Weyl-Heisenberg frames for subspaces of $L^2(R)$.

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Abstract

A Weyl-Heisenberg frame $\{E_{mb}T_{na}g\}_{m,n\in Z} = \{e^{2\pi imb(\cdot)}g(\cdot - na)\}_{m,n\in Z}$ for $L^2(R)$ allows every function $f \in L^2(R)$ to be written as an infinite linear combination of translated and modulated versions of the fixed function $g \in L^2(R)$. In the present paper we find sufficient conditions for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $\overline{span}\{E_{mb}T_{na}g\}_{m,n\in Z}$, which, in general, might just be a subspace of $L^2(R)$. Even our condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $L^2(R)$ is significantly weaker than the previous known conditions. The results also shed new light on the classical results concerning frames for $L^2(R)$, showing for instance that the condition $G(x) := \sum_{n\in Z} |g(x - na)|^2 > A > 0$ is not necessary for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $\overline{span}\{E_{mb}T_{na}g\}_{m,n\in Z}$. Our work is inspired by a recent paper by Benedetto and Li [1], where the relationship between the zero-set of the function G and frame properties of the set of functions $\{g(\cdot - n)\}_{n\in Z}$ is analyzed.

1 Preliminaries and notation.

Let $\mathcal{H}$ denote a separable Hilbert space with the inner product $< \cdot, \cdot >$ linear in the first entry. Let I denote a countable index set.

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We say that $\{g_i\}_{i\in I} \subseteq \mathcal{H}$ is a *frame* (for $\mathcal{H}$) if there exist constants $A, B > 0$ such that

$$A||f||^2 \leq \sum_{i\in I} |<f, g_i>|^2 \leq B||f||^2, \ \forall f \in \mathcal{H}.$$

In particular a frame for $\mathcal{H}$ is complete, i.e., $\overline{span}\{g_i\}_{i\in I} = \mathcal{H}$. In case $\{g_i\}_{i\in I}$ is not complete, $\{g_i\}_{i\in I}$ can still be a frame for the subspace $\overline{span}\{g_i\}_{i\in I}$; in that case we say that $\{g_i\}_{i\in I}$ is a *frame sequence.* The numbers A, B that appear in the definition of a frame are called *frame bounds.*

Orthonormal bases and, more generally, *Riesz bases*, are frames. Recall that $\{g_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$ if $\overline{span}\{g_i\}_{i\in I} = \mathcal{H}$ and

$$\exists A, B > 0: \quad A\sum_{i\in I} |c_i|^2 \leq ||\sum_{i\in I} c_i g_i||^2 \leq B\sum_{i\in I} |c_i|^2, \quad \forall \{c_i\}_{i\in I} \in \ell^2(I).$$

If $\{g_i\}_{i\in I}$ is a Riesz basis for $\overline{span}\{g_i\}_{i\in I}$, we say that $\{g_i\}_{i\in I}$ is a *Riesz sequence.*

The present paper deals with frames having a special structure: all elements are translated and/or modulated versions of a single function. Let $L^2(R)$ denote the Hilbert space of functions on the real line which are square integrable with respect to the Lebesgue measure. First, define the following operators on functions $f \in L^2(R)$:

$$\text{Translation by } \ a \in R: \quad (T_a f)(x) = f(x-a), \ x \in R.$$

$$\text{Modulation by } b \in R: \quad (E_b g)(x) = e^{2\pi ibx} f(x), \ x \in R.$$

A frame for $L^2(R)$ of the form $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is called a *Weyl-Heisenberg frame* (or Gabor frame). For a collection of different papers concerning those frames we refer to the monograph [5].

Sufficient conditions for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $L^2(R)$ has been known for about 10 years. The basic insight was provided by Daubechies [3]. A slight improvement was proved in [6]:

Theorem 1.1: *Let $g \in L^2(R)$ and suppose that*

$$(1) \quad \exists A, B > 0 : \quad A \leq \sum_{n \in Z} |g(x-na)|^2 \leq B \text{ for a.e. } x \in R$$

$$(2) \quad lim_{b \to 0} \sum_{k \neq 0} || \sum_{n \in Z} T_{na} g T_{na+\frac{k}{b}} \overline{g} ||_\infty = 0.$$

Then there exists $b_0 > 0$ such that $\{E_{mb}T_{na}g\}_{m,n \in Z}$ is a Weyl-Heisenberg frame for $L^2(R)$ for all $b \in]0; b_0[$.

