s}^1(\mathbb{R}^{2n})$  into  $M_{v_s}^1(\mathbb{R}^n)$  and the inversion formula (50) in Theorem 13 holds in  $M_{v_s}^1(\mathbb{R}^n)$ . (iv)  $M_{v_s}^1(\mathbb{R}^n)$  is invariant under the action of the metaplectic group  $\text{Mp}(2n, \mathbb{R})$ .

**Proof.** (i) Immediately follows from formula (57) using Lemma 1. The statement (ii) follows from Proposition 11.3.2 in [26]; (iii) follows from Corollary 11.3.4 in [26]. Properties (iv) and (v) have been established in Proposition 11.3.2 of [26]. ■

## 5 Spectral Properties

We are going to use the results above to compare the spectral properties of  $A$  and  $\tilde{A}$ . We assume throughout that the operators  $A$  and  $\tilde{A}$  are defined on some dense subspace of  $L^2(\mathbb{R}^n)$  and  $L^2(\mathbb{R}^{2n})$ , respectively.

### 5.1 General results

The following result is very useful for the study of the eigenvectors of the LW operators:

**Lemma 16** Let  $(\phi_j)_{j \in F}$  be an arbitrary orthonormal basis of  $L^2(\mathbb{R}^n)$ ; setting  $\Phi_{j,k} = \mathcal{U}_{\phi_j} \phi_k$  the family  $\{\Phi_{j,k} : (j,k) \in F \times F\}$  forms an orthonormal basis of  $L^2(\mathbb{R}^{2n})$ , i.e.  $L^2(\mathbb{R}^{2n}) = \bigoplus_j \mathcal{H}_{\phi_j}$  (Hilbert sum).

**Proof.** It is sufficient to prove the result for  $\mathcal{U}_{\phi_j}$ . Since the  $\mathcal{U}_{\phi_j}$  are isometries the vectors  $\Phi_{j,k}$  form an orthonormal system. Let us show that if  $\Psi \in L^2(\mathbb{R}^{2n})$  is orthogonal to the family  $(\Phi_{j,k})_{j,k}$  (and hence to all the spaces  $\mathcal{H}_{\phi_j}$ ) then it is the zero vector; it will follow that  $(\Phi_{j,k})_{j,k}$  is a basis. Assume that  $((\Psi|\Phi_{jk})) = 0$  for all  $j, k$ . Since we have

$$((\Psi|\Phi_{jk})) = ((\Psi|\mathcal{U}_{\phi_j}\phi_k)) = (\mathcal{U}_{\phi_j}^*\Psi|\phi_k)$$

this means that  $\mathcal{U}_{\phi_j}^*\Psi = 0$  for all  $j$  since  $(\phi_j)_j$  is a basis. In view of Theorem 13(ii) we thus have  $P_{\phi_j}\Psi = 0$  for all  $j$  so that  $\Psi$  is orthogonal to all  $\mathcal{H}_{\phi_j}$ . ■

**Theorem 17** (i) *The eigenvalues of the operators  $A$  and  $\tilde{A}$  are the same;* (ii) *Let  $\psi$  be an eigenvector of  $A$ :  $A\psi = \lambda\psi$ . Then  $\Psi = \mathcal{U}_\phi\psi$  is an eigenvector of  $A$  corresponding to the same eigenvalue:  $\tilde{A}\Psi = \lambda\Psi$ .* (ii) *Conversely, if  $\Psi$  is an eigenvector of  $A$  then  $\psi = \mathcal{U}_\phi^*\Psi$  is an eigenvector of  $A$  corresponding to the same eigenvalue.*

**Proof.** (i) That every eigenvalue of  $A$  also is an eigenvalue of  $\tilde{A}$  is clear: if  $A\psi = \lambda\psi$  for some  $\psi \neq 0$  then

$$\tilde{A}(\mathcal{U}_\phi\psi) = \mathcal{U}_\phi A\psi = \lambda\mathcal{U}_\phi\psi$$

and  $\Psi = \mathcal{U}_\phi\psi \neq 0$ ; this proves at the same time that  $\mathcal{U}_\phi\psi$  is an eigenvector of  $A$  because  $\mathcal{U}_\phi$  has kernel  $\{0\}$ . (ii) Assume conversely that  $\tilde{A}\Psi = \lambda\Psi$  for  $\Psi \in L^2(\mathbb{R}^{2n})$ ,  $\Psi \neq 0$ , and  $\lambda \in \mathbb{R}$ . For every  $\phi$  we have

$$A\mathcal{U}_\phi^*\Psi = \mathcal{U}_\phi^*\tilde{A}\Psi = \lambda\mathcal{U}_\phi^*\Psi$$

hence  $\lambda$  is an eigenvalue of  $A$  and  $\psi$  an eigenvector if  $\psi = \mathcal{U}_\phi^*\Psi \neq 0$ . We have  $\mathcal{U}_\phi\psi = \mathcal{U}_\phi\mathcal{U}_\phi^*\Psi = P_\phi\Psi$  where  $P_\phi$  is the orthogonal projection on the range  $\mathcal{H}_\phi$  of  $\mathcal{U}_\phi$ . Assume that  $\psi = 0$ ; then  $P_\phi\Psi = 0$  for every  $\phi \in \mathcal{S}(\mathbb{R}^n)$ , and hence  $\Psi = 0$  in view of Lemma 16. ■

The reader is urged to remark that the result above is quite general: it does not make any particular assumptions on the operator  $A$  (in particular it is not assumed that  $A$  is self-adjoint), and the multiplicity of the eigenvalues can be arbitrary.

