GABOR (SUPER)FRAMES WITH HERMITE FUNCTIONS

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Abstract. We investigate vector-valued Gabor frames (sometimes called Gabor superframes) based on Hermite functions H_n. Let $\mathbf{h} = (H_0, H_1, \dots, H_n)$ be the vector of the first $n+1$ Hermite functions. We give a complete characterization of all lattices $\Lambda \subseteq \mathbb{R}^2$ such that the Gabor system $\{e^{2\pi i\lambda_2 t}\mathbf{h}(t-\lambda_1) : \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^{n+1})$. As a corollary we obtain sufficient conditions for a single Hermite function to generate a Gabor frame and a new estimate for the lower frame bound. The main tools are growth estimates for the Weierstrass σ-function, a new type of interpolation problem for entire functions on the Bargmann-Fock space, and structural results about vector-valued Gabor frames.

1. Introduction

Given a function $g \in L^2(\mathbb{R})$ and a lattice $\Lambda \subset \mathbb{R}^2$, we study the frame property of the set $\{e^{2\pi i\lambda_2 t} g(t-\lambda_1) : \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$. Precisely, write $\pi_\lambda g = e^{2\pi i\lambda_2 t} g(t-\lambda_1)$ for the time-frequency shift by $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. Then we call the set $\mathcal{G}(g, \Lambda) = \{\pi_\lambda g : \lambda \in \Lambda\}$ a *Gabor frame* or *Weyl-Heisenberg frame*, whenever there exist constants $A, B > 0$ such that, for all $f \in L^2(\mathbb{R})$,

$$A\|f\|^2_{L^2(\mathbb{R})} \le \sum_{\lambda\in\Lambda} |\langle f, \pi_\lambda g\rangle_{L^2(\mathbb{R})}|^2 \le B\|f\|^2_{L^2(\mathbb{R})}\,. \tag{1}$$

Gabor frames originate in quantum mechanics through J. von Neumann and in information theory through D. Gabor [Ga] and nowadays have many applications in signal processing. A large body of results describes the structure of Gabor frames and provides sufficient conditions of a qualitative nature for $\mathcal{G}(g, \Lambda)$ to form a (Gabor) frame, see [D, Gr1] for details and references.

We will also study vector-valued Gabor frames. In this case the Hilbert space is $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^n)$ consisting of all vector-valued functions $\mathbf{f}(t) = \big(f_1(t), \dots, f_n(t)\big)$ with the natural inner product

$$\langle \mathbf{f}, \mathbf{g}\rangle = \sum_{j=1}^n \int_{-\infty}^{\infty} f_j(t)\overline{g_j(t)}dt = \sum_{j=1}^n \langle f_j, g_j\rangle. \tag{2}$$

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The *time-frequency shifts* π_z, $z = (x, \xi)$ act coordinate-wise by

$$\pi_z \mathbf{f}(t) = e^{2i\pi\xi t}\mathbf{f}(t-x)\,. \tag{3}$$

The vector-valued Gabor system $\mathcal{G}(\mathbf{g}, \Lambda) = \{\pi_\lambda \mathbf{g} : \lambda \in \Lambda\}$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^n)$, if there exist constants $A, B > 0$ such that

$$A\|\mathbf{f}\|^2_{L^2(\mathbb{R},\mathbb{C}^n)} \le \sum_{\lambda\in\Lambda} |\langle \mathbf{f}, \pi_\lambda \mathbf{g}\rangle|^2 \le B\|\mathbf{f}\|^2_{L^2(\mathbb{R},\mathbb{C}^n)}, \quad \forall \mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^n). \tag{4}$$

Vector-valued Gabor frames were introduced under the name "superframes" in signal processing by R. Balan [B1] in the context of "multiplexing" and studied in [B2, HL] for their own sake. The idea of "multiplexing" is to encode n independent signals (functions) $f_j \in L^2(\mathbb{R}), j = 1, \dots, n$, as a single sequence that captures the time-frequency information of each f_j. Fixing suitable windows $g_j \in L^2(\mathbb{R})$ and using vector-valued notation $\mathbf{f} = (f_1, \dots, f_n)$, one then considers the sequence of numbers $\langle \mathbf{f}, \pi_\lambda \mathbf{g}\rangle = \sum_{j=1}^n \langle f_j, \pi_\lambda g_j\rangle$ for $\lambda \in \Lambda$, i.e., the inner product in $L^2(\mathbb{R}, \mathbb{C}^n)$. Roughly speaking, these numbers then measure the time-frequency content of the whole $\mathbf{f}$ at the point λ in the time-frequency plane.

Now one requires that $\mathbf{f}$ is completely determined by these inner products and that there exists a stable reconstruction. This requirement leads to the definition of a vector-valued Gabor frame (4).

The general problem of characterizing *all* lattices Λ for which $\mathcal{G}(g, \Lambda)$ is a frame seems to be extremely difficult. In fact, this problem is solved only for three classes of basis functions, namely for the Gaussians $H_0(t) = e^{-at^2}, a > 0$, in [L, SW], for the hyperbolic secant $g(t) = (\cosh at)^{-1}$ in [JS], and for the one-sided exponential function $g(t) = e^{-a|t|}\chi_{[0,\infty)}(t)$ [J3]. For the Gaussian, our understanding is based on the connection between the frame property of $\mathcal{G}(g, \Lambda)$ and a classical interpolation problem in the Bargmann-Fock space of entire functions. The case of the hyperbolic secant is reduced to the case of the Gaussian.

For other choices of the g, where the connection to complex analysis is missing, the conditions for $\mathcal{G}(g, a\mathbb{Z} \times b\mathbb{Z})$ to form a frame become very different and often rather intriguing [J4].

Even less is known about Gabor superframes. Necessary density conditions were studied in [B2], sufficient density conditions can be derived from coorbit theory [FG] and from the sampling theory on the Heisenberg group [Fu]. In particular, these results imply that for any sufficiently dense lattice Λ and a mild condition on the vector $\mathbf{g}$ the Gabor system $\mathcal{G}(\mathbf{g}, \Lambda)$ is a Gabor superframe.

In this article we study the frame property of $\mathcal{G}(\mathbf{h}, \Lambda)$ in the case when $\mathbf{h}$ is the vector of the first $n + 1$ Hermite function $\mathbf{h} = (H_0, \dots, H_n)$. In the scalar-valued case, the study of Gabor frames with the Gaussian window H_0 is natural, because the Gaussian minimizes the uncertainty principle. Likewise, for vector-valued frames, the study of Gabor superframes with the Hermite window $\mathbf{h}$ is natural, because the first Hermite functions are the unique orthonormal set $\{f_0, \dots, f_n\}$ in $L^2(\mathbb{R})$ of size $n+1$ satisfying the normalizations $\int_{-\infty}^{\infty} t|f_j(t)|^2 dt = 0$

and $\int_{-\infty}^{\infty} \xi |\hat{f}_j(\xi)|^2 d\xi = 0$, $j = 0, \dots, n$, such that the uncertainty

$$\sum_{j=0}^{n} \left(\int_{-\infty}^{\infty} t^2 |f_j(t)|^2 dt + \int_{-\infty}^{\infty} \xi^2 |\hat{f}_j(\xi)|^2 d\xi \right) \tag{5}$$

is minimized. Another motivation comes again from signal processing where Hermite functions are used, see [HMS].

The case of the Hermite vector window has been already investigated by Führ by employing techniques related to sampling in the space of bandlimited functions on the Heisenberg group. He proved the following result.

Theorem [Fu]. *Let* $\mathbf{h} = (H_0, \dots, H_n)$. *There exists a constant* $C > 0$ *with the following property: If the diameter of the (smallest) fundamental domain of* Λ *is less that* $C(n+1)^{-1/2}$, *then* $\mathcal{G}(\mathbf{h}, \Lambda)$ *is a frame for* $L^2(\mathbb{R}, \mathbb{C}^{n+1})$.

Unfortunately nothing can be said about the constant C within such an approach. Führ uses the so-called "oscillation method" which is used for existence results, but in general does not yield sharp results.