The proof of Theorem 1.1 is based on the following identity, valid for all continuous functions f with compact support whenever g satisfies (1):

$$
\begin{aligned}
(3) \qquad & \sum_{m,n \in Z} | < f, E_{mb}T_{na}g > |^2 \\
= \; & \frac{1}{b} \int |f(x)|^2 G(x) dx \\
+ \; & \frac{1}{b} \sum_{k \neq 0} \int \overline{f(x)} f(x-k/b) \sum_{n \in Z} g(x-na) \overline{g(x-na-k/b)} dx.
\end{aligned}
$$

An estimate of the second term in (3) now shows that $\{E_{mb}T_{na}g\}_{m,n \in Z}$ is actually a frame for all values of b for which

$$(4) \quad \sum_{k \neq 0} || \sum_{n \in Z} T_{na} g T_{na+\frac{k}{b}} \overline{g} ||_\infty < A.$$

A more recent result can be found in [4] : in Theorem 2.3 it is proved that if (1) is satisfied and there exists a constant $D < A$ such that

$$(5) \quad \sum_{k \neq 0} \sum_{n \in Z} |g(x-na) g(x-na-\frac{k}{b})| \leq D \text{ for } a.e. \; x \in R,$$

then $\{E_{mb}T_{na}g\}_{m,n \in Z}$ is a frame for $L^2(R)$ with bounds $\frac{A-D}{b}, \frac{B+D}{b}$. The reader should observe that [4] does not provide us with a generalization of the results in [3], [6] in a strict sense: there are cases where (5) is satisfied but (4) is not, and vice versa. The main point is that other conditions (that are easy to check) for $\{E_{mb}T_{na}g\}_{m,n \in Z}$ to be a frame can be derived from (5), cf. Theorem 2.4 in [4].

Define the *Fourier Transform* $\mathcal{F}(f) = \hat{f}$ of $f \in L^1(R)$ by

$$\hat{f}(y) = \int f(x) e^{-2\pi i y x} dx.$$

As usual we extend the Fourier Transform to an isometry from $L^2(R)$ onto $L^2(R)$. We denote the inverse Fourier transformation of $g \in L^2(R)$ by $\mathcal{F}^{-1}g$ or $\check{g}$. It is important to observe the following comutator relations, valid for all $a \in R$:

$$\mathcal{F}T_a = E_{-a}\mathcal{F}, \quad \mathcal{F}E_a = T_a\mathcal{F}.$$

We need a result from [2]. The basic insight was provided by Benedetto and Li [1], who treated the case $a = 1$.

Theorem 1.2: *Let $g \in L^2(R)$. Then $\{T_{na}g\}_{n\in Z}$ is a frame sequence with bounds A, B if and only if*

$$0 < aA \leq \sum_{n\in Z} |\hat{g}(\frac{x+n}{a})|^2 \leq aB \text{ for a.e. } x \text{ for which} \sum_{n\in Z} |\hat{g}(\frac{x+n}{a})|^2 \neq 0.$$

In that case $\{T_{na}g\}_{n\in Z}$ is a Riesz sequence if and only if the set of x for which $\sum_{n\in Z} |\hat{g}(\frac{x+n}{a})|^2 = 0$ has measure zero.

Theorem 1.2 leads immediately to an equivalent condition to (1). Define the function G and its kernel N_G by

$$G : R \to [0, \infty], \quad G(x) := \sum_{n\in Z} |g(x - na)|^2,$$
$$N_G = \{x \in R \mid G(x) = 0\}.$$

Corollary 1.3: *$\{E_{\frac{n}{a}}g\}_{n\in Z}$ is a frame sequence with bounds A, B if and only if*

$$0 < \frac{A}{a} \leq \sum_{n\in Z} |g(x - na)|^2 \leq \frac{B}{a} \text{ for a.e. } x \in R - N_G.$$

In that case $\{E_{\frac{n}{a}}g\}_{n\in Z}$ is a Riesz sequence iff N_G has measure zero.

