

# A New Approach to the $\star$ -Genvalue Equation

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**Abstract.** We show that the eigenvalues and eigenfunctions of the star-genvalue equation can be completely expressed in terms of the corresponding eigenvalue problem for the quantum Hamiltonian. Our methods make use of a Weyl-type representation of the star-product and of the properties of the cross-Wigner transform, which appears as an intertwining operator.

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## 1. Introduction and Motivation

One of the key equations in the deformation quantization theory of Bayen et al. [1,2] is, no doubt, the star-genvalue (for short  $\star$ -genvalue) equation  $H \star \Psi = E\Psi$  where  $\star$  is the Moyal–Groenewold product [1,2,6]. In this Letter we show that the  $\star$ -genvalue equation can be completely solved in terms of the usual eigenvalue/eigenfunction problem  $\widehat{H}\psi = E\psi$  where  $\widehat{H}$  is the Weyl operator with symbol  $H$  (and vice versa). The underlying idea is simple: we first rewrite the equation  $H \star \Psi = E\Psi$  in the form

$$H\left(x + \frac{1}{2}i\hbar\partial_p, p - \frac{1}{2}i\hbar\partial_x\right)\Psi(x, p) = E\Psi(x, p),$$

where  $H(x + \frac{1}{2}i\hbar\partial_p, p - \frac{1}{2}i\hbar\partial_x)$  is the Weyl operator with symbol

$$\mathbb{H}(z, \zeta) = H\left(x - \frac{1}{2}\zeta_p, p + \frac{1}{2}\zeta_x\right).$$

We next show that the solutions of this equation and those of  $\widehat{H}\psi = E\psi$  can be obtained from each other using a family of intertwining operators (which is countable when  $\widehat{H}$  is essentially self-adjoint); these operators are up to a normalization

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factor, the cross-Wigner transforms  $\psi \mapsto W(\psi, \phi)$  where  $\phi$  describes the set of eigenfunctions of  $\widehat{H}$ . Our approach is inspired by previous work [3] of one of us on the time-dependent Torres-Vega [7] Schrödinger equation in phase space.

**Notation**

We will write  $z = (x, p)$  where  $x \in \mathbb{R}^n$  and  $p \in (\mathbb{R}^n)^*$ . Operators  $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$  are usually denoted by  $\widehat{A}, \widehat{B}, \dots$  while operators  $\mathcal{S}(\mathbb{R}^{2n}) \rightarrow \mathcal{S}'(\mathbb{R}^{2n})$  are denoted by  $\widetilde{A}, \widetilde{B}, \dots$ . The Greek letters  $\psi, \phi, \dots$  stand for functions defined on  $\mathbb{R}^n$  while their capitalized counterparts  $\Psi, \Phi, \dots$  denote functions defined on  $\mathbb{R}^{2n}$ . We will make use of the symplectic Fourier transform which is defined for  $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$  by the formula

$$\Psi_{\sigma}^{\hbar}(z) = F_{\sigma}^{\hbar} \Psi(z) = \left( \frac{1}{2\pi \hbar} \right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar} \sigma(z, z')} \Psi(z') dz',$$

where  $\sigma(z, z') = p \cdot x' - p' \cdot x$  is the standard symplectic form on  $\mathbb{R}^n \times (\mathbb{R}^n)^* \cong \mathbb{R}^{2n}$  (the dot  $\cdot$  stands for the duality bracket; in practice  $p \cdot x$  can be viewed as the usual Euclidean scalar product under the identification  $(\mathbb{R}^n)^* \cong \mathbb{R}^n$ ). The symplectic Fourier transform is involutive:  $F_{\sigma}^{\hbar} \circ F_{\sigma}^{\hbar}$  is the identity on  $\mathcal{S}'(\mathbb{R}^{2n})$ .