In this paper, we give a complete characterization of vector-valued frames with Hermite functions. If the lattice is given as $\Lambda = A\mathbb{Z}^2$ for some invertible, real-valued 2×2-matrix A, let $s(\Lambda) = |\det A|$ be the area of the fundamental domain of Λ. Our main result can be formulated as follows.

Theorem 1.1. *Let* $\mathbf{h} = (H_0, \dots, H_n)$ *be the vector of the first* $n+1$ *Hermite functions. Then* $\mathcal{G}(\mathbf{h}, \Lambda)$ *is a frame for* $L^2(\mathbb{R}, \mathbb{C}^{n+1})$, *if and only if*

$$s(\Lambda) < \frac{1}{n+1}. \tag{6}$$

By specializing to the n-th coordinate of $\mathbf{h}$, we obtain a condition for scalar-valued Gabor frames with Hermite functions. The following result was already announced in [GrL].

Proposition 1.2. *If* $s(\Lambda) < \frac{1}{n+1}$, *then* $\mathcal{G}(H_n, \Lambda)$ *is a frame for* $L^2(\mathbb{R})$.

For $n = 0$, the case of the Gaussian, we recover (the lattice part of) the results in [L, SW]. Our method of proof yields several new results about the dual window. On the one hand, we construct a dual window for $\mathcal{G}(H_n, \Lambda)$ with Gaussian decay in time and frequency; on the other hand, we derive a new estimate for the lower frame bound of $\mathcal{G}(H_0, \Lambda)$. Furthermore, we discuss an example for $n = 1$ which suggests that the sufficient conditions of Proposition 1.2 are sharp for all n.

In order to prove these results we combine the techniques of Gabor analysis and complex-analytic methods. The (now) classical structural results related to the scalar Gabor frame systems can be formulated for the vector case as well. They lead to an interpolation problem in the Fock space of entire functions.

This problem is not "purely holomorphic": the values of linear combinations of functions from the Fock space and their derivatives are prescribed in the lattice points; however, the coefficients of such combination are antiholomorphic polynomials. The classical methods of complex analysis cannot be applied for problems of this kind. Fortunately, in our particular case, i.e., for the lattices of sufficiently small density, one may use well-developed machinery of the elliptic functions.

The paper is organized as follows. In Section 2 we collect the necessary facts related to vector-valued frames, in particular, the construction of the frame operator and the vector-valued version of the structural theorems: Janssen's representation and the Wexler-Raz biorthogonality criteria. The complex analytic tools are presented in Section 3, they include basics about Fock spaces and the growth estimates of the Weierstrass σ-function. Section 4 contains the proofs of Theorem 1.1 and Proposition 1.2. We also prove a result (Proposition 4.3) indicating that the density condition in Proposition 1.2 might be sharp. The rest of Section 4 contains estimates of dual windows for the Hermitian frames and of the lower frame bound.

2. General Gabor Superframes

2.1. **Scalar case.** To derive the criterion for Gabor superframes from Hermite functions (Theorem 1.1), we need some general notion and results related to Gabor superframes such as Janssen's representation of the frame operator and Wexler-Raz identities. Although these are among the fundamental results about Gabor frames, they have not yet been formulated for Gabor superframes. For convenience and later reference we formulate them explicitly for Gabor frames on $\mathbb{R}^d$ (instead of $\mathbb{R}$) in this section. Since the vector-valued case is a simple consequence of the scalar-valued case, we defer the (easy) proofs to the Appendix.

Let $\Lambda = A\mathbb{Z}^{2d}$, be a lattice in $\mathbb{R}^{2d}$, where A is a non-singular real $2d \times 2d$-matrix. Let $s(\Lambda) = |\det|A$ be the volume of a fundamental domain of Λ. The *adjoint lattice* is defined by the commutant property as

$$\Lambda^\circ = \{\mu \in \mathbb{R}^{2d} : \pi(\lambda)\pi(\mu) = \pi(\mu)\pi(\lambda) \text{ for all } \lambda \in \Lambda\}\,. \tag{7}$$

If $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, then $\Lambda^\circ = \beta^{-1}\mathbb{Z}^d \times \alpha^{-1}\mathbb{Z}^d$. In general, for $\Lambda = A\mathbb{Z}^{2d} \subset \mathbb{R}^{2d}$, then

$$\Lambda^\circ = \mathcal{J}(A^T)^{-1}\mathbb{Z}^{2d}\,, \tag{8}$$

where A^T is the transpose of A and $\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ (consisting of $d \times d$ blocks) is the matrix defining the standard symplectic form [FK].

Furthermore, from (8) we see that

$$s(\Lambda^\circ) = s(\Lambda)^{-1}\,. \tag{9}$$

The density of Λ is defined as $d(\Lambda) = s(\Lambda)^{-1}$, so that $d(\Lambda)$ coincides with the usual notions of density.

In order to make the exposition self-contained we recall the (now) standard results about Gabor frames. We refer the reader to Ch.7 in [Gr1] for the complete proofs.

Given two functions (windows) $g, \gamma \in L^2(\mathbb{R}^d)$, the associated Gabor *frame operator* is defined to be

$$Sf = S^{\Lambda}_{g,\gamma} f = \sum_{\lambda \in \Lambda} \langle f, \pi_\lambda g \rangle \pi_\lambda \gamma. \tag{10}$$

For arbitrary $g, \gamma \in L^2(\mathbb{R}^d)$ the right-hand side in (10) defines a continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ with weak*-convergence of the sum.

Under slight conditions on g and γ, for example if both g and γ belong to the Feichtinger algebra M^1, the series in (10) converges in the usual $L^2(\mathbb{R}^d)$-norm. Recall that a function g on $\mathbb{R}^d$ belongs to the Feichtinger algebra $M^1(\mathbb{R}^d)$, if

$$\|g\|_{M^1} := \int_{\mathbb{R}^{2d}} |\langle g, \pi_z \varphi \rangle| \, dz < \infty \,, \tag{11}$$

where $\varphi(t) = 2^{d/4} \, e^{-\pi t^2}$ is the L^2-normalized Gaussian. This condition is met if, for example, both g and its Fourier transform $\hat{g}$, decay sufficiently fast, see [Gr1], Ch. 7 and also [Gr] for discussion and the proofs.

The convergence of (10) follows from the following lemma taken from [Gr1] (the statement is not stated as explicitly as we want it, but follows from combining Propositions 11.1.4, 12.1.11, and 12.2.1).

Lemma 2.1. *If $g \in M^1(\mathbb{R}^d)$, then, for each lattice Λ*

$$\sum_{\lambda \in \Lambda} |\langle f, \pi_\lambda g \rangle|^2 \le n(\Lambda) \, \|g\|^2_{M^1} \, \|f\|_2^2 \tag{12}$$

$$\text{and} \qquad \Big\| \sum_{\lambda \in \Lambda} c_\lambda \pi_\lambda g \Big\|_2 \le n(\Lambda)^{1/2} \, \|g\|_{M_1} \, \|c\|_2 \,, \tag{13}$$

where $n(\Lambda) = \kappa \max_{k \in \mathbb{Z}^{2d}} \operatorname{card}\,(\Lambda \cap (k + [0,1]^{2d}))$ for some absolute constant $\kappa > 0$.

In other words if $g \in M^1(\mathbb{R}^d)$, then for each choice of Λ the sequence $\mathcal{G}(g, \Lambda)$ is a *Bessel sequence*, i.e.

$$\sum_{\lambda \in \Lambda} |\langle f, \pi_\lambda g \rangle|^2 \le \text{Const} \, \|f\|_2^2.$$

Moreover, the upper frame bound of $\mathcal{G}(g, \Lambda)$ can be estimated by $n(\Lambda) \, \|g\|^2_{M^1}$.

By a theorem of Rieffel [R] and Janssen [J2] the frame operator can be represented as a sum of time-frequency shifts over the adjoint lattice:

$$Sf = S^{\Lambda}_{g,\gamma} f = s(\Lambda)^{-1} \sum_{\mu \in \Lambda^\circ} \langle \gamma, \pi_\mu g \rangle \pi_\mu f. \tag{14}$$

For $g, \gamma \in M^1(\mathbb{R}^d)$ the series in the right-hand side converges both in $L^2(\mathbb{R}^d)$-norm and also in operator norm in $L^2(\mathbb{R}^d)$.

