

Frames of Irregular Translates

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Abstract

In this note we study properties of a set of irregular translates of a function in $L^2(\mathbb{R}^d)$. This is achieved by looking at a set of exponentials restricted to a set $E \subset \mathbb{R}^d$ with frequencies in a countable set Λ . The results are obtained by analyzing which properties of this set of exponentials are preserved when multiplied by the Fourier transform of a function $h \in L^2(E)$. This in turn gives information on the set of Λ -translates of h . In particular we study frame and Riesz basis properties. Using density results due to Beurling, we prove the existence and give ways to construct frames by irregular translates.

Key words: Frames, Riesz bases, irregular translates, irregular sampling
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1 Introduction

Signal processing tools and algorithms are central in the technology of the 21st century. These tools and algorithms are used in digital instruments that have become indispensable in everyday life. There is a wide spectrum of devices ranging from medical applications to mass consumer gadgets, such as cameras, Smart Phones, MP3-players or high resolution TV. They could not exist without the recent development of sophisticated tools and techniques.

Signal processing is an area that for over 50 years belonged almost exclusively to engineering. Recently the digital revolution has produced a considerable increase of the need for more mathematics to tackle difficult problems, and for the design of new algorithms and the refinement of existing ones. This created a rich and fruitful interaction between both fields.

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One example of this is the concept of *frame* [14] which has been established as an important background for sampling theory and signal processing. Different from bases, frame decompositions are redundant. This property is advantageous in many applications such as de-noising, error robustness or sparsity.

Frames of translates [9] are an important class of frames that have a special structure. Here, one generating function h is shifted to create the analyzing family of elements, $\{h(x - kb)\}$. This topic was investigated in [8] and can be used in Gabor or wavelet theory [5]. On the other side these frames are very important in the theory of Shift invariant Spaces (SIS) [7, 12] that are central in Approximation, Sampling and Wavelet theory.

A typical example of frames of translates are filterbanks, which have constant shapes. For example the phase vocoder [13] corresponds to such a filterbank with regular shifts, which is often used in signal processing applications like time stretching. Introducing irregular shifts gives rise to a generalization of this analysis/synthesis system. Irregular frames of translates were investigated for example in [2]. But there are still many open questions for this case.

Frames of translates are connected to the concept of Gabor multipliers [15] by the Kohn-Nirenberg correspondence. These operators by themselves form an interesting subclass of time-variant filters [16]. Also in this case irregular shifts are interesting [3], for example in case of non-uniform frequency sampling as in a scale adapted to human perception [4].

Translates of a given function become exponentials multiplied by a fixed function through the Fourier transform. Due to the Plancherel Theorem metric properties are preserved. In this note we study which properties of a set of exponentials of irregular frequencies are preserved when they are multiplied by a fixed function h , and we characterize those functions h that preserve these properties. This gives properties of the set of translates of the inverse Fourier transform of h . On the other side, using density results due to Beurling [6], we prove the existence, and provide a method to construct, frame sequences of irregular translates.

Many of the results throughout the paper are probably expected and not surprising to people working in the field, but to our knowledge they were never explicitly written down.

The organization of the paper is as follows. In Section 2 we set the notation, give basic definitions and state known results. In Section 3 we give conditions in order that a set of exponentials that form a frame, remains a frame when multiplied by an appropriate function. Finally in section 4 we apply these results to the study of frames of translates.

2 Preliminaries and Notation

Throughout the article K will denote a countable index set.

Let \mathcal{H} be a separable Hilbert space. A sequence $\{\psi_k\}_{k \in K} \subseteq \mathcal{H}$ is a *frame*

for \mathcal{H} if there exist positive constants A and B that satisfy

$$A\|f\|^2 \leq \sum_{k \in K} |\langle f, \psi_k \rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}.$$

If $A = B$ then it is called a *tight frame*. If $\{\psi_k\}_{k \in K}$ satisfies the right inequality in the above formula, it is called a *Bessel sequence*.

A sequence $\{\psi_k\}_{k \in K}$ is a *Riesz basis* for \mathcal{H} if it is complete in \mathcal{H} and there exist positive constants A and B such that for every finite scalar sequence $\{c_k\}$ one has

$$A \sum |c_k|^2 \leq \left\| \sum c_k \psi_k \right\|^2 \leq B \sum |c_k|^2.$$

We say that $\{\psi_k\}_{k \in K}$ is a *frame sequence* if it is a frame for the space it spans, and it is a *Riesz sequence* if it is a Riesz basis for the space it spans.

For a closed subspace $\mathcal{V} \subseteq \mathcal{H}$ denote the projection on it by $\mathcal{P}_{\mathcal{V}}$. A sequence $\{\phi_k\}_{k \in K} \subseteq \mathcal{H}$ is an *outer frame* [1] for a closed subspace $\mathcal{V} \subseteq \mathcal{H}$ if $\mathcal{P}_{\mathcal{V}}(\phi_k)$ is a frame for \mathcal{V} .

For two sets $F \subseteq E \subseteq \mathbb{R}^d$ we use the notation

$$\widetilde{L^2(F)}^{(E)} := \{f \in L^2(E) : f(x) = 0 \text{ for a.e. } x \in E \setminus F\}. \quad (1)$$

This set is isomorphic to $L^2(F)$ using $\varphi : \widetilde{L^2(F)}^{(E)} \rightarrow L^2(F)$ where $\varphi(f) = f|_F$.