**2. The  $\star$ -Genvalue Equation: Short Review**

In view of Schwartz’s kernel theorem every linear continuous operator  $\widehat{A} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$  can be represented, for  $\psi \in \mathcal{S}(\mathbb{R}^n)$ , in the form  $\widehat{A}\psi(x) = \langle \mathcal{K}_A(x, \cdot), \psi \rangle$  with  $\mathcal{K}_A \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ . By definition the contravariant (Weyl) symbol of  $\widehat{A}$  is the tempered distribution  $A$  defined by the Fourier transform

$$a(x, p) = \left\langle e^{-\frac{i}{\hbar} p(\cdot)}, \mathcal{K}_{\widehat{A}} \left( x + \frac{1}{2}(\cdot), x - \frac{1}{2}(\cdot) \right) \right\rangle. \tag{1}$$

Assume that  $\widehat{B} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ ; then the product  $\widehat{C} = \widehat{A} \circ \widehat{B}$  exists and its Weyl symbol is given by the Moyal product

$$a \star b(z) = \left( \frac{1}{4\pi \hbar} \right)^{2n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{\frac{i}{2\hbar} \sigma(u, v)} a \left( z + \frac{1}{2}u \right) b \left( z - \frac{1}{2}v \right) dudv. \tag{2}$$

The main observation we will exploit in this paper is the following: if we write  $a = H$  and  $b = \Psi$  then we can write

$$H \star \Psi = \widetilde{H} \Psi, \tag{3}$$

where

$$\widetilde{H} = H \left( x + \frac{1}{2}i\hbar \partial_p, p - \frac{1}{2}i\hbar \partial_x \right)$$

is a certain pseudodifferential operator on  $\mathcal{S}(\mathbb{R}^{2n})$  we are going to identify. Let us view the linear operator  $\tilde{H} : \Psi \mapsto H \star \Psi$  on  $\mathcal{S}(\mathbb{R}^{2n})$  as a Weyl operator. Using formula (2), the kernel of  $\tilde{H}$  is the distribution

$$\mathcal{K}_{\tilde{H}}(z, y) = \left(\frac{1}{2\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\sigma(u, z-y)} H\left(z - \frac{1}{2}u\right) du. \tag{4}$$

It follows, using (1) and the Fourier inversion formula, that the contravariant symbol of  $\tilde{H}$  is given by

$$\mathbb{H}(z, \zeta) = \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\zeta \cdot \eta} \mathcal{K}_{\tilde{H}}\left(z + \frac{1}{2}\eta, z - \frac{1}{2}\eta\right) d\eta.$$

Performing the change of variables  $u = 2z + \eta - z'$  in the integral in (4) we get

$$\mathcal{K}_{\tilde{H}}\left(z + \frac{1}{2}\eta, z - \frac{1}{2}\eta\right) = \left(\frac{1}{2\pi\hbar}\right)^{2n} e^{\frac{2i}{\hbar}\sigma(z, \eta)} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\sigma(\eta, z')} H\left(\frac{1}{2}z'\right) dz';$$

setting  $H(\frac{1}{2}z') = H_{1/2}(z')$  the integral is  $(2\pi\hbar)^n$  times the symplectic Fourier transform  $F_{\sigma}^{\hbar} H_{1/2}(-\eta) = (H_{1/2})_{\sigma}(-\eta)$  so that

$$\begin{aligned} \mathbb{H}\left(\frac{1}{2}z, \zeta\right) &= \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\zeta \cdot \eta} e^{\frac{i}{\hbar}\sigma(z, \eta)} (H_{1/2})_{\sigma}(-\eta) d\eta = \\ &= \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\sigma(z - J\zeta, \eta)} (H_{1/2})_{\sigma}(\eta) d\eta, \end{aligned}$$

where  $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$  is the standard symplectic matrix. Since the second equality is the inverse symplectic Fourier transform of  $(H_{1/2})_{\sigma}$  calculated at the point  $z + J\zeta$  we finally get

$$\mathbb{H}(z, \zeta) = H\left(x - \frac{1}{2}\zeta_p, p + \frac{1}{2}\zeta_x\right); \tag{5}$$

we are viewing here  $\zeta = (\zeta_x, \zeta_p)$  as the dual variable of  $z = (x, p)$ ; this justifies formula (3) interpreting  $\tilde{H}$  as the quantized Hamiltonian obtained from  $\mathbb{H}$  by the quantization rule

$$(x, p) \mapsto \left(x + \frac{1}{2}i\hbar\partial_p, p - \frac{1}{2}i\hbar\partial_x\right). \tag{6}$$

