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Foreword

In 1946, D. Gabor suggested using a regularly spaced, discrete set of time and frequency translates of a single gaussian as a basis for expanding square-integrable functions. Such an expansion can be thought of intuitively as a type of Fourier decomposition, in which the coefficients measure (broadly speaking) the frequency components of the part of the function contributed by a particular interval of time—"the musical score", in N.C. deBruijn's evocative term. This far-reaching idea, forms of which arose independently in quantum mechanics and electrical engineering, has been extensively developed in recent years. A major force behind the current advances has been the general theory of frames—systems of elements, like Gabor's, into which an arbitrary element in a Hilbert space can be stably expanded, generally not uniquely.

The interest in frames formed by translates in time and frequency of a single function stems from both practice and theory, and has grown in range and sophistication. On the abstract side, as proposed in 1987 by R. Howe and T. Steger, questions about them can be formulated in terms of operator algebras. The view from Fourier analysis suggests uncertaintly principles that govern how well functions can be localized in time and in frequency. The desire to use such systems in practical computation leads to questions about their density, rates of convergence, and quality of approximation. There are also new applications.

Many of these aspects and approaches are discussed in the present volume, which gives a comprehensive picture of the current state of this area.

Henry Landau
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Introduction

Hans G. Feichtinger and Thomas Strohmer

Within the last decade, if not earlier, Gabor analysis has established itself as a rich and fertile field of mathematical analysis.

Both the understanding of the mathematical foundations and the efficiency of algorithms related to Weyl–Heisenberg families, whether frames or Riesz bases, have reached a level so that it makes sense to investigate applications of all kinds. Various new directions within Gabor analysis arose from the increased interest of mathematicians working in other areas such as pseudodifferential operators, operator algebras, or in applications such as wireless communication. In many cases Gabor systems are used, often under different names, in an ad hoc manner, and closer investigations of possible links to the systematic theory of Gabor analysis are likely to open new possibilities. At the same time, as part of a broad and systematic development of the field, various fundamental questions of Gabor analysis have been tackled recently—questions that have not been considered thus far.

After the “first Gabor book” *Gabor Analysis and Algorithms: Theory and Applications* (H. G. Feichtinger and T. Strohmer, eds., Birkhäuser, Boston, 1998) and the beautiful, self-contained description of the mathematical foundations of Gabor analysis by K. Gröchenig (*Foundations of Time-frequency Analysis*, Birkhäuser, Boston, 2001), the present book documents some of the ongoing research activities and provides insight into the richness of the field and its potential for further development. In discovering connections to established theories and relevant new applications, an exciting future of Gabor analysis can be predicted for years to come.

Gabor analysis has frequently become the meeting point for interdisciplinary research activities because it combines mathematical breadth and openness toward a variety of applications outside of mathematics. We hope that the reader will appreciate the efforts of the authors involved in this volume to support these claims. The book contains survey chapters as well as a number of results that have not been published previously. We are confident that both experts and novices will find the book useful for their work.

1.1 Recent Trends in Gabor Analysis

To give our readers an idea of some interesting recent developments in Gabor analysis, let us describe in a nutshell a (certainly subjective) selection of problem circles which have come to our attention during the last two years. The diversity of topics, many of which are linked to directions covered by one of the chapters, is another indication for the potential of this branch of mathematical analysis.

Following the majority of publications on the subject in the last several years, one may come to the conclusion that the central problem of Gabor analysis is to find out whether or not a Weyl–Heisenberg family generated from a fixed Gabor atom g along a time-frequency lattice of the form $a\mathbb{Z}^d \times b\mathbb{Z}^d$ is a frame in $L^2(\mathbb{R}^d)$. Usually one has to assume that g satisfies some regularity conditions and that the lattice constants (a, b) are sufficiently small in order to obtain a positive answer. Likewise, one may ask whether a triple (g, a, b) generates a Riesz basis for its closed linear span. One discovers that this is true under the same conditions on g for sufficiently large lattice constants (a, b). Admittedly, finding a complete description for some fixed atom g, or even better, a comprehensive class of atoms, is typically a mathematically challenging problem. So far it has been solved only for the Gaussians (frames occur if and only if $ab < 1$) and a few other special cases.