When $E = \mathbb{R}^d$ we will just write $L^2(E)$ in place of $\widetilde{L^2(E)}^{(\mathbb{R}^d)}$.

Frames of exponentials have been studied in [14]. Conditions on a discrete set Λ such that $\{e^{-2\pi i \lambda x}\}_{\lambda \in \Lambda}$ is a frame or a Riesz basis for $L^2(E)$, where $E \subseteq \mathbb{R}$ is a bounded interval, have been given in [21], [17], [22], [18]. For the case that E is a finite union of certain intervals it is known that such sets Λ exist [19]. In higher dimensions, there exists results for particular sets E [20], [23].

For $\lambda \in \mathbb{R}^d$, we denote by e_λ the function defined by $e_\lambda(x) = e^{-2\pi i \lambda x}$.

Let $\Lambda = \{\lambda_k\}_{k \in K}$ be a sequence in \mathbb{R}^d . Throughout the paper when we say that a set of exponentials $\{e_{\lambda_k} : k \in K\}$ is a frame (or a Riesz basis) of $L^2(E)$ we will mean that the set $\{e_{\lambda_k} \chi_E\}_{k \in K}$ has that property. Here χ_E stands for the indicator function of E .

2.1 Existence of irregular exponentials frames

In [6] Beurling gave sufficient conditions on a discrete set Λ in \mathbb{R}^d , in order that the associated exponentials $\{e_\lambda\}_{\lambda \in \Lambda}$ form a frame when restricted to a ball. These conditions are given in terms of density. In this section we review those results that will be used later.

Definition 2.1. *A set Λ is separated if*

$$\inf_{\lambda \neq \lambda'} \|\lambda - \lambda'\| > 0.$$

There are many notions for the density of a set Λ . We start with definitions that are due to Beurling.

Definition 2.2.

1. A lower uniform density $D^-(\Lambda)$ of a separated set $\Lambda \subset \mathbb{R}^d$ is defined as

$$D^-(\Lambda) = \lim_{r \rightarrow \infty} \frac{\nu^-(r)}{(2r)^d}$$

where $\nu^-(r) := \min_{y \in \mathbb{R}^d} \#(\Lambda \cap (y + [-r, r]^d))$, where $\#(Z)$ denotes the cardinal of the set Z .

2. An upper uniform density $D^+(\Lambda)$ of a separated set Λ is defined as

$$D^+(\Lambda) = \lim_{r \rightarrow \infty} \frac{\nu^+(r)}{(2r)^d}$$

where $\nu^+(r) := \max_{y \in \mathbb{R}^d} \#(\Lambda \cap (y + [-r, r]^d))$.

3. If $D^-(\Lambda) = D^+(\Lambda) = D(\Lambda)$, then Λ is said to have uniform Beurling density $D(\Lambda)$.

Remark. The limits in the definitions of $D^-(\Lambda)$ and $D^+(\Lambda)$ exist.

As an example, let $d > 0$ and $\Lambda = \{\lambda_j : j \in \mathbb{Z}\} \subset \mathbb{R}$ be separated sequence such that there exists $L > 0$ with $|\lambda_j - \frac{j}{d}| \leq L$, for all $j \in \mathbb{Z}$. Then $D^-(\Lambda) = D^+(\Lambda) = d$.

For the one dimensional case, Beurling proved the following theorem.

Theorem 2.3. (*Beurling*) Let $\Lambda \subset \mathbb{R}$ be separated, $a > 0$ and $\Omega = [-\frac{a}{2}, \frac{a}{2}]$. If $a < D^-(\Lambda)$ then $\{e_{\lambda} \chi_{\Omega}\}_{\lambda \in \Lambda}$ is a frame for $L^2(\Omega)$.

This previous result however is only valid in one dimension. For higher dimensions, Beurling introduced the following notion:

Definition 2.4. The gap ρ of the set Λ is defined as

$$\rho = \rho(\Lambda) = \inf \left\{ r > 0 : \bigcup_{\lambda \in \Lambda} B_r(\lambda) = \mathbb{R}^d \right\}$$

Equivalently, the gap ρ can be defined as

$$\rho = \rho(\Lambda) = \sup_{x \in \mathbb{R}^d} \inf_{\lambda \in \Lambda} |x - \lambda|.$$

It is not difficult to show that if Λ has gap ρ , then $D^-(\Lambda) \geq \frac{1}{2\rho}$. For a separated set Λ , and for the case where Ω is the ball $B_r(0)$ of radius r centered at the origin, Beurling proved the following result:

Theorem 2.5 (*Beurling*). Let $\Lambda \subset \mathbb{R}^d$ be separated, and $\Omega = B_r(0)$. If $\rho < 1/4$, then $\{e_{\lambda} \chi_{\Omega}\}_{\lambda \in \Lambda}$ is a frame for $L^2(\Omega)$.

Note that actually the same set of exponentials is also a frame for $L^2(B_r(x))$ for any vector $x \in \mathbb{R}^d$.

Using these results, in order to construct a frame of exponentials of $L^2(E)$ for a bounded set $E \subset \mathbb{R}^d$, it is enough to find a separated sequence Λ in \mathbb{R}^d with gap $\rho < 1/r$ with r the radius of a ball containing E .

3 Frames by Multiplication

We begin with the following trivial fact.