### 3. $\tilde{H}$ -Calculus

There is another very fruitful way of interpreting the Weyl operators  $\tilde{H} = H \star \Psi$ . Let us return to the expression (2) with  $a = H$  and  $b = \Psi$ ; performing the changes of variable  $u = 2(z' - z)$  and  $v = z_0$  this formula can be rewritten as

$$\tilde{H}\Psi(z) = \left(\frac{1}{2\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} \left[ \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\sigma(z_0, z')} H(z') dz' \right] e^{-\frac{i}{\hbar}\sigma(z, z_0)} \Psi\left(z - \frac{1}{2}z_0\right) dz_0.$$

Observing that the integral between brackets is  $(2\pi\hbar)^n$  times the symplectic Fourier transform of  $H$  we can rewrite this formula in the form

$$\tilde{H}\Psi(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} H_{\sigma}^{\hbar}(z_0) \tilde{T}(z_0) \Psi(z) dz_0, \tag{7}$$

where  $\tilde{T}(z_0)$  is the operator defined by

$$\tilde{T}(z_0)\Psi(z) = e^{-\frac{i}{\hbar}\sigma(z, z_0)} \Psi\left(z - \frac{1}{2}z_0\right). \tag{8}$$

Formula (7) is strongly reminiscent of the representation

$$\hat{H}\psi = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} H_{\sigma}^{\hbar}(z_0) \hat{T}(z_0) \psi dz_0 \tag{9}$$

of a Weyl operator  $\hat{H}$  in terms of its covariant symbol  $H_{\sigma}^{\hbar} = F_{\sigma}^{\hbar} H$  and of the Heisenberg–Weyl operator

$$\hat{T}(z_0)\psi(x) = e^{\frac{i}{\hbar}(p_0 \cdot x - \frac{1}{2}p_0 \cdot x_0)} \psi(x - x_0),$$

except that  $\tilde{T}(z_0)$  is allowed to act on functions of the phase space variable  $z$  and not only of  $x$ . This feeling is amplified when one notes (after a straightforward calculation) that the operators  $\tilde{T}(z_0)$  obey the relations

$$\tilde{T}(z_0 + z_1) = e^{-\frac{i}{2\hbar}\sigma(z_0, z_1)} \tilde{T}(z_0) \tilde{T}(z_1) \tag{10}$$

$$\tilde{T}(z_1) \tilde{T}(z_0) = e^{-\frac{i}{\hbar}\sigma(z_0, z_1)} \tilde{T}(z_0) \tilde{T}(z_1), \tag{11}$$

which are similar to those satisfied by the Heisenberg–Weyl operators. These facts suggest that  $\tilde{T}(z_0, t) = e^{\frac{it}{\hbar}} \tilde{T}(z_0)$  defines a unitary representation of the Heisenberg group. Let us prove this is indeed the case. For this we will need the linear mapping  $W_{\phi} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^{2n})$  defined by

$$W_{\phi}\psi = (2\pi\hbar)^{n/2} W(\psi, \phi), \tag{12}$$

where  $\phi$  denotes an arbitrary function in  $S(\mathbb{R}^n)$  such that  $\|\phi\|_{L^2} = 1$ , and  $W(\psi, \phi)$  is the cross-Wigner distribution. Explicitly

$$W_\phi \psi(z) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}p \cdot y} \psi\left(x + \frac{1}{2}y\right) \overline{\phi\left(x - \frac{1}{2}y\right)} dy. \tag{13}$$

In view of Moyal's identity

$$(W(\psi, \phi) | W(\psi', \phi'))_{L^2(\mathbb{R}^{2n})} = \left(\frac{1}{2\pi\hbar}\right)^n (\psi | \psi')_{L^2(\mathbb{R}^n)} (\phi | \phi')_{L^2(\mathbb{R}^n)},$$

the operator  $W_\phi$  extends into an isometry of  $L^2(\mathbb{R}^n)$  onto a subspace  $\mathcal{H}_\phi$  of  $L^2(\mathbb{R}^{2n})$ ; we are going to see in a moment  $\mathcal{H}_\phi$  is closed in  $L^2(\mathbb{R}^{2n})$ , but let us first give a formula for the adjoint  $W_\phi^*$  of  $W_\phi$ . We have

$$W_\phi^* \Psi(z) = \left(\frac{2}{\pi\hbar}\right)^{n/2} \int_{\mathbb{R}^n} e^{\frac{2i}{\hbar}p \cdot (x-y)} \phi(2y-x) \Psi(y, p) dp dy \tag{14}$$