Let us specialize the results above to the case where  $\tilde{A}$  is (essentially) self-adjoint:

**Corollary 18** *Suppose that  $A$  is a self-adjoint operator on  $L^2(\mathbb{R}^n)$  and that each of the eigenvalues  $\lambda_0, \lambda_1, \dots, \lambda_j, \dots$  has multiplicity one. Let  $\psi_0, \psi_1, \dots, \psi_j, \dots$*

be a corresponding sequence of orthonormal eigenvectors. Let  $\Psi_j$  be an eigenvector of  $\tilde{H}$  corresponding to the eigenvalue  $\lambda_j$ . There exists a sequence  $(\alpha_{j,k})_k$  of complex numbers such that

$$\Psi_j = \sum_{\ell} \alpha_{j,\ell} \Psi_{j,\ell} \quad \text{with} \quad \Psi_{j,\ell} = \mathcal{U}_{\psi_\ell} \psi_j \in \mathcal{H}_j \cap \mathcal{H}_\ell. \quad (60)$$

**Proof.** We know from Theorem 17 above that  $A$  and  $\tilde{A}$  have same eigenvalues and that  $\Psi_{j,k} = W_{\psi_k} \psi_j$  satisfies  $\tilde{H} \Psi_{j,k} = \lambda_j \Psi_{j,k}$ . Since  $A$  is self-adjoint its eigenvectors  $\psi_j$  form an orthonormal basis of  $L^2(\mathbb{R}^n)$ ; it follows from Lemma 16 that the  $\Psi_{j,k}$  form an orthonormal basis of  $L^2(\mathbb{R}^{2n})$ , hence there exist non-zero scalars  $\alpha_{j,k,\ell}$  such that  $\Psi_j = \sum_{k,\ell} \alpha_{j,k,\ell} \Psi_{k,\ell}$ . We have, by linearity and using the fact that  $\tilde{A} \Psi_{k,\ell} = \lambda_k \Psi_{k,\ell}$ ,

$$\tilde{A} \Psi_j = \sum_{k,\ell} \alpha_{j,k,\ell} \tilde{A} \Psi_{k,\ell} = \sum_{k,\ell} \alpha_{j,k,\ell} \lambda_k \Psi_{k,\ell}.$$

On the other hand we also have  $\tilde{A} \Psi_j = \lambda_j \Psi_j$ ,

$$\tilde{H} \Psi_j = \lambda_j \Psi_j = \sum_{j,k} \alpha_{j,k,\ell} \lambda_j \Psi_{k,\ell}$$

and this is only possible if  $\alpha_{j,k,\ell} = 0$  for  $k \neq j$ ; setting  $\alpha_{j,\ell} = \alpha_{j,j,\ell}$  formula (60) follows. (That  $\Psi_{j,\ell} \in \mathcal{H}_j \cap \mathcal{H}_\ell$  is clear using the definition of  $\mathcal{H}_\ell$  and the sesquilinearity of the cross-Wigner transform.) ■

## 5.2 Shubin classes

Shubin has introduced in [46] very convenient symbol classes for studying global hypoellipticity. These ‘‘Shubin classes’’ are defined as follows: let  $H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  ( $m_0, m_1 \in \mathbb{R}$  and  $0 < \rho \leq 1$ ) be the complex vector space of all functions  $a \in C^\infty(\mathbb{R}^{2n})$  for which there exists a number  $R \geq 0$  such that for  $|z| \geq R$  we have

$$C_0 |z|^{m_0} \leq |a(z)| \leq C_1 |z|^{m_1} \quad , \quad |\partial_z^\alpha a(z)| \leq C_\alpha |a(z)| |z|^{-\rho|\alpha|} \quad (61)$$

for some constants  $C_0, C_1, C_\alpha \geq 0$ ; we are using here multi-index notation  $\alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbb{N}^n$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_{2n}$ , and  $\partial_z^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} \partial_{y_1}^{\alpha_{n+1}} \dots \partial_{y_n}^{\alpha_{2n}}$ . We notice that the Shubin classes are invariant under linear changes of variables: if  $f \in GL(2n, \mathbb{R})$  and  $a \in H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  then  $a \circ f \in H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ . In particular they are invariant under linear symplectic transformations.

We denote by  $HG_\rho^{m_1, m_0}(\mathbb{R}^n)$  the class of operators  $A$  with  $\tau$ -symbols  $a_\tau$  belonging to  $H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ ; this means that for every  $\tau \in \mathbb{R}$  there exists  $a_\tau \in H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  such that

$$Au(x) = \left(\frac{1}{2\pi}\right)^n \iint_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a_\tau((1-\tau)x + \tau y, \xi) u(y) dy d\xi;$$

choosing  $\tau = \frac{1}{2}$  this means, in particular, that every operator with Weyl symbol  $a \in H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  is in  $HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ . Conversely, the condition  $a \in H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  is also sufficient, because if  $a_\tau \in H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  is true for some  $\tau$  then it is true for all  $\tau$ .

Shubin [46] (Chapter 4) proves the following result:

**Proposition 19 (Shubin)** *Let  $A \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  with  $m_0 > 0$ . If  $A$  is formally self-adjoint, that is if  $(A\psi|\phi) = (\psi|A\phi)$  for all  $\psi, \phi \in C_0^\infty(\mathbb{R}^n)$ , then: (i)  $A$  is essentially self-adjoint and has discrete spectrum in  $L^2(\mathbb{R}^n)$ ; (ii) There exists an orthonormal basis of eigenfunctions  $\phi_j \in \mathcal{S}(\mathbb{R}^n)$  ( $j = 1, 2, \dots$ ) with eigenvalues  $\lambda_j \in \mathbb{R}$  such that  $\lim_{j \rightarrow \infty} |\lambda_j| = \infty$ .*

This result has the following consequence for LW operators:

**Corollary 20** *Let  $A \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  be formally self-adjoint. Then the LW operator  $\tilde{A}$  has discrete spectrum  $(\lambda_j)_{j \in \mathbb{N}}$  and  $\lim_{j \rightarrow \infty} |\lambda_j| = \infty$  and the eigenfunctions of  $\tilde{A}$  are in this case given by  $\Phi_{jk} = \mathcal{U}_{\phi_j} \phi_k$  where the  $\phi_j$  are the eigenfunctions of  $A$ ; (iv) We have  $\Phi_{jk} \in \mathcal{S}(\mathbb{R}^{2n})$  and the  $\Phi_{jk}$  form an orthonormal basis of  $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ .*

**Proof.** It is an immediate consequence of Theorem 17 using the proposition above. ■

### 5.3 Gelfand triples

Dirac already emphasized in his fundamental work [9] the relevance of *rigged Hilbert spaces* for quantum mechanics. Later Schwartz provided an instance of rigged Hilbert spaces based on his class of test functions and on tempered distributions. Later Gelfand and Shilov formalized the construction of Schwartz and Dirac and introduced what is nowadays known as *Gelfand triples*. The prototypical example of a Gelfand triple is  $(\mathcal{S}(\mathbb{R}^n), L^2(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ . In the last decade Feichtinger and some of his collaborators (see [16, 10, 5]) emphasized the relevance of the Gelfand triple  $(M_{v_s}^1(\mathbb{R}^n), L^2(\mathbb{R}^n), M_{1/v_s}^\infty(\mathbb{R}^n))$  in time-frequency analysis. An important feature of Gelfand triples is the

existence of a kernel theorem, as we explained in Subsection 2.3. In the present investigation these classes of Gelfand triples will allow us to treat the case of the continuous spectrum of selfadjoint operators.

