

# A Deformation Quantization Theory for Non-Commutative Quantum Mechanics

Nuno Costa Dias    Maurice de Gosson  
Franz Luef    João Nuno Prata

April 9, 2010

## Abstract

We show that the deformation quantization of non-commutative quantum mechanics previously considered by Dias and Prata can be expressed as a Weyl calculus on a double phase space. We study the properties of the star-product thus defined, and prove a spectral theorem for the star-genvalue equation using an extension of the methods recently initiated by de Gosson and Luef.

**Mathematics Subject Classification (2000):** 47G30, 81S10

**Keywords:** deformation quantization; non-commutative quantum mechanics; Weyl operators, spectral properties.

## 1 Introduction

The incorporation of space-time non-commutativity in quantum field theory was considered originally by Snyder [46, 47], Heisenberg [32], Pauli [40] and Yang [50] as a means to regularize the ubiquitous divergences in quantum field theories. However, the development of renormalization techniques and certain undesirable features of non-commutative field theories - such as the breakdown of Lorentz invariance - have hindered further research in this direction. It was only more recently that new interest in the topic has emerged, due to important developments in various approaches to the quantization of gravity. The most prominent result was the discovery that the low energy

effective theory of a D-brane in the background of a Neveu-Schwarz-Neveu-Schwarz B field lives on a space with spatial non-commutativity [14, 23, 43]. This result reinforces the widely accepted idea that the Planck length constitutes a lower bound on the precision of a measurement of position [42].

To implement these ideas at the level of quantum field theory, one usually assumes [4, 15, 17, 38, 49] that the space-time coordinates do not commute, either in a canonical way

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad (1)$$

or in a Lie-algebraic way

$$[x^\mu, x^\nu] = iC_\beta^{\mu\nu} x^\beta, \quad (2)$$

where  $\theta^{\mu\nu}$  and  $C_\beta^{\mu\nu}$  are real constants for  $\mu, \nu, \beta = 0, 1, \dots, d$ . In general one chooses  $\theta^{0i} = 0$  (or  $C_\beta^{0i} = 0$ ) (i.e. time is an ordinary commutative parameter) to avoid problems with the lack of unitarity.

Equations (1,2) can be regarded as a particular case of the more general deformation  $[x^\mu, x^\nu] = if^{\mu\nu}(x)$ , where  $f^{\mu\nu}(x)$  are real functions of  $x$  such that antisymmetry and the Jacobi identity are respected [15]. This occurs when the laboratory frame has a space-time dependent motion. As suggested in [34] it is likely that  $f^{\mu\nu}$  have a fixed value in the cosmic microwave background radiation frame, which may be considered as approximately fixed in the celestial sphere. For this reason we shall adopt the simpler canonical situation (1) in this work. Moreover, there is an additional bonus. This choice will simplify drastically the theory in technical terms, since - as we shall see - one is able to construct symplectomorphisms to the usual canonical (Heisenberg) algebra.

Adopting this point of view many authors have addressed the problem of testing the existence of space-time non-commutativity in nature by resorting to a (non-relativistic, one particle) quantum mechanical approximation to the full quantum field theory. This generalization of quantum mechanics obtained by considering canonical extensions of the Heisenberg algebra is usually referred to as non-commutative quantum mechanics (NCQM) [27]. Many physical applications of NCQM have been considered, in particular the quantum Hall effect [16, 22, 48], quantum mechanics on the non-commutative sphere [39], gravitational quantum well [10], non-commutative supersymmetric quantum mechanics [31], non-commutative Chern-Simmons theory [21, 48], the Landau problem [33], and the hydrogen atom [13].

We shall thus consider time to be a commutative parameter and the remaining spatial coordinates to be canonically non-commutative. We shall also assume non-commuting momenta. There are several reasons for this. First of all, it seems more symmetrical. But, perhaps more importantly, there are various instances where momentum non-commutativity leads to important effects. In [7, 10] it was shown that momentum non-commutativity yields larger corrections than the configurational counterpart, and it is therefore more susceptible to experimental verification. Moreover, recent results in the context of non-commutative quantum cosmology [4, 5, 6] have revealed that momentum non-commutativity may be a decisive ingredient for the solution of the singularity problem of black holes.

Here and in a series of papers, we will develop various mathematical aspects of NCQM, notably the deformation quantization, spectral results [28], the geometry of the uncertainty principle [11, 35, 36] and the connection with modulation spaces [24, 30].

In Dias and Prata [2, 3] two of us have discussed various aspects of NCQM related to Flato–Sternheimer deformation quantization [8, 9]. In this paper we propose an operator theoretical approach, based on previous work de Gosson and Luef [28] (the remaining two of us). In that article it was shown that the Moyal–Groenewold product  $a \star b$  of two functions on  $\mathbb{R}^{2n}$  can be interpreted in terms of a Weyl calculus on  $\mathbb{R}^{2n}$ . In fact,

$$a \star b = \tilde{A}b \tag{3}$$

where  $\tilde{A}$  is the phase space operator with Weyl symbol  $\tilde{a}$  defined on  $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$  by

$$\tilde{a}(z, \zeta) = a(z - \frac{1}{2}J\zeta) \tag{4}$$

( $J$  is the standard symplectic matrix).

In this paper we show that this redefinition of the starproduct can be modified so that it leads to a natural notion of deformation quantization for the NCQM associated with an antisymmetric matrix of the type

$$\Omega = \begin{pmatrix} \hbar^{-1}\Theta & I \\ -I & \hbar^{-1}N \end{pmatrix} \tag{5}$$

where  $\Theta$  and  $N$  measure the non-commutativity in the position and momentum variables, respectively. We define a new starproduct  $\star_\Omega$  by replacing formula (3) by

$$a \star_\Omega b = \tilde{A}_\Omega b \tag{6}$$

where  $\tilde{A}_\Omega$  is the operator with Weyl symbol

$$\tilde{a}_\Omega(z, \zeta) = a(z - \frac{1}{2}\Omega\zeta). \quad (7)$$

Of course (7) reduces to (3) when  $\Theta = N = 0$ .