(this formula is obtained by a straightforward calculation using the identity  $(W_\phi \psi | \Psi)_{L^2(\mathbb{R}^{2n})} = (\psi | W_\phi^* \Psi)_{L^2(\mathbb{R}^n)}$ ).

**PROPOSITION 1.** *The range  $\mathcal{H}_\phi$  of  $W_\phi$  is closed, and hence a Hilbert space.*

*Proof.* Set  $P_\phi = W_\phi W_\phi^*$  where  $W_\phi^*$  is the adjoint of  $W_\phi$ ; we have  $P_\phi = P_\phi^*$  and  $P_\phi P_\phi^* = P_\phi$  hence  $P_\phi$  is an orthogonal projection. Since  $W_\phi^* W_\phi$  is the identity on  $L^2(\mathbb{R}^n)$  the range of  $W_\phi^*$  is  $L^2(\mathbb{R}^n)$  and that of  $P_\phi$  is therefore precisely  $\mathcal{H}_\phi$ . Since the range of a projection is closed, so is  $\mathcal{H}_\phi$ .  $\square$

This result, together with formula (11) shows that  $\tilde{T}(z_0)$  and  $\hat{T}(z_0)$  are unitarily equivalent representations of the Heisenberg group  $\mathbf{H}_n$ ; the irreducibility of the representation  $\tilde{T}(z_0) : \mathbf{H}_n \rightarrow \mathcal{H}_\phi$  follows from von Neumann's uniqueness theorem for the projective representations of the CCR.

Let us return to the operator  $\tilde{H} = H \star$ . A straightforward calculation showing that  $W_\phi$  satisfies the intertwining relations

$$\begin{aligned} x \star W_\phi \psi &= \left(x + \frac{1}{2}i\hbar\partial_p\right) W_\phi \psi = W_\phi(x\psi) \\ p \star W_\phi \psi &= \left(p - \frac{1}{2}i\hbar\partial_x\right) W_\phi \psi = W_\phi(-i\hbar\partial_x\psi). \end{aligned}$$

An educated guess is then that we have more generally

**PROPOSITION 2.** (i) *The operator  $W_\phi$  intertwines the operators  $\tilde{T}(z_0)$  and  $\hat{T}(z_0)$ :*

$$W_\phi(\hat{T}(z_0)\psi) = \tilde{T}(z_0)W_\phi\psi; \tag{15}$$

(ii) *We also have*

$$\tilde{H}W_\phi = W_\phi \widehat{H} \quad \text{and} \quad W_\phi^* \tilde{H} = \widehat{H} W_\phi^*. \tag{16}$$

*Proof.* Making the change of variable  $y = y' + x_0$  in the definition (13) of  $W_\phi$  we get

$$W_\phi(\widehat{T}(z_0)\psi, \phi)(z) = e^{-\frac{i}{\hbar}\sigma(z, z_0)} W_\phi \psi \left( z - \frac{1}{2}z_0 \right),$$

which is precisely (15). Applying  $W_\phi$  to both sides of (9), we get

$$W_\phi \widehat{H}\psi = \left( \frac{1}{2\pi\hbar} \right)^n \int_{\mathbb{R}^{2n}} H_\sigma^{\hbar}(z_0) W_\phi[\widehat{T}(z_0)\psi] dz_0$$

and hence

$$W_\phi \widehat{H}\psi = \left( \frac{1}{2\pi\hbar} \right)^n \int_{\mathbb{R}^{2n}} H_\sigma^{\hbar}(z_0)[\tilde{T}(z_0)W_\phi\psi] dz_0,$$

which is the first equality (16) in view of formula (7). To prove the second equality it suffices to apply this first equality to  $W_\phi^* \tilde{H} = (\tilde{H}^* W_\phi)^*$ . □