The main idea underlying the notion of Gelfand triple is the observation, that a triple of spaces – consisting of the Hilbert space itself, a small (topological vector) space contained in the Hilbert space, and its dual – allows a much better description of the spectrum. The main appeal of the notion of Banach triple is, in our context, the fact that we can even take a Banach space, namely the modulation space  $M_{v_s}^1(\mathbb{R}^n)$ .

**Definition 21** *A (Banach) Gelfand triple  $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$  consists of a Banach space  $\mathcal{B}$  which is continuously and densely embedded into a Hilbert space  $\mathcal{H}$ , which in turn is  $w^*$ -continuously and densely embedded into the dual Banach space  $\mathcal{B}'$ .*

In this setting the inner product on  $\mathcal{H}$  extends in a natural way to a pairing between  $\mathcal{B}$  and  $\mathcal{B}'$  producing an anti-linear functional  $F$  of the same norm. The framework of the Gelfand triple  $(\mathcal{S}(\mathbb{R}^n), L^2(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$  or more generally of  $(M_{v_s}^1(\mathbb{R}^n), L^2(\mathbb{R}^n), M_{1/v_s}^\infty(\mathbb{R}^n))$  allows one to formulate a spectral theorem for selfadjoint operators on  $\mathcal{S}(\mathbb{R}^n)$  or  $M_{v_s}^1(\mathbb{R}^n)$ . If  $F(A\psi) = \lambda F(\psi)$  holds for all  $\psi \in M_{v_s}^1(\mathbb{R}^n)$  or in  $\mathcal{S}(\mathbb{R}^n)$  in the distributional sense, then  $\lambda$  is called a *generalized eigenvalue* to the *generalized eigenvector* of the selfadjoint operator  $A$ . For a given generalized eigenvalue  $\lambda \in \mathbb{C}$  we denote by  $E_\lambda$  be the set of all generalized eigenvectors  $F$  in  $M_{1/v_s}^\infty(\mathbb{R}^n)$  or  $\mathcal{S}(\mathbb{R}^n)$ , respectively. The set of all generalized eigenvalues  $\cup_\lambda E_\lambda$  is called *complete*, if for any  $\psi, \phi$  in  $M_{1/v_s}^\infty(\mathbb{R}^n)$  or  $\mathcal{S}(\mathbb{R}^n)$  such that  $F(\psi) = F(\phi)$  for all  $F \in \cup_\lambda E_\lambda$ , then  $\psi = \phi$ .

**Theorem 22** *Let  $T$  be a selfadjoint operator on  $M_{v_s}^1(\mathbb{R}^n)$  or  $\mathcal{S}(\mathbb{R}^n)$ . Then all generalized eigenvalues  $\lambda$  are real numbers and  $L^2(\mathbb{R}^n)$  can be written as a direct sum of Hilbert spaces  $\mathcal{H}(\lambda)$  such that  $E_\lambda \subset \mathcal{H}_\lambda$ , and such that the  $\lambda$ -component of  $Tf$  is given by  $(Af)_\lambda = \lambda f$  for all  $M_{v_s}^1(\mathbb{R}^n)$  or  $\mathcal{S}(\mathbb{R}^n)$ . Moreover, the set of generalized eigenvectors  $\cup_\lambda E_\lambda$  is complete.*

As an illustration we treat generalized eigenvectors of the translation operator  $T_x f(y) = f(y - x)$ . We interpret the characters  $\chi_\omega(x) = e^{-2\pi i \omega \cdot x}$  as generalized eigenvectors for the translation operator  $T_x$  on  $M_{v_s}^1(\mathbb{R}^n)$ . Furthermore the set of generalized eigenvectors  $\{\chi_\omega : \omega \in \mathbb{R}^n\}$  is complete by Plancherel's theorem, i.e., if the Fourier transform  $\hat{f}(\omega) = \langle \chi_\omega, f \rangle$  vanishes for all  $\omega \in \mathbb{R}^n$  implies  $f \equiv 0$ . This suggests to think of the Fourier transform of  $f$  at frequency  $\omega$  as the evaluation of the linear functional  $\langle \chi_\omega, f \rangle$ .

Therefore the preceding theorem allows us to deal with the continuous spectrum as treated in Theorem 17 for the discrete spectrum.

## 5.4 Application to the Landau levels

As an illustration consider the harmonic oscillator Hamiltonian (2) of the Introduction. To simplify notation we take  $m = \omega = 1$  (this corresponds to the choice  $\gamma = \mu = 1$  for  $\tilde{A}^{\gamma,\mu}$ ); in addition we choose units in which  $\hbar = 1$ . In view of the results above the spectra of the harmonic oscillator Hamiltonian (2) and of the magnetic operator (3) are identical. The eigenvalues of the first are the numbers  $\lambda_k = k + \frac{1}{2}$  ( $k$  an integer). These are the well-known Landau energy levels [38]. The harmonic oscillator operator (2) satisfies the assumptions of Corollary 18. The normalized eigenvectors are the rescaled Hermite functions

$$\phi_k(x) = (2^k k! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} \mathcal{H}_k(x).$$

where

$$\mathcal{H}_k(x) = (-1)^{km} e^{x^2} \left(\frac{d}{dx}\right)^k e^{-x^2}$$

is the  $k$ -th Hermite polynomial. Using definition (45) of the wavepacket transform together with known formulae for the cross-Wigner transform of Hermite functions (Thangavelu [48], Chapter 1, Wong [51], Chapter 24, Theorem 24.1) one finds that the eigenvectors of the magnetic operator are linear superpositions of the functions

$$\Phi_{j+k,k}(z) = (-1)^j \frac{1}{\sqrt{2\pi}} \left(\frac{j!}{(j+k)!}\right)^{\frac{1}{2}} 2^{-\frac{k}{2}} z^k \mathcal{L}_j^k\left(\frac{1}{2}|z|^2\right) e^{-\frac{|z|^2}{4}}$$