Nevertheless this question should be seen as only *one of the basic questions* in the field, but *not* as the most essential problem. Indeed, if no time-frequency concentration of the overall Gabor system is required (i.e., the Gabor atom and its dual) it is easy to generate orthonormal bases of Weyl–Heisenberg type, e.g., using the box- or the sinc-function. This does not contradict the Balian–Low principle, which only implies that orthonormal bases of Weyl–Heisenberg type cannot have a good uniform time-frequency concentration.

However, if we go back to Gabor's original suggestions of 1946 concerning the choice of the Gaussian function, it is evident that he was motivated by the desire to use building blocks with optimal time-frequency concentration. Furthermore, he was apparently *hoping for uniqueness of coefficients* by choosing the Neumann lattice ($a = b = 1$), so "the" Gabor coefficients would be a very natural object for the interpretation of a signal. What has been overlooked in this approach is that the price to be paid in this situation is high instability (or, more precisely, the violation of both the frame and the Riesz basis property) and even worse, the fact that the behavior of Gabor coefficients (if they are well defined at all) is *not local* in a time-frequency sense. This problem is due to the properties of the dual function (in the sense of Bastiaans) which is notoriously nasty (not even in L^2), having very little smoothness and also poor decay.

From this point of view one may speculate that Dennis Gábor[1] himself would have been more satisfied with the situation that occurs by using a Gaussian for $ab < 1$, in conjunction with the use of the canonical dual $\widetilde{g}$. In this situation the uniqueness of the coefficients is enforced by the minimal norm property of the (canonical) frame coefficients. Since $\widetilde{g}$ is a Schwartz function as well, one has a very good time-frequency locality of the overall system and the Gabor coefficients actually describe the local time-frequency behavior of the functions that are analyzed. Hence, although Gabor systems, for which both g and $\widetilde{g}$ are well concentrated in the time-frequency sense, cannot be a (Riesz) basis for $L^2(\mathbb{R}^d)$ they are a lot more useful in many cases than orthonormal bases of box-functions or a system of Gaussians at critical density, i.e., with $a = b = 1$.

For a more refined description of smoothness (in the sense of Sobolev norms) or decay of functions, a suitable family of function spaces is needed. In the last few years it has turned out that the family of *modulation spaces* plays the analogous role in the context of Gabor analysis as the family of Besov and Triebel–Lizorkin spaces with respect to wavelet expansions. One may express this fact loosely by saying that the membership of functions or distributions in such modulation spaces can be completely described by the behavior of their Gabor coefficients with respect to "any good" Gabor frame. A precise mathematical description of this situation is possible using the concepts of Banach frames and Riesz projection bases respectively, which can be applied to families of vectors in Banach spaces (not only to Hilbert spaces, as the standard frame concept), for example, to the family of modulation spaces in our case. In the context of tempered distributions it is sufficient to use Gabor atoms, which belong to the Schwartz class themselves together with their dual atom. Such systems allow one to characterize the membership of a function f in the classical Schwartz space by polynomial decay (of any order) of its Gabor coefficients, just to give a typical example.

That the frame property (usually formulated in the L^2-context) already automatically implies that the dual atom $\widetilde{g}$ has essentially the same time-frequency concentration as the atom g itself, expressed by a weighted L^1-condition of its ambiguity function over phase space, is one of the important recent results in Gabor analysis (in its general form due to Gröchenig and Leinert).

There is also another important branch of Gabor analysis for which modulation spaces and their variants have become a natural tool, i.e., the theory of *pseudodifferential operators*. After first results studying the mapping properties of classical pseudodifferential operators between modulation spaces, nowadays various kinds of symbols are described according to

[1] In contrast to the usual spelling of the word "Gabor" as a technical term as in "Gabor analysis" the original Hungarian name is spelled with an accent, i.e., Gábor.

their membership in modulation spaces. The improvement of the classical Calderon–Vaillancourt theorem, due to Heil and Gröchenig, is certainly one of the highlights in this direction.

If we look closer at the situation encountered with "oversampled" (but time-frequency well-concentrated) Gabor systems, we recognize that despite the lack of linear independence, such "good" Gabor frames are not only suitable for the characterization of modulation spaces, but usually go hand in hand with a lot of robustness. One can show that they exhibit a certain stability toward small (uniform) jitter errors occurring at all the sampling points. Even more surprising, one can show that the dual atoms depend continuously on the lattice parameters. At the same time, various authors have begun to study more carefully the excess of Gabor frames, showing among other things that the terminology of "oversampling" is justified in the sense that it is possible to remove even infinite subfamilies from such a Gabor frame while preserving the frame property. The frames which are constituted by the remaining families are typical examples of irregular Gabor families, because the parameter set which is used to generate these frames in a coherent way is usually not a lattice anymore. Recently there has been much interest again in the generic case: Gabor families generated by well-spread point families in the time-frequency plane, sufficient conditions for forming a Gabor frame under various conditions, or stability investigations, leading to the observation that in most cases it is possible to allow some small uniform jitter error without losing the frame property.