Lemma 3.1. *For $E \subseteq \mathbb{R}^d$ and $h \in L^2(E)$ we have $\{h \cdot e_\lambda\} \in L^2(E)$ and $\|h \cdot e_\lambda\|_{L^2(E)} = \|h\|_{L^2(E)}$.*

Proof.

$$\|h \cdot e_\lambda\|_{L^2(E)}^2 = \int_E |h(x)|^2 |e_\lambda(x)|^2 dx = \int_E |h(x)|^2 dx = \|h\|_{L^2(E)}^2$$

□

Lemma 3.2. *Let E be a subset of \mathbb{R}^d of finite measure and $\{e_{\lambda_k}\}_{k \in K}$ be complete in $L^2(E)$. Let $h \in L^2(\mathbb{R}^d)$ such that $|\{t \in E : \hat{h}(t) = 0\}| = 0$. Then $\{\hat{h}e_{\lambda_k}\}$ is complete in $L^2(E)$.*

Proof. Assume $f \in L^2(E)$ and $\langle f, \hat{h}e_{\lambda_k} \rangle = 0$ for every $k \in K$. Then

$$\int_E f \overline{\hat{h}e_{\lambda_k}} = \int_E (f\bar{h}) \overline{e_{\lambda_k}} = 0 \text{ for every } k \in K.$$

By the completeness of $\{e_{\lambda_k}\}_{k \in K}$ in $L^2(E)$, this implies that $f\bar{h} = 0$ a.e. in E . Since $\hat{h} \neq 0$ a.e. in E , it follows that $f = 0$ a.e. in E . □

Proposition 3.3. *Let E be a subset of \mathbb{R}^d of finite measure and $h \in L^2(\mathbb{R}^d)$. Set $\Lambda = \{\lambda_k\}_{k \in K} \subseteq \mathbb{R}^d$ such that $\{e_{\lambda_k}\}_{k \in K}$ is a frame of $L^2(E)$. Then*

1. $\{\hat{h}e_{\lambda_k}\}_{k \in K}$ is a frame of $L^2(E)$ if and only if there exist constants A and B such that

$$0 < A \leq B < +\infty \quad \text{and} \quad A \leq |\hat{h}(t)| \leq B \quad \text{a.e. } t \in E. \quad (2)$$

2. If $h \in L^2(\mathbb{R})$ such that $|\{t \in E : \hat{h}(t) = 0\}| = 0$, then $\{\hat{h}e_{\lambda_k}\}$ is complete in $L^2(E)$.

Proof. Part 1.:

\implies

Assume that both $\{e_{\lambda_k}\}_{k \in K}$ and $\{\hat{h}e_{\lambda_k}\}_{k \in K}$ are frames of $L^2(E)$.

Assume that for every $A > 0$, there exists a set $U \subseteq E$ of positive measure such that $|\hat{h}(t)| < A$ for every $t \in U$.

For $n \in \mathbb{N}$, let $E_n = \{t \in E : |\hat{h}(t)| < \frac{1}{n}\}$. Note that $|E_n| > 0$ for every $n \in \mathbb{N}$. Define

$$f_n(t) = \begin{cases} \frac{1}{\sqrt{|E_n|}} & \text{for } t \in E_n \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

We have that $\|f_n\|_2 = 1$ for every $n \in \mathbb{N}$ and so $f_n \in L^2(E)$.

If α is the lower frame bound of $\{\hat{h}e_{\lambda_k}\}_{k \in K}$ and M is the upper frame bound of $\{e_{\lambda_k}\}_{k \in K}$, then

$$\begin{aligned} \alpha &\leq \sum_{k \in K} |\langle f_n, \hat{h}e_{\lambda_k} \rangle|^2 \\ &= \sum_{k \in K} |\langle f_n \bar{\hat{h}}, e_{\lambda_k} \rangle|^2 \leq M \|f_n \hat{h}\|_2^2 \\ &= M \int_{E_n} |f_n \hat{h}|^2 = \frac{1}{|E_n|} M \int_{E_n} |\hat{h}|^2 \leq \frac{M}{n^2} \rightarrow 0, \end{aligned}$$

which is a contradiction. So we can conclude that there exists an $A > 0$ such that $A \leq |\hat{h}(t)|$ a.e $t \in E$.

To prove the existence of the upper bound in (2), assume that for every $B > 0$ there exists a set $V \subseteq E$ of positive measure, such that $|\hat{h}(t)| > B$ for every $t \in V$.

For $s \in \mathbb{N}$, let $E_s = \{t \in E : |\hat{h}(t)| > s\}$. We have that $|E_s| > 0$ for every $s \in \mathbb{N}$. Define

$$f_s(t) = \begin{cases} \frac{1}{\sqrt{|E_s|}} & \text{for } t \in E_s \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Let m be the lower frame bound of $\{e_{\lambda_k}\}_{k \in K}$. Then

$$\begin{aligned} \sum_{k \in K} |\langle f_s, \hat{h}e_{\lambda_k} \rangle|^2 &= \\ &= \sum_{k \in K} |\langle f_s \bar{\hat{h}}, e_{\lambda_k} \rangle|^2 \geq m \|f_s \hat{h}\|_2^2 \\ &= m \int_{E_s} |f_s \hat{h}|^2 = \frac{1}{|E_s|} m \int_{E_s} |\hat{h}|^2 \geq ms^2 \rightarrow +\infty, \end{aligned}$$

which again is a contradiction, so there must exist a constant B that satisfies (2).