and  $\Phi_{j,j+k} = \overline{\Phi_{j+k,k}}$  for  $k = 0, 1, 2, \dots$ ; in the right-hand side  $z$  is interpreted as  $x + iy$  and

$$\mathcal{L}_j^k(x) = \frac{1}{j!} x^{-k} e^x \left(\frac{d}{dx}\right)^j (e^{-x} x^{j+k}), \quad x > 0$$

is the Laguerre polynomial of degree  $j$  and order  $k$ . In particular we recover the textbook result that the eigenspace of the ground state is spanned by the functions

$$\Phi_{0,k}(x, y) = (k! 2^{k+1} \pi)^{-1/2} (x - iy)^k e^{-\frac{1}{4}(x^2 + y^2)};$$

notice that this eigenspace is just  $\mathcal{H}_{\phi_0}$ . Finally we want to mention that the intertwining between the Weyl calculus and the Landau–Weyl calculus allows one to define annihilation and creation operators as in the case of the harmonic oscillator. Therefore our calculus provides us with natural operators that allow us to “move” between the eigenvectors of the Landau levels. We will come back to this issue in a forthcoming work.

## 6 Regularity and Hypoellipticity Results

We begin by stating a few boundedness results for Weyl and Landau–Weyl operators in modulation spaces. The main result of this section is Theorem 28 where we prove a global hypoellipticity result for Landau–Weyl operators whose symbol belong to the Shubin class  $H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ .

### 6.1 Global hypoellipticity

In [46] (Corollary 25.1, p. 186) Shubin has introduced the notion of global hypoellipticity (also see Boggiatto et al. [3], p. 70). This notion is more useful in quantum mechanics than the usual hypoellipticity because it incorporates the decay at infinity of the involved distributions.

**Definition 23** *We will say that a linear operator  $A : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$  is “globally hypoelliptic” if we have*

$$\psi \in \mathcal{S}'(\mathbb{R}^n) \text{ and } A\psi \in \mathcal{S}(\mathbb{R}^n) \implies \psi \in \mathcal{S}(\mathbb{R}^n). \quad (62)$$

Our discussion of pseudodifferential operators suggests the following refinement of the notion of global hypoellipticity to the setting of modulation spaces:

**Definition 24** *If  $A : M_{1/v_s}^\infty(\mathbb{R}^n) \longrightarrow M_{1/v_s}^\infty(\mathbb{R}^n)$  is a linear mapping, then it is “s-hypoelliptic” if we have*

$$\psi \in M_{1/v_{s_0}}^\infty(\mathbb{R}^n) \text{ and } A\psi \in M_{v_s}^1(\mathbb{R}^n) \implies \psi \in M_{v_s}^1(\mathbb{R}^n). \quad (63)$$

That this definition really provides us with a refinement of Definition 23 follows from the following observation:

**Lemma 25** *If  $A$  is s-hypoelliptic for every  $s \geq 0$  then it is globally hypoelliptic.*

**Proof.** Let  $\psi \in \mathcal{S}'(\mathbb{R}^n)$ ; in view of the second equality (20) there exists  $s_0$  such that  $\psi \in M_{1/v_{s_0}}^\infty(\mathbb{R}^n)$ . The condition  $A\psi \in M_{v_s}^1(\mathbb{R}^n)$  for every  $s$  then implies that  $\psi \in M_{v_s}^1(\mathbb{R}^n)$  for every  $s \geq 0$  hence our claim in view of the first equality (20). ■

Let us return to the Shubin classes we used in Section 5 when we studied spectral properties of Landau–Weyl operators. Using the properties of these classes Shubin ([46], Chapter IV, §23) constructs a (left) parametrix of  $A$ . i.e. a Weyl operator  $B \in G\Gamma_\rho^{-m_1, -m_0}(\mathbb{R}^n)$  such that  $BA = I + R$  where the kernel of  $R$  is in  $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ ; from the existence of such a parametrix follows readily that:

**Proposition 26** *Any Weyl operator  $A \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  is globally hypoelliptic.*

Note that the previous proposition of Shubin remains true for the case of modulation spaces, because all the arguments of his proof remain valid for this more general class of function spaces.

In [6] Fredholm properties of (localization) pseudodifferential operators on modulation spaces have been proved by Cordero and Gröchenig. These results provide natural generalizations of well-known results due to Shubin on global hypoellipticity. We invoke their results to get some classes of pseudodifferential operators that are  $s$ -hypoelliptic. We will use the following refinement of Proposition 26, also due to Shubin ([46], Chapter IV, §25):

**Proposition 27**  *$A \in HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  be such that  $\text{Ker } A = \text{Ker } A^* = \{0\}$ . Then there exists  $B \in HG_\rho^{-m_1, -m_0}(\mathbb{R}^{2n})$  such that  $BA = AB = I$  (i.e.  $B$  is the inverse of  $A$ ).*

In [6] a class of symbol classes  $\mathcal{M}_v$  is introduced, which for the weight  $v_s$  contains the classes  $HG_\rho^{m_1, m_0}(\mathbb{R}^{2n})$ . Therefore by Theorem 7.1 in [6] we get a result about the  $s$ -hypoellipticity for Landau–Weyl operators. The main result of this section is the following global hypoellipticity result:

**Theorem 28** *The Landau–Weyl operator  $\tilde{A}$  associated to an operator  $A \in G\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  such that  $\text{Ker } A = \text{Ker } A^* = \{0\}$  is  $s$ -hypoelliptic for each  $s \geq 0$  and hence also globally hypoelliptic.*

**Proof.** In view of Proposition 27 the operator  $A$  has an inverse  $B$  belonging to  $HG_\rho^{-m_1, -m_0}(\mathbb{R}^{2n})$ . In view of Corollary 7 the LW operator  $\tilde{B}$  is then an inverse of  $\tilde{A}$ . Assume now that  $\tilde{A}\Psi = \Phi \in \mathcal{S}(\mathbb{R}^{2n})$ ; then  $\Psi = \tilde{B}\Phi$ . The classes of symbols studied in Theorem 7.1 and Corollary 7.2 in [6] contain the Shubin classes  $G\Gamma_\rho^{m_1, m_0}(\mathbb{R}^n)$  as one sees by an elementary argument. Therefore it follows that any Weyl operator  $A \in H\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  is  $s$ -hypoelliptic for every  $s \geq 0$ , and thus globally hypoelliptic in view of Lemma 25. ■

Consequently, the Landau–Weyl operator  $\tilde{A}$  associated to an operator  $A \in G\Gamma_\rho^{m_1, m_0}(\mathbb{R}^n)$  such that  $\text{Ker } A = A^* = \{0\}$  is  $s$ -hypoelliptic for every  $s \geq 0$ , i.e. it is globally hypoelliptic.

