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Time–Frequency analysis of localization operators

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Abstract

We study a class of pseudodifferential operators known as time–frequency localization operators, Anti-Wick operators, Gabor–Toeplitz operators or wave packets. Given a symbol a and two windows φ_1, φ_2, we investigate the multilinear mapping from $(a, \varphi_1, \varphi_2) \in \mathscr{S}'(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d)$ to the localization operator $A_a^{\varphi_1,\varphi_2}$ and we give sufficient and necessary conditions for $A_a^{\varphi_1,\varphi_2}$ to be bounded or to belong to a Schatten class. Our results are formulated in terms of time–frequency analysis, in particular we use modulation spaces as appropriate classes for symbols and windows.

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1. Introduction

Time–frequency localization operators are a mathematical tool to define a restriction of functions to a region in time–frequency plane that is compatible with the uncertainty principle and to extract time–frequency features. In this sense they have been introduced and studied by Daubechies [10] and Ramanathan and Topiwala [29], and they are now extensively investigated as an important mathematical tool in signal analysis and other applications [17,35,37]. In other

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mathematical contexts, time–frequency localization operators have already been used as a quantization procedure ("Anti-Wick operators") by Berezin [1,30] or as an approximation of pseudodifferential operators ("wave packets") by Cordoba and Fefferman [8,9].

While the investigation of this particular class of pseudodifferential operators is usually carried out within "hard analysis", we will study localization operators as a part of time–frequency analysis. The very definition of a localization operator by means of the Schrödinger representation and the short-time Fourier transform suggests such an approach. Using function spaces associated to the short-time Fourier transform, the so-called modulation spaces, we will obtain the sharpest results for boundedness and Schatten class properties that are known so far. Furthermore, we will show that these results are in some sense optimal [11]. This is yet another example where modulation spaces are the appropriate function spaces.

To be more specific, we define the operators of translation and modulation by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega t} f(t). \tag{1}$$

For a fixed non-zero $g \in \mathscr{S}(\mathbb{R}^d)$ the short-time Fourier transform of $f \in \mathscr{S}'(\mathbb{R}^d)$ with respect to the window g is given by

$$V_g f(x,\omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i \omega t}\, dt. \tag{2}$$

Then the time–frequency localization operator $A_a^{\varphi_1,\varphi_2}$ with symbol a and windows φ_1, φ_2 is defined to be

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\, dx\, d\omega. \tag{3}$$

If $a = \chi_\Omega$ for some compact set $\Omega \subseteq \mathbb{R}^{2d}$ and $\varphi_1 = \varphi_2$, then $A_a^{\varphi_1,\varphi_2}$ is interpreted as the part of f that "lives on the set Ω" in the time–frequency plane. This is why $A_a^{\varphi_1,\varphi_2}$ is called a time–frequency localization operator. If $a \in \mathscr{S}'(\mathbb{R}^{2d})$ and $\varphi_1, \varphi_2 \in \mathscr{S}(\mathbb{R}^d)$, then (3) is a well-defined continuous operator from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$. If $\varphi_1(t) = \varphi_2(t) = e^{-\pi t^2}$, then $A_a = A_a^{\varphi_1,\varphi_2}$ is the classical Anti-Wick operator and the mapping $a \to A_a^{\varphi_1,\varphi_2}$ is interpreted as a quantization rule [1,30,37].

Note that the time–frequency shifts $(x,\omega,\tau) \mapsto \tau T_x M_\omega$, $(x,\omega) \in \mathbb{R}^{2d}$, $|\tau| = 1$, define the Schrödinger representation of the Heisenberg group; for a deeper understanding of localization operators it is therefore natural to use the mathematical tools associated to time–frequency shifts (see [19,23]).

Often it is more convenient to interpret the definition of $A_a^{\varphi_1,\varphi_2}$ in a weak sense, then (3) can be recast as

$$\langle A_a^{\varphi_1,\varphi_2} f, g \rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g \rangle = \langle a, \overline{V_{\varphi_1} f} V_{\varphi_2} g \rangle, \quad f, g \in \mathscr{S}(\mathbb{R}^d). \tag{4}$$

Here we will study localization operators as a multilinear mapping $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$. Our main interest focusses on the interplay between the roughness of the symbol a and the time–frequency concentration of the windows φ_j $(j = 1, 2)$.

To measure the time–frequency concentration of functions and distributions, we use norms and function spaces that are associated to the short-time Fourier transform, namely the class of *modulation spaces*. As special case we mention Feichtinger's algebra $M^1(\mathbb{R}^d)$ defined by the norm

$$\|f\|_{M^1} \coloneqq \|V_g f\|_{L^1(\mathbb{R}^{2d})} < \infty$$

for some (hence all) non-zero $g \in \mathscr{S}(\mathbb{R}^d)$ [12,23]. Its dual space $M^\infty(\mathbb{R}^{2d})$ is a very useful subspace of tempered distributions [30] and possesses the norm

$$\|f\|_{M^\infty} \coloneqq \sup_{(x,\omega) \in \mathbb{R}^{2d}} |V_g f(x, \omega)| < \infty.$$

Integrability conditions and decay estimates of the short-time Fourier transform occur naturally and inevitably in the study of localization operators and we will use the associated function spaces as symbol classes. While not as well known as the classical Hörmander classes or Shubin classes [30], modulation spaces appear to be the appropriate spaces for understanding time–frequency localization operators.

To give a flavor of the type of results, we formulate a simple sufficient condition for the L^2-boundedness of a localization operator.

Theorem 1.1. *If $a \in M^\infty(\mathbb{R}^{2d})$, and $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$, then $A_a^{\varphi_1,\varphi_2}$ is bounded on $L^2(\mathbb{R}^d)$, with operator norm at most*

$$\|A_a^{\varphi_1,\varphi_2}\|_{\mathrm{op}} \leqslant C\|a\|_{M^\infty} \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}.$$

Since M^∞ contains L^∞ and measures of the form $a = \sum_{k \in \mathbb{Z}^{2d}} a_k \delta_k$ with $(a_k) \in \ell^\infty$, Theorem 1.1 is a considerable improvement of the results in [3,17,36,37].

While it seems hopeless to find a characterization for the boundedness of $A_a^{\varphi_1,\varphi_2}$ for a fixed pair of windows, the condition $a \in M^\infty$ is optimal in the following sense.

Theorem 1.2. *If $A_a^{\varphi_1,\varphi_2}$ is bounded on $L^2(\mathbb{R}^d)$ uniformly with respect to* all *windows $\varphi_1, \varphi_2 \in M^1$*, i.e., *if there exists a constant $C > 0$ depending only on the symbol a such that, for all $\varphi_1, \varphi_2 \in \mathscr{S}(\mathbb{R}^d)$,*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_\infty} \leqslant C\|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}, \tag{5}$$

then $a \in M^\infty$.

In Section 3 we will derive a host of sufficient conditions for the boundedness and Schatten class of localization operators in terms of properties of the symbol a and

Table 1

Symbol	**Windows**		**Operator**
a	$\boldsymbol{\varphi}_1$	$\boldsymbol{\varphi}_2$	$\mathbf{A}_{\mathbf{a}}^{\varphi_1,\varphi_2}$
$L^\infty(\mathbb{R}^{2d})$	$M^1(\mathbb{R}^d)$	$M^1(\mathbb{R}^d)$	$B(L^2(\mathbb{R}^d))$
$M^\infty_{1/\tau_s}(\mathbb{R}^{2d})$	$M^1_{v_s}(\mathbb{R}^d)$	$M^1_{v_s}(\mathbb{R}^d)$	$B(M^{p,q}_m(\mathbb{R}^d))$
$H^{-s}(\mathbb{R}^{2d})$,	$M^2_{v_s}(\mathbb{R}^d)$	$M^2_{v_s}(\mathbb{R}^d)$	S_2
$M^{p,\infty}_{1/\tau_s}(\mathbb{R}^{2d})$,	$M^1_{v_s}(\mathbb{R}^d)$	$M^1_{v_s}(\mathbb{R}^d)$	S_p
$\mathscr{E}'(\mathbb{R}^{2d})$	$\mathscr{S}(\mathbb{R}^d)$	$\mathscr{S}(\mathbb{R}^d)$	S_1

the windows φ_1, φ_2. In most cases we use modulation spaces (defined in Section 2.2) as the appropriate classes of symbols and windows. Some known results and our new results are summarized in Table 1. The first line with anti-Wick symbol $a \in L^\infty \subseteq M^\infty$ is well known, see, e.g. [17,37]. The second line is the general version of Theorem 1.1 and will be proved in Section 3.1. The Hilbert–Schmidt result of line 3 was obtained in [4], the general condition for membership in the Schatten class S_p will be proved in Section 3.2, the trace class result in Section 3.3.

Our analysis of localization operators will heavily use the interplay between time–frequency methods and the Weyl calculus. The techniques used to prove the optimal results will combine the following ingredients.

(a) *Representation of the localization operator* $A_a^{\varphi_1,\varphi_2}$ *as a Weyl transform.* Let $W(g,f)$ be the cross-Wigner distribution as defined in (8). Then the Weyl transform L_σ of $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$ is defined by

$$\langle L_\sigma f, g \rangle = \langle \sigma, W(g,f) \rangle, \quad f, g \in \mathscr{S}(\mathbb{R}^d). \tag{6}$$

Every continuous operator from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ can be representated as a Weyl transform, and a calculation in [4,19,30] reveals that $A_a^{\varphi_1,\varphi_2} = L_{a*W(\varphi_2,\varphi_1)}$, so the (Weyl) symbol of $A_a^{\varphi_1,\varphi_2}$ is given by

$$\sigma = a * W(\varphi_2, \varphi_1). \tag{7}$$

(b) *Boundedness of pseudodifferential operators.* Formula (7) allows us to apply known results about pseudodifferential operators to investigate time–frequency localization operators. To obtain sharp results, we will use the improvement of the Calderòn–Vaillancourt Theorem in [24] which uses modulation spaces instead of the Hörmander classes as symbols.