### 4. Spectral Results

We will need the following result, which is quite interesting by itself:

LEMMA 3. *Let  $(\phi_j)_j$  be an arbitrary orthonormal basis of  $L^2(\mathbb{R}^n)$ ; the vectors  $\Phi_{j,k} = W_{\phi_j} \phi_k$  form an orthonormal basis of  $L^2(\mathbb{R}^{2n})$ .*

*Proof.* Since the  $W_{\phi_j}$  are isometries the vectors  $\Phi_{j,k}$  form an orthonormal system. It is sufficient to show that if  $\Psi \in L^2(\mathbb{R}^{2n})$  is orthogonal to the family  $(\Phi_{j,k})_{j,k}$  (and hence to all the spaces  $\mathcal{H}_{\phi_j}$ ) then  $\Psi = 0$ . Assume that  $(\Psi | \Phi_{jk})_{L^2(\mathbb{R}^{2n})} = 0$  for all  $j, k$ . Since we have

$$(\Psi | \Phi_{jk})_{L^2(\mathbb{R}^{2n})} = (\Psi | W_{\phi_j} \phi_k)_{L^2(\mathbb{R}^{2n})} = (W_{\phi_j}^* \Psi | \phi_k)_{L^2(\mathbb{R}^n)},$$

it follows that  $W_{\phi_j}^* \Psi = 0$  for all  $j$  since  $(\phi_j)_j$  is a basis; using the anti-linearity of  $W_\phi$  in  $\phi$  we have in fact  $W_\phi^* \Psi = 0$  for all  $\phi \in L^2(\mathbb{R}^n)$ . Let us show that this implies that  $\Psi = 0$ . In view of formula (14) for the adjoint of  $W_\phi$  the operator  $W_\phi^*$  has kernel

$$\Phi_x(y, p) = \left( \frac{2}{\pi\hbar} \right)^{n/2} e^{\frac{2i}{\hbar} p \cdot (x-y)} \phi(2y-x).$$

Let us fix  $x$ ; the property  $W_\phi^* \Psi = 0$  for all  $\phi$  is then equivalent to  $\langle \Psi, \Phi_x \rangle = 0$  for all  $\Phi_x \in \mathcal{S}(\mathbb{R}^{2n})$  (fixed  $x$ ) and hence  $\Psi = 0$ , which we set out to show. □

We now have everything we need to prove the main results of this Letter. We begin by stating the following general property:

**THEOREM 4.** *The following properties are true: (i) The eigenvalues of the operators  $\widehat{H}$  and  $\widetilde{H} = H\star$  are the same; (ii) Let  $\psi$  be an eigenfunction of  $\widehat{H}$ :  $\widehat{H}\psi = \lambda\psi$ . Then, for every  $\phi$ , the function  $\Psi = W_\phi\psi$  is an eigenfunction of  $\widetilde{H}$  corresponding to the same eigenvalue:  $\widetilde{H}\Psi = \lambda\Psi$ . (iii) Conversely, if  $\Psi$  is an eigenfunction of  $\widetilde{H}$  then  $\psi = W_\phi^*\Psi$  is an eigenfunction of  $\widehat{H}$  corresponding to the same eigenvalue.*

*Proof.* That every eigenvalue of  $\widehat{H}$  also is an eigenvalue of  $\widetilde{H}$  is clear: if  $\widehat{H}\psi = \lambda\psi$  for some  $\psi \neq 0$  then

$$\widetilde{H}(W_\phi\psi) = W_\phi\widehat{H}\psi = \lambda(W_\phi\psi)$$

and  $W_\phi\psi \neq 0$  because  $W_\phi$  is injective; this proves at the same time that  $W_\phi\psi$  is an eigenfunction of  $\widetilde{H}$ . Assume conversely that  $\widetilde{H}\Psi = \lambda\Psi$  for  $\Psi \neq 0$  and  $\lambda \in \mathbb{R}$ . For every  $\phi$  we have, using the second equality (16),