In this article we are going to rigorously justify the definition above and study the properties of this new starproduct  $\star_\Omega$ . The difficulty associated with the fact that the symplectic form associated with  $\Omega$  depends on  $\hbar$  will be resolved (we will show that  $a \star_\Omega b$  is well-defined as a starproduct thanks to supplementary conditions on  $\Theta$  and  $N$  which are physically meaningful). In fact  $\star_\Omega$  coincides with the starproduct defined (in terms of the generalized Weyl-Wigner map [18]) in Eqn. (21) of [2], and where it was shown that it is related to the standard starproduct. (In the same paper it was concluded in Eqn. (53) that the generalized starproduct between two polynomials can be represented as a kind of ‘‘Bopp shift’’ which also turns out to be identical with  $\star_\Omega$ ).

**Notation 1** *The generic point of phase space  $\mathbb{R}^{2n}$  is denoted  $z = (x, p)$ . We denote by  $\text{Sp}(2n, \mathbb{R})$  the standard symplectic group, defined as the group of linear automorphisms of  $\mathbb{R}^{2n}$  equipped with the symplectic form  $\sigma(z, z') = Jz \cdot z'$ ,  $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ . We use the standard notation  $\mathcal{S}(\mathbb{R}^m)$  and  $\mathcal{S}'(\mathbb{R}^m)$  for the Schwartz space of test functions on  $\mathbb{R}^m$  and its dual.*

## 2 Description Of the Problem

Let us begin by explaining what we mean by non-commutativity in the present context. The study of non-commutative field theories and their connections with quantum gravity (see [1, 17, 20, 43, 49] and the references therein) leads to the consideration of commutation relations of the type

$$[\tilde{z}_\alpha, \tilde{z}_\beta] = i\hbar\omega_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2n \quad (8)$$

where  $\Omega = (\omega_{\alpha\beta})_{1 \leq \alpha, \beta \leq 2n}$  is the  $2n \times 2n$  antisymmetric matrix defined by (5) where  $\Theta = (\theta_{\alpha\beta})_{1 \leq \alpha, \beta \leq n}$  and  $N = (\eta_{\alpha\beta})_{1 \leq \alpha, \beta \leq n}$  are antisymmetric matrices measuring the non-commutativity in the position and momentum variables. We have set here  $\tilde{z}_\alpha = \tilde{x}_\alpha$  if  $1 \leq \alpha \leq n$  and  $\tilde{z}_\alpha = \tilde{p}_{\alpha-n}$  if  $n+1 \leq \alpha \leq 2n$ ,

where

$$\tilde{x}_\alpha = x_\alpha + \frac{1}{2}i \sum_\beta \theta_{\alpha\beta} \partial_{x_\beta} + \frac{1}{2}i\hbar \partial_{p_\alpha} \quad (9)$$

$$\tilde{p}_\alpha = p_\alpha - \frac{1}{2}i\hbar \partial_{x_\alpha} + \frac{1}{2}i \sum_\beta \eta_{\alpha\beta} \partial_{p_\beta}. \quad (10)$$

It turns out that, as proved in [2],  $\Omega$  is invertible if

$$\theta_{\alpha\beta} \eta_{\gamma\delta} < \hbar^2 \quad \text{for } 1 \leq \alpha < \beta \leq n \text{ and } 1 \leq \gamma < \delta \leq n. \quad (11)$$

We will assume from now on that these conditions are satisfied; that this requirement is physically meaningful is well-known (it is fulfilled for instance in the case of the non-commutative quantum well; see for instance [10, 12]). Since we will be concerned with a deformation quantization with parameter  $\hbar \rightarrow 0$  we will furthermore assume that  $\Theta$  and  $N$  depend smoothly on  $\hbar$  in such a way that

$$\Theta(\hbar) = o(\hbar^2) \quad \text{and} \quad N(\hbar) = o(\hbar^2) \quad (12)$$

(recall that  $f(\hbar) = o(\hbar^m)$  means that  $\lim_{\hbar \rightarrow 0} (f(\hbar)/\hbar^m) = 0$ ). We thus have

$$\lim_{\hbar \rightarrow 0} \Omega = J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

(the standard symplectic matrix). It turns out that the conditions (12) are compatible with numerical results in [10, 12] where it is shown that the estimates  $\theta \leq 4 \times 10^{-40} m^2$  and  $\eta \leq 1.76 \times 10^{-61} kg^2 m^2 s^{-2}$  hold. Moreover, the analysis of non-commutative quantum mechanics in the context of dissipative open systems, reveals that a transition  $\theta \rightarrow 0$  occurs prior to  $\hbar \rightarrow 0$  [19].

These facts, and the theory developed in [28], suggests that we represent  $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_{2n})$  by the vector operator

$$\tilde{z} = z + \frac{1}{2}i\hbar \Omega \partial_z \quad (13)$$

which acts on functions defined on the phase space  $\mathbb{R}^{2n}$ . Notice that the conditions (12) show that in the limit  $\hbar \rightarrow 0$  we have the asymptotic formulae

$$\tilde{x}_\alpha = x_\alpha + \frac{1}{2}i\hbar \partial_{p_\alpha} + o(\hbar^2) \quad , \quad \tilde{p}_\alpha = p_\alpha - \frac{1}{2}i\hbar \partial_{x_\alpha} + o(\hbar^2). \quad (14)$$

The ‘‘quantization rules’’ (13) lead us to the consideration of pseudo-differential operators formally defined by (7).

The underlying symplectic structure we are going to use is defined as follows. We will denote by  $s$  a linear automorphism of  $\mathbb{R}^{2n}$  such that  $\sigma = s^*\omega$ ; equivalently  $sJs^T = \Omega$ . Thus  $s$  is a symplectomorphism  $s : (\mathbb{R}^{2n}, \sigma) \rightarrow (\mathbb{R}^{2n}, \omega)$ . Note that the mapping  $s$  is sometimes called the ‘‘Seiberg–Witten map’’ in the physical literature; its existence is of course mathematically a triviality (because it is just a linear version of Darboux’s theorem, see [26], §1.1.2). Writing  $s$  in block-matrix form  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$  the condition  $sJs^T = \Omega$  is equivalent to

$$AB^T - BA^T = \hbar^{-1}\Theta \quad , \quad CD^T - DC^T = \hbar^{-1}N \quad , \quad AD^T - BC^T = I.$$

Of course, the automorphism  $s$  is not uniquely defined: if  $s^*\omega = s'^*\omega$  then  $s^{-1}s' \in \text{Sp}(2n, \mathbb{R})$ . Also note that in the limit  $\hbar \rightarrow 0$  the matrices  $\hbar^{-1}\Theta$  and  $\hbar^{-1}N$  vanish and  $s$  becomes, as expected, symplectic in the usual sense, that is  $s \in \text{Sp}(2n, \mathbb{R})$ .