**Remark 29** *The condition  $A \in G\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{2n})$  does not imply that  $\tilde{A} \in G\Gamma_\rho^{m_1, m_0}(\mathbb{R}^{4n})$  as is seen by inspection of formula (33) for the symbol  $\tilde{a}$ .*

Let us illustrate this when the symbol  $a$  is a non-degenerate quadratic form:

**Example 30** *Let  $a$  be a positive-definite quadratic form on  $\mathbb{R}^{2n}$ :  $a(z) = \frac{1}{2}Mz \cdot z$  with  $M = M^T > 0$ . Then  $A$  is globally hypoelliptic; in fact  $a \in H\Gamma_1^{2,2}(\mathbb{R}^{2n})$  as is seen using an adequate diagonalization of  $M$ . The operator  $\tilde{A}$  is globally hypoelliptic. In particular the magnetic operator (3) is globally hypoelliptic.*

Notice that in this example we have recovered the global hypoellipticity of the magnetic operator obtained by Wong [52] using very different methods (the theory of special functions).

## 6.2 Regularity results for the Schrödinger equation

Let us apply some of the previous results to the study of regularity properties of the Schrödinger equations

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi \quad , \quad i\hbar \frac{\partial}{\partial t} \Psi = \tilde{H}\Psi.$$

Let the Hamiltonian function be a quadratic form:

$$H(z) = \frac{1}{2}Mz \cdot z \quad , \quad M = M^T.$$

The corresponding Hamiltonian flow consists of the linear symplectic mappings  $s_t = e^{tJM}$  and is hence a one-parameter subgroup of  $\mathrm{Sp}(2n, \mathbb{R})$ . It follows from the theory of covering spaces that there is a bijective correspondence between the one-parameter subgroups of the symplectic group  $\mathrm{Sp}(2n, \mathbb{R})$  and those of the metaplectic group  $\mathrm{Mp}(2n, \mathbb{R})$ ; let us denote this correspondence by  $\mu$ . Thus

$$\mu(s_t) = S_t$$

means that if  $(s_t)$  is a one-parameter subgroup of  $\mathrm{Sp}(2n, \mathbb{R})$  then  $(S_t)$  is the only one-parameter subgroup of  $\mathrm{Mp}(2n, \mathbb{R})$  whose projection is precisely  $(s_t)$ . We will similarly write

$$\tilde{\mu}(s_t) = \tilde{S}_t$$

where  $\tilde{S}_t \in \mathrm{Mp}(4n, \mathbb{R})$  is defined by formula (38).

The first part of following result is well-known:

**Proposition 31** *Let  $(s_t)$  be the Hamiltonian flow determined by the Hamilton equations  $\dot{z} = J\partial_z H(z) = Mz$ . The one parameter groups  $(S_t)$  and  $(\tilde{S}_t)$  defined by  $S_t = \mu(s_t)$  and  $\tilde{S}_t = \tilde{\mu}(s_t)$  satisfy the Schrödinger equations*

$$i\hbar \frac{\partial}{\partial t} S_t = H S_t \quad , \quad i\hbar \frac{\partial}{\partial t} \tilde{S}_t = \tilde{H} \tilde{S}_t \quad (64)$$

where  $H(x, -i\hbar\partial_x)$  and  $\tilde{H}$  are the Weyl and LW operators determined by the Hamiltonian function  $H$ .

**Proof.** That  $S_t$  satisfies the first equation (64) is a classical result (see for instance [20, 32, 33], for detailed accounts). That  $\tilde{S}_t$  satisfies the second equation immediately follows. ■

We next show that the spreading of wavefunction  $\Psi$  and its evolution in time can be controlled in terms of the spaces  $L^1_{v_s}(\mathbb{R}^{2n})$ .

**Proposition 32** *Let  $\Psi \in \mathcal{S}'(\mathbb{R}^{2n})$  is a solution of the Schrödinger equation*

$$i\hbar \frac{\partial \Psi}{\partial t} = \tilde{H} \Psi \quad , \quad \Psi(\cdot, 0) = \Psi_0. \quad (65)$$

*If  $\Psi_0 \in \mathcal{H}_\phi \cap L^1_{v_s}(\mathbb{R}^{2n})$  for some  $\phi$  then  $\Psi(\cdot, t) \in L^1_{v_s}(\mathbb{R}^{2n})$  for every  $t \in \mathbb{R}$ .*

**Proof.** Since  $\Psi_0 \in \mathcal{H}_\phi$  we have  $\Psi_0 = \mathcal{U}_\phi \psi_0$  for some  $\psi_0 \in L^2(\mathbb{R}^n)$ ; the condition  $\Psi_0 \in L^1_{v_s}(\mathbb{R}^{2n})$  implies that  $\psi_0 \in M^1_{v_s}(\mathbb{R}^n)$ . Let  $\psi$  be the unique solution of the Cauchy problem

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi \quad , \quad \psi(\cdot, 0) = \psi_0;$$

that solution is  $\psi = S_t \psi_0$  in view of Proposition 31, hence  $\psi(\cdot, t) \in M^1_{v_s}(\mathbb{R}^n)$  for every  $t \in \mathbb{R}$ . We claim that the (unique) solution of (65) with  $\Psi_0 \in \mathcal{H}_\phi \cap L^1_{v_s}(\mathbb{R}^{2n})$  is  $\Psi = \mathcal{U}_\phi \psi$ ; the proposition will follow in view of the definition of  $M^1_{v_s}(\mathbb{R}^n)$ . Set  $\Psi' = \mathcal{U}_\phi \psi$ . Since  $\tilde{H} \mathcal{U}_\phi = \mathcal{U}_\phi H(x, -i\hbar\partial_x)$  in view of the second equality (53) in Proposition 14, we have

$$i\hbar \frac{\partial \Psi'}{\partial t} = \mathcal{U}_\phi (i\hbar \frac{\partial}{\partial t} \psi) = \tilde{H} \mathcal{U}_\phi \psi = \tilde{H} \Psi'.$$

Now  $\Psi'(\cdot, 0) = \Psi_0$  hence  $\Psi' = \Psi$ . ■

## 7 Generalization; Application to Deformation Quantization

### 7.1 The operators $\tilde{A}^{\gamma,\mu}$

Let us show how to generalize the constructions above to the operators  $\tilde{A}^{\gamma,\mu}$  corresponding to the more general quantization rule (1).