$$\widehat{H}W_\phi^*\Psi = W_\phi^*\widetilde{H}\Psi = \lambda W_\phi^*\Psi,$$

hence  $\lambda$  is an eigenvalue of  $\widehat{H}$ ;  $W_\phi^*\Psi$  is an eigenfunction if it is different from zero. Let us prove this is indeed the case. We have  $W_\phi W_\phi^*\Psi = P_\phi\Psi$  where  $P_\phi$  is the orthogonal projection on the range  $\mathcal{H}_\phi$  of  $W_\phi$ . Assume that  $W_\phi^*\Psi = 0$ ; then  $P_\phi\Psi = 0$  for every  $\phi \in \mathcal{S}(\mathbb{R}^n)$ , and hence  $\Psi = 0$  in view of Lemma 3 above.  $\square$

*Remark 5.* The result above is quite general, because we do not make any assumption on the multiplicity of the (star)eigenvalues, nor do we assume that  $\widehat{H}$  is essentially self-adjoint. Notice that the proof actually works for arbitrary  $\phi \in \mathcal{S}'(\mathbb{R}^n)$  (For examples see the end of this section).

**COROLLARY 6.** *Suppose that  $\widehat{H}$  is an essentially self-adjoint operator on  $L^2(\mathbb{R}^n)$  and that each of the eigenvalues  $\lambda_0, \lambda_1, \dots, \lambda_j, \dots$  has multiplicity one. Let  $\psi_0, \psi_1, \dots, \psi_j, \dots$  be a corresponding sequence of orthonormal eigenfunctions. Let  $\Psi_j$  be an eigenfunction of  $\widetilde{H}$  corresponding to the eigenvalue  $\lambda_j$ . There exists a sequence  $(\alpha_{j,k})_k$  of complex numbers such that*

$$\Psi_j = \sum_{\ell} \alpha_{j,\ell} \Psi_{j,\ell} \quad \text{with} \quad \Psi_{j,\ell} = W_{\psi_\ell} \psi_j \in \mathcal{H}_j \cap \mathcal{H}_\ell. \tag{17}$$

*Proof.* We know from Theorem 4 above that  $\widehat{H}$  and  $\widetilde{H}$  have same eigenvalues and that  $\Psi_{j,k} = W_{\psi_k} \psi_j$  satisfies the eigenvalue equation  $\widetilde{H}\Psi_{j,k} = \lambda_j \Psi_{j,k}$ . Since  $\widehat{H}$  is self-adjoint and its eigenvalues are distinct, its eigenfunctions  $\psi_j$  form an orthonormal basis of  $L^2(\mathbb{R}^n)$ . It follows from Lemma 3 that the  $\Psi_{j,k}$  form an

orthonormal basis of  $L^2(\mathbb{R}^{2n})$ , hence there exist non-zero scalars  $\alpha_{j,k,\ell}$  such that  $\Psi_j = \sum_{k,\ell} \alpha_{j,k,\ell} \Psi_{k,\ell}$ . We have, by linearity and using the fact that  $\tilde{H}\Psi_{k,\ell} = \lambda_k \Psi_{k,\ell}$ ,

$$\tilde{H}\Psi_j = \sum_{k,\ell} \alpha_{j,k,\ell} \tilde{H}\Psi_{k,\ell} = \sum_{k,\ell} \alpha_{j,k,\ell} \lambda_k \Psi_{k,\ell}.$$

On the other hand we also have  $\tilde{H}\Psi_j = \lambda_j \Psi_j$  and hence

$$\tilde{H}\Psi_j = \lambda_j \Psi_j = \sum_{j,k} \alpha_{j,k,\ell} \lambda_j \Psi_{k,\ell},$$

which is only possible if  $\alpha_{j,k,\ell} = 0$  for  $k \neq j$ ; setting  $\alpha_{j,\ell} = \alpha_{j,j,\ell}$  formula (17) follows. (That  $\Psi_{j,\ell} \in \mathcal{H}_j \cap \mathcal{H}_\ell$  is clear using the definition of  $\mathcal{H}_\ell$  and the sesquilinearity of the cross-Wigner transform.)  $\square$