\Leftarrow)

Assume there exist positive constants $A, B > 0$ such that $A \leq |\hat{h}(t)| \leq B$ a.e. $t \in E$ and that $\{e_{\lambda_k}\}_{k \in K}$ is a frame of $L^2(E)$ with lower and upper frame bounds m and M respectively.

Since $\hat{h} \in L^\infty(E)$, $f\hat{h} \in L^2(E)$ and therefore for every $f \in L^2(E)$

$$\sum_{k \in K} |\langle f, \hat{h}e_{\lambda_k} \rangle|^2 = \sum_{k \in K} |\langle f \bar{\hat{h}}, e_{\lambda_k} \rangle|^2,$$

we have that

$$m \|f \bar{\hat{h}}\|_2^2 \leq \sum_{k \in K} |\langle f, \hat{h}e_{\lambda_k} \rangle|^2 \leq M \|f \bar{\hat{h}}\|_2^2 \quad \text{for every } f \in L^2(E).$$

But

$$\|f\bar{h}\|^2 \geq A^2\|f\|^2 \quad \text{and} \quad \|f\bar{h}\|^2 \leq B^2\|f\|^2,$$

so

$$mA^2\|f\|^2 \leq \sum_{k \in K} |\langle f, \hat{h}e_{\lambda_k} \rangle|^2 \leq MB^2\|f\|^2 \quad \text{for every } f \in L^2(E).$$

This completes the proof of part 1.

Part 2. is an immediate consequence of Lemma 3.2. □

Analogously, the following result can be proved:

Proposition 3.4. *Let E be a subset of \mathbb{R}^d of finite measure and $h \in L^2(\mathbb{R}^d)$. Set $\Lambda = \{\lambda_k\}_{k \in K} \subseteq \mathbb{R}^d$ such that $\{e_{\lambda_k}\}_{k \in K}$ is a tight frame of $L^2(E)$. Then $\{\hat{h}e_{\lambda_k}\}_{k \in K}$ is a tight frame of $L^2(E)$ if and only if there exists a positive constant A such that*

$$|\hat{h}(t)| = A \quad \text{a.e. } t \in E. \quad (5)$$

On the same lines, we can also obtain a similar result for Riesz bases instead of frames:

Proposition 3.5. *Let E be a subset of \mathbb{R}^d of finite measure and $h \in L^2(\mathbb{R}^d)$. Set $\Lambda = \{\lambda_k\}_{k \in K} \subseteq \mathbb{R}^d$ such that $\{e_{\lambda_k}\}_{k \in K}$ is a Riesz basis of $L^2(E)$. Then $\{\hat{h}e_{\lambda_k}\}_{k \in K}$ is a Riesz basis of $L^2(E)$ if and only if there exist constants A and B such that*

$$0 < A \leq B < +\infty \quad \text{and} \quad A \leq |\hat{h}(t)| \leq B \quad \text{a.e. } t \in E. \quad (6)$$

Proof. \implies)

Since $\{\hat{h}e_{\lambda_k}\}_{k \in K}$ is a Riesz basis of $L^2(E)$ it is a frame of $L^2(E)$, so by Proposition 3.3 there exist constants A and B such that inequality (6) holds.

\impliedby)

If there exist constants A and B such that inequality (6) holds, then by Proposition 3.3 $\{\hat{h}e_{\lambda_k}\}_{k \in K}$ is a frame of $L^2(E)$. Hence, for every $f \in L^2(E)$,

$$f = \sum_{k \in K} c_{\lambda_k} \hat{h}e_{\lambda_k}, \quad (7)$$

where $\{c_{\lambda_k}\}_{\lambda_k \in \Lambda} \in \ell^2(\Lambda)$. To see that the coefficients $\{c_{\lambda_k}\}_{\lambda_k \in \Lambda}$ in (7) are unique, we observe that since $|\hat{h}(t)| \geq A$ a.e. $t \in E$,

$$\frac{f}{\hat{h}} = \sum_{k \in K} c_{\lambda_k} e_{\lambda_k} \in L^2(E). \quad (8)$$

The result follows using that $\{e_{\lambda_k}\}_{k \in K}$ is a Riesz basis of $L^2(E)$. □

Proposition 3.6. *Let E be a subset of \mathbb{R}^d of finite measure and $h \in L^2(\mathbb{R}^d)$. Set $\Lambda = \{\lambda_k\}_{k \in K} \subseteq \mathbb{R}^d$ such that $\{e_{\lambda_k}\}_{k \in K}$ is a frame of $L^2(E)$. Then $\{\hat{h}e_{\lambda_k}\}_{k \in K}$ is a Bessel sequence of $L^2(E)$ if and only if there exists a constant $B > 0$ such that*

$$|\hat{h}(t)| \leq B \quad \text{a.e. } t \in E. \quad (9)$$

Proof. One can apply the same arguments as in proof of Proposition 3.3 part 1. \square

Proposition 3.7. *Let E be a subset of \mathbb{R}^d of finite measure and $h \in L^2(\mathbb{R}^d)$.*

If

1. *there exist constants A and B such that*

$$0 < A \leq B < +\infty \quad \text{and} \quad A \leq |\hat{h}(t)| \leq B \quad \text{a.e. } t \in E, \quad (10)$$

2. *$\{\hat{h}e_{\lambda_k}\}_{k \in K}$ is a frame of $L^2(E)$,*

then $\{e_{\lambda_k}\}_{k \in K}$ is a frame of $L^2(E)$.