(c) *Convolution relations of modulation spaces*. In view of (7) we need to understand convolution relations between modulation spaces and properties of the Wigner distribution. While there is now a large body of literature on modulation spaces and their applications, the analysis of localization operators poses new questions and requires new properties of these spaces. A large part of Section 2 is therefore devoted to the study of modulation spaces.

(d) *A new characterization of M^1*. The statement about necessity of Theorem 4.3 is related to a new representation theorem for Feichtinger's algebra M^1 in Section 5. This representation resembles the characterization of the Hardy space H^1 as the projective tensor product $H^2 \hat{\otimes} H^2$ and seems to be of independent interest.

Notation. We define $t^2 = t \cdot t$, for $t \in \mathbb{R}^d$, and $xy = x \cdot y$ is the scalar product on $\mathbb{R}^d$.

The Schwartz class is denoted by $\mathscr{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathscr{S}'(\mathbb{R}^d)$. We use the brackets $\langle f, g \rangle$ to denote the extension to $\mathscr{S}(\mathbb{R}^d) \times \mathscr{S}'(\mathbb{R}^d)$ of the inner product $\langle f, g \rangle = \int f(t)\overline{g(t)}\, dt$ on $L^2(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\omega) = \mathscr{F}f(\omega) = \int f(t) e^{-2\pi i t\omega}\, dt$, the involution g^* is $g^*(t) = \overline{g(-t)}$.

The singular values $\{s_k(L)\}_{k=1}^{\infty}$ of a compact operator $L \in B(L^2(\mathbb{R}^d))$ are the eigenvalues of the positive self-adjoint operator $\sqrt{L^*L}$. For $1 \leqslant p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in l^p. For consistency, we define $S_\infty := B(L^2(\mathbb{R}^d))$ to be the space of bounded operators on $L^2(\mathbb{R}^d)$. In particular, S_2 is the space of Hilbert–Schmidt operators, and S_1 is the space of trace class operators.

Throughout the paper, we shall use the notation $A \lesssim B$ to indicate $A \leqslant cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ if $A \leqslant cB$ and $B \leqslant kA$, for suitable $c, k > 0$.

2. Time–Frequency methods

First we summarize some concepts and tools of time–frequency analysis. Since these methods are now available in textbooks [19,23], we will make free use of those results and not strive to make the paper self-contained.

2.1. Short-time fourier transform (STFT) and wigner distribution

The time–frequency representations needed for the Weyl calculus and for localization operators are the *short-time Fourier transform* and the *Wigner distribution.*

The short-time Fourier transform (STFT) of a distribution $f \in \mathscr{S}'(\mathbb{R}^d)$ with respect to a non-zero window $g \in \mathscr{S}(\mathbb{R}^d)$ is

$$V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i \omega t}\, dt,$$

whereas the *cross-Wigner distribution* $W(f, g)$ of $f, g \in L^2(\mathbb{R}^d)$ is defined to be

$$W(f,g)(x,\omega) = \int f\Big(x + \frac{t}{2}\Big)\overline{g\Big(x - \frac{t}{2}\Big)} e^{-2\pi i \omega t}\, dt. \tag{8}$$

The quadratic expression $Wf = W(f,f)$ is usually called the Wigner distribution of f.

Both the STFT $V_g f$ and the Wigner distribution $W(f,g)$ are defined on many pairs of Banach spaces. For instance, they both map $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$ and $\mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d)$ into $\mathscr{S}(\mathbb{R}^{2d})$. Furthermore, they can be extended to a map from $\mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}'(\mathbb{R}^d)$ into $\mathscr{S}'(\mathbb{R}^{2d})$.

We first list some crucial properties of the STFT (for proofs, see [23, Chapter 3] and [25]).

Lemma 2.1. *Let* $f, g, f_j, g_j \in L^2(\mathbb{R}^d)$, $j = 1, 2$, *then we have*

(i) $V_g f(x,\omega) = (f \cdot T_x \bar{g})^\wedge(\omega) = e^{-2\pi i x\omega}(f * (M_\omega g)^*)(x)$.

(ii) (*STFT of time–frequency shifts*) *For* $y, \xi \in \mathbb{R}^d$, *we have*

$$V_g(M_\xi T_y f)(x,\omega) = e^{-2\pi i(\omega-\xi)y}(V_g f)(x-y, \omega-\xi) \tag{9}$$

$$V_{(M_\xi T_y g)}(M_\xi T_y f)(x,\omega) = e^{2\pi i(\xi x - \omega y)}(V_g f)(x,\omega) \tag{10}$$

(iii) (*Fourier transform of a product of STFTs*),

$$(V_{g_1} f_1 \overline{V_{g_2} f_2})^\wedge(x,\omega) = (V_{f_2} f_1 \overline{V_{g_2} g_1})(-\omega, x).$$

Note that (9) and (10) can be read backwards and yield a formula for the $2d$-dimensional time–frequency shift $M_\zeta T_z(V_g f)$, $z, \zeta \in \mathbb{R}^{2d}$.

To investigate the local properties of the STFT, we will need to compute the STFT of a STFT. Since the STFT of a function on $\mathbb{R}^{2d}$ is a function on $\mathbb{R}^{4d}$, we distinguish between the STFT $V_g f(x,\omega), (x,\omega) \in \mathbb{R}^{2d}$, of $f \in \mathscr{S}'(\mathbb{R}^d)$ and the STFT $\mathscr{V}_\Phi F(z,\zeta), (z,\zeta) \in \mathbb{R}^{4d}$ of $F \in \mathscr{S}'(\mathbb{R}^{2d})$. We write $z = (z_1, z_2) \in \mathbb{R}^{2d}$ and $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$, when necessary.

Lemma 2.2. *Fix a nonzero* $\varphi \in \mathscr{S}(\mathbb{R}^d)$ *and let* $f, g \in \mathscr{S}(\mathbb{R}^d)$.

(a) *Set* $\Phi = V_\varphi \varphi \in \mathscr{S}(\mathbb{R}^{2d})$. *Then the STFT of* $V_g f$ *with respect to the window* Φ *is given by*

$$\mathscr{V}_\Phi(V_g f)(z,\zeta) = e^{-2\pi i z_2 \zeta_2}\, \overline{V_\varphi g(-z_1 - \zeta_2, \zeta_1)}\, V_\varphi f(-\zeta_2, z_2 + \zeta_1). \tag{11}$$

(b) *Let* $\Phi = W(\varphi, \varphi) \in \mathscr{S}(\mathbb{R}^{2d})$. *Then the STFT of* $W(\varphi_2, \varphi_1)$ *with respect to the window* Φ *is given by*

$$\mathscr{V}_\Phi(W(\varphi_2, \varphi_1))(z,\zeta) = e^{-2\pi i z_2 \zeta_2}\, \overline{V_\varphi \varphi_1\left(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2}\right)}\, V_\varphi \varphi_2\left(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2}\right). \tag{12}$$

Proof. Before we calculate $\mathscr{V}_\Phi(V_g f)(z,\zeta) = \langle V_g f, M_\zeta T_z \Phi \rangle = (V_g f\, T_z \bar{\Phi})^\wedge(\zeta)$, we rewrite the time–frequency shift $M_\zeta T_z \Phi$. We use Lemma 2.1(ii) to evaluate $T_z\Phi$ and find that

$$T_z\Phi(x,\omega) = V_\varphi\varphi(x,\omega)(x-z_1,\omega-z_2) = e^{2\pi i(\omega - z_2)\cdot z_1} V_\varphi(M_{z_2}T_{z_1}\varphi)(x,\omega).$$

Now we substitute this expression into the formula for $\mathscr{V}_\Phi(V_g f)$, and after rearranging some terms we apply Lemma 2.1(iii):

$$\begin{aligned}
\mathscr{V}_\Phi(V_g f)(z,\zeta) &= \langle V_g f, M_\zeta T_z \Phi \rangle \\
&= \int\int_{\mathbb{R}^{2d}} V_g f(x,\omega)\overline{V_\varphi(M_{z_2}T_{z_1}\varphi)(x,\omega)} e^{-2\pi i[(x\zeta_1+\omega\zeta_2)+(\omega - z_2)z_1]}\,dx\,d\omega \\
&= e^{2\pi i z_1 z_2}\int\int_{\mathbb{R}^{2d}} V_g f(x,\omega)\overline{V_\varphi(M_{z_2}T_{z_1}\varphi)(x,\omega)} e^{-2\pi i[\zeta_1 x+(z_1+\zeta_2)\omega]}\,dx\,d\omega \\
&= e^{2\pi i z_1 z_2}(V_g f \cdot \overline{V_\varphi(M_{z_2}T_{z_1}\varphi)})^\wedge(\zeta_1, z_1+\zeta_2) \\
&= e^{2\pi i z_1 z_2}(V_{(M_{z_2}T_{z_1}\varphi)} f \cdot \overline{V_\varphi g})(-z_1-\zeta_2,\zeta_1) \\
&= e^{-2\pi i z_2 \zeta_2}(V_\varphi f)(-\zeta_2, z_2+\zeta_1)\overline{V_\varphi g(-z_1-\zeta_2,\zeta_1)}.
\end{aligned}$$

This proves (a).