We remark that the continuous spectrum could be dealt with in a similar fashion provided that one generalizes the transform  $W_\phi$  by allowing the “parameter” to be a tempered distribution (in which case the normalization condition  $\|\phi\|_{L^2} = 1$  does no longer make sense, of course); the same remark applies to the case where  $\widehat{H}$  is no longer essentially self-adjoint (cf. the remark following the proof of Theorem 4). To illustrate this, let us consider the two following typical examples (in dimension  $n = 1$ ):

- $H(x, p) = p$ . In this case  $\widehat{H} = -i\hbar\partial/\partial x$  is a symmetric operator and the equation  $\widehat{H}\psi = E\psi$  has solutions for every real value of  $E$ ; these solutions are the tempered distributions  $\psi(x) = C \exp(iEx/\hbar)$  ( $C$  any complex constant). A straightforward calculation shows that

$$W_\phi\psi(x, p) = C' e^{\frac{2i}{\hbar}(E-p)x} \overline{F\phi(p)},$$

where  $C'$  is a new constant and  $F\phi$  is the Fourier transform of  $\phi$ . If we let  $\phi$  range over  $\mathcal{S}'(\mathbb{R}^n)$  and use the fact that the Fourier transform is an automorphism of  $\mathcal{S}'(\mathbb{R}^n)$  we see that  $W_\phi\psi$  can be any distribution of the type

$$\Psi(x, p) = \Phi(p) e^{\frac{2i}{\hbar}(E-p)x},$$

with  $\Phi \in \mathcal{S}'(\mathbb{R}^n)$ ; these distributions are precisely the solutions of the equation

$$p \star \Psi = \left( p - \frac{1}{2} i \hbar \partial_x \right) \Psi = E \Psi,$$

as a straightforward calculation shows.

- $H(x, p) = x$ . Here  $\widehat{H}$  is the operator of multiplication by  $x$ ; this a symmetric operator without any eigenvalues and eigenfunctions. It is, however, self-adjoint, and the solutions of  $\widehat{H}\psi = E\psi$  are the distributions  $\psi = C\delta(x - E)$ ; one finds by an argument similar to that above that  $W_\phi\psi$  can be any distribution of the type

$$\Psi(x, p) = \Phi(x) e^{-\frac{2i}{\hbar}(E-x)p},$$

which is the general solution of the equation

$$x \star \Psi = \left( x + \frac{1}{2} i \hbar \partial_p \right) \Psi = E \Psi.$$

### 5. An Example and its Obvious Extension

As an illustration consider the harmonic oscillator Hamiltonian

$$H = \frac{1}{2} (p^2 + x^2). \tag{18}$$

In view of the results shown above the spectra of the operators  $\widehat{H}$  and  $\widetilde{H}$  are identical. Choosing for simplicity  $\hbar = 1$  the eigenvalues of  $\widehat{H}$  are the numbers  $\lambda_N = N + \frac{1}{2}$  with  $N = 0, 1, 2, \dots$ . The normalized eigenfunctions are the re-scaled Hermite functions

$$\psi_k(x) = (2^k k! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} \mathcal{H}_k(x), \tag{19}$$

where

$$\mathcal{H}_k(x) = (-1)^{km} e^{x^2} \left( \frac{d}{dx} \right)^k e^{-x^2}$$

is the  $k$ th Hermite polynomial. Using definition (12) of  $W_\phi$  together with known formulae for the cross-Wigner transform of Hermite functions (see for instance [9, Chap. 24, Theorem 24.1]) one finds that the eigenfunctions of  $\widetilde{H}$  are linear superpositions of the functions

$$\Psi_{j+k,k}(z) = (-1)^j \left( \frac{j!}{(j+k)!} \right)^{\frac{1}{2}} 2^{\frac{k}{2}+1} \zeta^{-k} \mathcal{L}_j^k(2|z|^2) e^{-|z|^2}, \tag{20}$$

where  $\zeta = x + ip$  and  $\Psi_{j,j+k} = \overline{\Psi_{j+k,k}}$  for  $k = 0, 1, 2, \dots$ ; here

$$\mathcal{L}_j^k(x) = \frac{1}{j!} x^{-k} e^x \left( \frac{d}{dx} \right)^j (e^{-x} x^{j+k}), \quad x > 0$$

is the Laguerre polynomial of degree  $j$  and order  $k$ . (For similar results see Bayen et al. [2].)