Proof. Let α and β be the lower respectively the upper frame bound of $\{\hat{h}e_{\lambda_k}\}_{k \in K}$. By condition 2 for every $f \in L^2(E)$ we can write

$$\begin{aligned} \sum_{k \in K} |\langle f, e_{\lambda_k} \rangle|^2 &= \sum_{k \in K} |\langle f, \frac{1}{\hat{h}} \hat{h} e_{\lambda_k} \rangle|^2 = \sum_{k \in K} |\langle f \frac{1}{\hat{h}}, \hat{h} e_{\lambda_k} \rangle|^2 \\ &\leq \beta \|f \frac{1}{\hat{h}}\|^2 \leq \frac{\beta}{A^2} \|f\|^2. \end{aligned}$$

Analogously we obtain

$$\sum_{k \in K} |\langle f, e_{\lambda_k} \rangle|^2 \geq \frac{\alpha}{B^2} \|f\|^2.$$

\square

Observation 3.8. *Proposition 3.7 remains true if we replace "frame" by "Riesz basis". If we replace condition 2 by the condition that there exists a positive constant A such that*

$$A \leq |\hat{h}(t)| \quad \text{a.e. } t \in E \quad (11)$$

then Proposition 3.7 is also true when we replace "frame" by "Bessel sequence".

The results can also be extended to frame sequences:

Proposition 3.9. *Let E be a subset of \mathbb{R}^d of finite measure and $h \in L^2(\mathbb{R}^d)$. Set $\Lambda = \{\lambda_k\}_{k \in K} \subseteq \mathbb{R}^d$ such that $\{e_{\lambda_k}\}_{k \in K}$ is a frame of $L^2(E)$. Let $F := \text{supp}(\hat{h}) \cap E$. Then*

$$1. \overline{\text{span}\{\hat{h}e_{\lambda_k}\}} = \widetilde{L^2(F)}^{(E)}.$$

2. $\{\hat{h}e_{\lambda_k}\}_{k \in K}$ is a frame sequence of $L^2(E)$ if and only if there exist constants A and B such that

$$0 < A \leq B < +\infty \quad \text{and} \quad A \leq |\hat{h}(t)| \leq B \quad \text{a.e. } t \in F. \quad (12)$$

3. Let \hat{h} be compactly supported. Then $\{\hat{h}e_{\lambda_k}\}_{k \in K}$ is a frame sequence of $L^2(\mathbb{R}^d)$ if and only if there exist constants A and B such that

$$0 < A \leq B < +\infty \quad \text{and} \quad A \leq |\hat{h}(t)| \leq B \quad \text{a.e. } t \in F. \quad (13)$$

Proof.

1. Clearly for each $k \in K$ we have $\hat{h}e_{\lambda_k} \in \widetilde{L^2(F)}^{(E)}$, and so

$$V := \overline{\text{span}\{\hat{h}e_{\lambda_k} : k \in K\}} \subseteq \widetilde{L^2(F)}^{(E)}$$

as this is a closed subspace.

On the other hand due to Lemma 3.2 $\overline{\text{span}\{\hat{h}e_{\lambda_k} : k \in K\}} = L^2(F) \cong \widetilde{L^2(F)}^{(E)}$. Therefore $V = \widetilde{L^2(F)}^{(E)}$.

2. Using the first part the second part is equivalent to

$\{\hat{h}e_{\lambda_k}\}_{k \in K}$ is a frame for $\widetilde{L^2(F)}^{(E)} \cong L^2(F)$ if and only if there exist constants A and B such that

$$0 < A \leq B < +\infty \quad \text{and} \quad A \leq |\hat{h}(t)| \leq B \quad \text{a.e. } t \in F.$$

This is just Proposition 3.3 applied to $L^2(F)$.

3. As F has finite measure, just choose a set of finite measure $E \supset F$ and apply part 2. Note that for $E \subseteq \mathbb{R}^d$ we have $\widetilde{L^2(F)}^{(E)} \cong \widetilde{L^2(F)}^{(\mathbb{R}^d)}$.

□

4 Application to Frames of Translates

Let $E \subseteq \mathbb{R}^d$ of finite measure.

$$P_E = \left\{ f \in L^2(\mathbb{R}^d) : \text{supp } \hat{f} \subseteq E \right\}.$$

Theorem 4.1. Let $\Lambda = \{\lambda_k\}_{k \in K} \subseteq \mathbb{R}^d$ such that $\{e_{\lambda_k}\}_{k \in K}$ is a frame for $L^2(E)$. Let $h \in P_E$. Then

1. $\{T_{\lambda_k} h\}_{k \in K}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$ if and only if there exists $B > 0$ such that $|\hat{h}(\omega)| \leq B$ a.e.
2. $\{T_{\lambda_k} h\}_{k \in K}$ is a frame for P_E if and only if there exist $B \geq A > 0$ such that $A \leq |\hat{h}(\omega)| \leq B$ for almost all $\omega \in E$.
3. $\{T_{\lambda_k} h\}_{k \in K}$ is a frame sequence in $L^2(\mathbb{R}^d)$ if and only if there exist $B \geq A > 0$ such that $A \leq |\hat{h}(\omega)| \leq B$ for a.e. $\omega \in \text{supp } \hat{h}$.