(b) was proved in [23, Lemma 14.5.1], related versions occur in [27,28]. □

2.2. Modulation spaces and other function spaces

Modulation spaces. The modulation space norms are a measure of the joint time–frequency distribution of $f \in \mathscr{S}'$. For their basic properties we refer, for instance, to [23, Chapters 11–13] and the original literature quoted there.

For the quantitative description of decay properties, we use weight functions on the time–frequency plane. In the sequel v will always be a continuous, positive, even, submultiplicative weight function (in short, a submultiplicative weight), i.e., $v(0) = 1$, $v(z) = v(-z)$, and $v(z_1+z_2) \leqslant v(z_1)v(z_2)$, for all $z, z_1, z_2 \in \mathbb{R}^{2d}$. A positive, even weight function m on $\mathbb{R}^{2d}$ is called *v-moderate* if $m(z_1+z_2) \leqslant Cv(z_1)m(z_2)$ for all $z_1, z_2 \in \mathbb{R}^{2d}$.

For our investigation of localization operators we will mostly use the polynomial weights defined by

$$v_s(z) = v_s(x,\omega) = \langle z \rangle^s = (1+x^2+\omega^2)^{s/2}, \quad z=(x,\omega)\in\mathbb{R}^{2d}, \tag{13}$$

$$\tau_s(z) = \tau_s(x,\omega) = \langle \omega \rangle^s = (1+\omega^2)^{s/2}. \tag{14}$$

Note that on $\mathbb{R}^{4d}$ we have

$$\tau_s(z,\zeta) = v_s(\zeta) \quad z,\zeta \in \mathbb{R}^{2d}.$$

Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, a v-moderate weight function m on $\mathbb{R}^{2d}$ of polynomial growth, and $1 \leqslant p,q \leqslant \infty$, the *modulation space* $M_m^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M_m^{p,q}$ is

$$\|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p \, dx \right)^{q/p} d\omega \right)^{1/p}.$$

If $p = q$, we write M_m^p instead of $M_m^{p,p}$, and if $m(z) \equiv 1$ on $\mathbb{R}^{2d}$, then we write $M^{p,q}$ and M^p for $M_m^{p,q}$ and $M_m^{p,p}$.

Then $M_m^{p,q}(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window g. Moreover, if m is v-moderate and $g \in M_v^1 \backslash \{0\}$, then $\|V_g f\|_{L_m^{p,q}}$ is an equivalent norm for $M_m^{p,q}(\mathbb{R}^d)$ (see [23, Theorem 11.3.7]). We always measure the $M_m^{p,q}$-norm with a fixed non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$ and repeatedly use the fact that for any $g_1 \in M_v^1(\mathbb{R}^d)$ the norm equivalence

$$\|f\|_{M_m^{p,q}} \asymp \|V_{g_1} f\|_{L_m^{p,q}}$$

holds. For technicalities when v grows faster than a polynomial we refer to [23, Chapter 11.4].

Among the modulation spaces the following well-known function spaces occur:

(i) $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.

(ii) Weighted L^2-spaces: if $\mu_s(x,\omega) = \langle x \rangle^s$, then

$$M_{\mu_s}^2(\mathbb{R}^d) = L_s^2(\mathbb{R}^d) = \{f : f(x)\langle x \rangle^s \in L^2(\mathbb{R}^d)\}.$$

(iii) Sobolev spaces: if $\tau_s(x,\omega) = \langle \omega \rangle^s$, then

$$M_{\tau_s}^2(\mathbb{R}^d) = H^s(\mathbb{R}^d) = \{f : \hat{f}(\omega)\langle \omega \rangle^s \in L^2(\mathbb{R}^d)\}.$$

(iv) Shubin–Sobolev spaces [4,30]: if $v_s(x,\omega) = \langle (x,\omega) \rangle^s$, then

$$M_{v_s}^2(\mathbb{R}^d) = L_s^2(\mathbb{R}^d) \cap H^s(\mathbb{R}^d) = Q_s(\mathbb{R}^d)$$

(v) Feichtinger's algebra: $M^1(\mathbb{R}^d) = S_0(\mathbb{R}^d)$.

(vi) The Schwartz class [25]: $\mathcal{S}(\mathbb{R}^d) = \bigcap_{s \geqslant 0} M_{v_s}^\infty(\mathbb{R}^d)$.

(vii) The space of tempered distributions [25]: $\mathcal{S}'(\mathbb{R}^d) = \bigcup_{s \geqslant 0} M_{1/v_s}^\infty(\mathbb{R}^d)$.

Roughly speaking, a weight in ω regulates the smoothness of $f \in M_m^{p,q}$, whereas a weight in x regulates the decay of $f \in M_m^{p,q}$.

A comparison of modulation spaces with Fourier–Lebesgue spaces and embedding theorems are contained in [21].

Potential spaces. For $s \in \mathbb{R}$ the Bessel kernel is

$$G_s = \mathscr{F}^{-1}\{(1+|\cdot|^2)^{-s/2}\}, \tag{15}$$

and the *potential space* [2,32] is defined by

$$W_s^p = G_s * L^p(\mathbb{R}^d) = \{f \in \mathscr{S}', f = G_s * g, g \in L^p\}$$

with norm $\|f\|_{W_s^p} = \|g\|_p$.

For comparison we list the following embeddings between potential and modulation spaces.

Lemma 2.3. *We have*

(i) *If* $p_1 \leqslant p_2$ *and* $q_1 \leqslant q_2$, *then* $M_m^{p_1,q_1} \hookrightarrow M_m^{p_2,q_2}$.
(ii) *For* $1 \leqslant p \leqslant \infty$ *and* $s \in \mathbb{R}$

$$W_s^p(\mathbb{R}^d) \hookrightarrow M_{\tau_s}^{p,\infty}(\mathbb{R}^d).$$

Proof. (i) See [23, Theorem 12.2.2].

(ii) Assume $s = 0$ first. Using Lemma 2.1(i) and Young's inequality, we find that

$$\begin{aligned}\|f\|_{M^{p,\infty}} \asymp \|V_g f\|_{L^{p,\infty}} &= \sup_{\omega \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p \, dx\right)^{1/p} \\ &= \sup_{\omega \in \mathbb{R}^d} \|(f * M_\omega g^*)\|_p \leqslant \sup_{\omega \in \mathbb{R}^d} \|f\|_p \|M_\omega g^*\|_1 \lesssim \|f\|_p.\end{aligned}$$

Consequently $L^p \subseteq M^{p,\infty}$. For arbitrary $s \in \mathbb{R}$ we observe that $W_s^p = G_s * L^p$ by definition and $M_{\tau_s}^{p,\infty} = G_s * M^{p,\infty}$ by [14]. So the embedding follows for all $s \in \mathbb{R}$. □

Wiener amalgam spaces ([13,15,18,20]). Let $g \in \mathscr{D}(\mathbb{R}^{2d})$ be a test function that satisfies $\sum_{(k,l) \in \mathbb{Z}^{2d}} T_{(k,l)} g \equiv 1$. Let $X(\mathbb{R}^{2d})$ be a Banach space of functions invariant under translations and with the property that $\mathscr{D} \cdot X \subset X$, e.g., L^p, $\mathscr{F}L^p$, or $L^{p,q}$. Then the *Wiener amalgam space* $W(X, L_m^{p,q})$ with local component X and global component $L_m^{p,q}$ is defined as the space of all functions or distributions for which the norm

$$\|f\|_{W(X,L_m^{p,q})} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \|f \cdot T_{(z_1,z_2)} g\|_X\right)^p m(z_1,z_2)^p \, dz_1\right)^{q/p} dz_2\Bigg)^{1/q}$$

is finite. Equivalently, $f \in W(X, L_m^{p,q})$ if and only if

$$\left(\sum_{l \in \mathbb{Z}^d} \left(\sum_{k \in \mathbb{Z}^d} \| f \cdot T_{(k,l)} g \|_X^p m(k,l)^p \right)^{q/p} \right)^{1/q} < \infty .$$

It can be shown that different choices of $g \in \mathscr{D}$ generate the same space and yield equivalent norms. If the local component is $\mathscr{F}L^1$, then we can express this norm by means of the STFT as follows:

$$\begin{aligned} \|f\|_{W(\mathscr{F}L^1, L_m^{p,q})} &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \| f \cdot T_{(z_1,z_2)} \bar{g} \|_{\mathscr{F}L^1}^p m(z_1,z_2)^p \, dz_1 \right)^{q/p} dz_2 \right)^{1/q} \\ &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^{2d}} |(f \cdot T_{(z_1,z_2)} \bar{g})^\wedge(\zeta)| \, d\zeta \right)^p m(z_1,z_2)^p \, dz_1 \right)^{q/p} dz_2 \right)^{1/q} \\ &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^{2d}} |V_g f(z_1,z_2,\zeta)| d\zeta \right)^p m(z_1,z_2)^p \, dz_1 \right)^{q/p} dz_2 \right)^{1/q} \end{aligned} \tag{16}$$

Remarks. 1. If X is a Banach algebra, then Hölder's inequality for amalgam spaces reads as follows [13]:

$$\|f \cdot g\|_{W(X,L^1)} \lesssim \|f\|_{W(X,L_m^{p,q})} \|g\|_{W(X,L_{1/m}^{p',q'})}. \tag{17}$$

2. Certain modulation spaces coincide with Wiener amalgam spaces, in particular, (16) implies that

$$M^1(\mathbb{R}^d) = W(\mathscr{F}L^1, L^1) \tag{18}$$

with equivalent norms (see also [12]).