Notice that the above-cited example can be generalized without difficulty to the case of arbitrary quadratic Hamiltonians of the type

$$H = \frac{1}{2} Mz \cdot z,$$

where  $M$  is a positive-definite symmetric matrix. In fact, in view of Williamson's diagonalization theorem [8] there exists a symplectic matrix  $S$  such that

$$M = S^T D S, \quad D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$$

where  $\Lambda$  is the diagonal matrix whose entries are the moduli  $\omega_j > 0$  of the eigenvalues  $\pm i\omega_j$  of  $JM$ . We thus have

$$H \circ S = \sum_{j=1}^n \frac{\omega_j}{2} (x_j^2 + p_j^2)$$

and  $\widehat{H \circ S} = \widehat{S} \widehat{H} \widehat{S}^{-1}$  where  $\widehat{S}$  is anyone of the two metaplectic operators associated with  $S$ . The eigenvalues of  $\widehat{H \circ S}$  and  $\widehat{H}$  are the same; they are the numbers

$$\lambda_{N_1, \dots, N_n} = \sum_{j=1}^n \left( N_j + \frac{1}{2} \right) \omega_j$$

and the eigenfunctions  $\psi_S$  of  $\widehat{H \circ S}$  and those,  $\psi$ , of  $\widehat{H}$  by the formula  $\psi_S = \widehat{S}\psi$ . Now, the eigenfunctions of  $\widehat{H}$  are tensor products of rescaled Hermite functions; using the fact that  $\psi_S = \widehat{S}\psi$  together with the symplectic covariance formula

$$W(\widehat{S}\psi, \widehat{S}\phi)(z) = W(\psi, \phi)(S^{-1}z)$$

satisfied by the cross-Wigner distributions, the eigenvalues of  $\widetilde{H} = H\star$  are calculated in terms of tensor products of the functions (20). We do not give the details of the calculations here since they are rather lengthy but straightforward.

### 6. Concluding Remarks

Due to limitation of length there are several aspects of our approach we have not discussed in this Letter. For instance, the methods we have developed should apply with a few modifications to more general phase space (for instance co-adjoint orbits). A perhaps even more exciting problem is the following, which is closely related to our previous results [4] on the relationship between the uncertainty principle and the topological notion of symplectic capacity. A rather straightforward extension of the methods we used in [4] to analyze Hardy’s uncertainty principle shows [5] that if

$$|W_\phi \psi(z)| \leq C e^{-\frac{1}{\hbar}(a|x|^2 + b|y|^2)} \quad \text{for } z \in \mathbb{R}^{2n}, \tag{21}$$

then we must have  $ab \leq 1$ . In particular we can have  $|W_\phi \psi(z)| \leq C e^{-\frac{1}{\varepsilon}|z|^2}$  only if  $\varepsilon \geq \hbar$ ; it follows that the Hilbert spaces  $\mathcal{H}_\phi$  do not contain any nontrivial function with compact support: assume in fact that  $\Psi \in \mathcal{H}_\phi$  is such that  $\Psi(z) = 0$  for  $|z| \geq R > 0$ . Then, given an arbitrary  $\varepsilon < \hbar$  one can find a constant  $C_\varepsilon$  such that  $|\Psi(z)| \leq C_\varepsilon e^{-\frac{1}{\varepsilon}|z|^2}$ , which is impossible since  $\Psi = W_\phi \psi$  for some  $\psi \in L^2(\mathbb{R}^n)$ . This suggests (taking Theorem 4 into account) that the solutions  $\Psi$  of the  $\star$ -genvalue equation cannot be too concentrated around a point in phase-space. In fact we conjecture that if an estimate of the type  $|\Psi(z)| \leq C e^{-\frac{1}{\hbar}Mz \cdot z}$  ( $M$  symmetric positive-definite) holds, then the symplectic capacity of the ellipsoid  $Mz \cdot z \leq \hbar$  must be at least  $\frac{1}{2}\hbar$ , in conformity with the uncertainty principle. We will come back to this important question in the near future.

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