Proof.

1. : Let $f \in L^2(\mathbb{R}^d)$.

$$\sum_{k \in K} |\langle f, T_{\lambda_k} h \rangle_{L^2(\mathbb{R}^d)}|^2 = \sum_{k \in K} |\langle \hat{f}, e_{\lambda_k} \hat{h} \rangle_{L^2(\mathbb{R}^d)}|^2 = \sum_{k \in K} |\langle \hat{f}, e_{\lambda_k} \hat{h} \rangle_{L^2(E)}|^2 = (*)$$

By Proposition 3.6 there exists $\beta > 0$ such that $(*) \leq \beta \cdot \|\hat{f}\|_{L^2(E)}^2$ if and only if there exists $B > 0$ such that $|\hat{h}(\omega)| \leq B$ for almost all $\omega \in E$. As $\text{supp } \hat{h} \subseteq E$ this is equivalent to: There exists $B > 0$ such that $|\hat{h}(\omega)| \leq B$ almost everywhere. Since $\|\hat{f}\|_{L^2(E)} \leq \|\hat{f}\|_{L^2(\mathbb{R}^d)}$ we have:

$$\exists \beta > 0 : \sum_{k \in K} |\langle f, T_{\lambda_k} h \rangle_{L^2(\mathbb{R}^d)}|^2 \leq \beta \cdot \|\hat{f}\|_{L^2(\mathbb{R}^d)}^2$$

is equivalent to

$$\exists B > 0 : |\hat{h}(\omega)| \leq B \text{ a.e.}$$

2. & 3.:

For $f \in P_E$ we have

$$\sum_{k \in K} |\langle f, T_{\lambda_k} h \rangle_{P_E}|^2 = \sum_{k \in K} |\langle \hat{f}, e_{\lambda_k} \hat{h} \rangle_{L^2(E)}|^2.$$

Note that $\|f\|_{P_E} = \|\hat{f}\|_{L^2(E)}$. So the statements are a direct consequence of Proposition 3.6 and Proposition 3.9.

□

This implies the following interesting Corollary.

Corollary 4.2. *Let $h \in P_E$ such that \hat{h} is continuous. Then there does not exist $\Lambda = \{\lambda_k\}_{k \in K} \subseteq \mathbb{R}^d$ such that $\{h(\cdot - \lambda_k)\}_{k \in K}$ is a frame of P_E .*

Example:

Consider the Paley Wiener space

$$P_{1/2} = \left\{ f \in L^2(\mathbb{R}) : \text{supp } \hat{f} \subseteq \left[-\frac{1}{2}, \frac{1}{2} \right] \right\},$$

which is generated by $\psi(x) = \frac{\sin \pi x}{\pi x}$.

The translates $\{\psi(\cdot - k)\}_{k \in \mathbb{Z}}$ are an orthonormal basis, in particular a frame for $P_{1/2}$.

If $h \in P_{1/2} \cap L^1(\mathbb{R})$ there does not exist $\Lambda = \{\lambda_k\}_{k \in K} \subseteq \mathbb{R}$ such that $\{h(\cdot - \lambda_k)\}_{k \in K}$ is a frame for $P_{1/2}$.

The result of Corollary 4.2 represents an obstacle for the applications, since any generator h in our construction will have poor decay and in consequence it will produce a big error if we need to truncate the expansions in terms of the translates of h .

However, this problem can be overcome for open bounded sets E if we generalize a trick that appears in [11] and [1] to this case. The price to pay for this, is a little bit of oversampling.

Theorem 4.3. *Let $E \subset \mathbb{R}^d$ be an open bounded set and $\Lambda = \{\lambda_k\}_{k \in K}$ a separated sequence in \mathbb{R}^d such that the exponentials $\{e_{\lambda_k}\}_{k \in K}$ form a frame of $L^2(E)$. Then, there exists a separated sequence $\Lambda' = \{\lambda'_k\}$ containing Λ and a function g of the Schwartz class, compactly supported in frequency, such that each function f in P_E has an expansion as*

$$f(x) = \sum_{k \in K} \alpha_k g(x - \lambda'_k), \quad (14)$$

where the sequence of coefficients $\{\alpha_k\}_{k \in K}$ is in $l_2(\Lambda')$ and the series converges uniformly, unconditionally and in $L^2(\mathbb{R}^d)$.

Proof. For a small $\delta > 0$ let $E_\delta = \{x \in \mathbb{R}^d : d(x, E) < \delta\}$.

Consider a function $g \in L^2(\mathbb{R}^d)$ such that its Fourier transform \hat{g} satisfies: \hat{g} is of class C^∞ , $0 \leq \hat{g} \leq 1$, $\hat{g}(\omega) = 1$ for $\omega \in E$, and $\hat{g} = 0$ for $\omega \in \mathbb{R}^d \setminus E_\delta$. Then \hat{g} is a Schwartz class function, and therefore g is, too.

Now we consider a separated sequence Λ' containing Λ . Clearly the sequence Λ' can be chosen so that the associated exponentials form a frame of $L^2(E_\delta)$. One simply adds sufficient points to Λ , to decrease its gap.

For $f \in P_E \subset P_{E_\delta}$, using the frame expansion, we have that

$$\hat{f}(\omega) = \sum_{\lambda' \in \Lambda'} \alpha_k e_{\lambda'_k}(\omega) \chi_{E_\delta}(\omega), \quad a.e. \omega \in \mathbb{R}^d,$$

with unconditional convergence.