We begin by noting that the operators  $\tilde{X}_j^{\gamma,\mu}$  and  $\tilde{Y}_j^{\gamma,\mu}$  are obtained from  $\tilde{X}_j$  and  $\tilde{Y}_j$  by conjugation with the metaplectic rescalings  $\tilde{S}^{\gamma,\mu}$  defined in Lemma 11:

$$\tilde{X}_j^{\gamma,\mu} = \tilde{S}^{\gamma,\mu} \tilde{X}_j (\tilde{S}^{\gamma,\mu})^{-1}, \quad \tilde{Y}_j^{\gamma,\mu} = \tilde{S}^{\gamma,\mu} \tilde{Y}_j (\tilde{S}^{\gamma,\mu})^{-1}. \quad (66)$$

(the proof is purely computational and is therefore omitted). These formulae suggest the following definition: for any LW operator  $\tilde{A}$  and  $(\gamma, \mu) \in \mathbb{R}^2$  such that  $\gamma\mu \neq 0$  we set

$$\tilde{A}^{\gamma,\mu} = \tilde{S}^{\gamma,\mu} \tilde{A} (\tilde{S}^{\gamma,\mu})^{-1}. \quad (67)$$

We have:

**Proposition 33** (i) *The contravariant symbol of  $\tilde{A}^{\gamma,\mu} : \mathcal{S}(\mathbb{R}^{2n}) \rightarrow \mathcal{S}'(\mathbb{R}^{2n})$  is the function*

$$\tilde{a}^{\gamma,\mu}(x, y; p_x, p_y) = a\left(\frac{\gamma}{2}x - \frac{1}{\mu}p_y, \frac{\mu}{2}y + \frac{1}{\gamma}p_x\right) \quad (68)$$

where  $a$  is the contravariant symbol of  $A$ . (ii) *We have*

$$\tilde{A}^{\gamma,\mu} = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma^{\gamma,\mu}(z) \tilde{T}^{\gamma,\mu}(z) dz \quad (69)$$

(Bochner integral) where  $a_\sigma^{\gamma,\mu} = \tilde{S}^{\gamma,\mu} a_\sigma$  and  $\tilde{T}^{\gamma,\mu}(z)$  is the unitary operator defined by

$$\tilde{T}^{\gamma,\mu}(z_0)\Psi(z) = e^{-\frac{i\gamma\mu}{2\hbar}\sigma(z, z_0)}\Psi(z - z_0). \quad (70)$$

**Proof.** Formula (68) follows from the symplectic covariance (30) of Weyl calculus taking (41) into account. Formula (69) follows, by a change of variables in definition (32) of  $\tilde{A}$ . ■

The following intertwining result is a straightforward consequence of Proposition 14:

**Corollary 34** (i) *The mapping  $\mathcal{U}_\phi^{\gamma,\mu} = \tilde{S}^{\gamma,\mu}\mathcal{U}_\phi$  is an isometry of  $L^2(\mathbb{R}^n)$  onto the closed subspace  $\mathcal{H}_\phi^{\gamma,\mu} = \tilde{S}^{\gamma,\mu}\mathcal{H}_\phi$  of  $L^2(\mathbb{R}^{2n})$ ; explicitly*

$$\mathcal{U}_\phi^{\gamma,\mu}\psi(z) = \left(\frac{\pi\gamma\mu\hbar}{2}\right)^{n/2} W(\psi, \phi)\left(\frac{1}{2}\gamma x, \frac{1}{2}\mu y\right). \quad (71)$$

(ii) The operator  $\tilde{A}^{\gamma,\mu}$  satisfies the intertwining formula

$$\tilde{A}^{\gamma,\mu}\mathcal{U}_\phi^{\gamma,\mu} = \mathcal{U}_\phi^{\gamma,\mu}\hat{A} \quad \text{with } \mathcal{U}_\phi^{\gamma,\mu} = \tilde{S}^{\gamma,\mu}\mathcal{U}_\phi. \quad (72)$$

**Proof.** (i)  $\mathcal{U}_\phi^{\gamma,\mu}$  is the compose of two isometries hence an isometry.  $\mathcal{H}_\phi^{\gamma,\mu}$  is closed because  $\mathcal{H}_\phi$  is, and  $\tilde{S}^{\gamma,\mu}$  is an isomorphism  $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ . (ii) Formula (72) immediately follows from the definitions of  $\tilde{A}^{\gamma,\mu}$  and  $\mathcal{U}_\phi^{\gamma,\mu}$  and the second intertwining formula (53). ■

## 7.2 The Moyal product and Deformation Quantization

The Moyal product plays a central role in deformation quantization of Flato and Sternheimer [1]. Let  $H$  be a Hamiltonian function and assume that  $\Psi \in \mathcal{S}(\mathbb{R}^n)$ ; the Moyal product [1]  $H \star_{\hbar} \Psi$  is defined by

$$(H \star_{\hbar} \Psi)(z) = \left(\frac{1}{4\pi\hbar}\right)^{2n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{\frac{i}{2\hbar}\sigma(u,v)} H(z + \frac{1}{2}u) \Psi(z - \frac{1}{2}v) dudv; \quad (73)$$

when  $\hbar = 1/2\pi$  it reduces to the twisted product  $\#$  familiar from standard Weyl calculus:  $H \star_{1/2\pi} \Psi = H\#\Psi$ .

