CONSTRUCTION AND RECONSTRUCTION OF TIGHT FRAMELETS AND WAVELETS VIA MATRIX MASK FUNCTIONS

MARCIN BOWNIK AND ZIEMOWIT RZESZOTNIK

ABSTRACT. The paper develops construction procedures for tight framelets and wavelets using matrix mask functions in the setting of a generalized multiresolution analysis (GMRA). We show the existence of a scaling vector of a GMRA such that its first component exhausts the spectrum of the core space near the origin. The corresponding low-pass matrix mask has an especially advantageous form enabling an effective reconstruction procedure of the original scaling vector. We also prove a generalization of the Unitary Extension Principle for an infinite number of generators. This results in the construction scheme for tight framelets using low-pass and high-pass matrix masks generalizing the classical MRA constructions. We prove that our scheme is flexible enough to reconstruct all possible orthonormal wavelets. As an illustration we exhibit a pathwise connected class of non-MSF non-MRA wavelets sharing the same wavelet dimension function.

1. INTRODUCTION AND PRELIMINARIES

The main aim of this work is to develop a constructive procedure for constructing tight framelets and wavelets from more primitive objects being low-pass and high-pass matrix mask functions. In the case of multiresolution analysis (MRA) wavelets such procedure is well studied and understood.

Usually, an MRA construction starts with a 1-periodic measurable function m, also called a low-pass mask and satisfying the quadrature-mirror equation

$$|m(\xi)|^2 + |m(\xi + 1/2)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}. \tag{1.1}$$

Under small regularity assumptions, such as m is Hölder continuous at 0 and $m(0) = 1$, one defines a refinable function $\varphi \in L^2(\mathbb{R})$ by

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m(2^{-j}\xi). \tag{1.2}$$

While φ might fail to be an orthogonal scaling function of an MRA, one can always obtain a tight frame wavelet $\psi \in L^2(\mathbb{R})$ using a high-pass mask h by

$$\hat{\psi}(\xi) = h(\xi/2)\hat{\varphi}(\xi/2), \qquad \text{where } h(\xi) = e^{2\pi i \xi}\overline{m(\xi + 1/2)}. \tag{1.3}$$

Date: September 7, 2006.

2000 *Mathematics Subject Classification.* Primary: 42C40.

The first author was partially supported by the NSF grant DMS-0441817. The second author was supported under the EU-project MEXT-CT-2004-517154. The research for this paper was carried during a one-year visit of the second author to the University of Oregon.

The fact that ψ is a tight framelet can be shown directly by employing the characterization equations [17, Section 7.1]. Alternatively, it is also a consequence of a Unitary Extension Principle of Ron and Shen [15, 20].

While Hölder continuity of m at 0 is a relatively weak assumption, some MRA wavelets can not be obtained by this scheme [14]. To circumvent this problem, Paluszyński et al. [18] introduced the class of low-pass filters satisfying

$$\lim_{n\to\infty}\prod_{j=n}^{\infty}|m(2^{-j}\xi)|=1 \qquad \text{for a.e. } \xi\in\mathbb{R}.$$

This is obviously much weaker condition than Hölder continuity. Moreover, any low-pass mask m of an MRA scaling function must satisfy it by the characterization of scaling functions of MRAs [17]. While the infinite product (1.2) might not be convergent, the authors of [18] proved that one can always construct a refinable function φ satisfying

$$\hat\varphi(\xi)=m(\xi/2)\hat\varphi(\xi/2) \qquad \text{a.e. } \xi\in\mathbb{R}.$$

This is because the product (1.2) converges after taking absolute values and one must only recover the phase factor of $\hat\varphi$ by using a mutiplier argument. As a consequence, the procedure of constructing MRA tight frame wavelets from [18] recovers all possible MRA wavelets. Similar ideas were used to prove the connectivity result for MRA wavelets by the Wutam Consortium [24].

Since MRA wavelets form only a special class among all orthonormal wavelets, one could ask whether similar construction and reconstruction procedures are possible for non-MRA wavelets. The most natural way of classifying non-MRA wavelets uses the wavelet dimension function

$$D_\psi(\xi)=\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}}|\hat\psi(2^j(\xi+k))|^2.$$

The wavelet dimension function has many interesting properties that were investigated by several authors [1, 2, 13, 17, 19, 21]. One of its fundamental properties says that D_ψ can be identified with the multiplicity function of the core space of a GMRA generated by the wavelet ψ. In particular, ψ is an MRA wavelet if and only if $D_\psi\equiv 1$. Moreover, by the result of Speegle and the authors [13] all possible wavelet dimension functions m are characterized by the following 4 conditions:

(D1) $m:\mathbb{R}\to\mathbb{N}\cup\{0\}$ is a measurable 1-periodic function,
(D2) $m(\xi)+m(\xi+1/2)=m(2\xi)+1$ for a.e. $\xi\in\mathbb{R}$,
(D3) $\sum_{k\in\mathbb{Z}}\mathbf{1}_\Delta(\xi+k)\ge m(\xi)$ for a.e. $\xi\in\mathbb{R}$, where

$$\Delta=\{\xi\in\mathbb{R}: m(2^{-j}\xi)\ge 1 \text{ for } j\in\mathbb{N}\cup\{0\}\},$$

(D4) $\liminf_{j\to\infty} m(2^{-j}\xi)\ge 1$ for a.e. $\xi\in\mathbb{R}$.

Note that we have intentionally omitted the integrability condition on m, since it is a consequence of (D1) and (D2) by Lemma 3.1 proved in this paper.

The above characterization opens the possibility of constructing wavelets and framelets from more general low-pass matrix masks than the standard scalar masks satisfying (1.1). In general, one starts with a matrix-valued 1-periodic function M satisfying

$$M(\xi)M^*(\xi)+M(\xi+1/2)M^*(\xi+1/2)=\Omega(2\xi) \qquad \text{for a.e. } \xi\in\mathbb{R}, \tag{1.4}$$

where

$$\Omega(\xi) = \operatorname{diag}(\mathbf{1}_{S_1}(\xi), \mathbf{1}_{S_2}(\xi), \ldots), \qquad S_j = \{\xi \in \mathbb{R} : m(\xi) \geq j\},\ j \in \mathbb{Z}.$$

Baggett et al. [4] showed that if the multiplicity function m is bounded and M satisfies some weak regularity assumptions then one can define a refinable vector function $\Phi = (\varphi_j)_{j\in J} \subset L^2(\mathbb{R})$, $J = \{1, \ldots, N\}$, by

$$\hat{\Phi}(\xi) = \left[\prod_{j=1}^{\infty} M(2^{-j}\xi)\right] e, \tag{1.5}$$

where $e = (1, 0, \ldots, 0)$. To make sure that the above product converges one must assume that M is Hölder continuous at 0 and $M(0)$ is a matrix having all zero entries except a single 1 in the upper-left corner. In general, Φ might not be a scaling vector of some GMRA in the sense that each φ_j is a quasi-orthogonal generator of

$$\mathcal{S}(\varphi_j) = \overline{\operatorname{span}}\{\varphi_j(\cdot - k) : k \in \mathbb{Z}\}$$

and $\mathcal{S}(\varphi_i) \perp \mathcal{S}(\varphi_j)$ for $i \neq j$. Nevertheless, the authors of [4] proved that by choosing an appropriate high-pass matrix mask H, one can always obtain a tight frame wavelet $\psi \in L^2(\mathbb{R})$ by setting

$$\hat{\psi}(\xi) = H(\xi/2)\hat{\Phi}(\xi/2) \qquad \text{a.e. } \xi \in \mathbb{R}. \tag{1.6}$$

More precisely, H is 1-periodic measurable $1 \times N$ matrix-valued function satisfying

$$H(\xi)H^*(\xi) + H(\xi + 1/2)H^*(\xi + 1/2) = 1 \qquad \text{a.e. } \xi \in \mathbb{T}, \tag{1.7}$$

$$M(\xi)H^*(\xi) + M(\xi + 1/2)H^*(\xi + 1/2) = 0 \qquad \text{a.e. } \xi \in \mathbb{T}. \tag{1.8}$$

This naturally leads to a fundamental problem of the theory of non-MRA wavelets, which asks whether it is possible to use the above scheme of low-pass and high-pass matrix masks to construct all orthonormal wavelets.

The goal of this paper is to give an affirmative answer to this problem. To give the idea of the level of difficulty behind this project one should realize that, a priori, no regularity assumption on the low-pass matrix mask functions can be assumed. Furthermore, an example in [13] demonstrates that the multiplicty function m could be unbounded which leads to a matrix-valued low-pass mask M of infinite size. Hence, the infinite product in (1.5) might not converge and special convergence procedures are needed to interpret such ill-defined expressions.

The starting point of this paper is the investigation of the properties of scaling vectors corresponding to the core space of a GMRA. Unlike the case of an MRA, where the scaling function is unique (up to a unimodular 1-periodic phase factor in the Fourier domain), there are many possible choices for scaling vectors for a GMRA. This has been traditionally considered as an impediment of a successful theory, since different choices of a scaling vector Φ could lead to totally different low-pass masks M satisfying

$$\hat{\Phi}(\xi) = M(\xi/2)\Phi(\xi/2) \qquad \text{a.e. } \xi \in \mathbb{R}. \tag{1.9}$$

It might seem that the only useful information extracted from (1.9) is a matrix analogue of the quadrature-mirror equation (1.4). Nevertheless, we show that abundance of choice is also a blessing if one carefully chooses generators of the scaling vector. The key idea is to choose the first generator φ_1 such that it exhausts the entire spectrum of the core space near the origin. Consequently, the remaining generators $\varphi_2, \varphi_3, \dots$ must be supported away from the origin in the Fourier domain. This leads to an especially advantageous form of the low-pass matrix mask that is described below.

Theorem 1.1. *Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is a GMRA such that $\dim_{V_0}$ is finite a.e. Then, there exists a scaling vector Φ such that the corresponding low-pass matrix mask $M = (m_{i,j})$ satisfies (1.4) and the first column of $M(\xi)$ has zeros in every entry, except the first where it has absolute value "almost equal" to 1 near the origin. More precisely, for a.e. $\xi \in \mathbb{R}$*

$$\exists N = N(\xi) < \infty \quad \text{such that } m_{i,1}(2^{-j}\xi) = 0 \qquad \text{for } i \geq 2,\ j > N, \quad \text{and} \tag{1.10}$$

$$\lim_{k\to\infty} \prod_{j=k}^{\infty} |m_{1,1}(2^{-j}\xi)| = 1. \tag{1.11}$$

In the case when a GMRA is associated with an orthonormal wavelet ψ, the almost everywhere finiteness of $\dim_{V_0}$ is automatically satisfied. Moreover, it is easy to verify that the high-pass filter H given by (1.6) must satisfy (1.7) and (1.8). The crux of our approach is the assertion claiming that one can reverse the above process. That is, given a low-pass and high-pass matrix masks as in Theorem 1.1, we can construct a tight frame wavelet ψ satisfying (1.6) and (1.9). More precisely, we have the following construction scheme for tight framelets.

Theorem 1.2. *Let m be a function satisfying (D1)–(D4). Let $M = (m_{i,j})$ be a matrix-valued function satisfying (1.4), (1.10), and (1.11). Then, there exists a refinable vector $\Phi \subset L^2(\mathbb{R})$ satisfying (1.9) and*

$$\lim_{j\to\infty} ||\hat{\Phi}(2^{-j}\xi)|| = 1 \qquad \text{a.e. } \xi \in \mathbb{R}.$$

Moreover, if a matrix function H satisfies

$$M^*(\xi)M(\xi+d) + H^*(\xi)H(\xi+d) = \Omega(\xi)\delta_{0,d} \qquad \text{for } d = 0, 1/2 \text{ and for a.e. } \xi \in \mathbb{T}, \tag{1.12}$$

then ψ given by (1.6) is a tight framelet.

The first key ingredient in the proof of Theorem 1.2 is the existence of a refinable vector Φ, which is a result of a special convergence procedure making sense out of potentially divergent infinite product in (1.5). The second ingredient is a generalization of the Unitary Extension Principle to a situation when a refinable vector Φ has infinitely many components. In particular, we also show that orthogonality of rows (that is, conditions (1.4), (1.7) and (1.8)) always implies the orthogonality of columns (i.e., condition (1.12)), but not vice versa. We also prove that if one wishes to construct a wavelet, then the orthogonality of rows must necessarily be imposed, whereas the weaker condition (1.12) is sufficient for constructing framelets.

The last part of the paper proves that the above scheme is flexible enough to reconstruct all possible wavelets ψ. A pivotal role in the reconstruction scheme is played by the concept of a multiplier. We say that a unimodular function ν is a *multiplier* associated to M if it satisfies

$$\nu(2\xi)\overline{\nu(\xi)}|m_{1,1,}(\xi)| = m_{1,1}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}.$$

Theorem 1.3. *Suppose ψ is an orthonormal wavelet and let $\{V_j\}_{j\in\mathbb{Z}}$ be its associated GMRA. Let Φ be the scaling vector provided by Theorem 1.1. Let M and H be the low-pass and high-pass matrix mask functions of Φ and ψ, respectively. Then, there exists a multiplier ν associated to M such that the scaling vector is recovered by*

$$\hat{\Phi}(\xi) = \lim_{N\to\infty} \nu(2^{-N}\xi) \prod_{j=1}^{N} M(2^{-j}\xi)e \qquad \text{for a.e. } \xi, \tag{1.13}$$

and the wavelet ψ is recovered by (1.6).

Finally, the paper ends with a selection of examples illustrating the inner workings of the construction and reconstruction procedures hidden behind the facades of Theorems 1.2 and 1.3. In particular, we give an example of a class of non-MSF and non-MRA wavelets such that all of its members share the same dimension function. We also prove that this class of wavelets is pathwise connected indicating that our techniques have a potential of attacking probably the most fundamental, yet recalcitrant, problem of the connectivity of the set of all orthonormal wavelets.

Despite the fact that all of our results are motivated by the classical case of dyadic dilations in $\mathbb{R}$, we will adopt a more general setting of expansive integer-valued dilations in $\mathbb{R}^n$. More specifically, we shall assume that we are given an $n\times n$ integer-valued matrix A that is expansive, i.e., all its eigenvalues have modulus greater than 1. For simplicity, its transpose will be denoted by B.

We recall that a sequence $\{D^j(V) : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^n)$ is called a *generalized multiresolution analysis* (GMRA) if

(M1) $T_k V = V$ for all $k \in \mathbb{Z}^n$,
(M2) $V \subset D(V)$,
(M3) $\overline{\bigcup_{j\in\mathbb{Z}} D^j(V)} = L^2(\mathbb{R}^n)$,
(M4) $\bigcap_{j\in\mathbb{Z}} D^j(V) = \{0\}$.

Here, the *dilation* operator D is given by $Df(x) = |\det A|^{1/2} f(Ax)$ for some $n\times n$ expansive integer-valued matrix A and the *translation* operator $T_k f(x) = f(x-k)$ for some $k \in \mathbb{Z}^n$.

As we can see, a GMRA is based on the *core space* V. Condition (M1) means that V is a shift-invariant (SI) space. If V satisfies (M2), then we call it *refinable*. Also, we shall often write V_j instead of $D^j(V)$.

We say that a finite family $\Psi = \{\psi^1,\dots,\psi^L\} \subset L^2(\mathbb{R}^n)$ is a wavelet if its associated affine system

$$\psi_{j,k}(x) = |\det A|^{j/2}\psi(A^j x - k), \qquad j\in\mathbb{Z}, k\in\mathbb{Z}^n, \psi\in\Psi$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. In the more general case, when the affine system is a frame or tight frame (with constant 1), we say that Ψ is a framelet or a tight framelet. The latter are characterized by the well-known equations that we list in (3.7) and (3.8). Moreover, the framelet Ψ is called *semi–orthogonal* if

$$\bigoplus_{j\in\mathbb{Z}} D^j(W) = L^2(\mathbb{R}^n), \qquad \text{where } W = \overline{\operatorname{span}}\{\psi(\cdot - k) : k\in\mathbb{Z}^n, \psi\in\Psi\}.$$

It turns out that every semi-orthogonal tight framelet comes from a GMRA. Indeed, for a finite family $\Psi \subset L^2(\mathbb{R}^n)$ we define its *space of negative dilates* V by

$$V = \overline{\operatorname{span}}\{\psi_{j,k} : j < 0, k \in \mathbb{Z}^n, \psi \in \Psi\}.$$

We say that a framelet Ψ is associated with a GMRA (or that it generates a GMRA) if its space of negative dilates V satisfies (M1)–(M4). It is not hard to check that if Ψ is a semi-orthogonal tight framelet then conditions (M1)–(M4) hold and, therefore, V is a core space of a GMRA.

Depending on the context, V and V_0 shall denote either a general SI space, or the core space of a GMRA, or the space of negative dilates.