There are many good arguments to look at Gabor analysis from an *abstract harmonic analysis* point of view. The basic concepts of time-frequency analysis in general are two commutative groups of operators, the time- and the frequency-shifts. Since these concepts are well defined for functions on arbitrary locally compact abelian (LCA) groups G, it is natural to describe Gabor analysis in the context of LCA groups, and to replace the time-frequency plane by the abstract *phase space* $G \times \widehat{G}$, where $\widehat{G}$ is the dual group. For example, the Janssen representation and the Walnut representation of the frame operator are indeed valid in this more general context. However, one should distinguish between results which are obtained from the corresponding statements for the Euclidean case just by a kind of transcription, and those which are becoming intrinsically harder, when formulated in this generality.

One of the benefits of the more abstract point of view is the fact that certain structural properties become more transparent. For example, the appearance of the lattice constants $(1/b, 1/a)$ can be understood, once it is clear that they generate the adjoint time-frequency lattice. On the other hand, the adjoint lattice Λ° is well defined for any lattice Λ in phase space. Moreover, this abstract understanding of Gabor analysis is even attractive for engineers: it avoids a repetitive treatment of the case of one and several variables, discrete or continuous, or the finite, i.e., discrete and periodic case.

In addition we mention that the more general context also naturally suggests formulating results in Gabor analysis for arbitrary lattices Λ within phase space, and not just for product lattices (often referred as "the separable case"), which are of the form $a\mathbf{Z}^d \times b\mathbf{Z}^d$, or more generally, $A\mathbf{Z}^d \times B\mathbf{Z}^d$ for invertible $d \times d$ matrices A and B. Again, such a viewpoint enables one to see things in their natural context, but also increases the freedom to design Gabor systems by allowing more flexibility in the choice of lattices.

Within Gabor analysis various interesting *numerical problems* arise. An important context is finite-dimensional Gabor analysis, i.e., the expansion of a vector of finite length into a double sum of Gabor atoms by choosing appropriate coefficients. Until a few years ago this problem had been considered a *numerically intensive* task, because of the non-orthogonality of Weyl–Heisenberg families. But meanwhile a deeper understanding of the structure of this problem has enabled the development of fast algorithms. Although Gabor families with good time-frequency concentration have to be linearly dependent, this does not necessarily imply that one is dealing with a "general linear problem" requiring the calculation of the pseudo-inverse of a huge matrix. On the contrary, the dual frame for a Gabor frame is known to be itself a Gabor frame, determined by the *dual Gabor atom* $\widetilde{g}$, which is the solution to the linear equation $S\widetilde{g} = g$, where S is the positive definite frame operator. This leads to a highly structured linear system of equations (due to the commutation relations satisfied by the frame operator), for which a number of interesting algorithmic approaches exist. Furthermore, the determination of the corresponding tight Gabor atom $h = S^{-1/2}g$ can be carried out in a numerically efficient way nowadays.

This situation allows us to carry out Gabor analysis for real-world signals (even two-dimensional "signals," images) with standard mathematical software, and to demonstrate the practical consequences of theoretical observations. It is also possible to implement operators such as Gabor multipliers or time-variant filters efficiently using these tools.

At the same time it has happened several times that observations resulting from numerical experiments have lead to the discovery of structural properties of Gabor systems. We believe that even more of these numerical experiments need to be carried out in the future, and that algorithmic questions (for example, in the direction of real-time applications) should play a larger role in the field in the near future.

Some of the most interesting recent applications of Gabor analysis are in the area of digital and wireless communication. For instance, the different ways to express the duality of Gabor families and the Ron–Shen principle establish deep connections between Gabor frame theory and Orthogonal Frequency Division Multiplexing (OFDM), which is an important transmission scheme for wireless communication. According to the Balian–Low principle one cannot have at the same time minimal redundancy of Gabor systems and good time-frequency localization. Thus the design of low redundancy Gabor systems is intimately related to the design of OFDM systems of high spectral efficiency.