We claim that

$$H \star_{\hbar} \Psi = \tilde{H}^{2,1}\Psi = H(x - \frac{1}{2}i\hbar\partial_p, p + \frac{1}{2}i\hbar\partial_x)\Psi. \quad (74)$$

The proof is similar to that of Theorem 6. Let us view  $\Psi \longmapsto H \star_{\hbar} \Psi$  as a Weyl operator, denoted by  $H\star_{\hbar}$ . Using formula (73) the distributional kernel of  $H\star_{\hbar}$  is given by

$$\mathcal{K}_{H\star_{\hbar}}(z, y) = \left(\frac{1}{2\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\sigma(u, z-y)} H(z - \frac{1}{2}u) du \quad (75)$$

hence, using (22) and the Fourier inversion formula, the contravariant symbol of  $H\star_{\hbar}$  is given by

$$\mathbb{H}(z, \zeta) = \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\zeta\cdot\eta} \mathcal{K}_{H\star_{\hbar}}(z + \frac{1}{2}\eta, z - \frac{1}{2}\eta) d\eta.$$

Using (75) and performing the change of variables  $u = 2z + \eta - z'$  we get

$$\mathcal{K}_{H\star_{\hbar}}(z + \frac{1}{2}\eta, z - \frac{1}{2}\eta) d\eta = \left(\frac{1}{2\pi\hbar}\right)^{2n} e^{\frac{2i}{\hbar}\sigma(z,\eta)} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\sigma(\eta, z')} H(\frac{1}{2}z') dz';$$

setting  $H(\frac{1}{2}z') = H_{1/2}(z')$  the integral is  $(2\pi\hbar)^n$  times the symplectic Fourier transform  $F_\sigma^\hbar H_{1/2}(-\eta) = (H_{1/2})_\sigma(-\eta)$  so that

$$\begin{aligned}\mathbb{H}(\tfrac{1}{2}z, \zeta) &= \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\zeta\cdot\eta} e^{\frac{i}{\hbar}\sigma(z,\eta)} (H_{1/2})_\sigma(-\eta) d\eta \\ &= \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\sigma(z+J\zeta,\eta)} (H_{1/2})_\sigma(\eta) d\eta\end{aligned}$$

Since the second equality is the inverse symplectic Fourier transform of  $(H_{1/2})_\sigma$  calculated at  $z + J\zeta$  we finally get

$$\mathbb{H}(z, \zeta) = H\left(x + \frac{1}{2}\zeta_p, p - \frac{1}{2}\zeta_x\right) \quad (76)$$

with  $\zeta = (\zeta_x, \zeta_p)$ .

An immediate consequence of these results is:

**Proposition 35** *The isometries  $\mathcal{U}_\phi^{2,1}$  defined by the formula*

$$\mathcal{U}_\phi^{2,1}\psi(z) = (\pi\hbar)^{n/2} W(\psi, \phi)(z) \quad (77)$$

*satisfy the intertwining relation*

$$(H \star_\hbar \Psi)\mathcal{U}_\phi^{2,1} = \mathcal{U}_\phi^{2,1}\widehat{H}.$$

**Proof.** Formula (77) is just (71) with  $\gamma = 2, \mu = 1$ . ■

## 8 Concluding Remarks

Due to limitations of length and time we have only been able to give a few applications of the theory of modulation spaces to the Landau–Weyl calculus. Modulation spaces and related topics have turned out to be the proper setting for the discussion of pseudodifferential operators in the last decade, see for instance the papers [25, 30] and the references therein; for related topics such as the spaces  $\ell^q(L^p)$  see for instance the work of Birman and Solomjak [2], Christ and Kiselev [4] or Simon [42]. Recently modulation spaces have also found various applications in the study of Schrödinger operators (see Cordero and Nicola [7, 8]).

## References

- [1] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation Theory and Quantization. I. Deformation of Symplectic Structures. *Annals of Physics* **111**, 6–110 (1978); II Physical Applications **110**, 111–151 (1978)
- [2] Birman, M. Sh. and Solomjak, M. Z.: Spectral theory of selfadjoint operators in Hilbert space, Mathematics and its Applications (Soviet Series), Translated from the 1980 Russian original by S. Khrushchëv and V. Peller, D. Reidel Publishing Co., Dordrecht (1987)
- [3] Boggiatto P., Buzano E., Rodino L., *Global Hypoellipticity and Spectral Theory*, Akademie Verlag, Math. Research **92** (1996)
- [4] Christ, M. and Kiselev, A.: Absolutely continuous spectrum for one-dimensional Schrödinger operators with slowly decaying potentials: some optimal results. *J. Amer. Math. Soc.*, **11**(4) 771–797 (1998)
- [5] Cordero, E. Feichtinger, H. G., and Luef F.: Banach Gelfand triples for Gabor analysis. In *'Pseudo-Differential Operators, Quantization and Signals.'* *C.I.M.E. 2006*, Lecture Notes in Mathematics. Springer (2007)
- [6] E. Cordero and K. Gröchenig. Symbolic calculus and Fredholm property for localization operators. *J. Fourier Anal. Appl.*, 12(3):371–392, 2006.
- [7] Cordero E. and Nicola F.: Strichartz estimates in Wiener amalgam spaces for the Schrödinger equation. *Math. Nachr.* **281**(1), 25–41 (2008)
- [8] E. Cordero and F. Nicola. Metaplectic representation on Wiener amalgam spaces and applications to the Schrödinger equation. *J. Funct. Anal.*, **254**(2), 506–534, 2008.
- [9] Dirac, P. A. M.: *The Principles of Quantum Mechanics* (Oxford Science Publications, fourth revised edition. (1999).
- [10] Dörfler, M., Feichtinger, H.G., Gröchenig, K.: Time-Frequency Partitions for the Gelfand Triple  $(S_0, L^2, S'_0)$ . *Math. Scand.*, **98**(1), 81–96 (2006)
- [11] Feichtinger, H. G.: On a new Segal algebra. *Monatsh. Math.*, **92**(4), 269–289 (1981)