Every shift-invariant space $V \subset L^2(\mathbb{R}^n)$ has a *set of generators* Φ, that is, a countable family of functions whose integer shifts form a tight frame (with constant 1) for V. Although this family is not unique, the function

$$\sigma_V(\xi) = \sum_{\varphi \in \Phi} |\hat{\varphi}(\xi)|^2$$

does not depend (except on a set of measure zero) on the choice of the family of generators. Here, the Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi\rangle}\, dx.$$

We call σ_V the *spectral function* of V. This notion was introduced by the authors in [11]. The basic property of σ is that it is additive on countable orthogonal sums and that $\sigma_{L^2(\mathbb{R}^n)} = 1$. The spectral function also behaves nicely under dilations since $\sigma_{D(V)}(\xi) = \sigma_V(B^{-1}\xi)$. Moreover, if V is generated by a single function φ then

$$\sigma_V(\xi) = \begin{cases} |\hat{\varphi}(\xi)|^2 (\sum_{k \in \mathbb{Z}^n} |\hat{\varphi}(\xi + k)|^2)^{-1} & \text{for } \xi \in \operatorname{supp} \hat{\varphi}, \\ 0 & \text{otherwise.} \end{cases}$$

There are several other equivalent ways of defining the spectral function. The original one involves the *range function* $\mathcal{J}$, that is, a mapping from the torus $\mathbb{T}^n$ to the set of closed subspaces of $\ell^2(\mathbb{Z}^n)$. It turns that there is a $1-1$ correspondence between SI spaces and measurable range functions $\mathcal{J}$ given by

$$V = \{f \in L^2(\mathbb{R}^n) : \mathcal{T} f(\xi) \in \mathcal{J}(\xi) \text{ for a.e. } \xi \in \mathbb{T}^n\},$$

see [9, Proposition 1.5]. Here, $\mathcal{T} : L^2(\mathbb{R}^n) \to L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n))$ is an isometric isomorphism given by $\mathcal{T} f(\xi) = (\hat{f}(\xi + k))_{k \in \mathbb{Z}^n}$, where $\mathbb{T}^n$ is identified with $[-1/2, 1/2)^n$. The spectral function σ_V can be equivalently defined by

$$\sigma_V(\xi + k) = ||P_{\mathcal{J}}(\xi) e_k||^2 \qquad \text{for } \xi \in \mathbb{T}^n \text{ and } k \in \mathbb{Z}^n,$$

where $\{e_k\}_{k \in \mathbb{Z}^n}$ denotes the standard basis of $\ell^2(\mathbb{Z}^n)$ and $P_{\mathcal{J}}(\xi)$ is an orthogonal projection of $\ell^2(\mathbb{Z}^n)$ onto $\mathcal{J}(\xi)$. Since these notions appear only in the proofs of Lemma 2.1 and Theorem 2.1, we ask the reader to consult [11] before accessing this part of the paper.

The spectral function also allows us to define the *dimension function* of V

$$\dim_V(\xi) = \sum_{k \in \mathbb{Z}^n} \sigma_V(\xi + k).$$

The dimension function (also called the multiplicity function) is integer–valued and additive on countable orthogonal sums as well. Moreover, the minimal number of functions needed to generate V is equal to the L^∞ norm of $\dim_V$. Again, we refer the reader to [9, 11] for the proofs of all these facts.

2. Scaling vectors for GMRA

The main goal of this section is to provide a constructive procedure for selecting a suitable set of generators Φ for the core space V_0 of a GMRA $\{V_j\}_{j\in\mathbb{Z}}$. The result of this procedure is a collection of (mutually orthogonal) quasi-orthogonal generators $\Phi = (\varphi_j)_{j\in J}$ called a scaling vector for V_0. (In particular, if $\{V_j\}_{j\in\mathbb{Z}}$ is a usual MRA, we obtain a single orthogonal generator of V_0, usually called a scaling function.) Quasi-orthogonality means that integer shifts of the generator form a tight frame for the corresponding SI space. Such a space, that is generated by just one function, say φ, is called a principal shift-invariant (PSI) space and is denoted by $\mathcal{S}(\varphi)$.

The main feature of our generator selecting procedure is that it distinguishes the first generator φ_1 as having a dominating effect on all remaining generators. More precisely, the first generator φ_1 is chosen so that it exhausts the entire spectrum of the core space V_0 in some neighborhood of the origin. The fact that the space V_0 is refinable and this exhaustion property of φ_1 leads to a very special form of a matrix mask of Φ, whose first column has zeros in every, but the first entry, near the origin, see Theorem 2.2.

Our procedure is somewhat reminiscent of the superfunction theory in the study of finitely generated shift-invariant (FSI) spaces by de Boor, DeVore, and Ron [7, 8]. Among other things, the authors proved [8, Result 1.2] that an approximation order of a FSI space can be realized by some PSI space generated by a single function ψ, called a "superfunction". Therefore, ψ has a dominating effect by providing the same approximation order as the whole FSI space generated by some finite collection of generators. This is analogous to our construction, where the first generator φ_1 makes all other generators to be innocuous near the origin in the Fourier domain, thus producing a special form of a matrix mask.

To achieve the above dominating effect we show the existence of a quasi-generator φ_0 of a SI space V_0 having the same spectral function as that of V_0 in a pre-specified localized region of the Fourier domain. Therefore, the generator φ_0 exhausts locally the space V_0 in that region.

Lemma 2.1. *Assume that $V_0 \subset L^2(\mathbb{R}^n)$ is SI. Let K be a measurable subset of $\mathbb{R}^n$ such that*

$$|K \cap (K+l)| = 0 \qquad \text{for all } l \in \mathbb{Z}^n \setminus \{0\}.$$

Let $\varphi = P_{V_0}(\check{\mathbf{1}}_K)$, where P_{V_0} is the orthogonal projection onto V_0. Define $\varphi_0 \in L^2(\mathbb{R}^n)$ by

$$\hat{\varphi}_0(\xi) = \begin{cases} \hat{\varphi}(\xi)(\sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(\xi+k)|^2)^{-1/2} & \xi \in \operatorname{supp}\hat{\varphi}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Then $\varphi_0 \in V_0$ is a quasi-orthogonal generator of $\mathcal{S}(\varphi_0)$ and

$$\sigma_{\mathcal{S}(\varphi_0)}(\xi) = |\hat{\varphi}_0(\xi)|^2 = \sigma_{V_0}(\xi) \qquad \text{for a.e. } \xi \in K. \tag{2.2}$$

Proof. Let $\varphi_K \in L^2(\mathbb{R}^n)$ be given by $\hat{\varphi}_K = \mathbf{1}_K$, and hence $\varphi = P_{V_0}\varphi_K$. Clearly, φ_0 is a quasi-orthogonal generator of the PSI space $\mathcal{S}(\varphi_0) = \mathcal{S}(\varphi) \subset V_0$. In particular,

$$|\hat{\varphi}_0(\xi)|^2 = \sigma_{\mathcal{S}(\varphi_0)}(\xi) = \sigma_{\mathcal{S}(\varphi)}(\xi).$$

Let $\mathcal{J}$ be the range function of V_0 with the corresponding projections $P_{\mathcal{J}}(\xi)$. Then for any $f \in L^2(\mathbb{R}^n)$ we have

$$\mathcal{T}(P_{V_0}f)(\xi) = P_{\mathcal{J}}(\xi)(\mathcal{T}f(\xi)) \qquad \text{for a.e. } \xi \in \mathbb{T}^n.$$

Hence, for a.e. $\xi \in \mathbb{T}^n$,

$$\mathcal{T}\varphi(\xi) = \mathcal{T}(P_{V_0}\varphi_K)(\xi) = P_{\mathcal{J}}(\xi)(\mathcal{T}\varphi_K(\xi)) = \begin{cases} P_{\mathcal{J}}(\xi)e_k & \xi + k \in K,\ k \in \mathbb{Z}^n, \\ 0 & \text{otherwise.} \end{cases}$$

Fix $k \in \mathbb{Z}^n$. If $\xi + k \in K$, $\xi \in \mathbb{T}^n$, and $\mathcal{T}\varphi(\xi) \neq 0$, then we necessarily have

$$\sigma_{\mathcal{S}(\varphi)}(\xi + k) = \frac{|\hat{\varphi}(\xi + k)|^2}{||\mathcal{T}\varphi(\xi)||^2} = \frac{|\langle \mathcal{T}\varphi(\xi), e_k\rangle^2}{||\mathcal{T}\varphi(\xi)||^2} = \frac{|\langle P_{\mathcal{J}}(\xi)e_k, e_k\rangle|^2}{||P_{\mathcal{J}}(\xi)e_k||^2} = ||P_{\mathcal{J}}(\xi)e_k||^2 = \sigma_{V_0}(\xi + k).$$

On the other hand, if $\xi + k \in K$, $\xi \in \mathbb{T}^n$, and $\mathcal{T}\varphi(\xi) = 0$, then

$$\sigma_{\mathcal{S}(\varphi)}(\xi + k) = 0 = ||P_{\mathcal{J}}(\xi)e_k||^2 = \sigma_{V_0}(\xi + k).$$

Since $k \in \mathbb{Z}^n$ is arbitrary, this proves (2.2). □

The next result provides a decomposition of any SI space V_0 as an orthogonal sum of carefully chosen PSI spaces, such that the first PSI space exhausts the spectrum of V_0 near the origin.

Theorem 2.1. *Assume that $V_0 \subset L^2(\mathbb{R}^n)$ is SI. Then there exists an orthogonal decomposition*

$$V_0 = \bigoplus_{j=1}^{\infty} \mathcal{S}(\varphi_j) \tag{2.3}$$

such that each φ_j is a quasi-orthogonal generator of $\mathcal{S}(\varphi_j)$ and

$$\operatorname{supp} \dim_{\mathcal{S}(\varphi_j)} = \{\xi \in \mathbb{R}^n : \dim_{V_0}(\xi) \geq j\} \qquad \text{for every } j \in \mathbb{N}. \tag{2.4}$$

Furthermore, the spectral function of $\mathcal{S}(\varphi_1)$ coincides with that of V_0 near the origin, i.e.,

$$\sigma_{\mathcal{S}(\varphi_1)}(\xi) = |\hat{\varphi}_1(\xi)|^2 = \sigma_{V_0}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{T}^n, \tag{2.5}$$

Proof. The existence of a decomposition satisfying (2.3) and (2.4) is already known, see [9]. The novelty of Theorem 2.1 lies in the fact that the first quasi-orthogonal generator φ_1 can be chosen to satisfy (2.5).

Let $\varphi_0 \in V_0$ be a quasi-orthogonal generator guaranteed by Lemma 2.1 with $K = \mathbb{T}^n$. That is,

$$\sigma_{\mathcal{S}(\varphi_0)}(\xi) = |\hat{\varphi}_0(\xi)|^2 = \sigma_{V_0}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{T}^n.$$

Define $E = \operatorname{supp}\dim_{V_0} \setminus \operatorname{supp}\dim_{\mathcal{S}(\varphi_0)}$. Consider two possible cases. If $|E| > 0$, then define a SI space $V = V_0 \cap \check{L}^2(E)$. Here,

$$\check{L}^2(E) = \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset E\}.$$

Let φ be a quasi-orthogonal generator of V such that

$$\operatorname{supp}\dim_{\mathcal{S}(\varphi)} = \{\xi \in \mathbb{R}^n : \dim_V(\xi) \geq 1\} = \{\xi \in E : \dim_{V_0}(\xi) \geq 1\} = E.$$

Since $\varphi_0 \in \check{L}^2(\mathbb{R}^n \setminus E)$, $\varphi \in \check{L}^2(E)$, and the set E is invariant under translations by $\mathbb{Z}^n$, $\varphi_1 = \varphi_0 + \varphi$ is also a quasi-orthogonal generator. Moreover, $\varphi_1 \in V_0$ since $\varphi_0, \varphi \in V_0$, and

$$\operatorname{supp}\dim_{\mathcal{S}(\varphi_1)} = \operatorname{supp}\dim_{\mathcal{S}(\varphi_0)} \cup \operatorname{supp}\dim_{\mathcal{S}(\varphi)} = \operatorname{supp}\dim_{\mathcal{S}(\varphi_0)} \cup E = \operatorname{supp}\dim_{V_0}.$$

Hence, (2.4) holds for $j = 1$. Since $\mathcal{S}(\varphi_0) \subset \mathcal{S}(\varphi_1) \subset V_0$, we have that for a.e. $\xi \in \mathbb{T}^n$

$$\sigma_{V_0}(\xi) \leq \sigma_{\mathcal{S}(\varphi_0)}(\xi) \leq \sigma_{\mathcal{S}(\varphi_1)}(\xi) \leq \sigma_{V_0}(\xi),$$

which proves (2.5). In the case of $|E| = 0$, we let $\varphi_1 = \varphi_0$. Trivially, (2.4) holds for $j = 1$ and (2.5) also holds.

Finally, it suffices to consider a SI space $V_0 \ominus \mathcal{S}(\varphi_1)$ and its decomposition guaranteed by the first part of Theorem 2.1. That is we have:

$$V_0 \ominus \mathcal{S}(\varphi_1) = \bigoplus_{j=2}^{\infty} \mathcal{S}(\varphi_j),$$

$$\operatorname{supp}\dim_{\mathcal{S}(\varphi_j)} = \{\xi \in \mathbb{R}^n : \dim_{V_0 \ominus \mathcal{S}(\varphi_1)}(\xi) \geq j-1\} = \{\xi \in \mathbb{R}^n : \dim_{V_0}(\xi) \geq j\} \qquad \text{for } j \geq 2.$$

Therefore, $\{\varphi_j\}_{j=1}^{\infty}$ is the sequence of quasi-orthogonal generators fulfilling (2.3)–(2.5). □

Theorem 2.1 leads naturally to the definition of a scaling vector for general SI spaces.

Definition 2.1. Suppose that V_0 is SI and for $j \in \mathbb{N}$ let

$$S_j = \{\xi \in \mathbb{R}^n : \dim_{V_0}(\xi) \geq j\}. \tag{2.6}$$

Let $J = \{j \in \mathbb{N} : |S_j| > 0\}$. Naturally,

$$J = \begin{cases} \{1, \dots, L\} & \text{if } L = \operatorname{ess\,sup}_{\xi \in \mathbb{R}^n} \dim_{V_0}(\xi) < \infty \\ \mathbb{N} & \text{otherwise.} \end{cases} \tag{2.7}$$

A *scaling vector* for V_0 is defined as

$$\Phi = (\varphi_j)_{j \in J},$$

where $\{\varphi_j\}_{j \in J}$ are quasi-orthogonal generators as in Theorem 2.1. The Fourier transform of Φ,

$$\hat{\Phi}(\xi) = (\hat{\varphi}_j(\xi))_{j \in J}$$

is treated as a column vector with values in $\ell^2(J)$, since

$$||\hat{\Phi}(\xi)||^2_{\ell^2(J)} = \sigma_{V_0}(\xi) \leq 1. \tag{2.8}$$

Also, define the *diagonal matrix function* of V_0 as

$$\Omega(\xi) = \begin{bmatrix} \mathbf{1}_{S_1}(\xi) & 0 & 0 & \dots \\ 0 & \mathbf{1}_{S_2}(\xi) & 0 & \dots \\ 0 & 0 & \mathbf{1}_{S_3}(\xi) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.9}$$

Since φ_j is a quasi-orthogonal generator of $\mathcal{S}(\varphi_j)$ and $\mathcal{S}(\varphi_j) \perp \mathcal{S}(\varphi_{j'})$ for $j \neq j'$, we have that for a.e. $\xi \in \mathbb{R}^n$

$$\sum_{k\in\mathbb{Z}^n} |\hat{\varphi}_j(\xi+k)|^2 = \mathbf{1}_{S_j}(\xi),$$

$$\sum_{k\in\mathbb{Z}^n} \hat{\varphi}_j(\xi+k)\overline{\hat{\varphi}_{j'}(\xi+k)} = 0 \qquad \text{for } j \neq j'.$$

Hence, in short

$$\sum_{k\in\mathbb{Z}^n} \hat{\Phi}(\xi+k)\hat{\Phi}^*(\xi+k) = \Omega(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{2.10}$$

The next result provides a characterization of elements of a SI space in terms of its scaling vector. Proposition 2.1 is an immediate consequence of the corresponding well-known result for PSI spaces; for example, see [18, Theorem 5.9].

Proposition 2.1. *Suppose that a SI space V_0 is decomposed as in (2.3) and each φ_j is a quasi-orthogonal generator of $\mathcal{S}(\varphi_j)$. Then $f \in V_0$ if and only if*

$$\hat{f}(\xi) = \sum_{j\in\mathbb{N}} r_j(\xi)\hat{\varphi}_j(\xi), \tag{2.11}$$

where convergence is in L^2, each r_j is $\mathbb{Z}^n$-periodic function in $L^2(S_j)$, and

$$||f||^2 = \sum_{j\in J} ||r_j||^2.$$

Moreover, the sequence $\{r_j\}_{j\in\mathbb{N}}$ of such functions is unique.

Consequently, note that the series (2.11) converges a.e. after choosing a suitable subsequence. In particular, if the fibers of the SI space V_0 are finitely dimensional, meaning that (2.12) holds, the convergence in (2.11) is also in the almost everywhere sense. This observation leads to a simple characterization of refinability of such SI spaces.

Lemma 2.2. *Suppose that V_0 is a SI space such that*

$$\dim_{V_0}(\xi) < \infty \qquad \text{for a.e. } \xi. \tag{2.12}$$

Suppose that Φ is a scaling vector of V_0, and $\{S_j\}_{j\in J}$ is given by (2.6) with J as in (2.7). Then the space V_0 is refinable if and only if

$$\hat{\Phi}(B\xi) = M(\xi)\hat{\Phi}(\xi), \tag{2.13}$$

where M is $\mathbb{Z}^n$-periodic matrix function with entries $m_{i,j} \in L^2(S_j)$, $i,j \in J$. Moreover, if such an M exists, then it is unique.

Proof. Note that condition (2.12) guarantees that $\hat{\Phi}(\xi)$ has finitely many non-zero entries and the matrix product in (2.13) is meaningful. By Proposition 2.1, (2.13) implies that each $\varphi_j \in D(V_0)$. Since $D(V_0)$ is SI, we must have $V_0 \subset D(V_0)$ and V_0 is refinable.

Conversely, if V_0 is refinable then the matrix M satisfying (2.13) is uniquely determined with the use of Proposition 2.1 for $f = D^{-1}\varphi_j$, $j \in \mathbb{N}$. □

A matrix function M satisfying (2.13) is often called a matrix mask function of Φ or a low-pass matrix mask. We are now ready to prove the main result of this section providing a description of a matrix mask corresponding to the scaling vector Φ for V_0.