2.3. Convolution relations for modulation spaces

In view of the relation between the multiplier a and the Weyl symbol (7), we need to understand the convolution relations between modulation spaces and some properties of the Wigner distribution.

We first show a convolution relation for modulation spaces in the style of Young's theorem. For further reference we formulate it for arbitrary submultiplicative weights v on $\mathbb{R}^{2d}$ and v-moderate weights m. We write $m_1(x) = m(x,0)$ and $m_2(\omega) = m(0,\omega)$ for the restrictions to $\mathbb{R}^d \times \{0\}$ and $\{0\} \times \mathbb{R}^d$, and likewise for v.

Proposition 2.4. *Let* $v(\omega)>0$ *be an arbitrary weight function on* $\mathbb{R}^d$ *and* $1\leqslant p,q,r,s,t\leqslant\infty$. *If*

$$\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}, \quad \textit{and} \quad \frac{1}{t}+\frac{1}{t'}=1,$$

then

$$M^{p,st}_{m_1\otimes v}(\mathbb{R}^d) * M^{q,st'}_{v_1\otimes v_2 v^{-1}}(\mathbb{R}^d) \hookrightarrow M^{r,s}_m(\mathbb{R}^d) \tag{19}$$

with norm inequality $\|f*h\|_{M^{r,s}_m} \lesssim \|f\|_{M^{p,st}_{m_1\otimes v}} \|h\|_{M^{q,st'}_{v_1\otimes v_2 v^{-1}}}$.

Proof. We measure the modulation space norm with respect to the Gaussian windows $g_0(x)=e^{-\pi x^2}$ and $g(x)=2^{-n/d}e^{-\pi x^2/2}=(g_0*g_0)(x)\in\mathscr{S}(\mathbb{R}^d)$. Recall that different windows yield equivalent norms for the modulation spaces, specifically,

$$\|f\|_{M^{p,q}_m} \asymp \|V_{g_0}f\|_{L^{p,q}_m} \asymp \|V_g f\|_{L^{p,q}_m}.$$

Using Lemma 2.1(*i*) and the identity $M_\omega(g_0^* * g_0^*)=M_\omega g_0^* * M_\omega g_0^*$, we express the STFT of $f*h$ as follows:

$$V_g(f*h)(x,\omega)=e^{-2\pi i x\omega}((f*h)*M_\omega g^*)(x)=e^{-2\pi i x\omega}((f*M_\omega g_0^*)*(h*M_\omega g_0^*))(x).$$

To estimate the $M^{p,q}_m$-norm of $f*h$, we first majorize m by $m(x,\omega)\leqslant m(x,0)v(0,\omega)=m_1(x)v_2(\omega)$ and then we use Young's inequality with $1/p+1/q-1=1/r$ in the x-variable and Hölder's inequality in the ω-variable. We obtain that

$$\begin{aligned}
&\|f*h\|_{M^{r,s}_m}\\
&\asymp \|V_g(f*h)\|_{L^{r,s}_m}\\
&\leqslant \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|(f*M_\omega g_0^*)*(h*M_\omega g_0^*)(x)|^r m_1(x)^r\,dx\right)^{s/r} v_2(\omega)^s\,d\omega\right)^{1/s}\\
&=\left(\int_{\mathbb{R}^d}\|(f*M_\omega g_0^*)*(h*M_\omega g_0^*)\|^s_{L^r_{m_1}} v_2(\omega)^s\,d\omega\right)^{1/s}\\
&\lesssim \left(\int_{\mathbb{R}^d}\|f*M_\omega g_0^*\|^s_{L^p_{m_1}}\|h*M_\omega g_0^*\|^s_{L^q_{v_1}} v_2(\omega)^s\,d\omega\right)^{1/s}\\
&\lesssim \left(\int_{\mathbb{R}^d}\|f*M_\omega g_0^*\|^{st}_{L^p_{m_1}} v(\omega)^{st}d\omega\right)^{\frac{1}{st}}\left(\int_{\mathbb{R}^d}\|h*M_\omega g_0^*\|^{st'}_{L^q_{v_1}}\frac{v_2(\omega)^{st'}}{v(\omega)^{st'}}\,d\omega\right)^{\frac{1}{st'}}\\
&\asymp \|f\|_{M^{p,st}_{m_1\otimes v}}\|h\|_{M^{q,st'}_{v_1\otimes v_2 v^{-1}}}. \qquad \square
\end{aligned}$$

Remarks. 1. Despite the large number of indices, the statement of this proposition has some intuitive meaning: a function $f \in M^{p,q}$ behaves like $f \in L^p$ and $\hat{f} \in L^q$; so the parameters related to the x-variable behave like those in Young's theorem for convolution, whereas the parameters related to ω behave like Hölder's inequality for pointwise multiplication.

2. A special case of Proposition 2.4 with a different and longer proof is contained in [33].

2.4. An estimate for the Wigner distribution

We next calculate the modulation space norm of a cross-Wigner distribution.

Proposition 2.5. *If* $1 \leqslant p \leqslant \infty$, $s \geqslant 0$, $\varphi_1 \in M^1_{v_s}(\mathbb{R}^d)$ *and* $\varphi_2 \in M^p_{v_s}(\mathbb{R}^d)$, *then* $W(\varphi_2, \varphi_1) \in M^{1,p}_{\tau_s}(\mathbb{R}^{2d})$, *and*

$$\|W(\varphi_2,\varphi_1)\|_{M^{1,p}_{\tau_s}} \lesssim \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^p_{v_s}}. \tag{20}$$

Proof. As usual, we prove (20) for $\varphi_1, \varphi_2 \in \mathscr{S}$ first and then obtain the full result by an approximation argument.

Let $g \in \mathscr{S}(\mathbb{R}^d)$ and set $\Phi = W(g,g) \in \mathscr{S}(\mathbb{R}^{2d})$. If $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$, we write $\tilde{\zeta} = (\zeta_2, -\zeta_1)$. Then Lemma 2.2(b) says that

$$|\mathscr{V}_\Phi(W(\varphi_2,\varphi_1))(z,\zeta)| = |V_g\varphi_1(z + \tfrac{\tilde{\zeta}}{2})||V_g\varphi_2(z - \tfrac{\tilde{\zeta}}{2})|.$$

Consequently

$$\|W(\varphi_2,\varphi_1)\|_{M^{1,p}_{\tau_s}} \asymp \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |V_g\varphi_1(z + \tfrac{\tilde{\zeta}}{2})|\ |V_g\varphi_2(z - \tfrac{\tilde{\zeta}}{2})|\, dz\right) \langle \zeta \rangle^{sp}\, d\zeta\right)^{1/p}.$$

After the change of variables $z \mapsto z - \tilde{\zeta}/2$, the integral over z becomes the convolution $(|V_g\varphi_1| * |V_g\varphi_2|^*)(\tilde{\zeta})$, and observing that $\tau_s(z,\zeta) = \langle \zeta \rangle^s = v_s(\zeta) = v_s(\tilde{\zeta})$, we obtain

$$\begin{aligned}\|W(\varphi_2,\varphi_1)\|_{M^{1,p}_{\tau_s}} &\lesssim \left(\int \int_{\mathbb{R}^{2d}} (|V_g\varphi_1| * |V_g\varphi_2^*|)(\tilde{\zeta}) \langle \zeta \rangle^{sp}\, d\zeta\right)^{1/p} \\ &= \|\,|V_g\varphi_1| * |V_g\varphi_2^*|\,\|_{L^p_{v_s}} \\ &\lesssim \|V_g\varphi_1\|_{L^1_{v_s}} \|V_g\varphi_2\|_{L^p_{v_s}} \asymp \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^p_{v_s}},\end{aligned}$$

where we have used Young's convolution inequality in the last step. □

3. Sufficient conditions for boundedness and Schatten class

Using the tools of time–frequency analysis developed in the previous section, we now analyze the properties of localization operators with symbols in a modulation space. We will reduce this problem to the corresponding problem for the Weyl calculus, and so we first recall a boundedness and trace class result for the Weyl calculus in terms of modulation spaces.

Theorem 3.1. (i) *If* $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, *then* L_σ *is bounded on* $M^{p,q}(\mathbb{R}^d)$, $1 \leqslant p, q \leqslant \infty$, *with a uniform estimate* $\|L_\sigma\|_{\mathrm{op}} \lesssim \|\sigma\|_{M^{\infty,1}}$ *for the operator norm. In particular*, L_σ *is bounded on* $L^2(\mathbb{R}^d)$.

(ii) *If* $\sigma \in M^1(\mathbb{R}^{2d})$, *then* $L_\sigma \in S_1$ *and* $\|L_\sigma\|_{S^1} \lesssim \|\sigma\|_{M^1}$.
(iii) *If* $1 \leqslant p \leqslant 2$ *and* $\sigma \in M^p(\mathbb{R}^{2d})$, *then* $L_\sigma \in S_p$ *and* $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^p}$.
(iv) *If* $2 \leqslant p \leqslant \infty$ *and* $\sigma \in M^{p,p'}(\mathbb{R}^{2d})$, *then* $L_\sigma \in S_p$ *and* $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^{p,p'}}$.