Now, because of the properties of g and the fact that $\text{supp}(\hat{f}) \subset E$ we can write

$$\hat{f}(\omega) = \sum_k \alpha_k e_{\lambda'_k}(\omega) \hat{g}(\omega), \quad a.e. \omega \in \mathbb{R}^d. \quad (15)$$

Taking the inverse Fourier transform in (15) we have

$$f(x) = \sum_k \alpha_k g(x - \lambda'_k), \quad a.e. x \in \mathbb{R}^d,$$

where the convergence is unconditional in $L^2(\mathbb{R})$.

Moreover, by Cauchy Schwartz,

$$\left| \sum_{|k| \leq N} \alpha_k g(x - \lambda'_k) \right|^2 \leq \|\alpha\|_2^2 \sum_{|k| \leq N} |g(x - \lambda'_k)|^2.$$

The uniform convergence is therefore a straightforward consequence of the decay of g and the fact that the sequence Λ' is separated. \square

Notice that the function g is not in P_E and its translates by elements in $\{\lambda'_k\}$ do not form a frame sequence. However its orthogonal projections on P_E is a frame of P_E .

Proposition 4.4. *With the assumptions of Theorem 4.3, $\{T_{\lambda'_k} g\}$ is an outer frame for P_E .*

Proof. Let \mathcal{P} denote the orthogonal projector onto P_E . For $f \in P_E$ we have,

$$\begin{aligned} \langle \mathcal{P}(T_{\lambda'_k} g), f \rangle &= \langle T_{\lambda'_k} g, \mathcal{P}f \rangle = \langle T_{\lambda'_k} g, f \rangle = \\ &= \langle e_{\lambda'_k} \hat{g}, \hat{f} \rangle = \int_{\mathbb{R}^d} (\hat{g} \bar{\hat{f}}) e_{\lambda'_k} = \overline{\int_E \hat{f} e_{-\lambda'_k}} \end{aligned}$$

Since $\text{supp}(\hat{f}) \subset E$ and $\{e_{-\lambda'_k}\}_k$ forms a frame of $L^2(E)$ we have that $\{\mathcal{P}(T_{\lambda'_k} g)\}_{k \in K}$ forms a frame for P_E . \square

The next result is about properties that are preserved under the action of convolution.

Proposition 4.5. *Let $\Lambda = \{\lambda_k\}_{k \in K} \subseteq \mathbb{R}^d$ such that $\{e_{\lambda_k}\}_{k \in K}$ is a frame for $L^2(E)$. Let $f, g \in P_E$. Then*

1. *If $\{T_{\lambda_k} f\}_{k \in K}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$, and $\{T_{\lambda_k} g\}_{k \in K}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$, then $\{T_{\lambda_k}(f * g)\}_{k \in K}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$.*
2. *If $\{T_{\lambda_k} f\}_{k \in K}$ is a frame for P_E and $\{T_{\lambda_k} g\}_{k \in K}$ is a frame for P_E , then $\{T_{\lambda_k}(f * g)\}_{k \in K}$ is a frame for P_E .*
3. *If $\{T_{\lambda_k} f\}_{k \in K}$ is a frame sequence in $L^2(\mathbb{R}^d)$, and $\{T_{\lambda_k} g\}_{k \in K}$ is a frame sequence in $L^2(\mathbb{R}^d)$, then $\{T_{\lambda_k}(f * g)\}_{k \in K}$ is a frame sequence in $L^2(\mathbb{R}^d)$.*
4. *If $\{T_{\lambda_k} f\}_{k \in K}$ is a frame for P_E and $\{T_{\lambda_k}(f * g)\}_{k \in K}$ is a frame for P_E , then $\{T_{\lambda_k} g\}_{k \in K}$ is a frame for P_E .*

5. If $\{T_{\lambda_k} f\}_{k \in K}$ is a frame sequence in $L^2(\mathbb{R}^d)$ and $\{T_{\lambda_k}(f * g)\}_{k \in K}$ is a frame sequence in $L^2(\mathbb{R}^d)$, then $\{T_{\lambda_k} g\}_{k \in K}$ is a frame sequence in $L^2(\mathbb{R}^d)$.
6. Let $\{T_{\lambda_k}(f * g)\}_{k \in K}$ be a Bessel sequence in $L^2(\mathbb{R}^d)$. If there exists $C > 0$ such that $|\hat{f}(\omega)| \geq C$ a.e., then $\{T_{\lambda_k} g\}_{k \in K}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$.

Proof. First observe that $f * g \in P_E$.

1. By Theorem 4.1 there exist $B_1, B_2 > 0$ such that $|\hat{f}(\omega)| \leq B_1$ a.e. and $|\hat{g}(\omega)| \leq B_2$ a.e. Since $\widehat{f * g} = \hat{f} \hat{g}$, we have that $|\widehat{f * g}(\omega)| \leq B_1 B_2$ a.e. and the result follows.

Part 2 and part 3 can be proved analogously.