Theorem 2.2. *Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is a GMRA such that (2.12) holds. Let M be the matrix mask function as in Lemma 2.2. Then*

$$\sum_{d\in\mathcal{D}} M(\xi+d)M^*(\xi+d) = \Omega(B\xi), \tag{2.14}$$

where $\mathcal{D}$ consists of representatives of distinct cosets of $B^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. Moreover, the first column of $M(\xi)$ has zeros in every, but the first entry, near the origin in the sense that for a.e. $\xi\in\mathbb{R}^n$, there exists $N=N(\xi)$ such that

$$m_{i,1}(B^{-j}\xi)=0 \qquad \text{for } i\ge 2,\ j>N. \tag{2.15}$$

Furthermore, the upper-left corner of $M(\xi)$ has absolute value "almost equal" to 1 near the origin, i.e.,

$$\lim_{k\to\infty}\prod_{j=k}^{\infty}|m_{1,1}(B^{-j}\xi)| = 1 \qquad \text{for a.e. } \xi\in\mathbb{R}^n. \tag{2.16}$$

Proof. The condition on the support of M implies that $M(\xi)\Omega(\xi)=M(\xi)$. Hence, by (2.10),

$$\begin{aligned}\Omega(B\xi) &= \sum_{k\in\mathbb{Z}^n}\hat\Phi(B\xi+k)\hat\Phi^*(B\xi+k)\\ &= \sum_{k\in\mathbb{Z}^n} M(\xi+B^{-1}k)\hat\Phi(\xi+B^{-1}k)\hat\Phi^*(\xi+B^{-1}k)M^*(\xi+B^{-1}k)\\ &= \sum_{d\in\mathcal{D}} M(\xi+d)\Omega(\xi+d)M^*(\xi+d) = \sum_{d\in\mathcal{D}} M(\xi+d)M^*(\xi+d),\end{aligned}$$

which proves (2.14).

To show (2.15) we will use the fact [11, Lemma 2.7] that

$$\lim_{j\to\infty}|\hat\varphi_1(B^{-j}\xi)|^2 = \lim_{j\to\infty}\sigma_{V_0}(B^{-j}\xi) = \lim_{j\to\infty}\sigma_{V_j}(\xi) = 1 \qquad \text{a.e. } \xi\in\mathbb{R}^n. \tag{2.17}$$

Combining this with (2.8) yields that there exists $N=N(\xi)$ such that the first coordinate of $\hat\Phi(B^{-j}\xi)$ is non-zero and all others are zero for all $j\ge N$. By (2.13),

$$\hat\varphi_i(B^{-j+1}\xi) = m_{i,1}(B^{-j}\xi)\hat\varphi_1(B^{-j}\xi) \qquad \text{for } j\ge N,\ i\in\mathbb{N}.$$

Since $\hat\varphi_i(B^{-j+1}\xi)=0$ for $i\ge 2$ and $\hat\varphi_1(B^{-j}\xi)\neq 0$ for $j>N$, we have (2.15).

To show (2.16) it suffices to observe that for every $l>k\ge N$,

$$\hat\varphi_1(B^{-k}\xi) = \hat\varphi_1(B^{-l}\xi)\prod_{j=k+1}^{l} m_{1,1}(B^{-j}\xi)$$

By (2.17),

$$|\hat\varphi_1(B^{-k}\xi)| = \lim_{l\to\infty}|\hat\varphi_1(B^{-l}\xi)|\prod_{j=k+1}^{l}|m_{1,1}(B^{-j}\xi)| = \prod_{j=k+1}^{\infty}|m_{1,1}(B^{-j}\xi)|,$$

which proves (2.16) by letting $k \to \infty$. □

Theorem 2.2 shows that the matrix mask function M must satisfy the analogue (2.14) of the usual quadrature-mirror equation of a low-pass filter $m(\xi)$ of an MRA, i.e.,

$$\sum_{d\in\mathcal{D}} |m(\xi+d)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{T}^n. \tag{2.18}$$

In addition, the first column of M must be of a special form $(m_{1,1}(\xi), 0, 0, \ldots)$ with $|m_{1,1}(\xi)| \approx 1$ for ξ near 0. Moreover, (2.14) implies that

$$\sum_{d\in\mathcal{D}} |m_{1,1}(\xi+d)|^2 \le \mathbf{1}_{S_1}(B\xi) \le 1 \qquad \text{for a.e. } \xi \in \mathbb{T}^n.$$

For these reasons $m_{1,1}(\xi)$ plays a role similar to that of a usual low-pass filter and it has a dominating effect on the entire matrix mask function M. These issues are further explored in Section 4, where the procedure for reconstructing a scaling vector from its matrix mask is presented. We should also emphasize that conditions (2.14)–(2.15) are only necessary and not sufficient for guaranteeing that M is a matrix mask of some scaling vector. This a simple consequence of the usual MRA case, where (2.18) and (2.16) alone are not enough to produce the scaling function and some extra conditions, such as Lawton's or Cohen's conditions are needed [17, 18].

Next, we will look at semi-orthogonal wavelets associated to a GMRA. In [12] we pointed out that one can always find a semi-orthogonal wavelet (possibly with infinite number of generators) associated to any GMRA. To be more precise, let us state the following

Theorem 2.3. *Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is a GMRA such that (2.12) holds. Then there exists a semi-orthogonal wavelet $\{\psi_j\}_{j\in\tilde{J}} \subset L^2(\mathbb{R}^n)$ such that*

$$W_0 := V_1 \ominus V_0 = \bigoplus_{j\in\tilde{J}} \mathcal{S}(\psi_j), \tag{2.19}$$

$$\operatorname{supp}\dim_{\mathcal{S}(\psi_j)} = \tilde{S}_j := \{\xi \in \mathbb{R}^n : \dim_{W_0}(\xi) \ge j\}$$

Here, $\tilde{J}$ is either $\{1, \ldots, N\}$ or $\mathbb{N}$.

Conversely, suppose that we have a semi-orthogonal wavelet $\Psi = (\psi^{\tilde{j}})_{\tilde{j}\in\tilde{J}}$, where $\tilde{J} = \{1, \ldots, N\}$ is finite, which is associated with a GMRA $\{V_j\}_{j\in\mathbb{Z}}$; that is (2.19) holds. Equivalently, a GMRA $\{V_j\}_{j\in\mathbb{Z}}$ associated to Ψ is given by

$$V_j = \operatorname{span}\{D^i T_k \psi^{\tilde{j}} : i < j,\ k \in \mathbb{Z}^n,\ \tilde{j} \in \tilde{J}\} \qquad \text{for } j \in \mathbb{Z}.$$

Let Ψ be the column vector defined as $\Psi = (\psi_j)_{j\in\tilde{J}}$. By Proposition 2.1, there exists a matrix function $H(\xi) = \{h_{i,j}(\xi)\}_{i\in\tilde{J}, j\in J}$ such that

$$\hat{\Psi}(B\xi) = H(\xi)\hat{\Phi}(\xi),$$

and $h_{i,j} \in L^2(S_j)$. Let $\tilde{\Omega}$ be the diagonal matrix function corresponding to Ψ, i.e.,

$$\tilde{\Omega}(\xi) = \operatorname{diag}\{\mathbf{1}_{\tilde{S}_j}(\xi) : j \in \tilde{J}\}, \qquad \text{where } \tilde{S}_j = \operatorname{supp}\dim_{\mathcal{S}(\psi^j)}. \tag{2.20}$$

Then we have the following description of a matrix mask function H corresponding to a semi-orthogonal wavelet Ψ, often called a high-pass matrix mask.

Proposition 2.2. *Suppose Ψ is a semi-orthogonal wavelet associated with a GMRA $\{V_j\}_{j\in\mathbb{Z}}$. Let M and H be the matrix mask functions as above. Then*

$$\sum_{d\in\mathcal{D}} H(\xi+d)H^*(\xi+d) = \tilde{\Omega}(B\xi), \tag{2.21}$$

$$\sum_{d\in\mathcal{D}} M(\xi+d)H^*(\xi+d) = \sum_{d\in\mathcal{D}} H(\xi+d)M^*(\xi+d) = 0. \tag{2.22}$$

Proof. The condition on the support of H implies that $H(\xi)\Omega(\xi) = H(\xi)$. Hence, by (2.10),

$$\begin{aligned}\tilde{\Omega}(B\xi) &= \sum_{k\in\mathbb{Z}^n} \hat{\Psi}(B\xi+k)\hat{\Psi}^*(B\xi+k)\\ &= \sum_{k\in\mathbb{Z}^n} H(\xi+B^{-1}k)\hat{\Phi}(\xi+B^{-1}k)\hat{\Phi}^*(\xi+B^{-1}k)H^*(\xi+B^{-1}k)\\ &= \sum_{d\in\mathcal{D}} H(\xi+d)\Omega(\xi+d)H^*(\xi+d) = \sum_{d\in\mathcal{D}} H(\xi+d)H^*(\xi+d),\end{aligned}$$

which proves (2.21).

Likewise,

$$\begin{aligned}0 &= \sum_{k\in\mathbb{Z}^n} \hat{\Phi}(B\xi+k)\hat{\Psi}^*(B\xi+k)\\ &= \sum_{k\in\mathbb{Z}^n} M(\xi+B^{-1}k)\hat{\Phi}(\xi+B^{-1}k)\hat{\Phi}^*(\xi+B^{-1}k)H^*(\xi+B^{-1}k)\\ &= \sum_{d\in\mathcal{D}} M(\xi+d)\Omega(\xi+d)H^*(\xi+d) = \sum_{d\in\mathcal{D}} M(\xi+d)H^*(\xi+d),\end{aligned}$$

which proves (2.22). □

We can now summarize the content of this section in a simple flow diagram below.

semi-orthogonal wavelet Ψ

↓

GMRA $\{V_j\}_{j\in\mathbb{Z}}$ generated by Ψ

↓ Theorem 2.1

scaling vector Φ for V_0

↓ Theorem 2.2 + Proposition 2.2

low-pass matrix mask M satisfying (2.14)–(2.16)

+

high-pass matrix mask H satisfying (2.21)–(2.22)

Note that the second arrow from above does not provide a unique scaling vector Φ, since the decomposition of Theorem 2.1 is not unique. Although the other two arrows give a unique correspondence, the above procedure yields a multitude of low-pass and high-pass matrix masks for a fixed semi-orthogonal wavelet. This fact makes the problem of reconstructing scaling vector and semi-orthogonal wavelet from their corresponding low-pass and high-pass matrix masks a highly non-trivial task. This will be explored in Sections 4 and 5, where the appropriate reconstruction procedure is provided, see Theorem 5.4.

We end this section with remarks involving orthogonality of columns versus rows for combined matrix function of low-pass and high-pass masks.

Remark 2.1. Let T be the combined matrix mask function of a scaling vector Φ and the associated semi-orthogonal wavelet Ψ given by

$$T(\xi) = \begin{bmatrix} M(\xi+d_1) & \dots & M(\xi+d_q) \\ H(\xi+d_1) & \dots & H(\xi+d_q) \end{bmatrix},$$

where $d_1, \dots, d_q$ are representatives of distinct cosets of $B^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. Note that for a.e. $\xi \in \mathbb{T}^n$, $T(\xi)$ has only a finite number of non-zero entries. More precisely, there are only $\sum_{d\in\mathcal{D}} \dim_{V_0}(\xi+d)$ non-zero columns and $\dim_{V_0}(B\xi)+\dim_{W_0}(B\xi)$ non-zero rows. Let $\tilde{T}(\xi)$ be a finite sub-matrix of $T(\xi)$ consisting of only these columns and rows. The consistency equation of Baggett, see [5, 11], says that

$$\dim_{V_1}(B\xi) = \sum_{d\in\mathcal{D}} \dim_{V_0}(\xi+d) = \dim_{V_0}(B\xi)+\dim_{W_0}(B\xi) \qquad \text{for a.e. } \xi \in \mathbb{T}^n, \tag{2.23}$$

which implies that the matrix $\tilde{T}(\xi)$ is square for a.e. $\xi \in \mathbb{T}^n$. Moreover, Theorem 2.2 and Proposition 2.2 imply that that the rows of the matrix $\tilde{T}(\xi)$ are mutually orthogonal and normalized, that is $\tilde{T}(\xi)$ is a unitary matrix, see also [4, Theorem 2.5]. Consequently, the columns of $\tilde{T}(\xi)$ are mutually orthogonal and normalized.

Remark 2.2. One could consider the combined matrix mask function $T(\xi)$ corresponding to a more general situation when Ψ is a framelet obtained by a similar procedure. In this case, the rows of $T(\xi)$ do not have to be mutually orthogonal, anymore. In fact, the sub-matrix of non-zero rows and columns $\tilde{T}(\xi)$ does not have to be square, since we can have much more generators in Ψ, and hence more rows in the matrix mask function $H(\xi)$. It turns out that unlike the situation of semi-orthogonal wavelets, where orthogonality of rows of $T(\xi)$ is necessary, orthogonality of columns plays a critical role for general framelets. This will be explored in the next section.

3. Unitary Extension Principle

The Unitary Extension Principle, and its generalizations such as Oblique Extension Principle, are powerful tools in constructing tight framelets [15, 20]. Since these techniques are used for constructing framelets with many desired properties such as smoothness, compact support, vanishing moments, etc., the Unitary Extension Principle is very often stated with some very mild and convenient regularity assumptions on a refinable function φ.

Since the interest of our work lies mainly in L^2 theory of framelets and wavelets, it is imperative to avoid any regularity assumptions, regardless of their mildness, limiting the applicability of our results. Furthermore, we are also forced to study situations where we are given a refinable vector consisting of infinite number of functions. Since these two problems were not adequately addressed yet, we provide an extension of Unitary Extension Principle, that is perfectly adapted to the L^2 theory. We start with a definition of a refinable vector consisting of potentially infinitely many functions.

Definition 3.1. We say that $\Phi = (\varphi_j)_{j\in J} \subset L^2(\mathbb{R}^n)$ is a refinable vector, where $J = \{1,\dots,N\}$ or $J = \mathbb{N}$, if

$$\hat{\Phi}(B\xi) = M(\xi)\hat{\Phi}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{3.1}$$

where $M = (m_{i,j})_{i,j\in J}$ is a matrix of $\mathbb{Z}^n$-periodic, measurable functions.

In order to make sense of (3.1) in the case when $J = \mathbb{N}$, we assume additionally that

$$\sum_{j\in J} \mathbf{1}_{R_j}(\xi) < \infty \qquad \text{for a.e. } \xi \in \mathbb{T}^n, \tag{3.2}$$

where

$$R_j = \operatorname{supp} \dim_{\mathcal{S}(\varphi_j)} .$$

Note that we can always assume that $\operatorname{supp} m_{i,j} \subset R_j$, since the values of $m_{i,j}$ outside of R_j do not affect (3.1).

Remark 3.1. Condition (3.2) is a technical matter that allows us to talk meaningfully about the equation (3.1). However, if the matrix $M(\xi)$ has only finitely many non-zero entries for a.e. $\xi \in \mathbb{R}^n$, then (3.1) makes sense right away. Moreover, in this simple case, (3.2) follows from (3.1).

Theorem 3.1 is a generalization of the Unitary Extension Principle of Ron and Shen [20] to a situation when a refinable vector Φ is infinite. We note that the original result of Ron and Shen, in the case when Φ is finite, requires certain mild decay assumptions on Φ, see [15, 20]. Nevertheless, Theorem 3.1 shows that these decay assumptions are unnecessary and they can be safely removed.

Theorem 3.1. *Suppose $\Phi = (\varphi_j)_{j\in J}$ is a refinable vector with a mask M such that*

$$\sum_{j\in J} ||\varphi_j||^2 = \int_{\mathbb{R}^n} \|\hat{\Phi}(\xi)\|^2 \, d\xi < \infty \tag{3.3}$$

and

$$\lim_{j\to\infty} ||\hat{\Phi}(B^{-j}\xi)|| = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{3.4}$$

Suppose also that $\Psi = (\psi_j)_{j\in\tilde{J}}$, where $\tilde{J} = \{1, \dots, N\}$ is finite, is given by

$$\hat{\Psi}(B\xi) = H(\xi)\hat{\Phi}(\xi), \tag{3.5}$$

where $H = (h_{i,j})_{i\in\tilde{J},\ j\in J}$ is $\mathbb{Z}^n$-periodic, measurable matrix function satisfying

$$M^*(\xi)M(\xi + d) + H^*(\xi)H(\xi + d) = \Omega(\xi)\delta_{0,d} \qquad \text{for a.e. } \xi, \tag{3.6}$$

and for any $d \in \mathcal{D}$.

Then $\Psi \subset L^2(\mathbb{R}^n)$ is a tight framelet.

Proof. It suffices to verify that $\Psi \subset L^2(\mathbb{R}^n)$ satisfies characterization equations for tight framelets

$$\sum_{j\in\mathbb{Z}} ||\hat{\Psi}(B^j\xi)||^2 = 1 \qquad \text{for a.e. } \xi. \tag{3.7}$$

$$\sum_{j=0}^{\infty} \hat{\Psi}^*(B^j\xi)\hat{\Psi}(B^j(\xi + q)) = 0 \qquad \text{for a.e. } \xi, \text{ and all } q \in \mathbb{Z}^n \setminus B\mathbb{Z}^n. \tag{3.8}$$

Note that for any $j \in \mathbb{Z}$,

$$\begin{aligned} ||\hat{\Psi}(B^j\xi)||^2 + ||\hat{\Phi}(B^j\xi)||^2 &= \hat{\Psi}^*(B^j\xi)\hat{\Psi}(B^j\xi) + \hat{\Phi}^*(B^j\xi)\hat{\Phi}(B^j\xi) \\ &= \hat{\Phi}^*(B^{j-1}\xi)H^*(B^{j-1}\xi)H(B^{j-1}\xi)\hat{\Phi}(B^{j-1}\xi) \\ &\quad + \hat{\Phi}^*(B^{j-1}\xi)M^*(B^{j-1}\xi)M(B^{j-1}\xi)\hat{\Phi}(B^{j-1}\xi) \\ &= \hat{\Phi}^*(B^{j-1}\xi)\Omega(B^{j-1}\xi)\hat{\Phi}(B^{j-1}\xi) = ||\hat{\Phi}(B^{j-1}\xi)||^2, \end{aligned} \tag{3.9}$$

where in the last step we used that $\operatorname{supp}\hat{\varphi}_i \subset S_i$. Therefore,

$$\int_{\mathbb{R}^n} ||\hat{\Psi}(\xi)||^2 d\xi = (|\det A| - 1)\int_{\mathbb{R}^n} ||\hat{\Phi}(\xi)||^2 d\xi < \infty,$$

and the fact that $\Psi \subset L^2(\mathbb{R}^n)$ is forced by (3.5) and (3.6).