Proof: The inequality

$$0 < \frac{A}{a} \leq \sum_{n\in Z} |g(x - na)|^2 \leq \frac{B}{a} \text{ for a.e. } x \in R - N_G$$

holds if and only if

$$(6) \qquad 0 < \frac{A}{a} \le \sum_{n\in Z} |g([x-n]a)|^2 \le \frac{B}{a} \;\text{ for } a.e.\ x \in R - N_G.$$

By Theorem 1.2, (6) is equivalent to $\{T_{\frac{n}{a}}\check{g}\}_{n\in Z}$ being a frame sequence with bounds A, B. Applying the Fourier transformation this is equivalent to $\{E_{\frac{n}{a}}g\}_{n\in Z}$ being a frame sequence with bounds A, B.
Q.E.D.

2 The results.

In the rest of the paper we concentrate on Weyl-Heisenberg frames $\{E_{mb}T_{na}g\}_{m,n\in Z}$. Our first result gives a sufficient condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame sequence. Our condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $L^2(R)$ is significantly weaker than the conditions mentioned in section 1.

Let $L^2(R - N_G)$ denote the set of functions in $L^2(R)$ that vanishes at N_G.

Theorem 2.1: *Let $g \in L^2(R)$, $a, b > 0$ and suppose that*

$$(7) \quad A := \inf_{x\in[0,a]-N_G} \left[\sum_{n\in Z} |g(x-na)|^2 - \sum_{k\neq 0} \Big| \sum_{n\in Z} g(x-na)\overline{g(x-na-\frac{k}{b})}\Big| \right] > 0$$

$$(8) \quad B := \sup_{x\in[0,a]} \sum_{k\in Z} \Big| \sum_{n\in Z} g(x-na)\overline{g(x-na-\frac{k}{b})}\Big| < \infty.$$

Then $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame for $L^2(R-N_G)$ with bounds $\frac{A}{b}, \frac{B}{b}$.

Proof: First, observe that $\overline{span}\{E_{mb}T_{na}g\}_{m,n\in Z} \subseteq L^2(R - N_G)$. Now consider a function $f \in L^2(R - N_G)$ which is bounded and has support in a compact set. The Heil-Walnut argument (3) is valid under the assumption (8) and it gives that

$$\begin{aligned}
(3) \qquad & \sum_{m,n\in Z} | < f, E_{mb}T_{na}g > |^2 \\
= \; & \frac{1}{b}\int |f(x)|^2 \sum_{n\in Z} |g(x-na)|^2 dx \\
+ \; & \frac{1}{b}\sum_{k\neq 0} \int \overline{f(x)} f(x-k/b) \sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)} dx.
\end{aligned}$$

We want to estimate the second term above. For $k \in Z$, define

$$H_k(x) := \sum_{n\in Z} T_{na}g(x)\overline{T_{na+k/b}g(x)}.$$

First, observe that

$$\begin{aligned}
& \textstyle\sum_{k\neq 0}|T_{-k/b}H_k(x)| \\
= & \textstyle\sum_{k\neq 0}|T_{-k/b}\sum_{n\in Z} T_{na}g(x)\overline{T_{na+k/b}g(x)}| \\
= & \textstyle\sum_{k\neq 0}|\sum_{n\in Z} T_{na-k/b}g(x)\overline{T_{na}g(x)}| \\
= & \textstyle\sum_{k\neq 0}|\sum_{n\in Z} T_{na+k/b}g(x)\overline{T_{na}g(x)}| \\
= & \textstyle\sum_{k\neq 0}|\sum_{n\in Z} \overline{T_{na+k/b}g(x)}T_{na}g(x)| \\
= & \textstyle\sum_{k\neq 0}|H_k(x)|
\end{aligned}$$