### 3 Definition of the starproduct $\star_\Omega$

Let  $\omega$  be the symplectic form on  $\mathbb{R}^{2n}$  defined by  $\omega(z, z') = z \cdot \Omega^{-1}z'$ ; it coincides with the standard symplectic form  $\sigma$  when  $\Omega = J$ .

We will need the two following unitary transformations:

- The  $\Omega$ -symplectic transform  $F_\Omega$  defined, for  $a \in \mathcal{S}(\mathbb{R}^{2n})$ , by

$$F_\Omega a(z) = \left(\frac{1}{2\pi\hbar}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\omega(z, z')} a(z') dz'; \quad (15)$$

it extends into an involutive automorphism of  $\mathcal{S}'(\mathbb{R}^{2n})$  (also denoted by  $F_\Omega$ ) and whose restriction to  $L^2(\mathbb{R}^{2n})$  is unitary;

- The unitary operator  $\tilde{T}_\Omega(z_0)$  defined, for  $\Psi \in \mathcal{S}'(\mathbb{R}^{2n})$  by the formula

$$\tilde{T}_\Omega(z_0)\Psi(z) = e^{-\frac{i}{\hbar}\omega(z, z_0)}\Psi\left(z - \frac{1}{2}z_0\right). \quad (16)$$

Notice that when  $\Omega = J$  we have  $\tilde{T}_\Omega(z_0) = \tilde{T}(z_0)$  where  $\tilde{T}(z_0)$  is defined by formula (8) in [28].

Let us express the operator  $\tilde{A}_\Omega = a(z + \frac{1}{2}i\hbar\Omega\partial_z)$  in terms of  $F_\Omega a$  and  $\tilde{T}_\Omega(z_0)$ .

**Proposition 2** Let  $\tilde{A}_\Omega$  be the operator on  $\mathbb{R}^{2n}$  with Weyl symbol

$$\tilde{a}_\Omega(z, \zeta) = a(z - \frac{1}{2}\Omega\zeta). \quad (17)$$

We have

$$\tilde{A}_\Omega = \left(\frac{1}{2\pi\hbar}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\Omega a(z) \tilde{T}_\Omega(z) dz. \quad (18)$$

**Proof.** Let us denote by  $\tilde{B}$  the right-hand side of (18). We have, setting  $u = z - \frac{1}{2}\Omega z_0$ ,

$$\begin{aligned} \tilde{B}\Psi(z) &= \left(\frac{1}{2\pi\hbar}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\Omega a(z_0) e^{-\frac{i}{\hbar}\omega(z, z_0)} \Psi(z - \frac{1}{2}\Omega z_0) dz_0 \\ &= \left(\frac{2}{\pi\hbar}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\Omega a[2(z - u)] e^{\frac{2i}{\hbar}\omega(z, u)} \Psi(u) du \end{aligned}$$

hence the kernel of  $\tilde{B}$  is given by

$$K(z, u) = \left(\frac{2}{\pi\hbar}\right)^n |\det \Omega|^{-1/2} F_\Omega a[2(z - u)] e^{\frac{2i}{\hbar}\omega(z, u)}.$$

It follows that the Weyl symbol  $\tilde{b}$  of  $\tilde{B}$  is given by

$$\begin{aligned} \tilde{b}(z, \zeta) &= \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\zeta \cdot \zeta'} K(z + \frac{1}{2}\Omega\zeta', z - \frac{1}{2}\Omega\zeta') d\zeta' \\ &= \left(\frac{2}{\pi\hbar}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\zeta \cdot \zeta'} F_\Omega a(2\zeta') e^{-\frac{2i}{\hbar}\omega(z, \zeta')} d\zeta' \end{aligned}$$

that is, using the obvious relation

$$\zeta \cdot \zeta' + 2\omega(z, \zeta') = \omega(2z - \Omega\zeta, \zeta')$$

together with the change of variables  $z' = 2\zeta'$ ,

$$\begin{aligned} \tilde{b}(z, \zeta) &= \left(\frac{2}{\pi\hbar}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\omega(2z - \Omega\zeta, \zeta')} F_\Omega a(2\zeta') d\zeta' \\ &= \left(\frac{1}{2\pi\hbar}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\omega(z - \frac{1}{2}\Omega\zeta, z')} F_\Omega a(z') dz' \end{aligned}$$

that is, using the fact that  $F_\Omega F_\Omega$  is the identity,

$$\tilde{b}(z, \zeta) = a(z - \frac{1}{2}\Omega\zeta) = \tilde{a}_\Omega(z, \zeta)$$

which concludes the proof. ■

The result above motivates the following definition:

**Definition 3** Let  $a \in \mathcal{S}'(\mathbb{R}^{2n})$  and  $b \in \mathcal{S}(\mathbb{R}^{2n})$ . The  $\Omega$ -starproduct of  $a$  and  $b$  is the element of  $\mathcal{S}'(\mathbb{R}^{2n})$  defined by

$$a \star_{\Omega} b = \tilde{A}_{\Omega} b. \quad (19)$$

Note that it is not yet clear from the definition above that  $\star_{\Omega}$  is a bona fide starproduct. For instance, while it is obvious that  $1 \star_{\Omega} b = b$  (because the operator  $\tilde{A}_{\Omega}$  with symbol  $a = 1$  is the identity), the formula  $b \star_{\Omega} 1 = b$  is certainly not, and it is even less clear that  $\star_{\Omega}$  is associative!

## 4 A New Star-Product Is Born...

It turns out that we can reduce the study of the newly defined starproduct to that of the usual Groenewold–Moyal product  $\star$ . For this we will need Lemma 4 below.