Next, we claim that

$$\lim_{j\to\infty} ||\hat{\Phi}(B^j\xi)|| = 0 \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{3.10}$$

Otherwise, due to monotonicity of the sequence $(||\hat{\Phi}(B^j\xi)||)_{j\in\mathbb{Z}}$, we could find $\delta > 0$ such that

$$E = \{\xi \in \mathbb{R}^n : ||\hat{\Phi}(B^j\xi)|| > \delta \text{ for all } j \in \mathbb{Z}\}$$

has a positive measure. Since $BE = E$, E must have infinite Lebesgue measure and consequently

$$\int_{\mathbb{R}^n} ||\hat{\Phi}(\xi)||^2 d\xi \geq \int_E ||\hat{\Phi}(\xi)||^2 d\xi = \infty,$$

which contradicts (3.3). Thus, (3.10) holds and together with (3.4), (3.9) it implies (3.7).

Likewise for any $j \geq 1$,

$$
\begin{aligned}
\hat{\Psi}^*(B^j\xi)\hat{\Psi}(B^j(\xi+q)) + \hat{\Phi}^*(B^j\xi)\hat{\Phi}(B^j(\xi+q)) \\
=&\hat{\Phi}^*(B^{j-1}\xi)H^*(B^{j-1}\xi)H(B^{j-1}(\xi+q))\hat{\Phi}(B^{j-1}(\xi+q)) \\
&+ \hat{\Phi}^*(B^{j-1}\xi)M^*(B^{j-1}\xi)M(B^{j-1}(\xi+q))\hat{\Phi}(B^{j-1}(\xi+q)) \\
=&\hat{\Phi}^*(B^{j-1}\xi)\Omega(B^{j-1}\xi)\hat{\Phi}(B^{j-1}(\xi+q)) \\
=&\hat{\Phi}^*(B^{j-1}\xi)\hat{\Phi}(B^{j-1}(\xi+q)),
\end{aligned}
$$

where in the penultimate step we used $\mathbb{Z}^n$-periodicity of M and H. The same calculation for $j = 0$ together with the observation that

$$H^*(B^{-1}\xi)H(B^{-1}(\xi+q)) + M^*(B^{-1}\xi)H(B^{-1}(\xi+q)) = 0$$

yields that

$$\hat{\Psi}^*(\xi)\hat{\Psi}(\xi+q) + \hat{\Phi}^*(\xi)\hat{\Phi}(\xi+q) = 0.$$

Combining these identities with

$$\lim_{j\to\infty} |\hat{\Phi}^*(B^j\xi)\hat{\Phi}(B^j(\xi+q))| \leq \lim_{j\to\infty} ||\hat{\Phi}(B^j\xi)||||\hat{\Phi}(B^j(\xi+q))|| = 0 \qquad \text{for a.e. } \xi \in \mathbb{R}^n,$$

proves (3.8).

□

In the next section we will use Theorem 3.1 to give a general construction procedure of tight framelets. To this end, it is convenient to prove the following fact about functions satisfying an inequality reminiscent of Baggett's consistency equation.

Lemma 3.1. *Suppose that $m : \mathbb{R}^n \to [0, \infty)$ is $\mathbb{Z}^n$-periodic, measurable function such that*

$$\sum_{d\in D} m(\xi+d) \leq m(B\xi) + M \qquad \text{for a.e. } \xi \in \mathbb{T}^n, \tag{3.11}$$

for some $M \geq 0$. Then m is integrable over its period and

$$\int_{\mathbb{T}^n} m(\xi)d\xi \leq M/(|\det A| - 1). \tag{3.12}$$

Heuristically, Lemma 3.1 seems to be trivial. Integrating (3.11) over $\mathbb{T}^n$ yields

$$|\det A| \int_{\mathbb{T}^n} m(\xi)d\xi \leq \int_{\mathbb{T}^n} m(\xi)d\xi + M.$$

Unfortunately, we do not know a priori whether m is integrable and a much more complicated argument is necessary. Despite its simplicity, we could not find Lemma 3.1 in the existing literature and therefore we provide its proof.

Proof. For an integer $N \geq 0$, let $R_N(\xi)$ be the "Riemann sum" of m of depth N given by

$$R_N(\xi) = R_N^m(\xi) := \frac{1}{|\det A|^N} \sum_{\epsilon_0,\dots,\epsilon_{N-1}\in\mathcal{D}} m\Big(\xi + \sum_{i=0}^{N-1} B^{-i}\epsilon_i\Big).$$

It is clear that $R_N(\xi)$ is measurable and $B^{-N}\mathbb{Z}^n$-periodic, since all the sums of the form $\sum_{i=0}^{N-1} B^{-i}\epsilon_i$, where $\epsilon_0, \dots, \epsilon_{N-1} \in \mathcal{D}$, are representatives of distinct cosets of $B^{-N}\mathbb{Z}^n/\mathbb{Z}^n$. Here, $\mathcal{D}$ consists as usual of representatives of distinct cosets of $B^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. By (3.11),

$$|\det A| R_N(\xi) \le R_{N-1}(B\xi) + M \qquad \text{for any } N \ge 1.$$

Hence, by iteration,

$$m(B^N\xi) = R_0(B^N\xi) \ge |\det A|^N R_N(\xi) - M\frac{|\det A|^N - 1}{|\det A| - 1}.$$

Take any $C > M/(|\det A| - 1)$ and let $\delta = C - M/(|\det A| - 1)$. Then

$$\{\xi \in \mathbb{R}^n : R_N(\xi) \ge C\} \subset \{\xi \in \mathbb{R}^n : m(B^N\xi) \ge \delta |\det A|^N\}.$$

For a fixed $K > 0$, let $R'_N(\xi) = R_N^{m'}(\xi)$, where m' is a truncation of m at height K given by $m'(\xi) = \min(m(\xi), K)$. It is clear that each $R'_N(\xi)$ is $B^{-N}\mathbb{Z}^n$-periodic, measurable and bounded by K. Furthermore,

$$\begin{aligned} |\{\xi \in \mathbb{T}^n : R'_N(\xi) \ge C\}| &\le |\{\xi \in \mathbb{T}^n : m(B^N\xi) \ge \delta |\det A|^N\}| \\ &= |\{\xi \in \mathbb{T}^n : m(\xi) \ge \delta |\det A|^N\}| \to 0 \qquad \text{as } N \to \infty. \end{aligned} \tag{3.13}$$

Since R'_N's are bounded, there exists a subsequence $\{N_i\}$ such that $\{R'_{N_i}(\xi)\}$ converges pointwise a.e. to some $f(\xi)$. Since $f(\xi)$ must be periodic with respect to every lattice $B^{-N}\mathbb{Z}^n \subset B^{-N+1}\mathbb{Z}^n$, and $\bigcup_{N=0}^{\infty} B^{-N}\mathbb{Z}^n$ is dense, $f(\xi)$ must be a constant function, $f(\xi) = C_0$. By (3.13), $0 \le C_0 \le M/(|\det A| - 1)$. Moreover,

$$C_0 = \int_{\mathbb{T}^n} f(\xi)d\xi = \lim_{i\to\infty} \int_{\mathbb{T}^n} R'_{N_i}(\xi)d\xi = \int_{\mathbb{T}^n} m'(\xi)d\xi = \int_{\mathbb{T}^n} \min(m(\xi), K)d\xi.$$

Hence, letting $K \to \infty$ allows to obtain (3.12) by the monotone convergence theorem. □

4. Construction of tight framelets

The main goal of this section is to provide a general reconstruction procedure for scaling vectors and semi-orthogonal wavelets from their corresponding low-pass and high-pass matrix masks. Hence, the goal is to reverse the flow of Section 2 by starting with a low-pass matrix mask function M satisfying conditions (2.14)–(2.16). Theorem 2.2 shows that this is a perfectly reasonable assumption, since any matrix mask function of a scaling vector must satisfy them. The key ingredient of our approach is a rather complicated procedure yielding a refinable vector Φ corresponding to the mask M, see Theorem 4.2. This, combined with the Unitary Extension Principle and appropriate conditions on a high-pass mask H, yields a tight framelet Ψ (see Theorem 4.3).

In general, we can only expect that Ψ is a tight framelet. However, if we know a priori that our low-pass and high-pass matrix masks correspond to some semi-orthogonal wavelet Ψ, then we prove that our procedure is flexible enough to recover Ψ itself. In particular, every orthogonal wavelet Ψ can be obtained by our recovery procedure via low-pass and high-pass matrix masks manipulations. This will be shown in the following section.

To start the construction of tight framelets we must recall a characterization of the dimension function associated to a GMRA proved in [11].

Theorem 4.1. *Suppose $(V_j)_{j\in\mathbb{Z}}$ is a GMRA. Then the dimension function of the core space V_0, $m(\xi) = \dim_{V_0}(\xi)$, satisfies the following conditions*
(D1) $m : \mathbb{R}^n \to \mathbb{N} \cup \{0, \infty\}$ is a measurable $\mathbb{Z}^n$-periodic function;
(D2) $\sum_{d\in\mathcal{D}} m(\xi + d) \geq m(B\xi)$ for a.e. $\xi \in \mathbb{R}^n$;
(D3) $\sum_{k\in\mathbb{Z}^n} \mathbf{1}_\Delta(\xi + k) \geq m(\xi)$ for a.e. $\xi \in \mathbb{R}^n$, where

$$\Delta = \{\xi \in \mathbb{R}^n : m(B^{-j}\xi) \geq 1 \text{ for } j \in \mathbb{N} \cup \{0\}\};$$

(D4) $\liminf_{j\to\infty} m(B^{-j}\xi) \geq 1$ for a.e. $\xi \in \mathbb{R}^n$.
Conversely, if m satisfies (D1)–(D4), then there exists a GMRA $(V_j)_{j\in\mathbb{Z}}$ such that $\dim_{V_0}(\xi) = m(\xi)$.

Our construction is based on a function m that satisfies conditions (D1)–(D4) of the above theorem. However, to ensure the existence of a tight framelet we shall add two more assumptions. Namely,
(D5) $m \in L^1(\mathbb{T}^n)$;
and
(D6) m is finite a.e. and there is $N \in \mathbb{N}$ such that for a.e. $\xi \in \mathbb{T}^n$ we have

$$\sum_{d\in\mathcal{D}} m(\xi + d) \leq m(B\xi) + N.$$

To motivate these final conditions we include the following

Proposition 4.1. *Let $\Psi \subset L^2(\mathbb{R}^n)$ be a tight framelet and V_0 its space of negative dilates. If $(V_j)_{j\in\mathbb{Z}}$ forms a GMRA, then $m = \dim_{V_0}$ satisfies (D5) and (D6).*

Proof. First we conduct the standard orthogonalization procedure. That is, for $j \in \mathbb{Z}$ we define $W_j := V_{j+1} \ominus V_j$ and observe that

$$\bigoplus_{j\in\mathbb{Z}} W_j = L^2(\mathbb{R}^n). \tag{4.1}$$

Clearly, W_0 is a shift-invariant space generated by $\{\psi - P_{V_0}\psi\}_{\psi\in\Psi}$, where P_{V_0} is the orthogonal projection on V_0. By Theorem 2.1 we can find quasi-orthogonal generators $\Phi = \{\varphi_1, \varphi_2, \ldots\}$ for W_0 as in Theorem 2.1. Since our tight framelet Ψ consists of a finite number of functions, Φ has a finite number of non-zero elements as well, say N. Condition 4.1 assures that Φ is a semi-orthogonal wavelet. This allows us to calculate the spectral function of V_0 in terms of Φ. Indeed, a formula from [11] gives us

$$\sigma_{V_0}(\xi) = \sum_{\varphi\in\Phi} \sum_{j>0} |\hat{\varphi}(B^j\xi)|^2.$$

After integrating the above formula we obtain that

$$\int_{\mathbb{R}^n} \sigma_{V_0} = \sum_{\varphi\in\Phi} \|\varphi\|^2/(|\det A| - 1) \leq N/(|\det A| - 1).$$

Since $\int_{\mathbb{R}^n} \sigma_{V_0} = \int_{\mathbb{T}^n} m$, this shows that (D5) is satisfied.

In order to justify (D6) we use basic properties of the dimension function that are given in [11]. Since $V_1 = V_0 \oplus W_0$, we get that $\sum_{d\in\mathcal{D}} m(B^{-1}\xi + d) = m(\xi) + \dim_{W_0}(\xi)$. But W_0 has N generators, therefore $\dim_{W_0} \leq N$ and (D6) follows. □

Remark 4.1. In order to construct our GMRA only conditions (D1)-(D5) are going to be used. We shall also show that (D6) is necessary and sufficient to guarantee the existence of a "high pass filter" that will be used to define our framelet. We also want to point out, that (D5) follows from (D6), as was shown in Lemma 3.1.

In short, our construction is guided by the standard procedure. We are going to consider a matrix mask function M that satisfies conditions (2.14)–(2.16). Then we will construct a corresponding refinable vector Φ and use Unitary Extension Principle to obtain an associated tight framelet Ψ.

We start equipped with a function m that satisfies conditions (D1)–(D5). Then, we define the sets S_j, for $j \in \mathbb{N}$, by a formula analogous to (2.6), that is

$$S_j = \{\xi \in \mathbb{R}^n : m(\xi) \geq j\}. \tag{4.2}$$

Let $J = \{i \in \mathbb{N} : |S_j| > 0\}$. Hence, $J = \{1, 2, \ldots, L\}$ or $J = \mathbb{N}$. The sets S_j, $j \in J$, are used to define the diagonal matrix function Ω as in (2.9). This allows us to consider a matrix mask function M with periodic entries $m_{i,j} \in L^2(S_j)$, $i, j \in J$, that satisfies conditions (2.14)–(2.16). In order to find the corresponding refinable vector we shall use the ideas of [18]. First, we will modify M to assure that the product of the dilates of M is convergent. Then, we shall use multipliers to recover the solution to the original problem. To proceed in this direction we need the following basic lemma about multipliers.

Lemma 4.1. *Let μ be a unimodular measurable function on $\mathbb{R}^n$ (that is, $\mu : \mathbb{R}^n \to \mathbb{T}$). If B is an expansive matrix, then there exists a unimodular measurable function ν such that*

$$\nu(B\xi)\overline{\nu(\xi)} = \mu(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{4.3}$$

Proof. It is well known, that for any expansive matrix B there is an ellipsoid $\mathcal{E}$ such that $\mathcal{E} \subset B(\mathcal{E})$, see e.g. [10, Lemma 2.2]. It follows, that for $W = B(\mathcal{E}) \setminus \mathcal{E}$ we have $\bigcup_{j\in\mathbb{Z}} B^j(W) = \mathbb{R}^n$. Therefore, it is enough to define a unimodular function ν on W and then extend it to $\mathbb{R}^n$ using equation (4.3). □

The mentioned modification of the matrix mask function M is very simple. The most important entry of M is $m_{1,1}$. Let μ be a phase of $m_{1,1}$. That is, μ is a unimodular measurable function such that

$$\mu(\xi)|m_{1,1}(\xi)| = m_{1,1}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{4.4}$$

A *multiplier* associated to the mask M is any unimodular measurable function ν satisfying (4.3) and (4.4). The *modified mask* is

$$M' := \bar{\mu} M \tag{4.5}$$

and the corresponding refinable vector is given by

$$\hat{\Phi}'(\xi) := \lim_{N\to\infty} \left[\prod_{j=1}^{N} M'(B^{-j}\xi)\right] e, \tag{4.6}$$

where e is the vector (1,0,0,...). Finally, the refinable vector Φ corresponding to the mask M will be given by $\hat{\Phi} := \nu\hat{\Phi}'$. In order to establish that Φ is refinable let us show a series of lemmas.

Lemma 4.2. *The vector function $\hat{\Phi}'$ in (4.6) is well defined.*

Proof. We need to show, that the limit in (4.6) does exist for a.e. $\xi \in \mathbb{R}^n$. We want to point out, that although $\Pi_{j=1}^{\infty} M'(B^{-j}\xi)$ may not exist, we are only interested in the first column of this matrix. Later, we will prove that under some natural assumptions this product matrix exists and all of its columns but the first must be zero, see (5.1).

By (2.15), for a.e. $\xi \in \mathbb{R}^n$ we can find $N(\xi)$ such that the first column of $M'(B^{-j}\xi)$ has only one non-zero entry (the first one) for all $j > N(\xi)$. Therefore, for $N > N(\xi)$ we have

$$\big[\Pi_{j=1}^{N} M'(B^{-j}\xi)\big]e = \big[\Pi_{j=1}^{N(\xi)} M'(B^{-j}\xi)\big]\big[\Pi_{j=N(\xi)+1}^{N} |m_{1,1}(B^{-j}\xi)|\big]e.$$

Thus,

$$\hat{\Phi}'(\xi) := \lim_{N\to\infty} p_N(\xi)v(\xi),$$

where $v(\xi) = \big[\Pi_{j=1}^{N(\xi)} M'(B^{-j}\xi)\big]e$ and $p_N(\xi) = \Pi_{j=N(\xi)+1}^{N} |m_{1,1}(B^{-j}\xi)|$. Since condition (2.14) guarantees that $|m_{1,1}| \le 1$, we see that $\{p_N(\xi)\}$ is a bounded decreasing sequence and our claim follows. □

Lemma 4.3. *The vector function $\hat{\Phi}'$ in (4.6) satisfies $\hat{\Phi}'(B\xi) = M'(\xi)\hat{\Phi}'(\xi)$.*

Proof. From (D5) it follows that our function m is finite a.e. Therefore, condition (2.14) implies that for a.e. $\xi \in \mathbb{T}^n$ the matrix $M'(\xi)$ has only finitely many non-zero terms. This allows us to see that

$$\hat{\Phi}'(B\xi) = \lim_{N\to\infty}\Big(M'(\xi)\big[\Pi_{j=1}^{N} M'(B^{-j}\xi)\big]e\Big) = M'(\xi)\lim_{N\to\infty}\Big(\big[\Pi_{j=1}^{N} M'(B^{-j}\xi)\big]e\Big) = M(\xi)\hat{\Phi}'(\xi).$$

□

Lemma 4.4. *The vector function $\hat{\Phi}'$ in (4.6) satisfies $\lim_{N\to\infty}\|\hat{\Phi}'(B^{-N}\xi)\| = 1$, for a.e. $\xi \in \mathbb{R}^n$.*

Proof. By (2.15), for a.e. $\xi \in \mathbb{R}^n$ we can find $N(\xi)$ such that for all $N > N(\xi)$ we have

$$\hat{\Phi}'(B^{-N}\xi) = \big(\Pi_{j=N+1}^{\infty} |m_{1,1}(B^{-j}\xi)|\big)e.$$

Therefore, our claim follows from (2.16).