The proof of (i) can be found in [24] and [23, Theorem 14.5.2], see also [31,33], (ii) is proved in [22], whereas (iii) and (iv) follow by interpolation from the first two statements, since $[M^1, M^2]_\theta = M^p$ for $1 \leqslant p \leqslant 2$, and $[M^{\infty,1}, M^{2,2}]_\theta = M^{p,p'}$ for $2 \leqslant p \leqslant \infty$ [14,16]. We note that (i) improves the theorem of Calderòn–Vaillancourt [6], whereas (ii) is an improvement of a result of Daubechies [9] and Hörmander [26].

3.1. Boundedness

Based on the tools developed in the previous section, we establish the following boundedness result. Its proof is now deceptively simple.

Theorem 3.2. *Let* $s \geqslant 0$, $a \in M^\infty_{1/\tau_s}(\mathbb{R}^{2d})$, $\varphi_1, \varphi_2 \in M^1_{v_s}(\mathbb{R}^d)$. *Then* $A_a^{\varphi_1,\varphi_2}$ *is bounded on* $M^{p,q}(\mathbb{R}^d)$ *for all* $1 \leqslant p, q \leqslant \infty$, *and the operator norm satisfies the uniform estimate*

$$\|A_a^{\varphi_1,\varphi_2}\|_{\mathrm{op}} \lesssim \|a\|_{M^\infty_{1/\tau_s}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^1_{v_s}}.$$

Proof. We use an appropriate convolution relation to show that the Weyl symbol $a * W(\varphi_2, \varphi_1)$ of $A_a^{\varphi_1,\varphi_2}$ is in $M^{\infty,1}$. If $\varphi_1, \varphi_2 \in M^1_{v_s}(\mathbb{R}^d)$, then by (20), we have $W(\varphi_2, \varphi_1) \in M^1_{\tau_s}(\mathbb{R}^{2d})$. Applying Proposition 2.4 in the form $M^\infty_{1/\tau_s} * M^1_{\tau_s} \subseteq M^{\infty,1}$, we obtain that the Weyl symbol $\sigma = a * W(\varphi_2, \varphi_1) \in M^{\infty,1}$. The result now follows from Theorem 3.1(i). $\square$

Remark. To compare Theorem 3.2 to existing results, we recall that the standard condition for $A_a^{\varphi_1,\varphi_2}$ to be bounded is $a \in L^\infty(\mathbb{R}^{2d})$, see [37]. A more subtle result of Feichtinger and Nowak [17] shows that the condition $a \in W(M, L^\infty)$ is sufficient for boundedness. Since we have the proper embeddings $L^\infty \subset W(M, L^\infty) \subset M^\infty \subset M^\infty_{1/\tau_s}$

for $s \geqslant 0$, Theorem 3.2 appears as a significant improvement. A special case of Theorem 3.2 follows also from Toft's work [33].

Since $\tau_s(z,\zeta) = \langle \zeta \rangle^s$ depends only on the frequency variable, the condition $a \in M^\infty_{1/\tau_s}$ describes the admissible roughness of a, while in some sense a remains bounded in z. On the other hand, if we allow the symbol a to grow in both time and frequency—by choosing the "full" weight $v_s = \langle (z,\zeta) \rangle^s$—then we obtain a negative result.

Proposition 3.3. *For any $s > 0$ there exist symbols $a \in M^\infty_{1/v_s}(\mathbb{R}^{2d})$ and windows $\varphi_1, \varphi_2 \in \mathscr{S}(\mathbb{R}^d)$ such that $A_a^{\varphi_1,\varphi_2}$ is unbounded on $L^2(\mathbb{R}^d)$.*

Proof. We choose $\varphi_1(t) = \varphi_2(t) = e^{-\pi t^2}$ and $a(z) = v_s(z) = \langle z \rangle^s$ and set $A_s = A_a^{\varphi_1,\varphi_2}$. By definition of the Shubin–Sobolev space Q_s [30, Section 25.3], we have

$$Q_s(\mathbb{R}^d) := \{ f \in \mathscr{S}'(\mathbb{R}^d) \colon A_s f \in L^2(\mathbb{R}^d) \} = A_s^{-1} L^2(\mathbb{R}^d),$$

with norm $\|f\|_{Q_s} := \|A_s f\|_{L^2}$. Furthermore, A_s is one-to-one from $Q_s(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Consequently, if $f \in L^2(\mathbb{R}^d) \backslash Q_s(\mathbb{R}^d)$, then $A_s f \notin L^2(\mathbb{R}^d)$, and thus A_s must be unbounded on $L^2(\mathbb{R}^d)$.

It remains to be shown that $a(z) = v_s(z) = \langle z \rangle^s$ belongs to $M^\infty_{1/v_r}(\mathbb{R}^{2d})$ for any $r \geqslant s$. So we estimate the STFT of v_s with respect to some $g \in \mathscr{S}(\mathbb{R}^{2d}) \backslash \{0\}$:

$$\begin{aligned} \|\mathscr{V}_g v_s\|_{L^\infty_{1/v_r}} &= \sup_{(z,\zeta) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}} \frac{1}{\langle z,\zeta \rangle^r} \left| \int_{\mathbb{R}^{2d}} \langle t \rangle^s \bar{g}(t-z)\, e^{-2\pi i t \zeta}\, dt \right| \\ &\leqslant \sup_{z \in \mathbb{R}^{2d}} \frac{1}{\langle z \rangle^r} \int_{\mathbb{R}^{2d}} \langle t-z \rangle^s \langle z \rangle^s |\bar{g}(t-z)|\, dt \\ &= \sup_{z \in \mathbb{R}^{2d}} \frac{\langle z \rangle^s}{\langle z \rangle^r} \int_{\mathbb{R}^{2d}} \langle t \rangle^s |\bar{g}(t)|\, dt < \infty. \end{aligned}$$

So $v_s \in M^\infty_{1/v_r}$ and we are done. □

These results demonstrate that bounded symbols with negative smoothness may still yield bounded localization operators, provided that the roughness of a is compensated by a suitable time–frequency localization frequency of the windows. On the other hand, a smooth unbounded symbol cannot, in general, yield a bounded operator. In view of the characterization of $\mathscr{S}'(\mathbb{R}^{2d}) = \bigcup_{s \geqslant 0} M^\infty_{1/v_s}(\mathbb{R}^{2d})$, the class M^∞ is the largest space of tempered distributions to yield bounded localization operators (see also Section 4).

3.2. Schatten class conditions

We next investigate the Schatten class properties of localization operators with symbols in a modulation space. Combining Proposition 2.4 with Theorem 3.1, we derive an almost optimal condition for $A_a^{\varphi_1,\varphi_2} \in S_p$.

Theorem 3.4. (i) *If* $1 \leqslant p \leqslant 2$, *then the mapping* $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ *is bounded from* $M^{p,\infty}_{1/\tau_s}(\mathbb{R}^{2d}) \times M^1_{v_s}(\mathbb{R}^d) \times M^p_{v_s}(\mathbb{R}^d)$ *into* S_p, *in other words,*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{p,\infty}_{1/\tau_s}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^p_{v_s}}.$$

(ii) *If* $2 \leqslant p \leqslant \infty$, *then the mapping* $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ *is bounded from* $M^{p,\infty}_{1/\tau_s} \times M^1_{v_s} \times M^{p'}_{v_s}$ *into* S_p, *and*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{p,\infty}_{1/\tau_s}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^{p'}_{v_s}}.$$

Proof. (i) If $\varphi_1 \in M^1_{v_s}(\mathbb{R}^d)$ and $\varphi_2 \in M^p_{v_s}(\mathbb{R}^d)$, then $W(\varphi_2, \varphi_1) \in M^{1,p}_{\tau_s}(\mathbb{R}^{2d})$ by (20). Since $a \in M^{p,\infty}_{1/\tau_s}$, the convolution relation $M^{p,\infty}_{1/\tau_s} * M^{1,p}_{\tau_s} \subseteq M^p$ of Proposition 2.4 implies that the Weyl symbol $\sigma = a * W(\varphi_2, \varphi_1)$ is in M^p. The result now follows from Theorem 3.1(iii).

(ii) is proved similarly by using the convolution relation $M^{p,\infty}_{1/\tau_s} * M^{1,p'}_{\tau_s} \subseteq M^{p,p'}$ and Theorem 3.1(iv). □

Using an easy embedding theorem, we can prove a slightly weaker statement with a more familiar symbol class. The following consequence was already derived in [4, Theorem 4.7].

Corollary 3.5. *Let* $a \in W^p_{-s}(\mathbb{R}^{2d})$ *for some* $s \geqslant 0$, $1 \leqslant p \leqslant \infty$, *and* $\varphi_1, \varphi_2 \in M^1_{v_s}(\mathbb{R}^d)$. *Then*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{W^p_{-s}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^1_{v_s}}.$$

Proof. The statement follows from the embeddings $W^p_{-s} \hookrightarrow M^{p,\infty}_{1/\tau_s}$ (Lemma 2.3) and $M^1_{v_s} \hookrightarrow M^p_{v_s}$. □

Remark. By using other convolution relations provided by Proposition 2.4, interpolation and embedding properties of modulation spaces, one may derive many variations of Theorem 3.4. We only mention two small modifications that might be of interest.