Given a frame $\mathcal{G}(g, \Lambda)$ we say that $\gamma \in L^2(\mathbb{R}^d)$ is a *dual window* if $S^{\Lambda}_{g,\gamma} = I$

Based on representation (14) one obtains a criterion for the system $\mathcal{G}(g, \Lambda)$ to form a frame in $L^2(\mathbb{R}^d)$:

Proposition 2.2 (Wexler-Raz biorthogonality relation)**.** *Assume that both $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(\gamma,\Lambda)$ are Bessel sequences in $L^2(\mathbb{R}^d)$. Then $\mathcal{G}(g,\Lambda)$ is a frame in $L^2(\mathbb{R}^d)$ with dual window γ if and only if*

$$\frac{1}{s(\Lambda)}\langle \gamma, \pi_\mu g\rangle = \delta_{0,\mu}, \ \textit{for } \mu \in \Lambda^\circ. \tag{15}$$

We remark that for $g,\gamma \in M^1(\mathbb{R}^d)$ the sequences $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(\gamma,\Lambda)$ always are Bessel sequences for any choice of Λ by Lemma 2.1.

The Wexler-Raz relations yield an estimate for the lower frame bound that deserves to be better known.

Corollary 2.3. *Assume that $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ with some dual $\gamma \in M^1(\mathbb{R}^d)$. Then the optimal lower frame bounded $A_{\mathrm{opt}} = \|S_{g,g}^{-1}\|_{L^2\to L^2}^{-1}$ satisfies*

$$A_{\mathrm{opt}} \geq n(\Lambda)^{-1}\, \|\gamma\|_{M^1}^{-2}\,. \tag{16}$$

Proof. Since $f = S^\Lambda_{g,\gamma} f = \sum_{\lambda\in\Lambda} \langle f, \pi_\lambda g\rangle\, \pi_\lambda \gamma$, Lemma 2.1 implies that

$$\|f\|_2^2 \leq n(\Lambda)\|\gamma\|_{M^1}^2 \sum_{\lambda\in\Lambda} |\langle f, \pi_\lambda g\rangle|^2\,,$$

whence the estimate for the lower frame bound follows. □

2.2. **Structure of Vector-Valued Gabor Frames.** For the case of vector-valued Gabor system the frame-type operator is defined as

$$S\mathbf{f} = S^\Lambda_{\mathbf{g},\boldsymbol{\gamma}}\mathbf{f} = \sum_{\lambda\in\Lambda} \langle \mathbf{f}, \pi_\lambda \mathbf{g}\rangle \pi_\lambda \boldsymbol{\gamma}. \tag{17}$$

The convergence properties are the same as in the scalar case.

Proposition 2.4. (Janssen representation for Gabor superframes) *Let the windows $\boldsymbol{\gamma} = (\gamma_j)_1^n, \mathbf{g} = (g_j)_1^n \in L^2(\mathbb{R}^d, \mathbb{C}^n)$ be such that $\gamma_j, g_j \in M^1$, $j = 1,2,\dots,n$. Then the frame type operator $S_{\mathbf{g},\boldsymbol{\gamma}}$ associated to $(\boldsymbol{\gamma}, \mathbf{g}; \Lambda)$ can be written as*

$$S\mathbf{f} = S^\Lambda_{\boldsymbol{\gamma},\mathbf{g}}\mathbf{f} = \sum_{\mu\in\Lambda^\circ} \Gamma(\mu)\pi_\mu \mathbf{f}, \tag{18}$$

where $\Gamma(\mu)$ is the $n\times n$ matrix with entries

$$\Gamma(\mu)_{kl} = s(\Lambda)^{-1}\langle \gamma_k, \pi_\mu g_l\rangle \tag{19}$$

and the sum converges in the operator norm on $L^2(\mathbb{R}, \mathbb{C}^n)$.

Given a Gabor (super)frame $\mathcal{G}(\mathbf{g},\Lambda)$ with $\mathbf{g} \in L^2(\mathbb{R}^d,\mathbb{C}^n)$, we say that $\boldsymbol{\gamma} \in L^2(\mathbb{R}^d,\mathbb{C}^n)$ is a *dual window* (with respect to the lattice Λ) if

$$S^\Lambda_{\mathbf{g},\boldsymbol{\gamma}} = S^\Lambda_{\boldsymbol{\gamma},\mathbf{g}} = I. \tag{20}$$

Dual windows always exist, a special choice is given by the *canonical dual window* $\boldsymbol{\gamma}^\circ := \left(S^\Lambda_{\mathbf{g},\mathbf{g}}\right)^{-1}\mathbf{g}$; as in the scalar case the invertibility of $S^\Lambda_{\mathbf{g},\mathbf{g}}$ is an easy consequence of the frame property.

From Janssen's representation we obtain a criteria for the system $\mathcal{G}(\mathbf{g},\Lambda)$ forms a frame in $L^2(\mathbb{R}^d,\mathbb{C}^n)$ with dual window $\boldsymbol{\gamma}$, that is a vector analog of the Wexler-Raz condition.

Proposition 2.5. (Wexler-Raz biorthogonality). *Assume that both $\mathcal{G}(\mathbf{g},\Lambda)$ and $\mathcal{G}(\boldsymbol{\gamma},\Lambda)$ are Bessel sequences in $L^2(\mathbb{R}^d,\mathbb{C}^n)$. Then $\mathcal{G}(\mathbf{g},\Lambda)$ is a vector-valued frame in $L^2(\mathbb{R}^d,\mathbb{C}^n)$ with dual window $\boldsymbol{\gamma}$ if and only if*

$$\frac{1}{s(\Lambda)}\langle \gamma_l,\pi_\mu g_j\rangle = \delta_{0,\mu}\delta_{l,j} \quad \textit{for} \quad \mu\in\Lambda^\circ,\ j,l=1,2,\dots,n\,. \tag{21}$$

As in the scalar case we obtain Balan's necessary density condition [B2] by adjusting an argument of Janssen [J1].

Proposition 2.6. *(Density theorem.) If $\mathcal{G}(\mathbf{g},\Lambda)$ is a frame for $L^2(\mathbb{R}^d,\mathbb{C}^n)$, then $s(\Lambda)\le n^{-1}$.*

Proof. Let $\boldsymbol{\gamma}^\circ = S^{-1}_{\mathbf{g},\mathbf{g}}\mathbf{g}$ be the canonical dual window. Then $\mathbf{f}=\sum_{\lambda\in\Lambda}\langle \mathbf{f},\pi(\lambda)\boldsymbol{\gamma}^\circ\rangle\pi_\lambda\mathbf{g}$ holds for every $\mathbf{f}\in L^2(\mathbb{R},\mathbb{C}^n)$. If $\mathbf{f}$ has also the representation $\mathbf{f}=\sum_{\lambda\in\Lambda}c_\lambda\pi_\lambda\mathbf{g}$, then by [DS]

$$\sum_{\lambda\in\Lambda}|\langle \mathbf{f},\pi_\lambda\boldsymbol{\gamma}\rangle|^2 \le \sum_{\lambda\in\Lambda}|c_\lambda|^2\,.$$

We apply this argument to the trivial expansion $g = 1\cdot\mathbf{g}+\sum_{\lambda\neq 0}0\cdot\pi_\lambda\mathbf{g}$. Thus we obtain

$$|\langle \mathbf{g},\boldsymbol{\gamma}\rangle|^2 \le \sum_{\lambda\in\Lambda}|\langle \mathbf{g},\pi_\lambda\boldsymbol{\gamma}\rangle|^2\le 1.$$

The Wexler-Raz identities (21) yield

$$\langle \mathbf{g},\boldsymbol{\gamma}\rangle = \sum_{j=1}^{n}\langle g_j,\gamma_j\rangle = ns(\Lambda),$$

and the result follows. □

The following duality result is not needed in the sequel, but is included for completeness. It answers a question of our engineering colleague G. Matz, see e.g. [HMS].