□

Lemma 4.5. *The vector function $\hat{\Phi}'$ in (4.6) satisfies $\int_{\mathbb{R}^n} \|\hat{\Phi}'(\xi)\|^2\,d\xi < \infty$.*

Proof. For $N \in \mathbb{N}$ and a.e. $\xi \in \mathbb{R}^n$ let us consider the following matrix

$$M_N(\xi) = \big[\Pi_{j=1}^{N} M'(B^{-j}\xi)\big]\mathbf{1}_{B^N(\mathbb{T}^n)}(\xi). \tag{4.7}$$

We claim that for all $N \in \mathbb{N}$ and a.e. $\xi \in \mathbb{R}^n$

$$\sum_{k\in\mathbb{Z}^n} M_N(\xi+k)M_N^*(\xi+k) = \Omega(\xi). \tag{4.8}$$

Indeed, for N=1 we use (2.14) to obtain

$$\sum_{k\in\mathbb{Z}^n} M_1(\xi+k)M_1^*(\xi+k) = \sum_{k\in\mathbb{Z}^n} M(B^{-1}(\xi+k))M^*(B^{-1}(\xi+k))\mathbf{1}_{B(\mathbb{T}^n)}(\xi+k)$$

$$= \sum_{d\in\mathcal{D}} M(B^{-1}\xi+d)M^*(B^{-1}\xi+d) = \Omega(\xi).$$

To proceed with the induction we observe that

$$M_{N+1}(\xi) = M'(B^{-1}\xi)M'_N(B^{-1}\xi),$$

for $N \in \mathbb{N}$ and a.e. $\xi \in \mathbb{R}^n$. Therefore,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n} M_{N+1}(\xi+k)M^*_{N+1}(\xi+k) \\
&= \sum_{k\in\mathbb{Z}^n} M(B^{-1}(\xi+k))M_N(B^{-1}(\xi+k))M^*_N(B^{-1}(\xi+k))M^*(B^{-1}(\xi+k)) \\
&= \sum_{d\in\mathcal{D}}\sum_{l\in\mathbb{Z}^n} M(B^{-1}\xi+d)M_N(B^{-1}\xi+d+l)M^*_N(B^{-1}\xi+d+l)M^*(B^{-1}\xi+d)
\end{aligned}$$

$$= \sum_{d\in\mathcal{D}} M(B^{-1}\xi+d)\Omega(B^{-1}\xi+d)M^*(B^{-1}\xi+d) = \sum_{d\in\mathcal{D}} M(B^{-1}\xi+d)M^*(B^{-1}\xi+d) = \Omega(\xi),$$

what proves our claim (4.8). In order to use it, we observe that for all $k \in \mathbb{Z}^n$ and a.e $\xi \in \mathbb{R}^n$

$$\|M_N(\xi+k)e\|^2 \le \|M_N(\xi+k)\|^2 \le \|M_N(\xi+k)\|^2_{HS} = \mathrm{tr}[M_N(\xi+k)M^*_N(\xi+k)], \tag{4.9}$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt operator norm. The above estimate and (4.8) give us

$$\sum_{k\in\mathbb{Z}^n} \|M_N(\xi+k)e\|^2 \le \mathrm{tr}[\Omega(\xi)] = m(\xi). \tag{4.10}$$

Since $\lim_{N\to\infty}\big(M_N(\xi)e\big) = \hat{\Phi}'(\xi)$, we can use Fatou's lemma to conclude that

$$\sum_{k\in\mathbb{Z}^n} \|\hat{\Phi}'(\xi+k)\|^2 \le m(\xi). \tag{4.11}$$

By (D5) the function $m(\xi)$ is integrable over $\mathbb{T}^n$, thus

$$\int_{\mathbb{R}^n} \|\hat{\Phi}'(\xi)\|^2 d\xi = \int_{\mathbb{T}^n} \sum_{k\in\mathbb{Z}^n} \|\hat{\Phi}'(\xi+k)\|^2\, d\xi \le \int_{\mathbb{T}^n} m(\xi)d\xi < \infty. \tag{4.12}$$

□

Remark 4.2. We have that $\hat{\Phi}' = (\hat{\varphi}'_j)_{j\in J}$. Since $\int_{\mathbb{R}^n} \|\hat{\Phi}'(\xi)\|^2\, d\xi = \sum_{j\in J}\|\hat{\varphi}_j\|^2$, the above lemma shows that $\hat{\Phi}' \subset L^2(\mathbb{R}^n)$.

To reverse the procedure given in (4.5) we use Lemma 4.1 to find a multiplier ν associated to M and define our refinable vector Φ by setting

$$\hat{\Phi} := \nu\hat{\Phi}'. \tag{4.13}$$

The following result assures that such Φ has all of the properties that we need.

Theorem 4.2. *The vector function $\hat{\Phi}$ given in (4.13) satisfies conditions (3.1)-(3.4).*

Proof. By Lemma 4.3 and (4.3) together with (4.5) we get that

$$\hat{\Phi}(B\xi) = \nu(B\xi)\hat{\Phi}'(B\xi) = \nu(B\xi)M'(\xi)\hat{\Phi}'(\xi) = \nu(B\xi)\bar{\mu}(\xi)M(\xi)\bar{\nu}(\xi)\hat{\Phi}(\xi) = M(\xi)\hat{\Phi}(\xi).$$

therefore, (3.1) holds. As we mentioned before, (2.14) implies that the mask matrix $M(\xi)$ has only finitely non-zero terms. Thus, condition (3.2) is satisfied, by Remark 3.1. Since

$\|\hat{\Phi}(\xi)\| = \|\hat{\Phi}'(\xi)\|$ a.e., properties (3.3) and (3.4) follow immediately from Lemma 4.4 and Lemma 4.5. □

As an immediate consequence of Theorem 4.2 and the Unitary Extension Principle from the previous section, we obtain our framelet construction result.

Theorem 4.3. *Let m be a function that satisfies (D1)–(D5) with the sets S_j given by (4.2) and the corresponding matrix function Ω defined in (2.9). Let $M = (m_{i,j})_{i,j\in J}$ be a matrix mask function with periodic entries $m_{i,j} \in L^2(S_j)$, that satisfies (2.14)–(2.16). Then there is a refinable vector Φ such that*

$$\hat{\Phi}(B\xi) = M(\xi)\hat{\Phi}(\xi) \quad \text{for a.e.} \quad \xi \in \mathbb{R}^n. \tag{4.14}$$

Moreover, if a matrix function $H = (h_{i,j})_{i\in\tilde{J},j\in J}$ with $\tilde{J} = \{1, \dots, N\}$ finite and with periodic entries $h_{i,j} \in L^2(S_j)$ satisfies

$$M^*(\xi)M(\xi + d) + H^*(\xi)H(\xi + d) = \Omega(\xi)\delta_{0,d} \quad \text{for any } d \in \mathcal{D} \text{ and a.e. } \xi \in \mathbb{R}^n, \tag{4.15}$$

then $\Psi = (\psi^j)_{j\in\tilde{J}}$ given by

$$\hat{\Psi}(B\xi) = H(\xi)\hat{\Phi}(\xi) \tag{4.16}$$

is a tight framelet for $L^2(\mathbb{R}^n)$.

Proof. The existence of a refinable vector Φ satisfying (4.14) is a consequence of Theorem 4.2. In addition, if a matrix function H satisfies (4.15), then by Theorem 3.1, Ψ given by (4.16) is a tight framelet for $L^2(\mathbb{R}^n)$. □

The next theorem gives the necessary and sufficient conditions for the existence of the high-pass matrix mask H that satisfies (4.15).

Theorem 4.4. *Let m be any function satisfying (D1)–(D5). Let $\tilde{m} : \mathbb{R}^n \to \mathbb{N} \cup \{0, \infty\}$ be a measurable $\mathbb{Z}^n$-periodic function satisfying*

$$\sum_{d\in\mathcal{D}} m(\xi + d) = m(B\xi) + \tilde{m}(B\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{4.17}$$

Define the sets

$$S_j = \{\xi \in \mathbb{R}^n : m(\xi) \geq j\}, \qquad \tilde{S}_j = \{\xi \in \mathbb{R}^n : \tilde{m}(\xi) \geq j\}$$

and the corresponding matrix functions Ω and $\tilde{\Omega}$ by (2.9). Assume that $M = (m_{i,j})_{i,j\in J}$ is a matrix mask function with periodic entries $m_{i,j} \in L^2(S_j)$ that satisfies (2.14). Then the following are equivalent:
(i) m satisfies (D6),
(ii) There exists a matrix function $H = (h_{i,j})_{i\in\tilde{J},j\in J}$ with $\tilde{J}$ finite and with periodic entries $h_{i,j} \in L^2(S_j)$ satisfying

$$\sum_{d\in\mathcal{D}} H(\xi + d)H^*(\xi + d) = \tilde{\Omega}(B\xi), \qquad \text{a.e. } \xi \in \mathbb{R}^n, \tag{4.18}$$

$$\sum_{d\in\mathcal{D}} H(\xi + d)M^*(\xi + d) = 0 \qquad \text{a.e. } \xi \in \mathbb{R}^n. \tag{4.19}$$

(iii) There exists a matrix function $H = (h_{i,j})_{i\in\tilde{J}, j\in J}$ *with* $\tilde{J}$ *finite and with periodic entries* $h_{i,j} \in L^2(S_j)$ *satisfying (4.15).*

Moreover, if a matrix function H *satisfies (4.18) and (4.19) then it also satisfies (4.15). However, the converse is in general false.*

Remark 4.3. Let E be any measurable subset of $\mathbb{T}^n$ such that $\{E + d : d \in \mathcal{D}\}$ is a partition of $\mathbb{T}^n$ (modulo null sets). Then, it is easy to see using the periodicity of M and H that if (4.18) and (4.19) hold for a.e. $\xi \in E$, then they must hold for a.e. $\xi \in \mathbb{R}^n$.

Proof. First, suppose that a matrix function $H = (h_{i,j})_{i\in\tilde{J}, j\in J}$ has periodic entries $h_{i,j} \in L^2(S_j)$, satisfies (4.15) and the index set $\tilde{J}$ has N elements. Consider the combined matrix function

$$T(\xi) = \begin{bmatrix} M(\xi + d_1) & \dots & M(\xi + d_q) \\ H(\xi + d_1) & \dots & H(\xi + d_q) \end{bmatrix}. \tag{4.20}$$

By the support conditions and (4.15), the matrix $T(\xi)$ has precisely $\sum_{d\in\mathcal{D}} m(\xi + d)$ non-zero columns. Condition (4.15) says that these non-zero columns form an orthonormal system. On the other hand, (2.14) and the fact that $\tilde{J}$ has N elements imply that the matrix $T(\xi)$ has at most $m(B\xi) + N$ non-zero rows. Clearly, any collection of orthonormal vectors must be smaller than the dimension of the space where they live. Consequently, (D6) must hold. This shows (iii) $\implies$ (i).

Conversely, suppose that (D6) holds and consider a matrix function

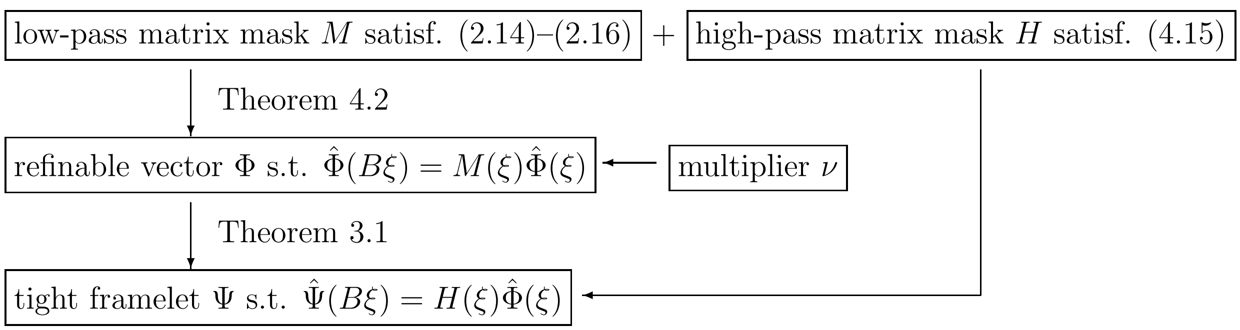
$$T'(\xi) = \begin{bmatrix} M(\xi + d_1) & \dots & M(\xi + d_q) \end{bmatrix}. \tag{4.21}$$

It is convenient to fix $\xi \in E$, where E is the same as in Remark 4.3. As before, by the support conditions, the matrix $T'(\xi)$ has at most $\sum_{d\in\mathcal{D}} m(\xi + d)$ non-zero columns. On the other hand, the matrix $T'(\xi)$ has $m(B\xi)$ non-zero rows forming an orthonormal system by (2.14). Therefore, for a fixed ξ, we have a finite submatrix $T''(\xi)$ with $c = \sum_{d\in\mathcal{D}} m(\xi + d)$ columns and $r = m(B\xi)$ orthonormal rows. Now, it suffices to find an extension of this submatrix to a unitary $c \times c$ matrix. Since $\tilde{m}(B\xi) = c - r \le N$ by (D6), at most N extra rows must be added. Define $[H(\xi + d_1) \dots H(\xi + d_q)]$ to be a matrix with rows indexed by $\tilde{J} = \{1, \dots, N\}$ such that the first $\tilde{m}(B\xi) = c - r$ rows of $[H(\xi + d_1) \dots H(\xi + d_q)]$ are formed by inserting the extra rows from a finite submatrix $T''(\xi)$ interspersed by zero columns, which were previously removed from the matrix $T'(\xi)$. The remaining rows (if any) of $[H(\xi + d_1) \dots H(\xi + d_q)]$ are defined to be zero. It is not hard to see that the these extra rows can be chosen in such a way that the resulting matrix function $[H(\xi + d_1) \dots H(\xi + d_q)]$ has measurable entries as a function of $\xi \in E$. As a result, the combined matrix function $T(\xi)$ has the same number of non-zero columns equal to $\sum_{d\in\mathcal{D}} m(\xi + d)$ as the number of non-zero rows equal to $m(B\xi) + \tilde{m}(B\xi)$ by (4.17). Furthermore, since the non-zero rows of $T(\xi)$ form an orthonormal sequence, the finite submatrix consisting of non-zero columns and rows must be unitary. Consequently, the constructed matrix H satisfies (4.18) and (4.19) for a.e. $\xi \in E$. By Remark 4.3 this shows that (i) $\implies$ (ii).

Next, if H is any matrix function as in (ii), then (2.14), (4.18), and (4.19) imply that the non-zero rows of the combined matrix function $T(\xi)$ form an orthonormal sequence. By (4.17) a finite submatrix consisting of non-zero columns and rows of $T(\xi)$ has the same number of non-zero columns as the number of non-zero rows and hence must be unitary. Since the rows of this finite submatrix are orthonormal, so are the columns, which implies that (4.15) holds. Therefore, (4.18) and (4.19) always imply (4.15). The converse implication is obviously false in general, since the combined matrix $T(\xi)$ may have a larger number of non-zero rows than non-zero columns and as a consequence the orthonormality of columns does not translate into orthonormality of rows. This proves (ii) $\Longrightarrow$ (iii) and completes the proof of Theorem 4.4. □

Remark 4.4. The origin of equation (4.17) is hidden in the consistency equation (2.23). Once we take $m = \dim_{V_0}$ and $\tilde{m} = \dim_{W_0}$, the connection becomes clear.

As a consequence of Theorems 4.3 and 4.4 we can deduce that any low-pass matrix mask function M satisfying (2.14)–(2.16) associated with the dimension function m satisfying (D1)–(D6) corresponds to some tight framelet Ψ via (4.14) and (4.16). To achieve this we must choose a high-pass mask matrix H such that the corresponding combined matrix function (4.20) has orthogonal columns, that is, (4.15) holds. Naturally, if we count on obtaining a semi-orthogonal wavelet Ψ, then the high-pass matrix mask H must satisfy a more restrictive conditions (4.18)–(4.19) resulting in row orthogonality of the combined matrix (4.20), see Theorem 5.2. Therefore, we can summarize the content of Theorem 4.3 in a simple flow diagram.