**Lemma 4** Let  $s$  be a linear automorphism such that  $\sigma = s^* \omega$  and define a automorphism  $M_s : \mathcal{S}'(\mathbb{R}^{2n}) \rightarrow \mathcal{S}'(\mathbb{R}^{2n})$  by

$$M_s \Psi(z) = \sqrt{|\det s|} \Psi(sz). \quad (20)$$

(hence  $M_s$  is unitary on  $L^2(\mathbb{R}^{2n})$ ). We have

$$M_s \tilde{A}_{\Omega} = \tilde{A}' M_s \quad (21)$$

where  $\tilde{A}' = \tilde{A}'_J$  corresponds to the operator  $\hat{A}'$  acting on  $L^2(\mathbb{R}^n)$  with Weyl symbol  $a'(z) = a(sz)$ , and hence

$$M_s(a \star_{\Omega} b) = \sqrt{|\det s|} (a' \star b') \quad (22)$$

where  $b'(z) = b(sz)$ .

**Proof.** Formula (22) immediately follows from formula (21). To prove formula (21) one first checks the identities

$$M_s \tilde{T}_{\Omega}(z_0) = \tilde{T}(s^{-1} z_0) M_s \quad , \quad M_s F_{\Omega} = F_J M_s$$

(the verification of which is purely computational and therefore left to the reader); using these identities we have

$$\begin{aligned}
M_s \tilde{A}_\Omega &= \left(\frac{1}{2\pi\hbar}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\Omega a(z_0) \tilde{T}(s^{-1}z_0) M_s dz_0 \\
&= \left(\frac{1}{2\pi\hbar}\right)^n |\det s| |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\Omega a(sz) \tilde{T}(z) M_s dz \\
&= \left(\frac{1}{2\pi\hbar}\right)^n |\det s|^{1/2} |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} M_s F_\Omega a(z) \tilde{T}(z) M_s dz \\
&= \left(\frac{1}{2\pi\hbar}\right)^n |\det s|^{1/2} |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_J(M_s a)(z) \tilde{T}(z) M_s dz \\
&= \tilde{A}' M_s.
\end{aligned}$$

■

The double equality

$$1 \star_\Omega a = a \star_\Omega 1 = a \tag{23}$$

now immediately follows from formula (22): we have

$$M_s(1 \star_\Omega a) = \sqrt{|\det s|} (1 \star a') = 1 \star M_s a = M_s a$$

hence we recover the equality  $1 \star_\Omega a = a$ ; similarly

$$M_s(a \star_\Omega 1) = \sqrt{|\det s|} (a' \star 1) = M_s a \star 1 = M_s a$$

hence  $a \star_\Omega 1 = a$ .

Let us now prove the associativity of the  $\Omega$ -starproduct:

**Proposition 5** *Assume that the starproducts  $a \star_\Omega (b \star_\Omega c)$  and  $(a \star_\Omega b) \star_\Omega c$  are defined. We then have*

$$a \star_\Omega (b \star_\Omega c) = (a \star_\Omega b) \star_\Omega c. \tag{24}$$

**Proof.** It is of course sufficient to show that

$$M_s [a \star_\Omega (b \star_\Omega c)] = M_s [(a \star_\Omega b) \star_\Omega c]. \tag{25}$$

We have, by repeated use of (22) together with the definition of  $M_s$ ,

$$\begin{aligned}
M_s [a \star_\Omega (b \star_\Omega c)] &= \sqrt{|\det s|} (a' \star (b \star_\Omega c)') \\
&= a' \star M_s (b \star_\Omega c) \\
&= \sqrt{|\det s|} [a' \star (b' \star c')].
\end{aligned}$$

A similar calculation yields

$$M_s [(a \star_\Omega b) \star_\Omega c] = \sqrt{|\det s|} [(a' \star b') \star c']$$

hence the equality (25) in view of the associativity of the Groenewold–Moyal product. ■

That we have a deformation of a Poisson bracket follows from the following considerations. Let us define an  $\Omega$ -Poisson bracket  $\{\cdot, \cdot\}_\Omega$  by

$$\{a, b\}_\Omega = -\omega(X_{a,\Omega}, X_{b,\Omega}) \quad (26)$$

where the vector fields  $X_{a,\Omega}$  and  $X_{b,\Omega}$  are given by

$$X_{a,\Omega} = \Omega \partial_z a \quad , \quad X_{b,\Omega} = \Omega \partial_z b. \quad (27)$$

In particular  $\{a, b\}_\Omega$  is the usual Poisson bracket  $\{a, b\}$  and  $X_{a,J}$ ,  $X_{a,J}$  are the usual Hamilton vector fields when  $\Theta = N = 0$ . We have the following asymptotic formula relating both notions of Poisson brackets:

$$\{a, b\}_\Omega = \{a, b\} + o(\hbar) \text{ for } \hbar \rightarrow 0. \quad (28)$$

In fact, by definitions (26) and (27),

$$\{a, b\}_\Omega = -X_{a,\Omega} \cdot \Omega^{-1} X_{b,\Omega} = -\Omega \partial_z a \cdot \partial_z b$$

that is

$$\{a, b\}_\Omega = \{a, b\} - \hbar^{-1} (\Theta \partial_x a \cdot \partial_x b + N \partial_p a \cdot \partial_p b)$$

from which (28) follows in view of the conditions (12).