That this is not only true "in principle" but that advanced methods from Gabor analysis may serve as a basis for the optimization of OFDM transmission pulses with significant performance gains has been verified by the recent release of a new short radio wave communication system of which we are aware.

Existing qualitative results on the time-frequency localization of Gabor frames and their associated canonical dual and tight frames certainly provide useful guidelines in applications. On the other hand, one has to admit that the mere existence of "some positive constant $C > 0$ such that ... " may often be insufficient information in order to prefer a given Gabor system over some other. For this reason we believe that it will be necessary to refine current time-frequency localization results to obtain more quantitative estimates for wide classes of Gabor systems, not only the very few ones which allow a complete analytic treatment. Let us recall in this context that most results concerning localization are concerned with canonical dual (and tight) frames, whereas in practice it may be advantageous to use alternative dual frames satisfying additional constraints that are perhaps not satisfied by the canonical dual frame (such as a fixed support length).

Examples like those given above and described in more detail in this book are typical of a situation where the *exchange* between application areas, and pure and applied mathematics is working out in a mutually fruitful way. Many questions arising from the practical problems lead to interesting theoretical issues, sometimes even to fundamental mathematical problems. On the other hand, some of the most theoretical parts of mathematics can be used to derive results in Gabor analysis which are relevant for certain applications and become useful later in the design of actual technical systems.

1.2 Outline of the Book

Heisenberg's Uncertainty Principle is at the core of time-frequency analysis and is one of the driving forces behind Gabor analysis. In Chapter 2, Karlheinz Gröchenig demonstrates that every time-frequency representation comes with its own version of the uncertainty principle. The classical uncertainty principle is an inequality that involves both a function f and its Fourier transform $\hat{f}$. Gröchenig develops a general approach to deriving new uncertainty principles by interpreting the pair $(f, \hat{f})$ as a time-frequency representation, replacing it by a different time-frequency representation such as the Wigner distribution and deriving the corresponding inequality. He shows that this approach can be extended to other classical uncertainty principles such as Hardy's theorem.

When does a triple (g, a, b), with $g \in L^2(\mathbb{R})$, time shift parameter a and frequency shift parameter b generate a Gabor frame? This is one of the basic, yet most difficult problems in Gabor theory. Even for windows as "elementary" as the Gaussian, hyperbolic secants, or (one-sided or two-sided) exponentials, this problem has proven to be highly non-trivial. In Chapter 3, A.J.E.M. Janssen is able to identify three classes of windows that do lead to Gabor frames under rather general conditions. The first two classes comprise non-negative windows that satisfy certain decay and convexity conditions, respectively. The crucial fact is that both window classes have Zak transforms which have only a few zeros per unit square. In the second part of Chapter 3, Janssen considers the difficult question for which a, b, c generates (χ_c, a, b) a Gabor frame where χ_c is the characteristic function on the interval $[0, c]$. A graphical illustration of the author's finding exhibits an appealing shape which is known as *Janssen's tie.*

In Chapter 4, Pete Casazza and Mark Lammers analyze the properties of the *bracket product* of two $L^2(\mathbb{R})$-functions f, g defined by

$$\langle f, g \rangle_a = \sum_{n \in \mathbb{Z}} f(t - na)\overline{g(t - na)}.$$

The authors show that, similar to the standard inner product, this inner product gives rise to a Bessel inequality, a Riesz representation theorem, and a Gram–Schmidt orthogonalization procedure. The authors then apply this to show that all operators associated to a Gabor system have a "compression" with regard to this function-valued inner product. It turns out that for this bracket product there is an equivalence between frames of translates and Gabor frames.

Chapters 5–7 focus on the interplay between Gabor analysis and operator theory. A variety of procedures in signal processing, such as time-frequency localization or thresholding algorithms, involve the pointwise multiplication of Gabor coefficients by some (characteristic) function. The corresponding linear operators are called Gabor multipliers. In Chapter 5, Hans Georg Feichtinger and Krzysztof Nowak present an introduction to the theory of Gabor multipliers. They describe how the properties of these operators are governed by the decay of the multiplier sequence, the time-frequency concentration properties of the Gabor window, and the time-frequency-lattice. These properties are described in terms of the mapping properties of the corresponding Gabor multiplier between modulation spaces, or membership in some operator ideal. Furthermore, the authors characterize the eigenvalue behavior of such operators.