Theorem 2.7 (Janssen-Ron-Shen duality)**.** *The Gabor system $\mathcal{G}(\mathbf{g},\Lambda)$ is a frame for $L^2(\mathbb{R}^d,\mathbb{C}^n)$, if and only if the union of Gabor systems $\bigcup_{j=1}^n\mathcal{G}(g_j,\Lambda^\circ)$ is a Riesz sequence for $L^2(\mathbb{R}^d)$.*

3. Complex Methods: Fock space and the Weierstrass sigma-function

The complex analytic techniques we use are concentrated around the Fock space of entire functions and precise estimates for the Weierstrass σ-function. These topics are closely related: roughly speaking σ-functions deliver examples of functions of "maximal possible growth" in the Fock space.

3.1. **Fock space.** In this subsection we recall the basic properties of the Fock space which are important for our applications. We refer the reader to [F], [Gr1] for detailed proofs and also for discussion of numerous application of this space in signal analysis and quantum mechanics.

Definition 3.1. *The Fock space $\mathcal{F}$ is the Hilbert space of all entire functions such that*

$$\|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm_z < \infty, \tag{22}$$

where dm is the planar Lebesgue measure.

The natural inner product in $\mathcal{F}$ is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{F}}$.

Below we list the properties of the Fock space, which will be used in the sequel.

(a) Each point evaluation is a bounded linear functional in $\mathcal{F}$, and the corresponding reproducing kernel is the function $w \mapsto e^{\pi \bar{z} w}$, precisely,

$$F(z) = \langle F(w), e^{\pi \bar{z} w} \rangle_{\mathcal{F}}, \qquad \forall F \in \mathcal{F}. \tag{23}$$

(b) The functions from $\mathcal{F}$ grow at most like the Gaussian (see e.g. [LS]), more precisely,

$$|F(z)| = o(1) e^{\frac{\pi}{2}|z|^2}, \text{ as } z \to \infty, \ F \in \mathcal{F}. \tag{24}$$

(c) The collection of monomials

$$e_n(z) = \left(\frac{\pi^n}{n!}\right)^{1/2} z^n, \ n = 0, 1, \dots \tag{25}$$

forms an orthonormal basis in $\mathcal{F}$.

(d) Define the Bargmann transform of a function $f \in L^2(\mathbb{R})$ by

$$f \mapsto \mathcal{B}f(z) = F(z) = 2^{1/4} e^{-\pi z^2/2} \int_{\mathbb{R}} f(t) e^{-\pi t^2} e^{2\pi t z} dt. \tag{26}$$

Proposition 3.1. *The Bargmann transform is a unitary mapping between $L^2(\mathbb{R})$ and $\mathcal{F}$.*

(e) The Hermite functions are defined by

$$H_n(t) = c_n e^{\pi t^2} \frac{d^n}{dt^n} e^{-2\pi t^2}, \ n = 0, 1, 2, \dots, . \tag{27}$$

where the coefficients c_n are chosen in order to have $\|H_n\|_2 = 1$. It is a classical result that the set of Hermite functions $\{H_n\}_{n=0}^{\infty}$ forms an orthonormal basis in $L^2(\mathbb{R})$.

Their image under the Bargmann transform is $\{e_n\}_{n=0}^{\infty}$ – the natural orthonormal basis in $\mathcal{F}$:

$$\mathcal{B}H_n(z) = e_n(z), \ n = 0, 1, \dots \ . \tag{28}$$

(f) In what follows we identify $\mathbb{C}$ and $\mathbb{R}^2$. In particular for each $\zeta = \xi + i\eta \in \mathbb{C}$ we write $\pi_\zeta = \pi_{(\xi,\eta)}$. Define the shift $\beta_\zeta : \mathcal{F} \to \mathcal{F}$ in the Fock space by

$$\beta_\zeta F(z) = e^{i\pi\xi\eta} e^{-\pi|\zeta|^2/2} e^{\pi\zeta z} F(z - \bar{\zeta}). \tag{29}$$

Then β_ζ is unitary on $\mathcal{F}$, and the Bargmann transform intertwines the Fock space shift and the time-frequency shift:

$$\beta_\zeta \mathcal{B} = \mathcal{B}\pi_\zeta. \tag{30}$$

Based on this proposition one can easily obtain inner product of a function $f \in L^2(\mathbb{R})$ with the time-frequency shifts of a Hermite function as in [BS]. This quantity is the short-time Fourier transform with respect to a Hermite function.

Proposition 3.2. *Let $f \in L^2(\mathbb{R})$ and $F(z) = \mathcal{B}f(z)$. Then, for all $\zeta \in \mathbb{C}$,*

$$\langle f, \pi_\zeta H_n\rangle_{L^2(\mathbb{R})} = \frac{1}{\sqrt{\pi^n n!}} e^{-i\pi\xi\eta} e^{-\pi|\zeta|^2/2} \sum_{k=0}^{n} \binom{n}{k} (-\pi\zeta)^k F^{(n-k)}(\bar{\zeta}). \tag{31}$$

Proof. Using the intertwining property (30), we have

$$\begin{aligned}
\langle f, \pi_\zeta H_n\rangle_{L^2(\mathbb{R})} &= \langle F, \beta_\zeta \mathcal{B} H_n\rangle_{\mathcal{F}} \\
&= \left(\frac{\pi^n}{n!}\right)^{1/2} e^{-i\pi\xi\eta}\, e^{-\pi|\zeta|^2/2} \langle F(z), e^{\pi\zeta z}(z-\bar{\zeta})^n\rangle \\
&= \left(\frac{\pi^n}{n!}\right)^{1/2} e^{-i\pi\xi\eta} e^{-\pi|\zeta|^2/2} \sum_{k=0}^{n} \binom{n}{k} (-\zeta)^k \langle F(z), z^{n-k} e^{\pi\zeta z}\rangle \\
&= \left(\frac{\pi^n}{n!}\right)^{1/2} e^{-i\pi\xi\eta} e^{-\pi|\zeta|^2/2} \sum_{k=0}^{n} \binom{n}{k} (-\zeta)^k \pi^{-n+k} \frac{d^{n-k}}{d\bar{\zeta}^{n-k}} \langle F(z), e^{\pi\zeta z}\rangle.
\end{aligned}$$

It remains to apply relation (23). □

Finally we give in this section a description of the space M^1 in the Bargmann transform terms. Since $|\langle f, \pi_{\bar{z}} H_0\rangle| = |\mathcal{B}f(z)|\, e^{-\pi|z|^2/2}$ (e.g., [Gr1] or [F]), we obtain the following.

Proposition 3.3. *A function $f \in L^2(\mathbb{R})$ belongs to M^1 if and only if its Bargmann transform $F(z) = \mathcal{B}f$ satisfies*

$$\|f\|_{M^1} = \int_{\mathbb{C}} |F(z)| e^{-\pi|z|^2/2} dm_z < \infty. \tag{32}$$

This is just a reformulation of condition (11).

3.2. **Estimates of σ-function.** In this section we collect definitions and known facts about the Weierstrass functions. They will be used in the next section.

Given two numbers $\omega_1, \omega_2 \in \mathbb{C}$ such that

$$\Im\Big(\omega_2/\omega_1\Big) > 0. \tag{33}$$

we consider the lattice $\Lambda = \{m_1\omega_1 + m_2\omega_2; m_1, m_2 \in \mathbb{Z}\} \subset \mathbb{C}$. The numbers $\omega_1, \omega_2 \in \mathbb{C}$ are called *periods* of the lattice Λ.

In the previous section we used lattices of the form $\Lambda = A\mathbb{Z}^2 \subset \mathbb{R}^2$, where A is an invertible 2×2 matrix. The two constructions coincide after the natural identification of $\mathbb{R}^2$ and $\mathbb{C}$, if $\det A > 0$. In this case ω_1, ω_2 correspond to the columns of A.