(a) If $a \in M^{1,p}_{1/\tau_s}$ and $\varphi_1 \in M^1_{v_s}$, $\varphi_2 \in M^{p'}_{v_s}$, then $A_a^{\varphi_1,\varphi_2}$ is of trace class, because $M^{1,p}_{1/\tau_s} * M^{1,p'}_{\tau_s} \subseteq M^1$. Comparing to Theorem 3.4(i), we see that this result allows us to use a window φ_2 with less time–frequency concentration, however, at the price of a slightly smaller symbol class.

(b) If $(a, \varphi_1, \varphi_2) \in M^{p,\infty}_{1/\tau_s} \times M^q_{v_s} \times M^r_{v_s}$, where $1/q + 1/r - 1 = 1/p$ and $1 \leqslant p \leqslant 2$, then $A_a^{\varphi_1,\varphi_2} \in S_p$. To see this, we observe that Theorem 3.4(i) also holds with the role of the windows reversed, i.e., for $(\varphi_1, \varphi_2) \in M^p_{v_s} \times M^1_{v_s}$. The result then follows from the interpolation property $[M^1_{v_s} \times M^p_{v_s}, M^p_{v_s} \times M^1_{v_s}]_\theta = M^q_{v_s} \times M^r_{v_s}$ with $1/q + 1/r - 1 = 1/p$.

3.3. Distributions with compact support as symbols

As an application we treat distributions with compact support. We shall prove that a distribution with compact support gives trace class operators, if the windows belong to suitable spaces. We write $\mathscr{E}'(\mathbb{R}^{2d})$ for the subspace of tempered distributions of compact support. It is well-known that every $a \in \mathscr{E}'(\mathbb{R}^{2d})$ can be represented as

$$a = \sum_{|\alpha| \leqslant m} \partial^\alpha f_\alpha \tag{21}$$

for compactly supported continuous functions f_α on $\mathbb{R}^{2d}$ [34, p. 263, Corollary 2]. For convenience and in contrast to the standard definition, we will call the integer m in (21) the order of $a \in \mathscr{E}'$.

Proposition 3.6. *Let $a \in \mathscr{E}'(\mathbb{R}^d)$ be of order m. Then its STFT satisfies the estimate*

$$|V_g a(x,\omega)| \leqslant C_N \langle x \rangle^{-N} \langle \omega \rangle^m \quad \forall N \in \mathbb{N},$$

and consequently $a \in M^{1,\infty}_{1/\tau_m}(\mathbb{R}^d)$.

Proof. Assume that $a \in \mathscr{E}'$ has a representation (21). We express the derivatives of a time–frequency shift of $g \in \mathscr{S}(\mathbb{R}^d)$ by means of Leibniz's rule as

$$\partial^\alpha (M_\omega T_x g)(t) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (2\pi i \omega)^\beta M_\omega T_x \partial_t^{\alpha-\beta} g(t),$$

and then use the obvious inequality $1 \leqslant \langle t \rangle^N \langle x \rangle^{-N} \langle t - x \rangle^N$ for $t, x \in \mathbb{R}^d$. After inserting these expressions into the STFT, we obtain

$$\begin{aligned} &|V_g a(x,\omega)| \\ &\quad \leqslant \sum_{|\alpha| \leqslant m} |\langle \partial^\alpha f_\alpha, M_\omega T_x g \rangle| = \sum_{|\alpha| \leqslant m} |\langle f_\alpha, \partial^\alpha (M_\omega T_x g) \rangle| \end{aligned}$$

$$\leqslant \sum_{|\alpha|\leqslant m}\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}|(2\pi i\omega)^{\beta}|\int_{\mathbb{R}^d}|f_\alpha(t)(M_\omega T_x\partial_t^{\alpha-\beta}g)(t)|\langle t\rangle^N\langle x\rangle^{-N}\langle t-x\rangle^N\,dt$$
$$\lesssim\langle\omega\rangle^m\langle x\rangle^{-N}\sum_{|\alpha|\leqslant m,\beta\leqslant\alpha}\|f_\alpha\langle\cdot\rangle^N\|_\infty\|\partial^{\alpha-\beta}g\langle\cdot\rangle^N\|_1.$$

It follows that $a\in M^{1,\infty}_{1/\tau_m}$. □

Corollary 3.7. *Assume that $a\in\mathscr{E}'(\mathbb{R}^{2d})$ is of order m and $\varphi_1,\varphi_2\in M^1_{v_m}(\mathbb{R}^d)$, then $A_a^{\varphi_1,\varphi_2}$ is a trace class operator.*

Proof. By Proposition 3.6 we have that $a\in M^{1,\infty}_{1/\tau_m}(\mathbb{R}^{2d})$. If $\varphi_1,\varphi_2\in M^1_{v_m}(\mathbb{R}^d)$, then Theorem 3.4(i) implies that $A_a^{\varphi_1,\varphi_2}$ is trace class. □

Remark. An estimate similar to Proposition 3.6 shows that any distribution of the form $a=\sum_{|\alpha|\leqslant m}\partial^\alpha\mu_\alpha$ with $\mu_\alpha\in M(\mathbb{R}^{2d})$ (bounded measures) belongs to M^∞_{1/τ_m}. If $\varphi_1,\varphi_2\in M^1_{v_m}(\mathbb{R}^d)$, then $A_a^{\varphi_1,\varphi_2}$ is bounded by Theorem 3.2. Once again we may use extremely rough symbols for localization operators provided that the windows are sufficiently smooth.

4. Necessary conditions

In this section we show that the sufficient conditions obtained in Theorems 3.2 and 3.4 (for $p=2$) are essentially optimal. This investigation requires a different approach and is based on the following local regularity property of the STFT. Again we formulate a general version for arbitrary submultiplicative weights v and v-moderate weights m.

Lemma 4.1. *Let $1\leqslant p,q\leqslant\infty$. If $f\in M^{p,q}_m(\mathbb{R}^d)$ and $g\in M^1_v(\mathbb{R}^d)$, then $V_gf\in W(\mathscr{F}L^1,L^{p,q}_m)(\mathbb{R}^{2d})$ with norm estimate*

$$\|V_gf\|_{W(\mathscr{F}L^1,L^{p,q}_m)}\lesssim\|f\|_{M^{p,q}_m}\|g\|_{M^1_v}. \tag{22}$$

Proof. Let $\varphi\in\mathscr{S}(\mathbb{R}^d)\backslash\{0\}$ and set $\Phi=V_\varphi\varphi\in\mathscr{S}(\mathbb{R}^{2d})$. We only consider $f,g\in\mathscr{S}(\mathbb{R}^d)$, the extension to arbitrary f,g is done by approximation.

We use the continuous amalgam norm (16) and then Lemma 2.2(a) to evaluate the $W(\mathscr{F}L^1,L^{p,q}_m)$-norm of V_gf.

$$\|V_gf\|_{W(\mathscr{F}L^1,L^{p,q}_m)}$$
$$\asymp\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^{2d}}|\mathscr{V}_{V_\varphi\varphi}(V_gf)(z_1,z_2,\zeta_1,\zeta_2)|\,d\zeta_1\,d\zeta_2\right)^p m(z_1,z_2)^p\,dz_1\right)^{q/p}dz_2\right)^{1/q}$$

$$= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^{2d}} |V_\varphi g(-z_1-\zeta_2,\zeta_1)|\,|V_\varphi f(-\zeta_2, z_2+\zeta_1)|\,d\zeta_1\,d\zeta_2\right)^p \right.\right.$$

$$\left.\times\, m(z_1,z_2)^p\,dz_1)^{q/p}\,dz_2\right)^{1/q}$$

$$= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^{2d}} |V_\varphi f(w_1,w_2)||V_\varphi g(w_1+z_1,w_2+z_2)|\,dw_1\,dw_2\right)^p \right.\right.$$

$$\left.\times\, m(z_1,z_2)^p\,dz_1)^{q/p}\,dz_2\right)^{1/q}$$

$$= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}(|V_\varphi f|^* * |V_\varphi g|)(z_1,z_2))^p m(z_1,z_1)^p\,dz_1\right)^{q/p} dz_2\right)^{1/q}$$

$$= \|\,|V_\varphi f|^* * |V_\varphi g|\,\|_{L^{p,q}_m}.$$

Now Young's theorem for mixed-norm spaces (see, e.g., [23, Proposition 11.1.3]) yields the desired estimate:

$$\|V_g f\|_{W(\mathscr{F}L^1,L^{p,q}_m)} \lesssim \|V_\varphi f\|_{L^{p,q}_m}\|V_\varphi g\|_{L^1_v} \asymp \|f\|_{M^{p,q}_m}\|g\|_{M^1_v}. \qquad \square$$

Lemma 4.1 improves the results in [16,23] which stated that $V_g f \in W(C, L^{p,q}_m)$. Clearly, more general versions of this lemma can be obtained by using other convolution relations in the last step of the proof.

As an immediate consequence we give an alternative proof of the main boundedness result for localization operators (Theorem 3.2). We now formulate it for arbitrary v-moderate weights m in place of the weights v_s and τ_s.

Corollary 4.2. *If* $a \in M^\infty(\mathbb{R}^{2d})$, $\varphi_1, \varphi_2 \in M^1_v(\mathbb{R}^d)$, *then* $A_a^{\varphi_1,\varphi_2}$ *is bounded on* $M^{p,q}_m(\mathbb{R}^d)$ *for* $1 \leqslant p, q \leqslant \infty$. *In particular, it is bounded on* $L^2(\mathbb{R}^d)$.