**Proposition 6** *We have*

$$a \star_\Omega b - b \star_\Omega a = i\hbar \{a, b\} + O(\hbar^2). \quad (29)$$

**Proof.** We have, since  $M_s$  is linear,

$$\begin{aligned} M_s(a \star_\Omega b - b \star_\Omega a) &= \sqrt{|\det s|} (a' \star b' - b' \star a') \\ &= \sqrt{|\det s|} (i\hbar \{a', b'\} + O(\hbar^2)) \end{aligned}$$

where, as usual,  $a'(z) = a(sz)$  and  $b'(z) = b(sz)$ . Now, by the chain rule and the relation  $J_s^T = s^{-1}\Omega$ ,

$$\begin{aligned} X_{a'} &= J_s^T \partial_z a(sz) = s^{-1} \Omega \partial_z a(sz) \\ X_{b'} &= J_s^T \partial_z b(sz) = s^{-1} \Omega \partial_z b(sz) \end{aligned}$$

and hence, using the identities  $\{a', b'\} = JX_{a'} \cdot X_{b'}$  and  $(s^T)^{-1}J^{-1}s^{-1} = \Omega^{-1}$ ,

$$\begin{aligned}\sqrt{|\det s|}\{a', b'\} &= -\sqrt{|\det s|}Js^{-1}\Omega\partial_z a(sz) \cdot s^{-1}\Omega\partial_z b(sz) \\ &= \sqrt{|\det s|}\partial_z a(sz) \cdot \Omega\partial_z b(sz) \\ &= -\sqrt{|\det s|}\Omega\partial_z a(sz) \cdot \Omega^{-1}(\Omega\partial_z b(sz)) \\ &= -M_s\omega(X_{a,\Omega}, X_{b,\Omega}).\end{aligned}$$

We have thus proven that

$$M_s(a \star_\Omega b - b \star_\Omega a) = -i\hbar M_s\omega(X_{a,\Omega}, X_{b,\Omega}) + O(\hbar^2).$$

From this and (28), Eqn. (29) follows. ■

More generally, using the approach above, it is easy to show that

$$f \star_\Omega g = \sum_{k \geq 0} B_{k,\Omega}(f, g)\hbar^k$$

where the  $B_{k,\Omega}(f, g)$  are bi-differential operators. In particular,  $B_{0,\Omega}(f, g) = 1$  and  $B_{1,\Omega}(f, g) = \frac{i}{2}\{f, g\}$ , but for  $k \geq 2$  they differ from those of the usual Moyal product. We leave these technicalities aside in this article.

## 5 The Intertwining Property

In [28] two of us defined a family of partial isometries  $W_\phi : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^{2n})$  indexed by  $\mathcal{S}(\mathbb{R}^n)$ , and intertwining the operator  $\tilde{A} = \tilde{A}_J$  and the usual Weyl operator  $\hat{A}$ :

$$\tilde{A}W_\phi = W_\phi\hat{A} \quad \text{and} \quad W_\phi^*\tilde{A} = \hat{A}W_\phi^*. \quad (30)$$

These intertwiners are defined by

$$W_\phi\psi = (2\pi\hbar)^{n/2}W(\psi, \phi) \quad (31)$$

where  $W(\psi, \phi)$  is the cross-Wigner distribution:

$$W(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}p \cdot y} \psi\left(x + \frac{1}{2}y\right) \overline{\phi\left(x - \frac{1}{2}y\right)} dy \quad (32)$$

and  $W_\phi^*$  denotes the adjoint of  $W_\phi$ .

The following result is an extension of Proposition 2 in [28].

**Theorem 7** *Let  $s$  be a linear automorphism of  $\mathbb{R}^{2n}$  such that  $s^*\omega = \sigma$ . (i) The mappings  $W_{s,\phi} : \mathcal{S}(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^{2n})$  defined by the formula:*

$$W_{s,\phi} = M_s^{-1}W_\phi \quad (33)$$

*are partial isometries  $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$  and we have*

$$\tilde{A}_\Omega W_{s,\phi} = W_{s,\phi} \hat{A}' \quad \text{and} \quad W_{s,\phi}^* \tilde{A}_\Omega = \hat{A}' W_{s,\phi}^* \quad (34)$$

*where  $\hat{A}'$  is the operator with Weyl symbol  $a' = a(sz)$  and  $W_{s,\phi}^*$  denotes the adjoint of  $W_{s,\phi}$ . (ii) The replacement of  $s$  by  $s'$  such that  $\sigma = s'^*\omega$  is equivalent to the replacement of  $\hat{A}'$  by  $\widehat{S}_\sigma^{-1} \hat{A}' \widehat{S}_\sigma$  and of  $W_{s,\phi}$  by  $W_{s,\widehat{S}_\sigma^{-1}\phi} \widehat{S}_\sigma^{-1}$  where  $\widehat{S}_\sigma$  is any of the two operators in the metaplectic group  $\text{Mp}(2n, \mathbb{R})$  whose projection on  $\text{Sp}(2n, \mathbb{R})$  is  $s_\sigma = s^{-1}s'$ .*

**Proof.** (i) We have, using the first formula (30) and definition (33),

$$\tilde{A}_\Omega W_{s,\phi} = M_s^{-1} \tilde{A}' M_s (M_s^{-1} W_\phi)$$

that is,

$$\tilde{A}_\Omega W_{s,\phi} = M_s^{-1} (\tilde{A}' W_\phi) = M_s^{-1} W_\phi \hat{A}' = W_{s,\phi} \hat{A}';$$

the equality  $W_{s,\phi}^* \tilde{A}_\Omega = \hat{A}' W_{s,\phi}^*$  is proven in a similar way. That  $W_{s,\phi}$  is a partial isometry is obvious since  $W_\phi$  is a partial isometry and  $M_s$  is unitary. (ii) We have  $s_\sigma = s^{-1}s' \in \text{Sp}(2n, \mathbb{R})$  hence  $a(s'z) = a(ss_\sigma z) = a'(s_\sigma z)$ . Let  $\hat{A}''$  be the operator with Weyl symbol  $a''(z) = a'(s_\sigma z)$ . In view of the symplectic covariance property of Weyl operators we have  $\hat{A}'' = \widehat{S}_\sigma^{-1} \hat{A}' \widehat{S}_\sigma$ . Similarly,

$$\begin{aligned} W_{s',\phi} \psi(z) &= M_s^{-1} (M_s M_{s'}^{-1} W_\phi) \psi(z) \\ &= M_s^{-1} W_\phi \psi(s_\sigma^{-1} z) \\ &= W_{s,\phi} \psi(s_\sigma^{-1} z) \end{aligned}$$

hence  $W_{s',\phi} \psi = W_{s,\widehat{S}_\sigma \phi} \widehat{S}_\sigma \psi$  in view of the symplectic covariance of the cross-Wigner transform (32); the result follows. ■

An important property of the mappings  $W_{s,\phi}$  is that they can be used to construct orthonormal bases in  $L^2(\mathbb{R}^{2n})$  starting from an orthonormal basis in  $L^2(\mathbb{R}^n)$ .