Let $Q = \{(x, y); 0 \le x < 1, 0 \le y < 1\}$. The parallelogram $\Pi_\Lambda = AQ$ with based on ω_1 and ω_2 is a fundamental domain for Λ (also called the *period parallelogram*). Its area can be expressed through the periods as

$$s(\Lambda) = \mathrm{Area}(\Pi_\Lambda) = \det A = \Im(\bar{\omega}_1\omega_2) = -\frac{i}{2}(\bar{\omega}_1\omega_2 - \bar{\omega}_2\omega_1). \tag{34}$$

Next consider the following Weierstrass functions:

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda'} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}, \tag{35}$$

$$\zeta(z) = \frac{1}{z} + \sum_{\omega\in\Lambda'} \left\{ \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right\}, \tag{36}$$

and

$$\sigma(z) = z \prod_{\omega\in\Lambda'} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{z^2}{2\omega^2}}, \tag{37}$$

where $\Lambda' = \Lambda \setminus \{0\}$. The lattice Λ plays a special role for the Weierstrass functions: $\mathcal{P}$ has poles of order two precisely on Λ, ζ has poles of order one, and σ has simple zeros precisely on Λ.

We list some basic properties of the Weierstrass functions. See, e.g., [A] for detailed proofs.

- The $\mathcal{P}$-function is elliptic, i.e.,

$$\mathcal{P}(z) = \mathcal{P}(z + \omega_k), \quad z \in \mathbb{C},\ k = 1, 2. \tag{38}$$

- The function ζ changes by a constant when its argument changes by a lattice point:

$$\eta_k := \zeta(z + \omega_k) - \zeta(z) = \mathrm{Const}\ ,\ k = 1, 2. \tag{39}$$

- The constants η_k satisfy the Legendre relation:

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i. \tag{40}$$

- The Weierstrass σ-function is quasi-periodic, it satisfies the following relation

$$\sigma(z + \omega_k) = -\sigma(z) e^{\eta_k z + \frac{1}{2}\eta_k\omega_k} \qquad k = 1, 2\,. \tag{41}$$

In order to understand the growth properties of the Weierstrass σ-function in dependence of the lattice Λ, we follow an elegant argument of Hayman [H]. He realized that, after a proper normalization and growth compensation, the absolute value of σ_Λ becomes a doubly periodic function. We include here this arguments in order to make our presentation self-contained.

Proposition 3.4. *Set*

$$\alpha(\Lambda) = \frac{i\pi}{\bar{\omega}_1\omega_2 - \bar{\omega}_2\omega_1} = \frac{\pi}{2s(\Lambda)}, \quad a(\Lambda) = \frac{1}{2}\frac{\eta_2\bar{\omega}_1 - \eta_1\bar{\omega}_2}{\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1}, \tag{42}$$

and

$$\sigma_\Lambda(z) = \sigma(z) e^{a(\Lambda) z^2}. \tag{43}$$

Then the function $|\sigma_\Lambda(z)| e^{-\alpha(\Lambda)|z|^2}$ *is periodic with periods* ω_1 *and* ω_2*.*

The periodicity implies a growth estimate for the modified sigma-function σ_Λ.

Proposition 3.5. *Set*

$$c(\Lambda) = \sup_{z \in \Pi_\Lambda} |\sigma_\Lambda(z)| e^{-\alpha(\Lambda)|z|^2}. \tag{44}$$

Then

$$|\sigma_\Lambda(z)| \le c(\Lambda)\, e^{\frac{\pi}{2s(\Lambda)}|z|^2}, \quad \forall z \in \mathbb{C}\,. \tag{45}$$

For each $\epsilon > 0$ *we have* [1]

$$|\sigma_\Lambda(z)| \asymp e^{\alpha(\Lambda)|z|^2}, \quad \textit{whenever } \textit{dist}(z, \Lambda) > \epsilon. \tag{46}$$

Proof. The function $|\sigma_\Lambda(z)| e^{-\alpha(\Lambda)|z|^2}$ is bounded in Π_Λ and has its only zeros at the vertices of Π_Λ. Hence it is bounded away from 0 on every compact subset of Π_Λ, which does not contain its vertices and relation (46) follows now by the periodicity. □

Proof of Proposition 3.4. It follows from (41) that

$$\left|\sigma(z+\omega_k) e^{a(\Lambda)(z+\omega_k)^2}\right| e^{-\alpha(\Lambda)|z+\omega_k|^2} = \left|\sigma(z) e^{a(\Lambda)z^2}\right| e^{-\alpha(\Lambda)|z|^2} e^{\Re A_k(z)},\ k = 1, 2,$$

where

$$A_k(z) = z\Big(\eta_k + 2a(\Lambda)\omega_k - 2\alpha(\Lambda)\bar{\omega}_k\Big) + \Big(\frac{1}{2}\eta_k\omega_k + a(\Lambda)\omega_k^2 - \alpha(\Lambda)|\omega_k|^2\Big),\ k = 1, 2. \tag{47}$$

An explicit calculation, based on relations (42) and (40) shows that

$$\eta_k + 2a(\Lambda)\omega_k - 2\alpha(\Lambda)\bar{\omega}_k = 0,\ k = 1, 2,$$

and also

$$\Re\Big(\frac{1}{2}\eta_k\omega_k + a(\Lambda)\omega_k^2 - \alpha(\Lambda)|\omega_k|^2\Big) = 0,\ k = 1, 2.$$

Proposition 3.4 now follows. □

[1] Here and in what follows the sign $\asymp$ means that the ratio of the left- and right- hand sides is located between two positive constants.

4. Frames with Hermite functions.

4.1. **Hermite superframes.** In this section we prove Theorem 1.1 and thus give a complete characterization of Gabor superframes with Hermite functions.

We divide the proof into several steps. First we will use the matrix form of the Wexler-Raz biorthogonality relations (21) and translate these relations to an interpolation problem on Fock space.

Let, as earlier, $\mathbf{h} = (H_j)_{j=0}^n$ be the vector-valued window consisting of the first $n+1$ Hermite functions. For each $\boldsymbol{\gamma} = (\gamma_j)_{j=0}^n \in L^2(\mathbb{R}, \mathbb{C}^{n+1})$ denote

$$G_j = \mathcal{B}\gamma_j,\ j = 0, 1, \dots, n. \tag{48}$$

Taking $\mathbf{g} = \mathbf{h}$ in the biorthogonality relations (21) and using Proposition 3.2, we can rewrite these relations as an interpolation problem for functions $G_j \in \mathcal{F}$:

$$\begin{aligned} \langle \gamma_j, \pi_\mu H_l \rangle_{L^2(\mathbb{R})} &= e^{-i\pi \Im\mu \Re\mu} e^{-\pi|\mu|^2/2} \frac{1}{\sqrt{\pi^l l!}} \sum_{k=0}^{l} \binom{l}{k} (-\pi\mu)^k G_j^{(l-k)}(\bar{\mu}) \\ &= s(\Lambda)\delta_{\mu,0}\delta_{j,l}, \quad j, l = 0, 1, \dots, n;\ \mu \in \Lambda^\circ. \end{aligned} \tag{49}$$

Thus $\mathcal{G}(\mathbf{h}, \Lambda)$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^{n+1})$, if and only if there exist functions $G_j \in \mathcal{F}$ satisfying (49) and if $\mathcal{G}(\boldsymbol{\gamma}, \Lambda)$ is a Bessel sequence in $L^2(\mathbb{R}, \mathbb{C}^{n+1})$.

This interpolation problem can be rewritten in a simpler way. Indeed, for each $\mu \in \Lambda^\circ \setminus \{0\}$ and $j \in \{0, 1, \dots, n\}$, the system (49) is a triangular linear system in the variables $\{G_j^{(m)}(\bar{\mu})\}_{m=0}^n$ with non-zero diagonal coefficients and zero right-hand side. Clearly, it has just zero solutions. Thus if $\mathbf{G} = (G_j)_{j=0}^n$ satisfies (49), then $G_j^{(m)}(\bar{\mu}) = 0,\ \ j, m = 0, 1, \dots, n,\ \ \mu \in \Lambda^\circ \setminus \{0\}$. For $\mu = 0$, the relations (49) take the form

$$\frac{1}{\sqrt{\pi^l l!}} G_j^{(l)}(0) = s(\Lambda)\delta_{j,l},\ j, l = 0, 1, \dots, n.$$

By adjusting the normalization of the G_j, we obtain the following statement.