Proof. *Step* 1. We show first that if $f \in M^{p,q}_m(\mathbb{R}^d)$ and $g \in M^{p',q'}_{1/m}(\mathbb{R}^d)$, then $\overline{V_{\varphi_1} f} \cdot V_{\varphi_2} g \in M^1(\mathbb{R}^{2d})$ with norm

$$\|\overline{V_{\varphi_1} f} \cdot V_{\varphi_2} g\|_{M^1} \lesssim \|\varphi_1\|_{M^1_v}\|\varphi_2\|_{M^1_v}\|f\|_{L^{p,q}_m}\|g\|_{L^{p',q'}_{1/m}}. \tag{23}$$

To see this, we apply Lemma 4.1 and find that $\overline{V_{\varphi_1} f} \in W(\mathscr{F}L^1, L^{p,q}_m)$ and $V_{\varphi_2} g \in W(\mathscr{F}L^1, L^{p',q'}_{1/m})$. Consequently Hölder's inequality (17) implies that $\overline{V_{\varphi_1} f} \cdot V_{\varphi_2} g \in W(\mathscr{F}L^1, L^1)$. Since $W(\mathscr{F}L^1, L^1) = M^1$ by (16), the claim is proved. The norm estimate follows from (22) and (17)

$$\begin{aligned}\|\overline{V_{\varphi_1}f}\cdot V_{\varphi_2}g\|_{M^1} &\asymp \|\overline{V_{\varphi_1}f}\cdot V_{\varphi_2}g\|_{W(\mathscr{F}L^1,L^1)}\\ &\leqslant \|V_{\varphi_1}f\|_{W(\mathscr{F}L^1,L^{p,q}_m)}\|V_{\varphi_2}g\|_{W(\mathscr{F}L^1,L^{p',q'}_{1/m})}\\ &\lesssim \|\varphi_1\|_{M^1_v}\|\varphi_2\|_{M^1_v}\|f\|_{L^{p,q}_m}\|g\|_{L^{p',q'}_{1/m}}.\end{aligned}$$

Step 2. By the previous step the action of $a\in M^\infty$ on $\overline{V_{\varphi_1}f}\cdot V_{\varphi_2}g$ is well-defined and (23) yields the norm estimate for $A_a^{\varphi_1,\varphi_2}$:

$$\begin{aligned}|\langle A_a^{\varphi_1,\varphi_2}f,g\rangle| = |\langle a,\overline{V_{\varphi_1}f}\cdot V_{\varphi_2}g\rangle| &\leqslant \|a\|_{M^\infty}\|\overline{V_{\varphi_1}f}\cdot V_{\varphi_2}g\|_{M^1}\\ &\lesssim \|a\|_{M^\infty}\|\varphi_1\|_{M^1_v}\|\varphi_2\|_{M^1_v}\|f\|_{M^{p,q}_m}\|g\|_{M^{p',q'}_{1/m}}.\end{aligned}$$

Since $M^{p',q'}_{1/m}$ is the dual space of $M^{p,q}_m$ [23, Theorem 11.3.6], this estimate implies that $A_a^{\varphi_1,\varphi_2}$ is bounded on $M^{p,q}_m$ with operator norm at most

$$\|A_a^{\varphi_1,\varphi_2}\|_{\mathrm{op}} \lesssim \|a\|_{M^\infty}\|\varphi_1\|_{M^1_v}\|\varphi_2\|_{M^1_v}$$

independent of p, q, and m. $\square$

In the remainder of this section we derive a converse of Theorem 3.2 and Corollary 4.2. Clearly, if we fix a pair of windows $\varphi_1,\varphi_2\in M^1$, then we cannot expect that $a\in M^\infty$. As an example consider the symbol

$$a := \sum_{|\alpha+\beta|\leqslant k}\frac{\partial}{\partial x^\alpha\partial\omega^\beta}\delta\in\mathscr{E}'(\mathbb{R}^{2d}),$$

and choose two windows $\varphi_1,\varphi_2\in\mathscr{S}(\mathbb{R}^d)$. Then by Proposition 3.6 $A_a^{\varphi_1,\varphi_2}$ is trace class, hence bounded, but $a\notin M^\infty(\mathbb{R}^{2d})$.

However, if we require that $A_a^{\varphi_1,\varphi_2}$ be well-defined and bounded for *all* pairs of windows in a certain class, e.g. in M^1, then we can deduce a converse of Theorem 3.2.

Theorem 4.3. *Let* $a\in\mathscr{S}'(\mathbb{R}^{2d})$ *and* $s\geqslant 0$. *If there exists a constant* $C = C(a) > 0$ *depending only on a such that*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_\infty}\leqslant C\|\varphi_1\|_{M^1_{v_s}}\|\varphi_2\|_{M^1_{v_s}} \tag{24}$$

for all $\varphi_1,\varphi_2\in\mathscr{S}(\mathbb{R}^d)$, *then* $a\in M^\infty_{1/\tau_s}$.

Proof. *Step* 1. We first compute the time–frequency shifts of $\overline{V_{\varphi_1}f}\cdot V_{\varphi_2}g$. Write $z=(z_1,z_2)\in\mathbb{R}^{2d}$, $\zeta=(\zeta_1,\zeta_2)\in\mathbb{R}^{2d}$. Using Lemma 2.1(ii), we obtain

$$\begin{aligned}
&T_{(z_1,z_2)}(\overline{V_{\varphi_1}f}\cdot V_{\varphi_2}g)(x,\omega)\\
&\quad=(\overline{V_{\varphi_1}f}V_{\varphi_2}g)(x-z_1,\omega-z_2)\\
&\quad=\ e^{-2\pi i(\omega-z_2)z_1}\,\overline{V_{\varphi_1}(M_{z_2}T_{z_1}f)}(x,\omega)e^{2\pi i(\omega-z_2)z_1}\,V_{\varphi_2}(M_{z_2}T_{z_1}g)(x,\omega)\\
&\quad=\ \overline{V_{\varphi_1}(M_{z_2}T_{z_1}f)}(x,\omega)\,V_{\varphi_2}(M_{z_2}T_{z_1}g)(x,\omega).
\end{aligned}\tag{25}$$

Similarly, we get

$$\begin{aligned}
M_{(\zeta_1,\zeta_2)}(\overline{V_{\varphi_1}f}V_{\varphi_2}g)(x,\omega)&=e^{2\pi i(x\zeta_1+\omega\zeta_2)}(\overline{V_{\varphi_1}f}\cdot V_{\varphi_2}g(x,\omega))\\
&=\overline{e^{-2\pi i(x\zeta_1+\omega\zeta_2)}\,V_{\varphi_1}f(x,\omega)}\cdot V_{\varphi_2}g(x,\omega)\\
&=\overline{V_{(M_{-\zeta_1}T_{\zeta_2}\varphi_1)}(M_{-\zeta_1}T_{\zeta_2}f)(x,\omega)}\cdot V_{\varphi_2}g(x,\omega).
\end{aligned}\tag{26}$$

The combination of (25) and (26) yields

$$M_\zeta T_z(\overline{V_{\varphi_1}f}V_{\varphi_2}g)=\overline{V_{(M_{-\zeta_1}T_{\zeta_2}\varphi_1)}(M_{-\zeta_1}T_{\zeta_2}M_{z_1}T_{z_2}f)}V_{\varphi_2}(M_{z_1}T_{z_2}g).\tag{27}$$

Step 2. Now fix $\varphi_1,\varphi_2,f,g\in\mathscr{S}(\mathbb{R}^d)$ and set $\Phi:=\overline{V_{\varphi_1}f}\cdot V_{\varphi_2}g$. Then $\Phi\in\mathscr{S}(\mathbb{R}^{2d})$. To verify that $a\in\mathscr{S}'(\mathbb{R}^{2d})$ is in $M^\infty_{1/\tau_s}(\mathbb{R}^{2d})$, we need to show that $\mathscr{V}_\Phi a\cdot\tau_s^{-1}$ is in $L^\infty(\mathbb{R}^{2d})$. Using the weak definition of $A_a^{\varphi_1,\varphi_2}$ and (27), the desired conclusion follows from the estimate

$$\begin{aligned}
|\mathscr{V}_\Phi a(z,\zeta)|&=|\langle a,M_\zeta T_z\Phi\rangle|\\
&=|\langle a,\overline{V_{(M_{-\zeta_1}T_{\zeta_2}\varphi_1)}(M_{-\zeta_1}T_{\zeta_2}M_{z_1}T_{z_2}f)}V_{\varphi_2}(M_{z_1}T_{z_2}g)\rangle|\\
&=|\langle A^{M_{-\zeta_1}T_{\zeta_2}\varphi_1,\varphi_2}(M_{-\zeta_1}T_{\zeta_2}M_{z_1}T_{z_2}f),M_{z_1}T_{z_2}g\rangle|\\
&\leqslant C\|M_{-\zeta_1}T_{\zeta_2}\varphi_1\|_{M^1_{v_s}}\|\varphi_2\|_{M^1_{v_s}}\|M_{-\zeta_1}T_{\zeta_2}M_{z_1}T_{z_2}f\|_2\|M_{z_1}T_{z_2}g\|_2\\
&\lesssim v_s(\zeta)\|\varphi_1\|_{M^1_{v_s}}\|\varphi_2\|_{M^1_{v_s}}\|f\|_2\|g\|_2<\infty,
\end{aligned}$$

for every $(z,\zeta)\in\mathbb{R}^{4d}$. In the last inequality we have used the hypothesis (24) on $A_a^{\varphi_1,\varphi_2}$. Since $v_s(\zeta)=\tau_s(z,\zeta)$, this means that $a\in M^\infty_{1/\tau_s}$ and the necessary condition is proved completely. □

Remark. Condition (24) implies that $A_a^{\varphi_1,\varphi_2}$ can be defined for *all* windows $\varphi_1,\varphi_2\in M^1_{v_s}$. However, we only know that $a\in\mathscr{S}'$, therefore $A_a^{\varphi_1,\varphi_2}$ cannot be defined a priori for arbitrary windows in $M^1_{v_s}$, hence we need the uniform estimate (24). The moral of Theorem 4.3 is that if $A_a^{\varphi_1,\varphi_2}$ is bounded on $L^2(\mathbb{R}^d)$ for *all* windows in $M^1_{v_s}$, then $a \in M^\infty_{1/\tau_s}$.