"Aspects of Gabor Analysis and Operator Algebra" is the title of the chapter by Jean-Pierre Gabardo and Deguang Han. In this chapter the authors use the connection between operator algebras generated by translation and modulation operators and Gabor analysis to study properties of Gabor frames. Using methods from von Neumann algebra theory they are able to derive conditions for a square-integrable function to generate a subspace Gabor frame and to characterize those subspace Gabor frames that admit a unique dual Gabor frame.

Pseudodifferential operators arise naturally in a variety of areas in mathematics, science, and engineering, including partial differential equations, quantum mechanics, and signal processing. Chapter 7, by Chris Heil, illustrates that Gabor frames provide a convenient tool for analyzing spectral properties of pseudodifferential and integral operators. Among other things he derives a sufficient condition of an integral operator or the symbol of a pseudodifferential operator which implies that the operator is trace-class. His results improve classical results due to Hörmander and Daubechies.

When using frames in applications we are confronted with the problem of the numerical computation of the inverse frame operator. Since in general the frame operator is acting in an infinite-dimensional Hilbert space, we have to approximate its inverse by finite-dimensional methods. In Chapter 8, Ole Christensen and Thomas Strohmer review and extend some methods for approximation of the inverse frame operator. Furthermore, they develop more specific methods for the computation of dual and tight Gabor frames.

The Balian–Low theorem implies that we cannot construct Gabor Riesz bases for $L^2(\mathbb{R})$ with good time-frequency localization. This limitation leads to the introduction of Wilson bases for $L^2(\mathbb{R})$. In applications we have to work with signals of finite length, i.e., signals defined on an interval. In Chapter 9, Kai Bittner investigates the construction of biorthogonal Wilson bases on the interval. His construction is based on a Zak transform for periodic functions and an unfolding operator for periodic Wilson bases. He also presents fast numerical algorithms for constructing these Wilson bases and their duals.

Chapter 10, by Jean-Pierre Antoine and Fabio Bagarello, is a fascinating application of time-frequency analysis and wavelet theory to quantum mechanics. Uncertainty relations constitute one of the most characteristic features of a quantum system. As an illustration of the phase space localization problem of quantum mechanical systems, Antoine and Bagarello study a two-dimensional electron gas in a magnetic field, such as encountered in the Fractional Quantum Hall Effect (FQHE). They analyze a general procedure for constructing an orthonormal basis for the lowest Landau level (i.e., the lowest energy level), and discuss the phase space localization problems encountered in this context. The authors then show how wavelet theory can help overcome some of these limitations. Detailed examples from wavelet analysis, such as the Littlewood–Paley and splines bases, are given. Furthermore, the authors exhibit a striking equivalence between FQHE states and basis vectors stemming from a multiresolution analysis.

Chapters 11 and 12 are concerned with the use of Gabor systems in important applications such as digital signal processing and wireless communication. In Chapter 11, Radu Balan and Ingrid Daubechies analyze two classes of optimization problems concerning the interaction between stochastic processes and Gabor systems. In the first case, signal encoding for transmission in noisy channels, the authors investigate how well a function f can be approximated from its noisy Gabor coefficients for optimally chosen (dual pairs of) window functions. In the second case, the signal approximation problem, they analyze how well a stochastic signal can be approximated by a Gabor-type expansion. The right functional spaces to study these problems are identified. The authors give explicit solutions in the Zak transform domain. These two problems are closely related and it turns out that the optimizer in both cases is generically ill-localized, similar to the Balian–Low phenomenon. A number of numerical examples are provided.

One of the most fascinating recent applications of Gabor theory is in the area of wireless communication. In Chapter 12, Helmut Bölcskei uses Gabor theory to analyze and design transmission systems for mobile wireless communication. One of the key factors determining the performance of wireless Orthogonal Frequency Division Multiplexing (OFDM) systems is the time-frequency localization of the transmitter and receiver pulse shaping filters. OFDM based on offset quadrature amplitude modulation (OQAM) circumvents the disadvantage of a standard OFDM system, namely that it cannot simultaneously provide maximal spectral efficiency and time-frequency well-localized pulse shapes (due to the Balian–Low theorem). The author establishes relations between OQAM/OFDM and Wilson and Gabor expansion. The pulse shaping filter design problem is studied in detail and several numerical examples are provided.

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