Proposition 4.1. *The Gabor system $\mathcal{G}(\mathbf{h}, \Lambda)$ is a Gabor superframe for $L^2(\mathbb{R}, \mathbb{C}^{n+1})$ if and only if there exist $n+1$ functions $\gamma_j \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(\gamma_j, \Lambda)$ is a Bessel sequence and $G_j = B\gamma_j, j = 0, \dots, n$, satisfy the (Hermite) interpolation problem*

$$G_j^{(\ell)}(\bar{\mu}) = \delta_{\mu,0}\, \delta_{j,\ell} \qquad \text{for } \mu \in \Lambda^\circ, j, \ell = 0, \dots, n\,. \tag{50}$$

Proof of Theorem 1.1. Sufficiency of (6). We use the Weierstrass functions described in Section 3.2. Construct the Weierstrass functions corresponding the lattice $\overline{\Lambda^\circ}$, as given by (7). Let the constants $\alpha(\overline{\Lambda^\circ}), a(\overline{\Lambda^\circ})$ be defined by (42) and the function $\sigma_{\overline{\lambda^\circ}}$ be defined by (43), where the σ-function is constructed for the lattice $\overline{\Lambda^\circ}$. Consider the function

$$S(z) = \left(\sigma_{\overline{\Lambda^\circ}}(z)\right)^{n+1} = \sigma(z)^{n+1} e^{a(\overline{\Lambda^\circ})(n+1)z^2}. \tag{51}$$

The zero set of S is $\overline{\Lambda^\circ}$, and the multiplicity of each zero is precisely $n+1$. In our case $\alpha(\overline{\Lambda^\circ}) = \frac{\pi}{2s(\Lambda^\circ)} = \frac{\pi}{2}s(\Lambda)$ and the growth estimates (45) and (46) can be rewritten as

$$|S(z)| \leq e^{\frac{\pi}{2}(n+1)s(\Lambda)|z|^2}, \text{ for all } z \in \mathbb{C}, \tag{52}$$

$$|S(z)| \asymp e^{\frac{\pi}{2}(n+1)s(\Lambda)|z|^2}, \quad \text{if dist } (z, \overline{\Lambda^\circ}) > \epsilon. \tag{53}$$

Now assume that $s(\Lambda) < (n+1)^{-1}$. Then the functions

$$S_m(z) = \frac{1}{z^{n+1-m}} S(z), \; m = 0, 1, \dots, n \tag{54}$$

belong to $\mathcal{F}$. They have zero of order $n+1$ at each $\mu \in \overline{\Lambda^\circ} \setminus \{0\}$, and also

$$S_m^{(l)}(0) = 0, \text{ for } 0 \leq l < m-1, \text{ and } S_m^{(m)}(0) \neq 0.$$

The solutions G_j to the interpolation problem (50) can be now found in the form

$$G_j = \sum_{m=j}^{n} c_{m,j} S_m, \quad j = 0, 1, \dots, n. \tag{55}$$

These functions have a zero of multiplicity $n+1$ at all points from $\overline{\Lambda^\circ} \setminus \{0\}$, while, for each j, the condition $G_j^{(l)}(0) = \delta_{j,l}$ leads one to a triangular system of linear equations with respect to the coefficients $\{c_{m,j}\}_{m=j}^n$ with non-zero diagonal entries. Clearly this system has a (unique) solution.

It remains to mention that each system $\mathcal{G}(\gamma_j, \Lambda)$ is a Bessel sequence. Each G_j inherits its growth from the functions S_j and from S, thus (52) and Proposition 3.3 imply that $\gamma_j \in M^1(\mathbb{R})$ for $j = 0, \dots, n$. Now apply Proposition 2.1.

Necessity. Let now $s(\Lambda) \geq (n+1)^{-1}$. The growth estimate (53) implies that

$$|S(z)| \geq \text{Const } e^{\frac{\pi}{2}|z|^2}, \quad \text{whenever dist } (z, \overline{\Lambda^\circ}) > \epsilon. \tag{56}$$

Now assume that $\mathcal{G}(\mathbf{h}, \Lambda)$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^{n+1})$. Then there exists a system of functions $G_0, G_1, \dots, G_n \in \mathcal{F}$ which satisfy the interpolation problem (50). The function G_n has zeros of multiplicity $n+1$ at $\Lambda^\circ \setminus \{0\}$ and a zero of multiplicity n at the origin. Therefore

$$\Phi(z) = \frac{zG_n(z)}{S(z)}$$

is an entire function. Estimates (56) and (24) yield

$$|\Phi(z)| = o(|z|), \; z \to \infty, \; \text{dist } (z, \overline{\Lambda^\circ}) > \epsilon.$$

By the maximum principle, the restriction dist $(z, \Lambda^\circ) > \epsilon$ can be removed, and Φ is in fact a bounded entire function. The Liouville theorem implies that Φ is a constant, or

$$G_n(z) = C \frac{S(z)}{z}.$$

But $z^{-1}S(z) \notin \mathcal{F}$, this follows again from (56). Therefore $C = 0$ and hence G_n is identically zero in contradiction to (50). This completes the proof of necessity. □

Remark. It follows from the density theorem (Proposition 2.6) that $\mathcal{G}(\mathbf{h},\Lambda)$ cannot be a frame for $s(\Lambda) > \frac{1}{n+1}$. Our proof shows that we must also exclude the case of the critical density $s(\Lambda) = \frac{1}{n+1}$. This is a Balian-Low type phenomenon (see the review [BHW]). We conjecture that in general, if $g_j \in M^1, j = 1, \dots, n$ and $\mathcal{G}(\mathbf{g},\Lambda)$ is a Gabor frame, then $s(\Lambda) < \frac{1}{n}$, but we will not pursue this question further in this work.

4.2. **Hermite frames.** Next we consider scalar-valued Gabor frames with Hermite functions.

We remark that if for some $\mathbf{g} \in L^2(\mathbb{R},\mathbb{C}^{n+1})$ and lattice $\Lambda \subset \mathbb{R}^2$ the system $\mathcal{G}(\mathbf{g},\Lambda)$ is a vector-valued frame in $L^2(\mathbb{R},\mathbb{C}^{n+1})$, then, trivially, for each $j = 0, 1, \dots, n$, the Gabor system $\mathcal{G}(g_j,\Lambda) = \{\pi_\lambda g_j, \lambda \in \Lambda\}$ is a frame for $L^2(\mathbb{R})$. More generally, for each $\mathbf{c} \in \mathbb{C}^{n+1}$, $\mathbf{c} \neq 0$, the system $\mathcal{G}(h,\Lambda) = \{\pi_\lambda h, \lambda \in \Lambda\}$ with $h = \sum_{j=0}^n c_j g_j$ also is a frame in $L^2(\mathbb{R})$. By applying this observation to the window $\mathbf{g} = (H_0, \dots, H_n)$ we obtain Proposition 1.2, which gives a condition for the one dimensional system with the Hermite window to be a frame for $L^2(\mathbb{R})$. Actually a slightly more general result related to linear combinations of Hermite functions holds true

Proposition 4.2. *Let $n \in \mathbb{Z}, n \geq 0$ and $h = \sum_{k=0}^n c_k H_k, \quad \sum_0^n |c_k| \neq 0$. If $s(\Lambda) < (n+1)^{-1}$, then $\mathcal{G}(h,\Lambda)$ is a frame for $L^2(\mathbb{R})$.*

Remark. Clearly the estimate above is not optimal in case $c_n = 0$.

Amazingly enough the sufficient density of Theorem 4.2 might be sharp, as is suggested by the following counter-example.