We would like to point out, that a refinable vector Φ in Theorem 4.3 is not unique. Indeed, the explicit formula for our choice of Φ is

$$\hat{\Phi}(\xi) = \nu(\xi) \lim_{N\to\infty} \left[\prod_{j=1}^{N} \bar{\mu}(B^{-j}\xi) M(B^{-j}\xi) \right] e = \lim_{N\to\infty} \nu(B^{-N}\xi) \prod_{j=1}^{N} M(B^{-j}\xi) e, \tag{4.22}$$

where $e = (1, 0, 0, \dots)$, μ is the phase of $m_{1,1}$ and ν is an arbitrary multiplier, i.e., a measurable unimodular function such that $\nu(B\xi)\overline{\nu(\xi)} = \mu(\xi)$. Recall that Lemma 4.1 guarantees the existence of such multipliers. Equivalently, we can define a multiplier associated to the mask M as any function ν satisfying

$$\nu(B\xi)\overline{\nu(\xi)} |m_{1,1,}(\xi)| = m_{1,1}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{4.23}$$

Indeed, if ν satisfies (4.23), then $\mu(\xi) = \nu(B\xi)\overline{\nu(\xi)}$ is a phase of $m_{1,1}$ and ν is its corresponding multiplier. Note that a phase μ satisfying (4.4) might not be unique if $m_{1,1}$ does not have a full support.

Since there are many possibilities for multipliers ν satisfying (4.23) we obtain a lot of choices for Φ. Moreover, these different choices generate distinct GMRA's. In general, if P is a matrix function such that $P(B\xi)^{-1}M(\xi)P(\xi) = M(\xi)$ a.e., then our $\hat{\Phi}$ can be replaced by $P\hat{\Phi}$. Nevertheless, we can loosely think that Φ is given by the standard product $\Pi_{j=1}^{\infty} M(B^{-j}\xi)$ that is applied to the vector e. Even better, it turns out that if the product is convergent, then this standard choice of Φ is valid.

Proposition 4.2. *If M is as in Theorem 4.3 and the product $\Pi_{j=1}^{\infty} M(B^{-j}\xi)$ is convergent for a.e. $\xi \in \mathbb{R}^n$, then Φ from Theorem 4.3 can be taken as*

$$\hat{\Phi}(\xi) := \left[\prod_{j=1}^{\infty} M(B^{-j}\xi)\right] e. \tag{4.24}$$

Proof. Since $\Pi_{j=1}^{\infty} M(B^{-j}\xi)$ is convergent a.e., we have that $\Pi_{j=1}^{\infty} m_{1,1}(B^{-j}\xi)$ is convergent a.e. as well. In particular,

$$\lim_{N\to\infty} \Pi_{j=N}^{\infty} m_{1,1}(B^{-j}\xi) = 1, \quad \text{a.e.} \tag{4.25}$$

By (4.3) this is equivalent to

$$\lim_{N\to\infty} \nu(B^{-N}\xi) \text{ exists for a.e. } \xi \in \mathbb{R}^n. \tag{4.26}$$

Let Φ be a refinable vector as in Theorem 4.3 and ν be any function as in Lemma 4.1. Define another function ν' satisfying the conclusions of Lemma 4.1 by

$$\nu'(\xi) = \overline{\alpha(\xi)}\nu(\xi), \qquad \text{where } \alpha(\xi) = \lim_{N\to\infty} \nu(B^{-N}\xi).$$

Indeed,

$$\nu'(B\xi)\overline{\nu'(\xi)} = \mu(\xi), \qquad \lim_{N\to\infty} \nu'(B^{-N}\xi) = 1 \text{ for a.e. } \xi \in \mathbb{R}^n.$$

Therefore, by (4.22),

$$\hat{\Phi}'(\xi) = [\Pi_{j=1}^{\infty} M(B^{-j}\xi)]e$$

is a refinable vector function obtained by the procedure of Theorem 4.3 with the multiplier ν'. This proves (4.24). □

5. Reconstruction of wavelets

The main goal of this section is to prove that every orthogonal wavelet can be reconstructed from its carefully chosen low-pass and high-pass matrix masks by the procedure described in the previous section. In order to achieve this, we will explore in more depth some subtle properties of the refinable vector Φ from Theorem 4.3.

Recall that by starting from a dimension function m and an appropriate matrix mask M, we obtained a refinable vector Φ in Section 4 and, therefore, also the associated GMRA. The standard issue in this type of constructions is the problem of "vanishing mass". In short, it may happen that $\int_{\mathbb{R}^n} ||\hat{\Phi}(\xi)||^2 d\xi < \int_{\mathbb{R}^n} m(\xi) d\xi$. In particular, Φ need not to be a scaling vector. Also, the GMRA that results from such procedure can have a strictly smaller dimension function (of its core space) than the original one that was used to start the construction. This feature was already observed in the classical MRA case on $\mathbb{R}$ with dilation by 2. The familiar Cohen's condition is one of the ways to assure that "no mass gets lost". It is crucial if one hopes to obtain a wavelet. However, as pointed in [18] in the dyadic scalar case, even if "some of the mass does vanish" one can still construct corresponding tight framelet. In the general case, the problem gains on complexity. Below, we give a simple necessary condition that is needed for preserving the "mass".

In order to achieve this preservation, one has to impose that the matrix M' given in (4.5) satisfies

$$\lim_{N\to\infty} \Pi_{j=1}^{N} M'(B^{-j}\xi) = \begin{bmatrix} * & 0 & 0 & \dots \\ * & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.1}$$

More precisely, for a.e. $\xi \in \mathbb{R}^n$ the limit does exist and is equal to a matrix whose only non-zero entries are in the first column. This also sheds a new light on (4.6), where we defined $\hat{\Phi}'$ as the first column of such a product.

We will show the necessity of the above condition in the following

Proposition 5.1. *Let m and Φ be as in Theorem 4.3. If*

$$\int_{\mathbb{R}^n} ||\hat{\Phi}(\xi)||^2 d\xi = \int_{\mathbb{T}^n} m(\xi) d\xi, \tag{5.2}$$

then (5.1) holds.

Proof. Let us show (5.1) by using calculations given in the proof of Lemma 4.5. First, we observe that (5.1) is equivalent to saying that $\lim_{N\to\infty} ||M_N(\xi)e_i|| \to 0$ for a.e. $\xi \in \mathbb{R}^n$, and every vector e_i, $i \geq 2$, of the standard basis. Also, we can use Φ' instead of Φ in our considerations.

By (4.12), the assumption (5.2) forces the inequality (4.11) to become an equality. Tracing back this fact through (4.10) and (4.9) we see that, eventually, one must have

$$\lim_{N\to\infty} (\mathrm{tr}[M_N(\xi)M_N^*(\xi)] - \|M_N(\xi)e\|^2) = 0,$$

for a.e. $\xi \in \mathbb{R}^n$. However, since $\mathrm{tr}[CD] = \mathrm{tr}[DC]$, the above becomes

$$\lim_{N\to\infty} \sum_{i\geq 2} \|M_N(\xi)e_i\|^2 = 0.$$

This shows the necessity of (5.1) and concludes the proof. □

In the next result we present the full conection between the "mass preservation" and the properties of our refinable vector Φ.

Theorem 5.1. *Let m and Φ be as in Theorem 4.3. Then, Φ is a scaling vector that generates a GMRA with the same dimension function as m if and only if (5.2) holds.*

Proof. Again, we can consider Φ' instead of Φ. If Φ' is a scaling vector then the dimension function of the corresponding GMRA is equal to $\sum_{k\in\mathbb{Z}^n} \|\hat{\Phi}'(\xi+k)\|^2$. Clearly, the assumption that this dimension function is the same as m implies that (5.2) holds.

On the other hand, assume that (5.2) is satisfied. From (4.12) and (4.11) it follows that

$$\sum_{k\in\mathbb{Z}^n} \|\hat{\Phi}'(\xi+k)\|^2 = m(\xi), \tag{5.3}$$

for a.e. $\xi \in \mathbb{T}^n$. By Proposition 5.1, applying Fatou's lemma to (4.8) yields

$$\sum_{k\in\mathbb{Z}^n} \hat{\Phi}'(\xi+k)\hat{\Phi}'^*(\xi+k) \le \Omega(\xi), \tag{5.4}$$

in the operator sense, for a.e. $\xi \in \mathbb{T}^n$. However, condition (5.3) simply says that

$$\operatorname{tr}\Big[\sum_{k\in\mathbb{Z}^n} \hat{\Phi}'(\xi+k)\hat{\Phi}'^*(\xi+k)\Big] = \operatorname{tr}[\Omega(\xi)] \qquad \text{a.e.}$$

Therefore, we must have an equality in (5.4). This shows that Φ' is a scaling vector. Moreover, (5.3) assures that the GMRA generated by Φ' has the dimension function equal to m. □

As a consequence of Theorem 5.1 we show that the procedure of Theorem 4.3 can result in a semi-orthogonal wavelet (with the expected size of generators) only if the combined matrix (4.20) has orthogonal rows. As a corollary, we conclude that the necessary condition for constructing orthogonal wavelets is that the high-pass filter H satisfies (4.18) and (4.19) with the diagonal matrix function $\tilde{\Omega}$ constantly equal to the identity matrix.

Theorem 5.2. *Suppose that m is a function that satisfies (D1)–(D6) and $\tilde{m}$ is given by (4.17). Suppose that M and H are low-pass and high-pass matrix masks as in Theorem 4.3. Let Ψ be the corresponding tight framelet. Then*

$$\int_{\mathbb{R}^n} ||\hat{\Psi}(\xi)||^2 d\xi \le \int_{\mathbb{T}^n} \tilde{m}(\xi) d\xi. \tag{5.5}$$

Moreover, if Ψ is a semi-orthogonal wavelet such that the equality holds in (5.5), then the high-pass filter H necessarily satisfies (4.18) and (4.19) with the diagonal matrix function $\tilde{\Omega}$ given by (2.20).

Proof. Let Φ be the refinable vector constructed in Theorem 4.3. Recall that the proof of Theorem 3.1 yields

$$\int_{\mathbb{R}^n} ||\hat{\Psi}(\xi)||^2 d\xi = (|\det A|-1)\int_{\mathbb{R}^n} ||\hat{\Phi}(\xi)||^2 d\xi < \infty. \tag{5.6}$$

On the other hand, since (D6) holds, condition (4.17) implies that $\tilde{m}$ is bounded. Therefore, we can integrate (4.17) over $\mathbb{T}^n$ to obtain

$$\int_{\mathbb{T}^n} \tilde{m}(\xi) d\xi = (|\det A|-1)\int_{\mathbb{T}^n} m(\xi) d\xi < \infty. \tag{5.7}$$

Combining (4.12), (5.6), and (5.7) yields (5.5).

In addition, suppose that Ψ is a semi-orthogonal wavelet such that the equality holds in (5.5). By Theorem 5.1, Φ is a scaling vector generating a GMRA $\{V_j\}_{j\in\mathbb{Z}}$ with the dimension function $\dim_{V_0} = m$. On the other hand, Ψ also generates a GMRA $\{V_j'\}_{j\in\mathbb{Z}}$ given by

$$V_j' = \operatorname{span}\{D^i T_k \psi^{\tilde{j}} : i < j,\ k \in \mathbb{Z}^n,\ \tilde{j} \in \tilde{J}\} \qquad \text{for } j \in \mathbb{Z}.$$

By [11, Corollary 4.3] the dimension function $\dim_{V_0'}$ of the core space V_0' can be computed explicitly and equals the wavelet dimension function D_Ψ. Consequently,

$$\int_{\mathbb{T}^n} \dim_{V_0'}(\xi) d\xi = \frac{1}{|\det A|-1}\int_{\mathbb{R}^n} ||\hat{\Psi}(\xi)||^2 = \int_{\mathbb{R}^n} ||\hat{\Phi}(\xi)||^2 = \int_{\mathbb{T}^n} \dim_{V_0}(\xi) d\xi < \infty. \tag{5.8}$$

On the other hand, by (4.16) $\Psi \subset V_1$ and hence $V_0' \subset V_0$. Thus, $\dim_{V_0'}(\xi) \le \dim_{V_0}(\xi)$ for a.e. ξ and (5.8) implies that

$$\dim_{V_0'}(\xi) = \dim_{V_0}(\xi) < \infty \qquad \text{for a.e. } \xi,$$

and hence $V_0' = V_0$. Therefore, the semi-ortohogonal wavelet ψ is associated with the GMRA $\{V_j\}_{j\in\mathbb{Z}}$. By Proposition 2.2 the high-pass matrix mask H satisfies claimed properties. □

As an immediate consequence of Theorem 5.2 we have

Corollary 5.1. *In addition to the assumptions of Theorem 5.2, assume that the equality holds in (D6). Hence, there is $N \in \mathbb{N}$ such that*

$$\sum_{d\in\mathcal{D}} m(\xi + d) = m(B\xi) + N \qquad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Let Ψ be the tight framelet as in Theorem 4.3. If $\Psi = (\psi^j)_{j=1}^N$ is an orthonormal wavelet, then the high-pass filter H necessarily satisfies (4.18) and (4.19) with the diagonal matrix function $\tilde{\Omega}(\xi) \equiv Id_{N\times N}$.

Proof. Our hypotheses imply that $\tilde{m}(\xi) \equiv N$ a.e. $\xi \in \mathbb{T}^n$. If $\Psi = (\psi^j)_{j=1}^N$ is an orthonormal wavelet, then

$$\int_{\mathbb{R}^n} ||\hat{\Psi}(\xi)||^2 d\xi = N = \int_{\mathbb{T}^n} \tilde{m}(\xi) d\xi.$$

By Theorem 5.2 and (2.20) H satisfies claimed properties since $\tilde{S}_j = \operatorname{supp}\dim_{\mathcal{S}(\psi^j)} = \mathbb{R}^n$ for $j = 1, \dots, N$. □

Finally, we will show that every scaling vector of a GMRA (that has an integrable dimension function of the core space) can be obtained by the procedure of Theorem 4.3 with an appropriate choice of a multiplier ν.

Theorem 5.3. *Suppose $\{V_j\}_{j\in\mathbb{Z}}$ is a GMRA such that $\dim_{V_0} \in L^1(\mathbb{T}^n)$. Let M be the low-pass matrix mask function of the scaling vector Φ for the core space V_0.*

(i) *Any multiplier ν' associated to M corresponds by Theorem 4.3 to some scaling vector Φ' with the same mask M (but generating not necessarily the same space V_0).*

(ii) *There exists a multiplier ν associated to M such that the scaling vector Φ in Theorem 4.3 is recovered by the product formula*

$$\hat{\Phi}(\xi) = \lim_{N\to\infty} \nu(B^{-N}\xi) \prod_{j=1}^{N} M(B^{-j}\xi)e \qquad \text{for a.e. } \xi. \tag{5.9}$$

Proof. Note that the dimension function $m = \dim_{V_0}$ of the core space V_0 satisfies the assumptions (D1)–(D5). Moreover, by Theorem 2.2, the matrix mask function M of Φ satisfies conditions (2.14)–(2.16). Therefore, a fixed multiplier ν' produces a refinable vector Φ' by the procedure of Theorem 4.3.

We are going to prove (i) and (ii) simultaneously. Observe that both Φ and Φ' satisfy the same refinable equation, which takes the form

$$\begin{aligned} \hat{\Phi}(B^{-N}\xi) &= m_{1,1}(B^{-N-1}\xi)\hat{\Phi}(B^{-N-1}\xi), \\ \hat{\Phi}'(B^{-N}\xi) &= m_{1,1}(B^{-N-1}\xi)\hat{\Phi}'(B^{-N-1}\xi), \end{aligned} \tag{5.10}$$

for sufficiently large $N > N(\xi)$ dependent on the choice of $\xi \in \mathbb{R}^n$. This is a simple consequence of the special form of the matrix mask M near the origin. Let $\hat{\varphi}_1$ and $\hat{\varphi}'_1$ be the first entries of $\hat{\Phi}$ and $\hat{\Phi}'$, respectively. By (5.10), the sequence $\{\hat{\varphi}'_1(B^{-N}\xi)/\hat{\varphi}_1(B^{-N}\xi)\}_{N>N(\xi)}$ must be constant whenever it is well-defined. Let $\alpha(\xi)$ be the constant value of this sequence. It is clear that $\alpha(B\xi) = \alpha(\xi)$. Moreover, the fact that $\hat{\Phi}$ and $\hat{\Phi}'$ have zeros in all but the first entry near the origin, (2.17), and Lemma 4.4, imply that $|\alpha(\xi)| = 1$.

Define another multiplier ν corresponding to the same matrix mask function M by

$$\nu(\xi) = \overline{\alpha(\xi)}\nu'(\xi).$$

Finally, let Φ'' be the refinable vector obtained by the procedure of Theorem 4.3 with the multiplier ν. By (5.10) and the previously mentioned special form of Φ and Φ' near the origin we have $\varepsilon > 0$ such that

$$\hat{\Phi}'(\xi) = \alpha(\xi)\hat{\Phi}(\xi) \qquad \text{for a.e. } |\xi| < \varepsilon.$$

On the other hand, by the product formula (4.22) we have that

$$\hat{\Phi}'(\xi) = \alpha(\xi)\hat{\Phi}''(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{5.11}$$

Since all functions Φ, Φ', and Φ'' satisfy the refinable equation with respect to the same matrix mask function M, we must necessarily have that $\Phi = \Phi''$. This completes the proof of part (ii).

To deduce part (i) observe that $\Phi = \Phi''$ together with (5.11) yields

$$\hat{\Phi}'(\xi) = \alpha(\xi)\hat{\Phi}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n$$

for some α such that $|\alpha(\xi)| = 1$ and $\alpha(B\xi) = \alpha(\xi)$ a.e. Therefore, we conclude that the refinable vector Φ' must be necessarily a scaling vector. Obviously, there is no guarantee that the SI space V'_0 generated by Φ' coincides with V_0 unless function α is $\mathbb{Z}^n$-periodic. This completes the proof of part (i) of Theorem 5.3. □

As an immediate consequence of Theorem 5.3 we have that every semi-orthogonal wavelet Ψ can be recovered by the procedure of Theorem 4.3. This is our main wavelet reconstruction result.