4. By Theorem 4.1,

$A_1 \leq |\hat{f}(\omega)| \leq B_1$ for almost all $\omega \in E$ and $A_2 \leq |\hat{f} \hat{g}(\omega)| \leq B_2$ for almost all $\omega \in E$. Hence

$$\frac{A_2}{B_1} \leq |\hat{g}(\omega)| \leq \frac{B_2}{A_1} \text{ for almost all } \omega \in E.$$

The proofs of 5. and 6. are analogous. □

The following proposition gives necessary and sufficient conditions in order that the union of frame-sequences of irregular translations is a frame sequence. The sufficient condition was proved first in [1]. We include a proof here for completeness.

Proposition 4.6. *Let $\{E_j\}_{j \in J}$ be a family of subsets of \mathbb{R}^d of finite measure such that $h_j \in P_{E_j}$ for all $j \in J$. Assume that $\{e_{\lambda_k} \chi_{E_j}\}_{k \in K}$ is a frame for $L^2(E_j)$ with frame bounds m_j and M_j for every $j \in J$. If $m = \inf_j m_j > 0$ and $M = \sup_j M_j < +\infty$, then*

$\{T_{\lambda_k} h_j\}_{k \in K, j \in J}$ is a frame for $P_{\bigcup_{j \in J} E_j}$ if and only if there exist constants $0 < p \leq P$ such that

$$p \leq \sum_{j \in J} |\hat{h}_j(w)|^2 \leq P \quad \text{a.e. in } \bigcup_{j \in J} E_j.$$

Proof. \Leftarrow)

Let $f \in P_{\bigcup_{j \in J} E_j}$.

$$\begin{aligned}
& \sum_{k \in K, j \in J} \left| \langle T_{\lambda_k} h_j, f \rangle_{L^2(\mathbb{R}^d)} \right|^2 \\
&= \sum_{k \in K, j \in J} \left| \langle e_{\lambda_k} \chi_{E_j} \hat{h}_j, \hat{f} \rangle_{L^2(\mathbb{R}^d)} \right|^2 = \sum_{k \in K, j \in J} \left| \langle e_{\lambda_k} \chi_{E_j} \hat{h}_j, \hat{f} \rangle_{L^2(E_j)} \right|^2 \\
&= \sum_{j \in J} \sum_{k \in K} \left| \langle e_{\lambda_k} \chi_{E_j}, \overline{\hat{h}_j} \hat{f} \rangle_{L^2(E_j)} \right|^2 \leq \sum_{j \in J} M_j \|\overline{\hat{h}_j} \hat{f}\|^2 \\
&\leq M \sum_{j \in J} \int_{\mathbb{R}^d} |\overline{\hat{h}_j} \hat{f}|^2(\omega) d\omega = M \int_{\mathbb{R}^d} \sum_{j \in J} |\hat{h}_j(\omega)|^2 |\hat{f}|^2(\omega) d\omega \\
&\leq MP \int_{\mathbb{R}^d} |\hat{f}|^2(\omega) d\omega = MP \|f\|^2.
\end{aligned}$$

The other inequality can be proved analogously.

\implies)

Let $\{T_{\lambda_k} h_j\}_{k \in K, j \in J}$ be a frame of $P_{\bigcup_{j \in J} E_j}$ and assume that for every $p > 0$ there exists a set $U \subseteq \bigcup_{j \in J} E_j$ of positive measure such that $\sum_{j \in J} |\hat{h}_j(w)|^2 < p$ for every $w \in U$.

For $n \in \mathbb{N}$ define $E_n = \{w \in \bigcup_{j \in J} E_j : \sum_{j \in J} |\hat{h}_j(w)|^2 < \frac{1}{n}\}$. Let $C_k = \{x \in \mathbb{R}^d : k-1 \leq \|x\| < k\}$. We can write $E_n = \bigcup_{k \in \mathbb{N}} E_n \cap C_k$. Since $|E_n| > 0$, there exists a $k_0 \in \mathbb{N}$ such that $|E_n \cap C_{k_0}| > 0$. Let $A_n = E_n \cap C_{k_0}$. Since $0 < |A_n| < +\infty$ we define

$$f_n(t) = \begin{cases} \frac{1}{\sqrt{|A_n|}} & \text{for } t \in A_n \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

Clearly $f_n \in L^2(\bigcup_{j \in J} E_j)$ for every $n \in \mathbb{N}$.

If α is the lower frame bound of $\{T_{\lambda_k} h_j\}_{\lambda_k \in \Lambda, j \in J}$ and M is the upper frame bound of $\{e_{\lambda_k} \chi_{E_j}\}_{k \in K}$, then

$$\begin{aligned}
\alpha &\leq \sum_{k \in K, j \in J} |\langle T_{\lambda_k} h_j, f_n \rangle|^2 = \sum_{k \in K, j \in J} \left| \langle e_{\lambda_k} \chi_{E_j} \hat{h}_j, \hat{f}_n \rangle \right|^2 \\
&= \sum_{k \in K, j \in J} \left| \langle e_{\lambda_k} \chi_{E_j}, \overline{\hat{h}_j} \hat{f}_n \rangle \right|^2 \leq \sum_{j \in J} M_j \|f_n \hat{h}_j\|^2 \\
&\leq M \sum_{j \in J} \int_{A_n} |f_n \hat{h}_j|^2(\omega) d\omega = \frac{1}{|A_n|} M \int_{A_n} \sum_{j \in J} |\hat{h}_j|^2(\omega) d\omega \\
&\leq \frac{M}{n} \rightarrow 0,
\end{aligned}$$

which is a contradiction. The other frame inequality can be proved analogously. \square

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