Proposition 4.3. *If $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ and $ab = 1/2$, then $\mathcal{G}(H_{2n+1},\Lambda)$ is not a Gabor frame for $L^2(\mathbb{R})$ for all integers $n \geq 0$.*

Proof. We use a Zak transform argument. For $a > 0$ the Zak transform is defined as

$$Z_a f(x,\xi) = \sum_{k=-\infty}^{\infty} f(x - ak) e^{2i\pi ak\xi}. \tag{57}$$

We refer the reader to [Gr1, Sec. 8.3] for a detailed discussion of its properties. In particular it is a unitary mapping between $L^2(\mathbb{R})$ and $L^2(Q_{a,1/a})$ – space of square summable functions in the rectangle $Q_{a,1/a} = [0,a) \times [0,1/a)$. For the case $ab = 1/2$ (more generally for the case $(ab)^{-1} \in \mathbb{Z}$) the Gabor frame operator is unitarily equivalent to a multiplication operator on $L^2(Q_{a,1})$ by means of the Zak transform. Precisely,

$$Z_a S_{H_n,H_n} f(x,\xi) = \big(|Z_a H_n(x,\xi)|^2 + |Z_a H_n(x - \frac{a}{2},\xi)|^2\big) Z_a f(x,\xi), \quad f \in L^2(\mathbb{R}).$$

Consequently $\mathcal{G}(H_n,\Lambda)$ is a frame for $L^2(\mathbb{R})$ if and only if

$$0 < \inf_{Q_{a,1}} \{ (|Z_a H_n(x,\xi)|^2 + |Z_a H_n(x-\frac{a}{2},\xi)|^2) \} \leq \sup_{Q_{a,1}} \{ (|Z_a H_n(x,\xi)|^2 + |Z_a H_n(x-\frac{a}{2},\xi)|^2) \} < \infty,$$

see relation (8.21) in [Gr1].

Clearly, $Z_a H_n(x,\xi)$ is a continuous function. Since H_{2n+1} is an odd function, its Zak transform satisfies $Z_a H_{n+1}(0,0) = Z_a H_{n+1}(\frac{a}{2},0) = 0$. This contradicts the above criterium, therefore $\mathcal{G}(H_{2n+1},\Lambda)$ cannot be a frame. □

In [GrL, Prop. 3.3] we showed that $\mathcal{G}(H_n,\Lambda)$ is a frame, if and only if there exists a $\gamma_n \in L^2(\mathbb{R})$, such that $\mathcal{G}(\gamma_n,\Lambda)$ is a Bessel system and the Bargmann transform $G_n = \mathcal{B}\gamma_n$ is in $\mathcal{F}$ and satisfies the interpolation problem

$$\sum_{k=0}^{n} \binom{n}{k} (-\pi\mu)^k G_n^{(n-k)}(\bar{\mu}) = \delta_{\mu,0} \qquad \forall \mu \in \Lambda^\circ . \tag{58}$$

Note that (58) coincides with (49) for $j = l = n$ and that for $s(\Lambda) < 1/(n+1)$ the function

$$G_n(z) = S_n(z) = cz^{-1}\sigma_{\Lambda^\circ}^{n+1}(z) \tag{59}$$

is a solution of (58) in $\mathcal{F}$.

If $s(\Lambda) \geq 1/(n+1)$, then we do not know whether (58) has any solution in $\mathcal{F}$. The difficulty is that this interpolation problem is not entirely holomorphic, and the standard complex variable methods do not seem sufficient to investigate this problem. In the light of Proposition 4.3 it is conceivable that the sufficient condition $s(\Lambda) < 1/(n+1)$ in Proposition 4.2 is also necessary.

4.3. **Norms and Decay of the Dual Window.** In the rest of this section we assume that $s(\Lambda) < \frac{1}{n+1}$, and we estimate the L^2 and M^1-norms of the dual window for the frame $\mathcal{G}(H_n,\Lambda)$. Though the estimates are simple, they seem to be new even for the Gaussian case $n = 0$.

Lemma 4.4. *Assume that $(n+1)s(\Lambda) < 1$ and let γ_n be the dual window with Bargmann transform G_n defined in* (59)*. Setting $\kappa = \frac{1-(n+1)s(\Lambda)}{3-(n+1)s(\Lambda)}$, then*

$$|\gamma_n(t)| + |\widehat{\gamma_n}(t)| \leq Ce^{-\pi\kappa t^2} . \tag{60}$$

Furthermore,

$$\|\gamma\|_{M_1} \leq 4c(\overline{\Lambda^\circ})^{n+1} \frac{1}{1-(n+1)s(\Lambda)}, \tag{61}$$

and

$$\|\gamma_n\|^2_{L^2(\mathbb{R})} \asymp \log \frac{1}{1-(n+1)s(\Lambda)}, \tag{62}$$

where $c(\Lambda)$ is defined in (44)*.*

Proof. The estimates involve a routine calculation with Gaussians and are a consequence of the growth estimate (45) of Proposition 3.5. Set $\rho = 1-(n+1)s(\Lambda)$.

For the decay property, we use the inversion formula for the short-time Fourier transform (e.g., [Gr1, Prop. 3.2.3]) which states

$$\gamma_n(t) = \int_{\mathbb{C}} \langle \gamma_n, \pi_z H_0\rangle \, \pi_z H_0(t)\, dm_z$$

with absolute convergence of the integral for each $t\in\mathbb{R}$. Since by (45)

$$|\langle \gamma_n, \pi_{\overline{z}} H_0\rangle| = |\mathcal{B}\gamma_n(z)| e^{-\pi|z|^2/2} \le C e^{-\pi\rho|z|^2/2}$$

and since $\pi_{x+i\xi}H_0(t) = e^{2\pi i\xi t}H_0(t-x)$, we obtain that

$$|\gamma_n(t)| \le C\int_{\mathbb{R}} e^{-\pi\rho\xi^2/2}\, d\xi \int_{\mathbb{R}} e^{-\pi\rho x^2/2} e^{-\pi(t-x)^2}\, dx\,.$$

The convolution of the two Gaussians in this integral is a multiple of the Gaussian $e^{-\pi\frac{\rho}{2+\rho}t^2}$ by the semigroup property of Gaussians with respect to convolution [Gr1, Lemma 4.4.5].

The M^1-norm of γ_n is readily estimated by

$$\begin{aligned}
\|\gamma_n\|_{M^1} &= \int_{\mathbb{C}} |G_n(z)|\, e^{-\pi|z|^2/2}\, dm_z \\
&\le c(\overline{\Lambda^\circ})^{n+1}\int_{\mathbb{C}} \frac{1}{|z|}\, e^{-\pi\rho|z|^2/2}\, dm_z \\
&= 2\pi c(\overline{\Lambda^\circ})^{n+1}\int_0^\infty e^{-\pi\rho r/2}\, dr = 4c(\overline{\Lambda^\circ})^{n+1}\frac{1}{\rho}\,.
\end{aligned}$$

For the estimate of $\|\gamma_n\|_2^2$ we have similarly

$$\|\gamma\|^2_{L^2(\mathbb{R})} = \|G\|^2_{\mathcal{F}} = \int_{\mathbb{C}} |z|^{-2}|\sigma_{\overline{\Lambda^\circ}}(z)|^{2(n+1)} e^{-\pi|z|^2} dm_z.$$

When $|z|\le 1$ the integrand is bounded uniformly, so this part does not bring essential contribution into the whole norm. It follows now from (46) that

$$\|\gamma\|^2_{L^2(\mathbb{R})} \asymp \int_{|z|>1} |z|^{-2} e^{-\pi\rho|z|^2} dm_z \asymp \int_1^\infty \frac{1}{r} e^{-\pi\rho r^2} dr \asymp -\log(1-(n+1)s(\Lambda))\,.$$

This is the desired inequality. □

Remarks. 1. In general, the dual window γ_n is not the canonical dual window (given by $S^{-1}_{H_n,H_n}H_n$). Janssen [J3] observed that for $n=0$ and $\Lambda = a\mathbb{Z}\times(aN)^{-1}\mathbb{Z}$ with a positive integer N the canonical dual window decays at most exponentially, whereas the dual window constructed by means of the σ-function possesses Gaussian decay.

2. The Gaussian decay (60) of the dual window of $\mathcal{G}(H_n,\Lambda)$ is usually expressed by saying that γ_n belongs to the Gelfand-Shilov space $S^{1/2}_{1/2}$. This is considered the smallest reasonable space of test functions in time-frequency analysis. An important consequence of this observation is the existence of universal Gabor frames, i.e., the frame $\mathcal{G}(H_n,\Lambda)$ is a (Banach) frame for all reasonable modulation spaces. We refer to [Gr3] for a precise statement and further details.