**Proposition 8** *Let  $(\phi_j)_{j \in F}$  be an arbitrary orthonormal basis of  $L^2(\mathbb{R}^n)$ ; the functions  $\Phi_{j,k} = W_{s,\phi_j} \phi_k$  with  $(j,k) \in F \times F$  form an orthonormal basis of  $L^2(\mathbb{R}^{2n})$ , and we have  $\Phi_{j,k} \in \mathcal{H}_j \cap \mathcal{H}_k$ , with  $\mathcal{H}_j = W_{s,\phi_j}(L^2(\mathbb{R}^n))$ .*

**Proof.** In [28] the property was proven for the mappings  $W_{\phi_j} = W_{I,\phi_j}$ ; the lemma follows since  $W_{s,\phi} = M_s^{-1}W_\phi$  and  $M_s$  is unitary. ■

## 6 The $\star_\Omega$ -Genvalue Equation: Spectral Results

Let us consider the star-genvalue equation for the star-product  $\star_\Omega$ :

$$a \star_\Omega \Psi = \lambda \Psi; \quad (35)$$

here  $a$  can be viewed as some Hamiltonian function whose properties are going to be described, and  $\Psi$  a “phase-space function”. Following definition (19) the study of this problem is equivalent to that of the eigenvalue equation

$$\tilde{A}_\Omega \Psi = \lambda \Psi \quad (36)$$

for the pseudo-differential operator  $\tilde{A}_\Omega$ . Using the intertwining relations (34) it is easy to relate the eigenvalues of  $\tilde{A}_\Omega$  to those of  $\hat{A}'$  following the lines in [28]; for instance one sees, adapting mutatis mutandis the proof of Theorem 4 in the reference, that the operators  $\tilde{A}_\Omega$  and  $\hat{A}'$  have the same eigenvalues (see Theorem 9 below). Note that it follows from Theorem 7(ii) that the eigenvalues of  $\hat{A}'$  do not depend on the choice of  $s$  such that  $s^*\omega = \sigma$ .

**Theorem 9** *The operators  $\tilde{A}_\Omega$  and  $\hat{A}'$  have the same eigenvalues. (i) Let  $\psi$  be an eigenvector of  $\hat{A}'$ :  $\hat{A}'\psi = \lambda\psi$ . Then  $\Psi = W_{s,\phi}\psi$  is an eigenvector of  $\tilde{A}_\Omega$  corresponding to the same eigenvalue:  $\tilde{A}_\Omega\Psi = \lambda\Psi$ . (ii) Conversely, if  $\Psi$  is an eigenvector of  $\tilde{A}_\Omega$  then  $\psi = W_{s,\phi}^*\Psi$  is an eigenvector of  $\hat{A}'$  corresponding to the same eigenvalue.*

**Proof.** That every eigenvalue of  $\hat{A}'$  also is an eigenvalue of  $\tilde{A}_\Omega$  is clear: if  $\hat{A}'\psi = \lambda\psi$  for some  $\psi \neq 0$  then

$$\tilde{A}_\Omega(W_{s,\phi}\psi) = W_{s,\phi}\hat{A}'\psi = \lambda W_{s,\phi}\psi$$

and  $\Psi = W_{s,\phi}\psi \neq 0$ ; this proves at the same time that  $W_{s,\phi}\psi$  is an eigenvector of  $\tilde{A}_\Omega$  because  $W_{s,\phi}$  is injective. (ii) Assume conversely that  $\tilde{A}_\Omega\Psi = \lambda\Psi$  for  $\Psi \in L^2(\mathbb{R}^{2n})$ ,  $\Psi \neq 0$ , and  $\lambda \in \mathbb{R}$ . For every  $\phi$  we have

$$\widehat{A}'W_{s,\phi}^*\Psi = W_{s,\phi}^*\tilde{A}_\Omega\Psi = \lambda W_{s,\phi}^*\Psi$$

hence  $\lambda$  is an eigenvalue of  $\widehat{A}'$  and  $\psi$  an eigenvector if  $\psi = W_{s,\phi}^*\Psi \neq 0$ . We have  $W_{s,\phi}\psi = W_{s,\phi}W_{s,\phi}^*\Psi = P_{s,\phi}\Psi$  where  $P_{s,\phi}$  is the orthogonal projection on the range  $\mathcal{H}_{s,\phi}$  of  $W_{s,\phi}$ . Assume that  $\psi = 0$ ; then  $P_{s,\phi}\Psi = 0$  for every  $\phi \in \mathcal{S}(\mathbb{R}^n)$ , and hence  $\Psi = 0$  in view of Proposition 8. ■

Let us give an application of the result above. Assume that the symbol  $a$  belongs to the Shubin class  $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ ; recall [44] that  $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$  ( $m_0, m_1 \in \mathbb{R}$  and  $0 < \rho \leq 1$ ) if  $a \in C^\infty(\mathbb{R}^{2n})$  and if there exist constants  $C_0, C_1 \geq 0$  and, for every  $\alpha \in \mathbb{N}^n$ ,  $|\alpha| \neq 0$ , a constant  $C_\alpha \geq 0$ , such that for  $|z|$  sufficiently large

$$C_0|z|^{m_0} \leq |a(z)| \leq C_1|z|^{m_1} \quad , \quad |\partial_z^\alpha a(z)| \leq C_\alpha|a(z)||z|^{-\rho|\alpha|}. \quad (37)$$

The following result (Shubin [44], Chapter 4) is important in our context:

**Theorem 10** *Let  $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$  be real, and  $m_0 > 0$ . Then the formally self-adjoint operator  $\widehat{A}$  with Weyl symbol  $a$  has the following properties: (i)  $\widehat{A}$  is essentially self-adjoint in  $L^2(\mathbb{R}^n)$  and has discrete spectrum; (ii) There exists an orthonormal basis of eigenfunctions  $\phi_j \in \mathcal{S}(\mathbb{R}^n)$  ( $j = 1, 2, \dots$ ) with eigenvalues  $\lambda_j \in \mathbb{R}$  such that  $\lim_{j \rightarrow \infty} |\lambda_j| = \infty$ .*