- [12] Feichtinger, H. G.: Banach spaces of distributions of Wiener's type and interpolation. In P. Butzer, S. Nagy, and E. Görlich, editors, *Proc. Conf. Functional Analysis and Approximation, Oberwolfach August 1980*, number 69 in *Internat. Ser. Numer. Math.*, pages 153–165. Birkhäuser Boston, Basel, 1981.
- [13] Feichtinger, H. G.: Modulation spaces on locally compact Abelian groups. Technical report, January (1983)
- [14] Feichtinger, H. G.: Modulation Spaces: Looking Back and Ahead. *Sampl. Theory Signal Image Process.*, **5(2)**, 109–140 (2006)
- [15] Feichtinger H. G and Gröchenig K.: Banach spaces related to integrable group representations and their atomic decompositions, I. *J. Funct. Anal.* **86**, 307–340 (1989)
- [16] Feichtinger H. G., Kozek, W.: Quantization of TF lattice-invariant operators on elementary LCA groups. In H. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms. Theory and Applications.*, Applied and Numerical Harmonic Analysis, pages 233–266, 452–488, Boston, MA, 1998. Birkhäuser Boston.
- [17] Feichtinger, H. G., Luef F.: Wiener Amalgam Spaces for the Fundamental Identity of Gabor Analysis. *Collect. Math.* **57**, 233–253 (2006)
- [18] de Gosson, M.: Symplectically Covariant Schrödinger Equation in Phase Space. *J. Phys.A:Math. Gen.* **38**, 9263–9287 (2005)
- [19] de Gosson, M.: The Weyl Representation of Metaplectic operators. *Letters in Mathematical Physics* **72**, 129–142 (2005)
- [20] de Gosson, M.: *Symplectic Geometry and Quantum Mechanics*. Birkhäuser, Basel, series “Operator Theory: Advances and Applications” (subseries: “Advances in Partial Differential Equations”), Vol. **166**, 2006
- [21] de Gosson, M., de Gosson, S.: An extension of the Conley–Zehnder Index, a product formula and an application to the Weyl representation of metaplectic operators. *J. Math. Phys.*, **47** (2006)
- [22] de Gosson, M.: Metaplectic Representation, Conley–Zehnder Index, and Weyl Calculus on Phase Space. *Rev. Math. Phys.* **19(10)**, 1149–1188 (2007)

- [23] de Gosson, M.: Explicit Calculation of the Maslov-type Index Occurring in Gutzwiller’s Trace Formula; Application to the Metaplectic Group. Max-Planck-Institut für Mathematik preprint server: Preprint MPIM2007-151. <http://www.mpim-bonn.mpg.de/preprints/retrieve>
- [24] Gröchenig K.: An uncertainty principle related to the Poisson summation formula. *Stud. Math.* **121**(1), 87–104 (1996)
- [25] Gröchenig K., Heil C.: Modulation spaces and pseudodifferential operators. *Integral Equations Operator Theory*, **34**(4), 439–457 (1999)
- [26] Gröchenig, K.: *Foundations of Time-Frequency Analysis*. Appl. Numer. Harmon. Anal., Birkhäuser, Boston, MA, 2001
- [27] Gröchenig, K., “Composition and spectral invariance of pseudodifferential operators on modulation spaces. *Journal d’Analyse Mathématique* **98**, 65–82, (2006)
- [28] Gröchenig, K.: Time-Frequency Analysis on Sjöstrand’s Class. *Rev. Mat. Iberoamericana* **22**(2), 703–724, (2006)
- [29] Gröchenig, K., Zimmermann, G.: Hardy’s theorem and the short-time Fourier transform of Schwartz functions. *J. London Math. Soc.* (2)**63**, 205–214 (2001)
- [30] Gröchenig, K., Heil, C.: Counterexamples for Boundedness of Pseudodifferential Operators. *Osaka J. Math.*, **41**, 1–11 (2004)
- [31] Grossmann, A.: Parity operators and quantization of  $\delta$ -functions. *Commun. Math. Phys.* **48**, 191–193 (1976)
- [32] Guillemin V., Sternberg, S.: *Geometric Asymptotics*. Math. Surveys Monographs **14**, Amer. Math. Soc., Providence R.I., 1978
- [33] Guillemin V., Sternberg, S.: *Symplectic Techniques in Physics*. Cambridge University Press, Cambridge, Mass., 1984
- [34] Hardy, G. H.: A theorem concerning Fourier transforms. *J. London. Math. Soc.* **8**, 227–231 (1933)
- [35] Hofer, H., Zehnder, E.: *Symplectic Invariants and Hamiltonian Dynamics*. Birkhäuser Advanced texts (Basler Lehrbücher) Birkhäuser Verlag, (1994)

- [36] Hörmander, L.: The analysis of linear partial differential operators HV, Springer Verlag, 1983–1985, (1982)
- [37] Ivrii, V.: Microlocal Analysis and Precise Spectral Asymptotics. Springer Monographs in Mathematics, (1998)
- [38] Landau, L.D., Lifshitz E.M.: *Quantum Mechanics: Nonrelativistic Theory*. Pergamon Press, (1997)
- [39] Polterovich, L.: *The Geometry of the Group of Symplectic Diffeomorphisms*. Lectures in Mathematics, Birkhäuser, 2001
- [40] Reed M., Simon, B.: *Methods of Modern Mathematical Physics*. Academic Press, New York (1972).
- [41] Royer, A.: Wigner functions as the expectation value of a parity operator. *Phys. Rev. A* **15**, 449–450 (1977)
- [42] Simon, B.: *Trace ideals and their applications*. Cambridge University Press, Cambridge, 1979.
- [43] Simon, B.: The classical limit of quantum partition functions. *Comm. Math. Phys.*, **71(3)**, 247–276 (1980).
- [44] Sjöstrand, J.: An algebra of pseudodifferential operators, *Math. Res. Lett.* 1(2), 185–192, (1994).
- [45] Sjöstrand, J.: Wiener type algebras of pseudodifferential operators, in *Séminaire sur les Équations aux Dérivées Partielles, 1994–1995*, École Polytech., Palaiseau, Exp. No. IV, 21 (1995).
- [46] Shubin, M.A.: *Pseudodifferential Operators and Spectral Theory*. Springer-Verlag, first edition 1987; second edition 2001 [original Russian edition in Nauka, Moskva, 1978]
- [47] Smith, T.B.: Wavefunctions on phase space. *J. Phys. A: Math. Gen.* **39**, 1469–1484 (2006)
- [48] Thangavelu, S.: *Lectures on Hermite and Laguerre Expansions*. Princeton University Press, 1993
- [49] Williamson, J.: On the algebraic problem concerning the normal forms of linear dynamical systems. *Amer. J. of Math.* **58**, 141–163 (1936)

- [50] Trèves, F.: *Introduction to Pseudo-differential and Fourier Integral Operators* (two Volumes). University Series in Mathematics, Plenum Press (1980)
- [51] Wong, M.W.: *Weyl Transforms*. Springer-Verlag, 1998
- [52] Wong, M.W.: Weyl Transforms and a Degenerate Elliptic Partial Differential Equation. Proc. R. Soc. A **461**, 3863–3870 (2005).