Finally let us briefly describe how the lower frame bound of the Gabor frame $\mathcal{G}(H_0,\Lambda)$ with Gaussian window behaves, when the lattice approaches the critical size $s(\Lambda)=1$. Since the constants in Lemma 4.4 depend on (the excentricity of) the lattice Λ, we fix a lattice Λ with size $s(\Lambda)=1$ and study the behavior of lower frame bound of $\mathcal{G}(H_0,q\Lambda)$, as q tends to 1. As long as $q<1$, $\mathcal{G}(H_0,q\Lambda)$ is a frame by the classical results in [L, SW], however, when $q=1$, then $\mathcal{G}(H_0,q\Lambda)$ cannot be a frame by the Balian-Low theorem [BHW]. The upper frame bound can be controlled uniformly with Proposition 2.1, therefore the lower frame bound must A converge to 0 as $q\to 1$.

Proposition 4.5. *Assume that $s(\Lambda)=1$ and $q<1$. Let A_q denote the optimal lower frame bound of $\mathcal{G}(H_0,q\Lambda)$. Then*

$$A_q \geq c(1-q^2)^2 = c(1-s(q\Lambda))^2$$

for some constant independent of q.

Proof. Let γ_0 be the dual window defined by (59). Corollary 2.3 of the Wexler-Raz relations and Lemma 4.4 imply that

$$A_q \geq (n(q\Lambda)\|\gamma_0\|^2_{M^1})^{-1} \geq \Big(16n(q\Lambda)c(\overline{(q\Lambda)^\circ})^2\Big)^{-1}(1-s(q\Lambda))^2$$

Thus we need to show that the two constants n and c are bounded, as long as q is bounded away from 0, $q\geq 1/2$, say. Clearly the constant $n(q\Lambda)$, which measures the maximal number of lattice points in a unit cube (Lemma 2.1), can be bounded uniformly. As for c, which is the supremum of the σ-function over the fundamental parallelogram (44), we note that $(q\Lambda)^\circ = q^{-1}\Lambda^\circ$ and that $\sigma_{q^{-1}\Lambda}(z)=q^{-1}\sigma_\Lambda(qz)$ by (37), (40) and (42). Consequently, $\sup_{1/2\leq q\leq 1} c(q^{-1}\overline{\Lambda^\circ})<\infty$, and we are done. □

5. Appendix

In this appendix we show how the structural results about Gabor superframes can be derived from the corresponding well-known results for scalar Gabor frames.

Proof of Janssen's representation, Theorem 2.4. We look at the l-th component of $S\mathbf{f}$. Using definition (17) it can be written explicitly as

$$(S\mathbf{f})_l = \sum_{\lambda\in\Lambda}\langle \mathbf{f},\pi_\lambda\mathbf{g}\rangle\pi_\lambda\boldsymbol{\gamma}_l = \sum_{j=1}^n\sum_{\lambda\in\Lambda}\langle f_j,\pi_\lambda g_j\rangle\pi_\lambda\gamma_l = \sum_{j=1}^n S^\Lambda_{g_j,\gamma_l}f_j.$$

If all g_j, γ_j are in M^1, then Janssen's representation (18) can be applied to each of the frame-type operators occurring in the above sum and we obtain that

$$(S\mathbf{f})_l = s(\Lambda)^{-1}\sum_{\mu\in\Lambda^\circ}\sum_{j=1}^n\langle\gamma_l,\pi_\mu g_j\rangle\pi_\mu f_j = \sum_{\mu\in\Lambda^\circ}\big(\Gamma(\mu)\pi_\mu\mathbf{f}\big)_l, \tag{63}$$

as was to be shown. □

Proof of the Wexler-Raz biorthonality conditions, Prop. 2.5. We remark first that time-frequency shifts on a lattice are linearly independent in the following sense: if $c = (c_\mu)_{\mu\in\Lambda^\circ} \in \ell^\infty(\Lambda^\circ)$ and $\sum_{\mu\in\Lambda^\circ} c_m \pi(\mu) = 0$ as an operator from M^1 to $(M^1)^*$, then $c_\mu = 0$ for all $\mu \in \Lambda^\circ$. See, e.g., [Gr2, R]. If $S_{\mathbf{g},\gamma} = \mathrm{I}$, then by Janssen's representation

$$f_\ell = (S\mathbf{f})_\ell = s(\Lambda)^{-1} \sum_{\mu\in\Lambda^\circ} \sum_{j=1}^{n} \langle \gamma_\ell, \pi_\mu g_j\rangle \pi_\mu f_j \,,$$

and so the linear independence forces $s(\Lambda)^{-1}\langle\gamma_\ell, \pi_\mu g_j\rangle = \delta_{\mu,0}\delta_{j,\ell}$, or in short notation, $s(\Lambda)^{-1}\Gamma(\mu) = \delta_{\mu,0}\mathrm{I}$.

Conversely, if the biorthogonality condition $s(\Lambda)^{-1}\Gamma(\mu) = \delta_{\mu,0}\mathrm{I}$ holds, then obviously $S_{\mathbf{g},\gamma} = \mathrm{I}$. □

Proof of the Ron-Shen duality, Theorem 2.7. Assume first that $\mathcal{G}(\mathbf{g}, \Lambda)$ is a frame for $L^2(\mathbb{R}^d, \mathbb{C}^n)$. Then the canonical dual is given by $\boldsymbol{\gamma}^\circ = S^{-1}\mathbf{g}$, and by general frame theory $\boldsymbol{\gamma}^\circ$ satisfies $S_{\mathbf{g},\boldsymbol{\gamma}^\circ} = \mathrm{I}$. Furthermore, $\mathcal{G}(\boldsymbol{\gamma}^\circ, \Lambda)$ is again a frame for $L^2(\mathbb{R}^d, \mathbb{C}^n)$, in particular it is a Bessel sequence. If

$$f = \sum_{j=1}^{n} \sum_{\mu\in\Lambda^\circ} c_{j,\mu} \pi_\mu g_j \tag{64}$$

for some sequence $(c_{j,\mu}) \in \ell^2(\{1, \dots, n\} \times \Lambda)$, then by the Bessel property of $\mathcal{G}(g_j, \Lambda^\circ)$ for each j, we obtain that

$$\|f\|_2^2 = \|\sum_{j=1}^{n} \sum_{\mu\in\Lambda^\circ} c_{j,\mu}\pi_\mu g_j\| \le B \sum_{j=1}^{n} \sum_{\mu\in\Lambda^\circ} |c_{j,\mu}|_2^2 = B\|c\|_2^2 \,.$$

For the converse inequality we use the Wexler-Raz relations. If $f \in L^2(\mathbb{R}^d)$ is given by (64), then the coefficients are uniquely determined by

$$c_{j,\mu} = \langle f, \pi_\mu \gamma_j\rangle \,.$$

Again, since $\mathcal{G}(\gamma_j, \Lambda^\circ)$ possesses the Bessel property, we find that $\|c\|_2^2 \le A'\|f\|_2^2$. Altogether we have shown that the set $\bigcup_{j=1}^n \mathcal{G}(g_j, \Lambda^\circ)$ is a Riesz sequence.

Conversely, assume that $\bigcup_{j=1}^n \mathcal{G}(g_j, \Lambda^\circ)$ is a Riesz sequence that generates a subspace $\mathcal{K} \subseteq L^2(\mathbb{R}^d)$. Then there exists a biorthogonal basis $\{e_{j,\mu}\}$ contained in $\mathcal{K}$. By the invariance of the Gabor systems $\mathcal{G}(g_j, \Lambda^\circ)$, we find that the biorthogonal basis must be of the form $e_{j,\mu} = \pi_\mu\gamma_j$ for some $\gamma_j \in L^2(\mathbb{R}^d)$. By the general properties of Riesz bases, $\mathcal{G}(\gamma_j, \Lambda^\circ)$ is a Bessel sequence for each j, and, after some rescaling, the biorthogonality states that $s(\Lambda)^{-1}\langle\pi(\mu')\gamma_j, \pi_\mu g_\ell\rangle = \delta_{\mu,\mu'}\delta_{j,\ell}$. According to the Wexler-Raz relations, this implies that $\mathcal{G}(\mathbf{g}, \Lambda)$ is a frame for $L^2(\mathbb{R}^d, \mathbb{C}^n)$. □

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