Theorem 5.4. *Suppose Ψ is a semi-orthogonal wavelet. Let $\{V_j\}_{j\in\mathbb{Z}}$ be the GMRA generated by Ψ, and let Φ be a scaling vector for V_0. Let M and H be the matrix mask functions of Φ and Ψ, respectively. Then there exists a multiplier ν associated to M such that the scaling vector Φ is recovered by the product formula (5.9) and Ψ is recovered by (4.16).*

Proof. Theorem 5.3 guarantees that we can recover Φ. To get back Ψ we use Proposition 2.2. It implies that the matrix mask functions M and H satisfy (2.21) and (2.22). As we pointed out in Theorem 4.4, these two conditions force H to satisfy (4.15). Thus, we can use such H to obtain Ψ via (4.16). □

6. Examples and comments

We shall construct some framelets using the procedure that was described in Section 4. We remind the reader that the whole process starts from choosing a dimension function. In order to find a specific dimension function one can construct a wavelet set and calculate the associated dimension function. In this way all dimension functions can be obtained by the result of Speegle and the authors [13].

It is customary to test GMRA constructions on the original "non-MRA object", that is, the Journé wavelet ψ given by $\hat{\psi} = \mathbf{1}_W$, where $W = [-\frac{16}{7}, -2] \cup [-\frac{1}{2}, -\frac{2}{7}] \cup [\frac{2}{7}, \frac{1}{2}] \cup [2, \frac{16}{7}]$. The associated dimension function (so called, Journé dimension function) is

$$m(\xi) = \begin{cases} 2 & \text{for } \xi \in [-\frac{1}{7}, \frac{1}{7}] \\ 1 & \text{for } \xi \in [-\frac{1}{2}, -\frac{3}{7}] \cup [-\frac{2}{7}, -\frac{1}{7}] \cup [\frac{1}{7}, \frac{2}{7}] \cup [\frac{3}{7}, \frac{1}{2}] \\ 0 & \text{for } \xi \in [-\frac{3}{7}, -\frac{2}{7}] \cup [\frac{2}{7}, \frac{3}{7}] \end{cases}. \tag{6.1}$$

Since m is $\mathbb{Z}$-periodic we list only its values on the torus. The same convention shall be used for other periodic objects that appear in this section. Clearly, m satisfies conditions (D1)-(D6) that are stated in Theorem 4.1 and thereafter. The corresponding sets S_j of (4.2) are S_1 and S_2, where S_1 is a periodization of $[-\frac{1}{2}, -\frac{3}{7}] \cup [-\frac{2}{7}, \frac{2}{7}] \cup [\frac{3}{7}, \frac{1}{2}]$ and S_2 is a periodization of $[-\frac{1}{7}, \frac{1}{7}]$. The "identity" periodic matrix function Ω of (2.9) is, therefore,

$$\Omega(\xi) = \begin{bmatrix} \mathbf{1}_{[-\frac{1}{2}, -\frac{3}{7}] \cup [-\frac{2}{7}, \frac{2}{7}] \cup [\frac{3}{7}, \frac{1}{2}]}(\xi) & 0 \\ 0 & \mathbf{1}_{[-\frac{1}{7}, \frac{1}{7}]}(\xi) \end{bmatrix},$$

for $\xi \in \mathbb{T}$. All of this provides the ground for finding an appropriate matrix mask function M that satisfies conditions (2.14)-(2.16).

Example 6.1. Consider possibly the simplest case

$$M(\xi) = \begin{bmatrix} \mathbf{1}_{F_1}(\xi) & 0 \\ 0 & \mathbf{1}_{F_2}(\xi) \end{bmatrix}.$$

Then condition (2.14) is equivalent to saying that for $i = 1, 2$

$$2F_i \cup (2F_i + 1) = S_i \tag{6.2}$$

and the union is disjoint. Condition (2.15) is already satisfied, and (2.16) holds as long as F_1 contains a neighborhood of the origin. There are plenty of choices of such F_1 and F_2. Clearly, only (6.2) deserves our attention. In order to get it, one can pick a set whose periodization gives S_i, divide it by 2, periodize it, and finally check if the union in (6.2) is indeed disjoint. For example, we can take the set $[-\frac{4}{7}, -\frac{1}{2}] \cup [-\frac{2}{7}, \frac{2}{7}] \cup [\frac{1}{2}, \frac{4}{7}]$ (its periodization is S_1) and define F_1 as the periodization of $[-\frac{2}{7}, -\frac{1}{4}] \cup [-\frac{1}{7}, \frac{1}{7}] \cup [\frac{1}{4}, \frac{2}{7}]$. In a similar way, F_2 can be simply chosen as the periodization of $[-\frac{1}{14}, \frac{1}{14}]$. This gives us

$$M(\xi) = \begin{bmatrix} \mathbf{1}_{[-\frac{2}{7}, -\frac{1}{4}] \cup [-\frac{1}{7}, \frac{1}{7}] \cup [\frac{1}{4}, \frac{2}{7}]}(\xi) & 0 \\ 0 & \mathbf{1}_{[-\frac{1}{14}, \frac{1}{14}]}(\xi) \end{bmatrix}, \tag{6.3}$$

for $\xi \in \mathbb{T}$. Since M has non-negative entries we can skip the step of choosing appropriate multipliers (4.3) and (4.4). Instead, we go directly to calculating $\Pi_{j=1}^{\infty} M(2^{-j}\xi)$. Clearly,

$$\Pi_{j=1}^{\infty} M(2^{-j}\xi) = \begin{bmatrix} \mathbf{1}_{E_1}(\xi) & 0 \\ 0 & \mathbf{1}_{E_2}(\xi) \end{bmatrix},$$

where $E_i = \bigcap_{j=1}^{\infty} 2^j F_i$ for $i = 1, 2$. To find the intersection for $i = 1$, let us denote $[-\frac{2}{7}, -\frac{1}{4}] \cup [-\frac{1}{7}, \frac{1}{7}] \cup [\frac{1}{4}, \frac{2}{7}]$ by I, and observe that $F_1 = \bigcup_{k \in \mathbb{Z}} (I + k)$. Using only the fact that $I \subset 2I$ and that $(2F_1 + 1) \cap 2F_1 = \emptyset$ (see 6.2), we get by induction that $\bigcap_{j=1}^{N} 2^j F_i = \bigcup_{k \in \mathbb{Z}} (2I + 2^N k)$.

Therefore, $E_1 = 2I = [-\frac{4}{7}, -\frac{1}{2}] \cup [-\frac{2}{7}, \frac{2}{7}] \cup [\frac{1}{2}, \frac{4}{7}]$. In a similar way we find that $E_2 = [-\frac{1}{7}, \frac{1}{7}]$, but this information is irrelevant. Indeed, as we pointed out in (4.24), the refinable vector is just the first column of $\Pi_{j=1}^{\infty} M(2^{-j}\xi)$. Thus, we obtain that for $\xi \in \mathbb{R}$

$$\hat{\Phi}(\xi) = \begin{bmatrix} \mathbf{1}_{[-\frac{4}{7},-\frac{1}{2}]\cup[-\frac{2}{7},\frac{2}{7}]\cup[\frac{1}{2},\frac{4}{7}]}(\xi) \\ 0 \end{bmatrix}.$$

As we can see, this is an example of a situation, where our construction produces a GMRA, whose dimension function is strictly smaller than the original one. Indeed, the above refinable vector generates a GMRA, whose dimension function is equal to

$$d(\xi) := \mathbf{1}_{[-\frac{4}{7},-\frac{1}{2}]\cup[-\frac{2}{7},\frac{2}{7}]\cup[\frac{1}{2},\frac{4}{7}]}(\xi),$$

for $\xi \in \mathbb{T}$. In effect, we can simply think that $\hat{\Phi} = \mathbf{1}_{[-\frac{4}{7},-\frac{1}{2}]\cup[-\frac{2}{7},\frac{2}{7}]\cup[\frac{1}{2},\frac{4}{7}]}$ and $M(\xi) = \mathbf{1}_{[-\frac{2}{7},-\frac{1}{4}]\cup[-\frac{1}{7},\frac{1}{7}]\cup[\frac{1}{4},\frac{2}{7}]}(\xi)$ for $\xi \in \mathbb{T}$. This simplifies the task of finding corresponding framelets. If we want to find one, then we should consider a high-pass mask H that satisfies (3.6). In other words, the combined matrix

$$T(\xi) = \begin{bmatrix} M(\xi) & M(\xi + \frac{1}{2}) \\ H(\xi) & H(\xi + \frac{1}{2}) \end{bmatrix},$$

needs to have orthogonal columns, and the norm of the first column has to be equal to $d(\xi)$ for $\xi \in \mathbb{T}$. A rather obvious choice for H is $H(\xi) = d(\xi) - M(\xi) = \mathbf{1}_{[-\frac{4}{7},-\frac{1}{2}]\cup[-\frac{1}{4},-\frac{1}{7}]\cup[\frac{1}{7},\frac{1}{4}]\cup[\frac{1}{2},\frac{4}{7}]}(\xi)$ for $\xi \in \mathbb{T}$. This choice yields a framelet ψ given by (3.5), that is, $\hat{\psi} = \mathbf{1}_{[-\frac{8}{7},-1]\cup[-\frac{1}{2},-\frac{2}{7}]\cup[\frac{2}{7},\frac{1}{2}]\cup[1,\frac{8}{7}]}$.

As we mentioned in Corollary 5.4, our precedure allows us to recover all semi-orthogonal wavelets. Let us see, therefore, how to get back the Journé wavelet.

Example 6.2. The low-pass matrix mask function M that we need to consider is

$$M(\xi) = \begin{bmatrix} \mathbf{1}_{F_1}(\xi) & 0 \\ \mathbf{1}_{F_2}(\xi) & 0 \end{bmatrix},$$

where F_1 is the periodization of $[-\frac{2}{7}, -\frac{1}{4}] \cup [-\frac{1}{7}, \frac{1}{7}] \cup [\frac{1}{4}, \frac{2}{7}]$ and F_2 is the periodization of $[-\frac{4}{7}, -\frac{1}{2}] \cup [\frac{1}{2}, \frac{4}{7}]$. As we can see, condition (2.14) is equivalent to saying that (6.2) holds and, in addition, that $F_1 \cap F_2 = \emptyset$. We also have that

$$\Pi_{j=1}^{\infty} M(2^{-j}\xi) = \begin{bmatrix} \mathbf{1}_{E_1}(\xi) & 0 \\ \mathbf{1}_{E_2}(\xi) & 0 \end{bmatrix},$$

where $E_1 = \bigcap_{j=1}^{\infty} 2^j F_1$ and $E_2 = (2F_2) \cap \bigcap_{j=2}^{\infty} 2^j F_1$. Using similar arguments as previously, we obtain that $E_1 = [-\frac{4}{7}, -\frac{1}{2}] \cup [-\frac{2}{7}, \frac{2}{7}] \cup [\frac{1}{2}, \frac{4}{7}]$ and $E_2 = 2(F_2 \cap E_1) = [-\frac{8}{7}, -1] \cup [1, \frac{8}{7}]$. Thus, we get that

$$\hat{\Phi} = \begin{bmatrix} \mathbf{1}_{[-\frac{4}{7},-\frac{1}{2}]\cup[-\frac{2}{7},\frac{2}{7}]\cup[\frac{1}{2},\frac{4}{7}]} \\ \mathbf{1}_{[-\frac{8}{7},-1]\cup[1,\frac{8}{7}]} \end{bmatrix}.$$

The dimension function corresponding to this scaling vector is the same as the Journé dimension function. It is easy to check, that the high-pass matrix mask function H given for $\xi \in \mathbb{T}$ by

$$H(\xi) = \begin{bmatrix} \mathbf{1}_{[-\frac{1}{4},-\frac{1}{7}]\cup[\frac{1}{7},\frac{1}{4}]}(\xi) & \mathbf{1}_{[-\frac{1}{7},\frac{1}{7}]}(\xi) \end{bmatrix}.$$

satisfies (3.6) and yields the Journé wavelet via (3.5).

So far the above two examples yield MSF type tight framelets and wavelets. The next two examples show how to construct large classes of non-MSF tight framelets and wavelets originating from the Journé dimension function (6.1).

Example 6.3. Consider a generalization of the previous examples, where the low-pass matrix mask function M is given by

$$M(\xi) = \begin{bmatrix} \mathbf{1}_{F_1}(\xi) & 0 \\ m_1(\xi) & m_2(\xi) \end{bmatrix}, \tag{6.4}$$

where F_1 is the periodization of $[-\frac{2}{7}, -\frac{1}{4}] \cup [-\frac{1}{7}, \frac{1}{7}] \cup [\frac{1}{4}, \frac{2}{7}]$ and m_1, m_2 are $\mathbb{Z}$-periodic measurable functions. Condition (2.14) imposes certain restrictions on possible functions m_1 and m_2. That is, we must stipulate that for $\xi \in \mathbb{T}$

$$\mathbf{1}_{F_1}(\xi) m_1(\xi) + \mathbf{1}_{F_1}(\xi + 1/2) m_1(\xi + 1/2) = 0 \tag{6.5}$$

$$|m_1(\xi)|^2 + |m_2(\xi)|^2 + |m_1(\xi + 1/2)|^2 + |m_2(\xi + 1/2)|^2 = \mathbf{1}_{[-\frac{1}{14}, \frac{1}{14}] \cup [\frac{3}{7}, \frac{4}{7}]}(\xi) \tag{6.6}$$

Since F_1 and $F_1 + 1/2$ are disjoint, m_1 must vanish on F_1 by (6.5) and consequently m_1 must be supported on the periodization of the interval $[-\frac{1}{14}, \frac{1}{14}] + 1/2 = [\frac{3}{7}, \frac{4}{7}]$ by (6.6). Consequently, we must have for $\xi \in \mathbb{T}$

$$m_1(\xi) = v(\xi) \mathbf{1}_{[\frac{3}{7}, \frac{4}{7}]}(\xi), \qquad m_2(\xi) = v(\xi) \mathbf{1}_{[-\frac{1}{14}, \frac{1}{14}]}(\xi), \tag{6.7}$$

where v is an arbitrary $\mathbb{Z}$-periodic measurable function satisfying

$$|v(\xi)|^2 + |v(\xi + 1/2)|^2 = \mathbf{1}_{[-\frac{1}{14}, \frac{1}{14}]}(\xi), \qquad \text{for a.e. } \xi \in (-1/4, 1/4). \tag{6.8}$$

It is easy to verify that as long as conditions (6.7) and (6.8) hold, the matrix mask function M satisfies (2.14)-(2.16). Thus, we can apply Theorem 4.3. The corresponding refinable vector $\hat{\Phi}$ is the first column of the infinite product

$$\prod_{j=1}^{\infty} M(2^{-j}\xi) = \begin{bmatrix} \mathbf{1}_{E_1}(\xi) & 0 \\ * & \prod_{j=1}^{\infty} m_2(2^{-j}\xi) \end{bmatrix}, \tag{6.9}$$

where E_1 is again the periodization of $[-\frac{4}{7}, -\frac{1}{2}] \cup [-\frac{2}{7}, \frac{2}{7}] \cup [\frac{1}{2}, \frac{4}{7}]$. The lower left entry of the above matrix is represented by a more complicated infinite product which can be computed for some specific choices of the function v satisfying (6.8), see the next example.

The corresponding framelet can be found by choosing a high-pass matrix mask H satisfying (3.6). In addition, if we hope on obtaining a wavelet we should apply Theorem 5.2. Since in our case $\tilde{m} = 1$ a.e., we see that H has to be 1×2 matrix-valued and must satisfy conditions (4.18), (4.19), with the diagonal matrix function $\tilde{\Omega}(\xi) = [1]$. Then, a direct but tedious calculation shows, that modulo a unimodular $\mathbb{Z}$-periodic function, the high-pass matrix mask H is given by

$$\begin{aligned} H(\xi) = &\begin{bmatrix} \mathbf{1}_{[-\frac{1}{4}, -\frac{1}{7}] \cup [\frac{1}{7}, \frac{1}{4}]}(\xi) & \mathbf{1}_{[-\frac{1}{7}, -\frac{1}{14}] \cup [\frac{1}{14}, \frac{1}{7}]}(\xi) \end{bmatrix} \\ &+ e^{2\pi i \xi} v(\xi + 1/2) \begin{bmatrix} \mathbf{1}_{[-\frac{4}{7}, -\frac{1}{2}] \cup [\frac{1}{2}, \frac{4}{7}]}(\xi) & \mathbf{1}_{[-\frac{1}{14}, \frac{1}{14}]}(\xi) \end{bmatrix} \quad \text{for } \xi \in \mathbb{T}. \end{aligned}$$

Define $\psi \in L^2(\mathbb{R})$ by $\hat{\psi} = H(\xi)\hat{\Phi}(\xi)$. Then, by Theorem 4.3, ψ is a tight framelet for any choice of $\mathbb{Z}$-periodic function v satisfying (6.8).

Note that if we choose $v = \mathbf{1}_{[-\frac{1}{14},\frac{1}{14}]}$, then the matrix mask M given in (6.4) becomes the same as the one given in (6.3). In effect, we obtain the same tight framelet ψ as in Example 6.1, that is not a wavelet. While the fact that ψ is not a wavelet can be verified directly, it is also a consequence of Proposition 5.1. That is, the procedure of constructing refinable vector from low-pass matrix mask can only result in a scaling vector if all but the first column of the product matrix $\prod_{j=1}^{\infty} M(2^{-j}\xi)$ are zeros. Indeed, if ψ were a wavelet, then by (5.6), condition (5.2) would hold as well. Therefore, the mentioned proposition would imply that (5.1) must be satisfied. Since it is not, ψ is not a wavelet. On the other hand, if we choose $v = \mathbf{1}_{[\frac{3}{7},\frac{4}{7}]}$, then Example 6.2 shows that we obtain the usual Journé wavelet modified by a negligible unimodular phase factor.

In the next example we construct a large class of non-MSF non-MRA wavelets by an appropriate choice of functions v satisfying (6.8). Naturally, each wavelet in this class must share the dimension function of the Journé wavelet given by (6.1).