It follows that:

**Theorem 11** *Let  $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$  be real, and  $m_0 > 0$ . (i) The star-eigenvalue equation  $a \star_\Omega \Psi = \lambda\Psi$  has a sequence of real eigenvalues  $\lambda_j$  such that  $\lim_{j \rightarrow \infty} |\lambda_j| = \infty$ , and these eigenvalues are those of the operator  $\widehat{A}'$  with Weyl symbol  $a'(z) = a(sz)$ . (ii) The star-eigenvectors of  $a$  are in one-to-one correspondence with the eigenvectors  $\phi_j \in \mathcal{S}(\mathbb{R}^n)$  of  $\widehat{A}'$  by the formula  $\Phi_{k,j} = W_{s,\phi_k}\phi_j$ .*

**Proof.** It is an immediate consequence of Theorems 9 and 10. ■

## 7 Concluding Remarks...

The results using the generalized Weyl-Wigner map [18] seem to be quite general since they also apply to the case of nonlinear transformations of  $\mathbb{R}^{2n}$  (for a review see also section II of [2]). In particular a more general starproduct than the one of non-commutative quantum mechanics was obtained in [2] (see Eqn.(23) in this reference). A future project could be to extend the approach of the present paper to this case. Another important topic we have not addressed in this article is the characterization of the optimal symbol classes and function spaces associated with the star-product  $\star_{\Omega}$ . As two of us have shown elsewhere [29] Feichtinger's modulation spaces (see [24, 25] for a review) and the closely related Sjöstrand classes [45] are excellent candidates in the case of Landau-type operators (which are a variant of the operators  $\tilde{A}$  corresponding to the case  $\Omega = J$ ). It seems very plausible that these function spaces are likely to play an equally important role in the theory of the star-product  $\star_{\Omega}$ . Another future project concerns a discussion of the starproduct  $\star_{\Omega}$  and its connection to Rieffel's work in deformation quantization as outlined in [41], and the methods introduced in [37] in a different context.

**Acknowledgement 12** *Maurice de Gosson has been financed by the Austrian Research Agency FWF (Projektnummer P20442-N13). Nuno Costa Dias and João Nuno Prata have been supported by the grants PDTC/MAT/69635/2006 and PTDC/MAT/099880/2008 of the Portuguese Science Foundation (FCT). Franz Luef has been financed by the Marie Curie Outgoing Fellowship PIOF 220464.*

## References

- [1] Binz E., Honegger R., Rieckers A., Field-theoretic Weyl quantization as a strict and continuous deformation quantization. *Ann. Henri Poincaré* 5 (2004) 327–346
- [2] Bastos, C., Bertolami, O. Dias, N. C., Prata, J. N.: Weyl–Wigner Formulation of Noncommutative Quantum Mechanics. *J. Math. Phys.* **49** (2008) 072101

- [3] Bastos, C., Dias, N. C., Prata, J. N.: Wigner measures in non-commutative quantum mechanics. Preprint, arXiv:0907.4438v1 [math-ph] 25 jul 2009
- [4] Bastos, C., Bertolami, O., Dias, N. C., Prata, J. N.: The singularity problem and phase-space noncanonical noncommutativity. Preprint, arXiv:0912.4027v1 [hep-th] 20 dec 2009
- [5] Bastos, C., Bertolami, O., Dias, N. C., Prata, J. N.: Black holes and phase-space noncommutativity. Phys. Rev. **D 80** (2009) 124038
- [6] Bastos, C., Bertolami, O., Dias, N. C., Prata, J. N.: Phase-space non-commutative quantum cosmology. Phys. Rev. **D 78** (2008) 023516
- [7] Bastos, C., Bertolami: Berry phase in the gravitational quantum well and the Seiberg-Witten map. Phys. Lett. **A 372** (2008) 5556-5559
- [8] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation Theory and Quantization. I. Deformation of Symplectic Structures. Annals of Physics **111**, 6–110 (1978)
- [9] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: *Deformation Theory and Quantization. II Physical Applications*. Annals of Physics **110**, 111–151 (1978)
- [10] Bertolami, O., Rosa, J. G., de Aragão, C. M. L., Castorina, P., Zappalà, D.: Noncommutative gravitational quantum well. Phys. Rev. D (3) **72** (2005), no. 2, 025010–025018
- [11] Bolonek, K., Kosiński, P.: On uncertainty relations in noncommutative quantum mechanics. Phys. Lett. **B 547** (2002) 51–54
- [12] Carroll, S. M., Harvey, J. A., Kostelecký, V. A., Lane, C. D., Okamoto, T.: Noncommutative field theory and Lorentz violation. Phys. Rev. Lett. **87**, no. 14, 141601–141605 (2001)
- [13] Chaichian, M., Sheikh-Jabbari, M.M., Tureanu, A.: Hydrogen atom spectrum and the Lamb shift in noncommutative QED. Phys. Rev. Lett. **86** (2001) 2716
- [14] Connes, A., Douglas, M. R., Schwarz, A.: Noncommutative geometry and matrix theory: compactification on tori. JHEP **9802** (1998) 003

- [15] Daszkiewicz, M., Walczyk, C.J.: Newton equation for canonical, Lie-algebraic and quadratic deformation of classical space. *Phys. Rev.* **D 77** (2008) 105008
- [16] Dayi, O.F., Jellal, A.: Hall effect in noncommutative coordinates. *J. Math. Phys.* **43** (2002) 4592
- [17] Delduc, F., Duret, Q., Gieres, F., Lefrançois, M.: Magnetic fields in noncommutative quantum mechanics, *J. Phys: Conf. Ser.* **103** (2008) 012020
- [18] Dias, N. C., Prata, J. N.: Generalized Weyl-Wigner Map and Vey Quantum Mechanics. *J. Math. Phys.* **42** (2001) 5565–5579
- [19] Dias, N. C., Prata, J. N.: Exact master equation for a noncommutative Brownian particle, *Ann. Phys.* **324** (2009) 73–96
- [20] Douglas, M. R., Nekrasov, N. A. Noncommutative field theory. *Rev. Mod. Phys.* **73** 977–1029 (2001)
- [21] Dunne, G., Jackiw, R.: "Peierls substitution" and Chern-Simmons quantum mechanics. *Nucl. Phys. Proc. Suppl.* **C 33** (1993) 114–118
- [22] Duval, C., Horváthy, P.A.: Exotic Galilean symmetry in the noncommutative plane, and the Hall effect. *J. Phys. A: Math. Gen.* **34** (2001) 10097
- [23] Douglas, M. R., Hull, C. D-branes and the noncommutative torus. *JHEP* **9802** 008 (1998)
- [24] Feichtinger, H. G.: Modulation Spaces: Looking Back and Ahead. *Sampl. Theory Signal Image Process.*, 5(2), 109–140 (2006)
- [25] Feichtinger, H. G., Luef F.: Wiener Amalgam Spaces for the Fundamental Identity of Gabor Analysis. *Collect. Math.* 57, 233–253 (2006)
- [26] de Gosson, M. *Symplectic Geometry and Quantum Mechanics.* Birkhäuser, Basel, series "Operator Theory: Advances and Applications" (subseries: "Advances in Partial Differential Equations"), Vol. 166 (2006)