Finally we show that Theorem 3.4 is optimal in the case of Hilbert–Schmidt operators.

Theorem 4.4. *Let* $a \in \mathscr{S}'(\mathbb{R}^{2d})$. *If there exists a constant* $C = C(a) > 0$ *depending only on a such that*

$$||A_a^{\varphi_1,\varphi_2}||_{S_2} \leqslant C||\varphi_1||_{M^1}||\varphi_2||_{M^1} \tag{28}$$

for all $\varphi_1, \varphi_2 \in \mathscr{S}(\mathbb{R}^d)$, *then* $a \in M^{2,\infty}$.

Proof. Since $\mathscr{V}_\Phi a(z,\zeta) = (a * M_\zeta \Phi^*)(z)$ by Lemma 2.1(i), we need to show that (28) implies that

$$||a||_{M^{2,\infty}} = ||\mathscr{V}_\Phi a||_{L^{2,\infty}} = \sup_{\zeta \in \mathbb{R}^{2d}} ||a * M_\zeta \Phi^*||_2 < \infty$$

for some diligently chosen window $\Phi \in \mathscr{S}(\mathbb{R}^{2d})$.

We choose $\varphi \in \mathscr{S}(\mathbb{R}^d)\backslash\{0\}$ and set $\Phi = W(\varphi,\varphi)^* \in \mathscr{S}(\mathbb{R}^{2d})$. An easy calculation or [23, Proposition 4.3.2] shows that

$$M_\zeta W(\varphi,\varphi)(x,\omega) = W(M_{\frac{\zeta_1}{2}} T_{-\frac{\zeta_2}{2}}\varphi, M_{-\frac{\zeta_1}{2}} T_{\frac{\zeta_2}{2}}\varphi)(x,\omega).$$

for $\zeta = (\zeta_1,\zeta_2) \in \mathbb{R}^{2d}$. To alleviate notation, we set $\varphi_\zeta = M_{\frac{\zeta_1}{2}} T_{-\frac{\zeta_2}{2}}\varphi$ and note that $||M_{\frac{\zeta_1}{2}} T_{-\frac{\zeta_2}{2}}\varphi||_{M^1} = ||\varphi||_{M^1}$ for all $\zeta \in \mathbb{R}^{2d}$.

Finally recall that $||A_a^{\varphi_1,\varphi_2}||_{S_2} = ||a * W(\varphi_2,\varphi_1)||_2$, because $a * W(\varphi_2,\varphi_1)$ is the Weyl symbol of $A_a^{\varphi_1,\varphi_2}$.

Now we compute the $M^{2,\infty}$-norm of a.

$$\begin{aligned} ||a||_{M^{2,\infty}} &\asymp \sup_{\zeta \in \mathbb{R}^{2d}} ||a * M_\zeta W(\varphi,\varphi)||_2 = \sup_{\zeta \in \mathbb{R}^{2d}} ||a * W(\varphi_\zeta,\varphi_{-\zeta})||_2 \\ &= \sup_{\zeta \in \mathbb{R}^{2d}} ||A_a^{\varphi_\zeta,\varphi_{-\zeta}}||_{S_2} \leqslant C \sup_{\zeta \in \mathbb{R}^{2d}} ||\varphi_\zeta||_{M^1}||\varphi_{-\zeta}||_{M^1} = ||\varphi||^2_{M^1} < \infty. \end{aligned}$$

So $a \in M^{2,\infty}$, as announced. □

5. Further Applications — A New Characterization of M^1

In the proof of Theorem 4.3 we have seen that a tempered distribution $a \in \mathscr{S}'(\mathbb{R}^{2d})$ that is defined on certain products $\overline{V_{\varphi_1} f} \cdot V_{\varphi_2} g$ is already in M^∞. This suggests that these products of STFTs already span the predual M^1 of M^∞. In making this observation rigorous, we obtain a new characterization of $M^1(\mathbb{R}^{2d})$, which seems to be of independent interest.

To be specific, we note first that as a special case of (23) we have proved the estimate

$$\|\overline{V_{\varphi_1} f}\, V_{\varphi_2} g\|_{M^1} \lesssim \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1} \|f\|_2 \|g\|_2 \tag{29}$$

for all $\varphi_1, \varphi_2 \in M^1$ and $f, g \in L^2$.

Therefore the following definition makes sense.

Definition 5.1. Let B be the subspace of $L^2(\mathbb{R}^d)$ of all sums

$$f = \sum_{n\in\mathbb{N}} \overline{V_{\varphi_1^n} f_n}\, V_{\varphi_2^n} g_n,$$

such that

$$\sum_{n\in\mathbb{N}} \|\varphi_1^n\|_{M^1} \|\varphi_2^n\|_{M^1} \|f_n\|_2 \|g_n\|_2 < \infty$$

with the norm

$$\|f\|_B = \inf\left\{ \sum_{n\in\mathbb{N}} \|\varphi_1^n\|_{M^1} \|\varphi_2^n\|_{M^1} \|f_n\|_2 \|g_n\|_2 \colon f = \sum_{n\in\mathbb{N}} \overline{V_{\varphi_1^n} f_n}\, V_{\varphi_2^n} g_n \right\}. \tag{30}$$

Then B is a Banach space (see [5] or [18, Theorem 2.1]) and by (29) B is continuously embedded in $M^1(\mathbb{R}^{2d})$.

We will show that in fact $B = M^1$. For this we use Feichtinger's fundamental characterization of M^1 as the minimal Banach space that is isometrically invariant under time–frequency shifts [12]. Precisely, *if $B \subseteq \mathcal{S}'(\mathbb{R}^d)$ is a Banach space such that $M^1 \cap B \neq \{0\}$ and $\|M_\omega T_x f\|_B \leqslant C \|f\|_B$ for all $(x,\omega) \in \mathbb{R}^{2d}$ and all $f \in B$, then $M^1 \subseteq B$* [23, 12.1.9].

Theorem 5.2. *We have*

$$B = M^1(\mathbb{R}^{2d})$$

with equivalent norms.

Proof. Since $B \hookrightarrow M^1$, we need to show the reverse inclusion. In view of the minimality of M^1 quoted above, it suffices to show that time–frequency shifts are uniformly bounded on B.

(i) Let $F \in B$ and choose a representation $f = \sum_{n\in\mathbb{N}} \overline{V_{\varphi_1^n} f_n}\, V_{\varphi_2^n} g_n$, such that

$$\sum_{n\in\mathbb{N}} \|\varphi_1^n\|_{M^1} \|\varphi_2^n\|_{M^1} \|f_n\|_2 \|g_n\|_2 \leqslant 2\, \|f\|_B.$$

Using Lemma 2.1(ii) and (25) we have

$$T_{(z_1,z_2)}F = T_{(z_1,z_2)} \sum_{n\in\mathbb{N}} \overline{V_{\varphi_1^n} f_n} V_{\varphi_2^n} g_n = \sum_{n\in\mathbb{N}} \overline{V_{\varphi_1^n}(M_{z_2}T_{z_1}f_n)} V_{\varphi_2^n}(M_{z_2}T_{z_1}g_n).$$

Since time–frequency shifts are isometric isomorphisms on M^1 and L^2, we find that

$$||T_zF||_B \leqslant \sum_{n\in\mathbb{N}} ||\varphi_1^n||_{M^1}||\varphi_2^n||_{M^1}||M_{z_2}T_{z_1}f_n||_2||M_{z_2}T_{z_1}g_n||_2 \leqslant 2||f||_B.$$

(ii) Likewise for modulations we get

$$M_{(\zeta_1,\zeta_2)}F = e^{2\pi i(x\zeta_1+\omega\zeta_2)} \sum_{n\in\mathbb{N}} \overline{V_{\varphi_1^n} f_n} V_{\varphi_2^n} g_n = \sum_{n\in\mathbb{N}} \overline{V_{(M_{-\zeta_1}T_{\zeta_2}\varphi_1^n)}(M_{-\zeta_1}T_{\zeta_2}f_n)} V_{\varphi_2^n} g_n.$$

The same arguments as in (i) show that $||M_\zeta F||_B \leqslant 2||F||_B$. This suffices to conclude that M^1 is continuously embedded in B. □

Remark. Since STFTs behave in many aspects like analytic functions, as is expressed in Proposition 2.5, Theorem 5.2 bears a striking resemblance to the well-known factorization theorems for the Hardy space H^1: for instance, on the torus, a result of F. Riesz states that $f \in H^1$ if and only if $f = \sum_{n=1}^\infty g_n h_n$ with $g_n, h_n \in H^2$ and $\sum_{n=1}^\infty ||g_n||_{H^2}||h_n||_{H^2} < \infty$, see [7].

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While finishing the manuscript, we received a preprint of J. Toft who derives a special case of our Theorems 2.4 and 3.2 with different methods [33].

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