Example 6.4. Let w be an arbitrary $\mathbb{Z}$-periodic measurable function satisfying

$$|w(\xi)|^2 + |w(\xi + 1/2)|^2 = \mathbf{1}_{[-\frac{1}{14},-\frac{1}{28}]\cup[\frac{1}{28},\frac{1}{14}]}(\xi) \qquad \text{for a.e. } \xi \in (-1/4, 1/4). \tag{6.10}$$

Then, v given for $\xi \in \mathbb{T}$ by $v(\xi) = w(\xi) + \mathbf{1}_{[\frac{13}{28},\frac{15}{28}]}(\xi)$ satisfies (6.8). Define $\mathbb{Z}$-periodic functions

$$m_1(\xi) = w(\xi)\mathbf{1}_{[\frac{3}{7},\frac{13}{28}]\cup[\frac{15}{28},\frac{4}{7}]}(\xi) + \mathbf{1}_{[\frac{13}{28},\frac{15}{28}]}(\xi), \qquad m_2(\xi) = w(\xi)\mathbf{1}_{[-\frac{1}{14},-\frac{1}{28}]\cup[\frac{1}{28},\frac{1}{14}]}(\xi). \tag{6.11}$$

Finally, let M be given by (6.4). The same argument as in Example 6.3 shows that M satisfies (2.14)–(2.16) and hence, it is a low-pass matrix mask function. In fact, we obtain a proper subclass of low-pass matrix masks considered in the previous example. We can also choose a high-pass matrix mask H by emulating Example 6.3. That is, we define

$$\begin{aligned} H(\xi) &= \left[\mathbf{1}_{[-\frac{1}{4},-\frac{1}{7}]\cup[\frac{1}{7},\frac{1}{4}]}(\xi) \quad \mathbf{1}_{[-\frac{1}{7},-\frac{1}{14}]\cup[-\frac{1}{28},\frac{1}{28}]\cup[\frac{1}{14},\frac{1}{7}]}(\xi)\right] \\ &+ e^{2\pi i\xi} w(\xi+1/2)\left[\mathbf{1}_{[-\frac{4}{7},-\frac{15}{28}]\cup[\frac{15}{28},\frac{4}{7}]}(\xi) \quad \mathbf{1}_{[-\frac{1}{14},-\frac{1}{28}]\cup[\frac{1}{28},\frac{1}{14}]}(\xi)\right] \qquad \text{for } \xi \in \mathbb{T} \end{aligned}$$

The advantage of our choice of low-pass and high-pass matrix masks is twofold. First, the corresponding refinable vector Φ can be easily computed. Second, it can be shown that Φ is a scaling vector and the resulting tight framelet ψ is a wavelet. Indeed, note that

$$M(\xi/2)M(\xi/4) = \begin{bmatrix} \mathbf{1}_{F_1}(\xi/2)\mathbf{1}_{F_1}(\xi/4) & 0 \\ m_1(\xi/2)\mathbf{1}_{F_1}(\xi/4) + m_2(\xi/2)m_1(\xi/4) & 0 \end{bmatrix}, \tag{6.12}$$

since $m_2(\xi)m_2(\xi/2) = 0$ for a.e. $\xi \in \mathbb{R}$. Moreover,

$$\prod_{j=3}^{\infty} M(2^{-j}\xi) = \begin{bmatrix} \mathbf{1}_{E_1}(\xi/4) & 0 \\ * & 0 \end{bmatrix}, \tag{6.13}$$

where $E_1 = [-\frac{4}{7}, -\frac{1}{2}] \cup [-\frac{2}{7}, \frac{2}{7}] \cup [\frac{1}{2}, \frac{4}{7}]$. Consequently,

$$\hat{\Phi}(\xi) = \begin{bmatrix} \hat{\varphi}_1(\xi) \\ \hat{\varphi}_2(\xi) \end{bmatrix} = \begin{bmatrix} \mathbf{1}_{E_1}(\xi) \\ (m_1(\xi/2)\mathbf{1}_{F_1}(\xi/4) + m_2(\xi/2)m_1(\xi/4))\mathbf{1}_{E_1}(\xi/4) \end{bmatrix}.$$

Finally, a direct but tedious calculation shows that

$$\hat{\varphi}_2(\xi) = \mathbf{1}_{[-\frac{15}{14},-1]\cup[1,\frac{15}{14}]} + w(\xi/2)\mathbf{1}_{[-\frac{15}{7},-\frac{29}{14}]\cup[-\frac{8}{7},-\frac{15}{14}]\cup[\frac{15}{14},\frac{8}{7}]\cup[\frac{29}{14},\frac{15}{7}]}. \tag{6.14}$$

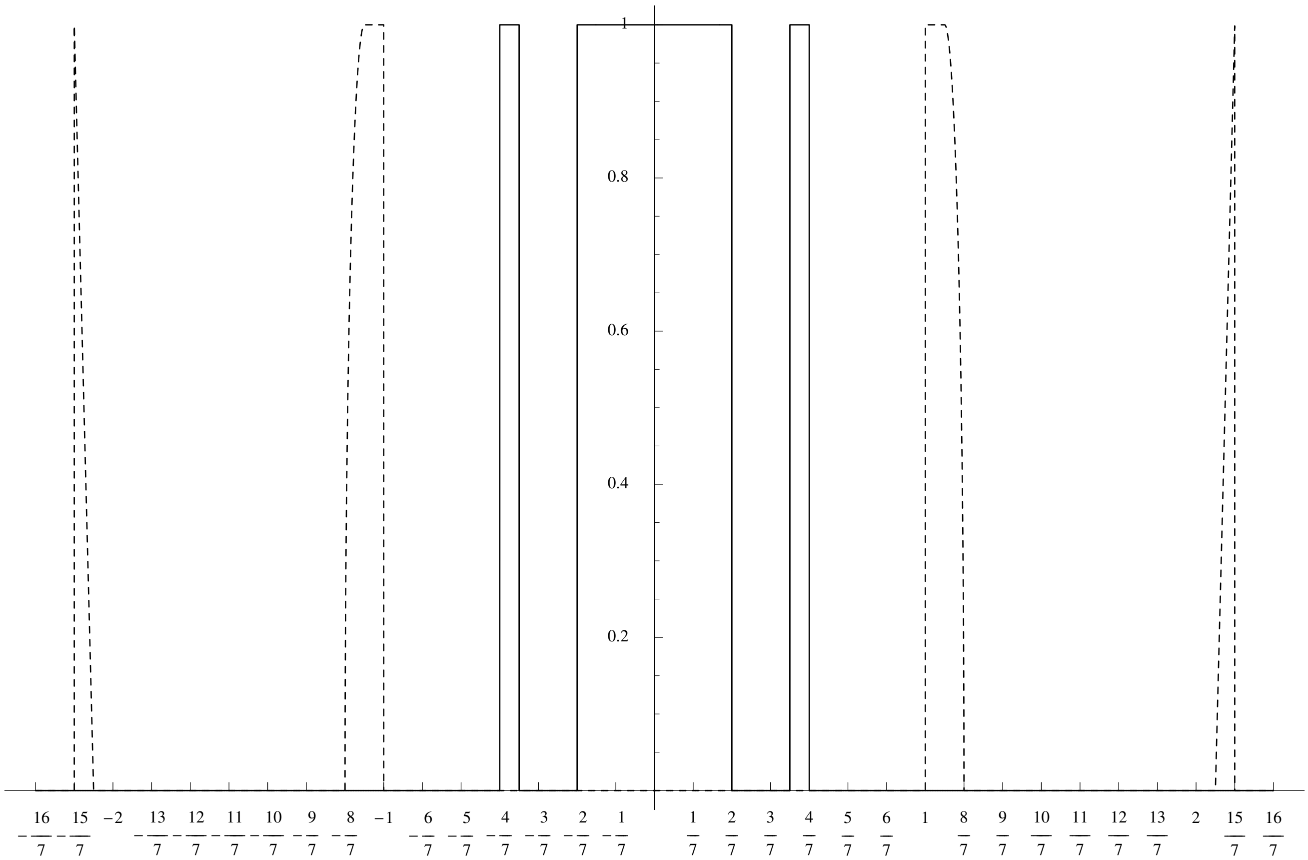


FIGURE 1. The graphs of $\hat{\varphi}_1$ (solid line) and a typical $\hat{\varphi}_2$ (dashed line).

To see that Φ is indeed a scaling vector it suffices to observe that

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}_2(\xi+k)|^2 = \mathbf{1}_{[-\frac{1}{7},\frac{1}{7}]}(\xi) \qquad \text{for a.e. } \xi \in [-1/2,1/2]$$

and check that (2.10) holds.

Finally, one can compute the formula for the corresponding wavelet $\psi = \psi_w$,

$$\begin{aligned}\hat{\psi}(\xi) = \mathbf{1}_{[-\frac{29}{14},-2]\cup[-\frac{1}{2},-\frac{2}{7}]\cup[\frac{2}{7},\frac{1}{2}]\cup[2,\frac{29}{14}]}(\xi) + w(\xi/4)\mathbf{1}_{[-\frac{30}{7},-\frac{29}{7}]\cup[-\frac{16}{7},-\frac{15}{7}]\cup[\frac{15}{7},\frac{16}{7}]\cup[\frac{29}{7},\frac{30}{7}]}(\xi)\\ +e^{\pi i\xi}w(\xi/2+1/2)\mathbf{1}_{[-\frac{15}{7},-\frac{29}{14}]\cup[-\frac{8}{7},-\frac{15}{14}]\cup[\frac{15}{14},\frac{8}{7}]\cup[\frac{29}{14},\frac{15}{7}]}(\xi).\end{aligned} \tag{6.15}$$

Figures 1 and 2 show graphs of a typical scaling vector $\hat{\Phi}$ and the corresponding wavelet $\hat{\psi}$.

Observe that the family of wavelets

$$\{\psi_w : w \text{ satisfies } (6.10)\}$$

is pathwise connected in $L^2(\mathbb{R})$. Indeed, given two $\mathbb{Z}$-periodic measurable functions w_0 and w_1 both satisfying (6.10), it is not difficult to construct a family $\{w_t\}_{t\in[0,1]}$ of functions satisfying (6.10) such that

$$w_s(\xi) \to w_t(\xi) \qquad \text{for a.e. } \xi \in \mathbb{T} \text{ as } s \to t.$$

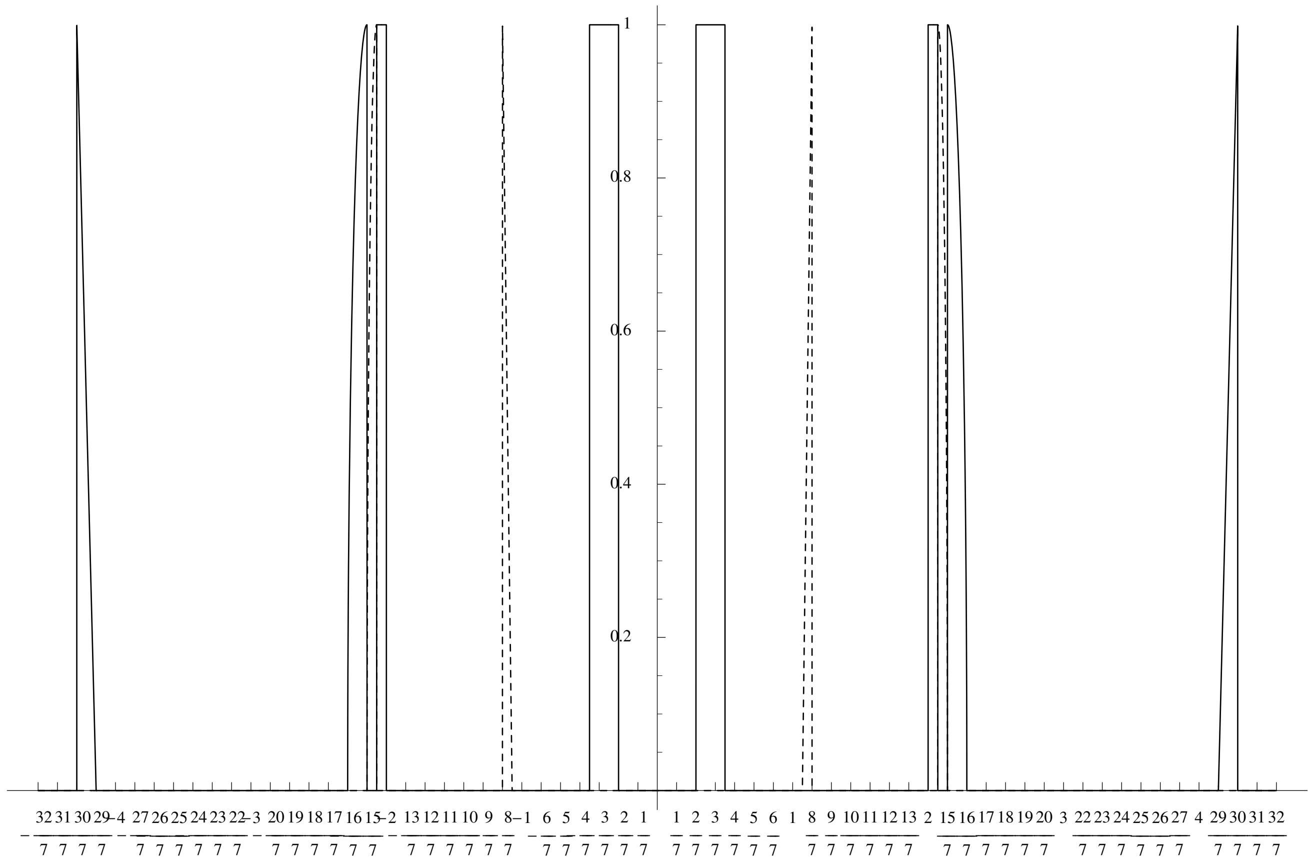


FIGURE 2. The graph of $\hat{\psi}$ (dashed line corresponds to the part that contains the phase factor $e^{\pi i \xi}$).

Then, by (6.15), we see that

$$\hat{\psi}_{w_s}(\xi) \to \hat{\psi}_{w_t}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R} \text{ as } s \to t.$$

Since $||\psi_{w_t}|| = 1$ for all $t \in [0,1]$, the map $t \mapsto \psi_{w_t}$ is the required continuous path.

Example 6.4 shows that a large class of wavelets can be constructed by the procedure of Theorem 4.3. Moreover, Theorem 5.4 shows that technically every imaginable wavelet can be obtained in that way. However, it is an open problem whether the same is true for all tight framelets.

Two serious difficulties arise when one wants to design a constructive method for obtaining all tight framelets on $\mathbb{R}^n$. The first problem is that it is not known if all such framelets are associated to a GMRA. This is often refered to as the "Baggett's problem". Baggett observed that a tight framelet Ψ generates a GMRA if and only if its space of negative dilates V satisfies

$$\bigcap_{j \in \mathbb{Z}} D^j(V) = \{0\}. \tag{6.16}$$

We have treated this problem with detail in [12]. An earlier result of the second author [22] assures that if the spectral function of V is integrable, then the above condition is satisfied.

It turns out that we can use Lemma 3.1 to improve on this result in the setting of the space of negative dilates.

Theorem 6.1. *Let Ψ be a tight framelet on $\mathbb{R}^n$ with its space of negative dilates V. If the set $\{\xi \in \mathbb{R}^n : \dim_V(\xi) < \infty\}$ has a positive (Lebesgue) measure, then (6.16) holds and Ψ generates a GMRA.*

Proof. As in the proof of Proposition 4.1, let $W = D(V) \ominus V$ and observe that since Ψ consists of a finite number of functions, W has a finite number of generators. That is, we have $\dim_W \le N$ for some $N \in \mathbb{N}$. The equation $D(V) = V \oplus W$ implies that

$$\sum_{d\in\mathcal{D}} m(B^{-1}\xi + d) = m(\xi) + \dim_W(\xi) \le m(\xi) + N, \tag{6.17}$$

where $m = \dim_V$. Thus, condition (3.11) of Lemma 3.1 is satisfied for such m. However, to apply Lemma 3.1 we need to show that m is finite a.e. This can be done using a simple ergodic argument.

Indeed, since the matrix $B : \mathbb{R}^n \to \mathbb{R}^n$ preserves the lattice $\mathbb{Z}^n$, it induces a measure preserving endomorphism $\tilde{B} : \mathbb{T}^n \to \mathbb{T}^n$. Moreover, $\tilde{B}$ is ergodic by [23, Corollary 1.10.1] because B is expansive. Define the set

$$E = \{\xi \in \mathbb{T}^n : m(\xi) < \infty\}.$$

The condition (6.17) implies that $\tilde{B}^{-1}E \subset E$. Since $\tilde{B}$ is measure preserving we must have $\tilde{B}^{-1}E = E$ (modulo null sets). Finally, by the ergodicity of $\tilde{B}$, we have either $|E| = 0$ or $|E| = 1$. Combining this with our hypothesis $|E| > 0$, proves that $m(\xi) < \infty$ for a.e. $\xi \in \mathbb{R}^n$.

Since all the assumptions of Lemma 3.1 are satisfied for our m, we get that $m \in L^1(\mathbb{T}^n)$. Equivalently, we have $\sigma_V \in L^1(\mathbb{R}^n)$. As we mentioned before, the latter implies that (6.16) holds by the result of the second author [22]. Therefore, Ψ generates a GMRA. ☐

If we consider an easier scenario and want to construct all tight framelets associated to a GMRA, we encounter the second difficulty. It is an open problem whether every tight framelet Ψ generating some GMRA $\{V_j\}_{j\in\mathbb{Z}}$ can be obtained by the procedure of Theorem 4.3 via the Unitary Extension Principle. In other words, is it possible to find appropriate matrix mask functions M and H resulting by the procedure of Theorem 4.3 in a tight framelet Ψ? This problem remains open even for tight framelets Ψ associated to an MRA, i.e., when $\dim_{V_0} \equiv 1$.

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Department of Mathematics, University of Oregon, Eugene, OR 97403–1222, USA
E-mail address: mbownik@uoregon.edu

Mathematical Institute, University of Wrocław, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail address: zioma@math.uni.wroc.pl
Current address: Numerical Harmonic Analysis Group, Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, A-1090 Wien, AUSTRIA