- [27] Gamboa, J., Loewe, M., Rojas, J.C.: Noncommutative quantum mechanics. *Phys. Rev.* **D 64** (2001) 067901
- [28] de Gosson, M., Luef F.: A new approach to the  $\star$ -genvalue equation. *Lett. Math. Phys.* **85**, 173–183 (2008)
- [29] de Gosson, M., Luef F.: Spectral and Regularity Properties of a Pseudo-Differential Calculus Related to Landau Quantization. To appear in *Journal of Pseudo-Differential Equations and Applications* (2010).
- [30] de Gosson, M., Luef F.: On the usefulness of modulation spaces in deformation quantization. *J. Phys. A: Math. Theor.* **42**, 315205–315221 (2009)
- [31] Harikumar, E., Kumar, V.S., Khare, A.: Supersymmetric quantum mechanics on noncommutative plane. *Phys. Lett.* **B 589** (2004) 155
- [32] Heisenberg, W.: Letter of Heisenberg to Peierls (1930), Wolfgang Pauli, *Scientific Correspondence*, Ed. Karl von Meyenn, Springer-Verlag, 1993.
- [33] Horváthy, P.A.: The noncommutative Landau problem. *Ann. Phys.* **299** (2002) 128
- [34] Kamoshita, J.-I.: Probing noncommutative space-time in the laboratory frame. *Eur. Phys. J.* **C 52** 451–457 (2007)
- [35] Kosiński, P., Bolonek, K.: Minimalization of uncertainty relations in noncommutative quantum mechanics. *Acta Phys. Polon.* **B 34** (2003) 2575–2588
- [36] Luef, F., de Gosson, M.: Symplectic capacities and the geometry of uncertainty: the irruption of symplectic topology in classical and quantum mechanics. *Phys. Rep.* (2009)
- [37] Luef, F., Manin, Yu. I.: Quantum theta functions and Gabor frames for modulation spaces. *Lett. Math. Phys.* **88**, no. 1-3, 131–161 (2009)
- [38] Martinetti, P. What kind of noncommutative geometry for quantum gravity? *Mod. Phys. Lett.* **A 20** 1315 (2005)
- [39] Nair, V.P., Polychronakos, A.P.: Quantum mechanics on the noncommutative plane and sphere. *Phys. Lett.* **B 505** (2001) 267–274

- [40] Pauli, W.: Letter of Pauli to Oppenheimer (1946), Wolfgang Pauli, Scientific Correspondence, Ed. Karl von Meyenn, Springer-Verlag, 1993.
- [41] Rieffel, M. A.: Deformation quantization for actions of  $\mathbb{R}^d$ . Mem. Am. Math. Soc., **506**, 1993
- [42] Rosenbaum, M., Vergara, J. D. The star-value equation and Wigner distributions in noncommutative Heisenberg algebra. Gen. Rel. Grav. **38** 607–624 (2006)
- [43] Seiberg, N., Witten, E.: String theory and noncommutative geometry. JHEP 09 (1999) 032
- [44] Shubin, M. A.: Pseudodifferential Operators and Spectral Theory. Springer-Verlag, first edition 1987; second edition 2001 [original Russian edition in Nauka, Moskva, 1978]
- [45] Sjöstrand, J.: An algebra of pseudodifferential operators, Math. Res. Lett. 1(2), 185–192, (1994)
- [46] Snyder, H.S.: Quantized space-time. Phys. Rev. 71 (1947) 38-41.
- [47] Snyder, H.S.: The electromagnetic field in quantized space-time. Phys. Rev. **72** (1947) 68–71.
- [48] Susskind, L.: The quantum Hall fluid and noncommutative Chern-Simmons theory. Preprint, arXiv: 0101029 [hep-th] 14 jan 2001
- [49] Szabo, R.: Quantum Field Theory on Noncommutative Spaces. Phys. Rep. **378** 207 (2003)
- [50] Yang, C. N.: On quantized space-time, Phys. Rev. **72** (1947) 874–874

**Author’s addresses:**

**Franz Luef**<sup>1</sup>: *Universität Wien, NuHAG*

---

<sup>1</sup>franz.luef@univie.ac.at

*Fakultät für Mathematik  
Wien 1090, Austria  
and  
Department of Mathematics  
UC Berkeley  
847 Evans Hall  
Berkeley  
CA 94720-3840, USA*

**João Nuno Prata<sup>2</sup> and Nuno Costa Dias<sup>3</sup>:**

*Departamento de Matemática. Universidade Lusófona de Humanidades  
e Tecnologias. Av. Campo Grande, 376,  
1749-024 Lisboa, Portugal  
and  
Grupo de Física Matemática,  
Universidade de Lisboa,  
Av. Prof. Gama Pinto 2,  
1649-003 Lisboa, Portugal*

**Maurice de Gosson<sup>4</sup>:** *Universität Wien, NuHAG  
Fakultät für Mathematik  
Wien 1090, Austria*

---

<sup>2</sup>joao.prata@mail.telepac.pt

<sup>3</sup>ncdias@meo.pt

<sup>4</sup>maurice.de.gosson@univie.ac.at