Now, by a slight modification of the argument in [4] Theorem 2.3,

$$\begin{aligned}
& |\textstyle\sum_{k\neq 0}\int \overline{f(x)}f(x-k/b)\sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)}dx| \\
\leq & \textstyle\sum_{k\neq 0}\int |f(x)|\cdot|T_{k/b}f(x)|\cdot|H_k(x)|dx \\
= & \textstyle\sum_{k\neq 0}\int |f(x)|\sqrt{|H_k(x)|}\cdot|T_{k/b}f(x)|\sqrt{|H_k(x)|}dx \\
\leq & \textstyle\sum_{k\neq 0}\Big(\int |f(x)|^2|H_k(x)|dx\Big)^{1/2}\Big(\int |T_{k/b}f(x)|^2|H_k(x)|dx\Big)^{1/2} \\
\leq & \Big(\textstyle\sum_{k\neq 0}\int |f(x)|^2|H_k(x)|dx\Big)^{1/2}\cdot\Big(\sum_{k\neq 0}\int |T_{k/b}f(x)|^2|H_k(x)|dx\Big)^{1/2} \\
= & \Big(\textstyle\int |f(x)|^2\sum_{k\neq 0}|H_k(x)|dx\Big)^{1/2}\cdot\Big(\int |f(x)|^2\sum_{k\neq 0}|T_{-k/b}H_k(x)|dx\Big)^{1/2} \\
= & \textstyle\int |f(x)|^2\sum_{k\neq 0}|H_k(x)|dx.
\end{aligned}$$

Note that $\sum_{k\neq 0}|H_k(x)| = \sum_{k\neq 0}|\sum_{n\in Z} T_{na}g(x)\overline{T_{na+k/b}g(x)}|$ is a periodic function with period a. By (3) and the assumption (7) we now have

$$\sum_{m,n\in Z} |< f, E_{mb}T_{na}g >|^2$$

$$\geq \frac{1}{b}\int |f(x)|^2 \left[\sum_{n\in Z}|g(x-na)|^2 - \sum\nolimits_{k\neq 0}|\sum_{n\in Z} g(x-na)\overline{g(x-na-\frac{k}{b})}|\right] dx$$
$$\geq \frac{A}{b}||f||^2.$$

Similary, by (3) and (8),

$$\sum_{m,n\in Z} |< f, E_{mb}T_{na}g >|^2$$
$$\leq \frac{1}{b}\int |f(x)|^2 \left[\sum_{n\in Z}|g(x-na)|^2 + \sum\nolimits_{k\neq 0}|\sum_{n\in Z} g(x-na)\overline{g(x-na-\frac{k}{b})}|\right] dx$$
$$= \frac{1}{b}\int |f(x)|^2 \sum\nolimits_{k\in Z}|\sum_{n\in Z} g(x-na)\overline{g(x-na-\frac{k}{b})}|$$
$$\leq \frac{B}{b}||f||^2.$$

Since those two estimates holds on a dense subset of $L^2(R-N_G)$, they hold on $L^2(R-N_G)$. Thus $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame for $L^2(R-N_G)$ with the desired bounds.
Q.E.D.

The advantage of Theorem 2.1 compared to the results in section 1 is that we compare the functions $\sum_{n\in Z}|g(x-na)|^2$ and $\sum_{k\neq 0}|H_k(x)|$ *pointwise* rather than assuming that the supremum of $\sum_{k\neq 0}|H_k(x)|$ is smaller than the infimum of $\sum_{n\in Z}|g(x-na)|^2$. It is easy to give concrete examples where Theorem 2.1 shows that $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame for $L^2(R)$ but where the conditions in section 1 are not satisfied:

Example: Let $a=b=1$ and define

$$g(x) = \begin{cases} 1+x & \text{if } x\in[0,1] \\ \frac{1}{2}x & \text{if } x\in[1,2[\\ 0 & \text{otherwise} \end{cases}$$

For $x\in[0,1[$ we have

$$G(x) = \sum_{n\in Z}|g(x-n)|^2 = g(x)^2 + g(x+1)^2 = \frac{5}{4}(x+1)^2$$

and

$$\sum_{k\neq 0}|\sum_{n\in Z} g(x-n)\overline{g(x-n-k)}| = (1+x)^2$$

so by Theorem 2.1 $\{E_m T_n g\}_{m,n\in Z}$ is a frame for $L^2(R)$ with bounds $A = \frac{1}{4}, B = \frac{5}{4}$. But $\inf_{x\in R} G(x) = \frac{5}{4}$ and

$$\sum_{k\neq 0}||\sum_{n\in Z} T_n g T_{n+k}\overline{g}||_\infty = 4,$$

so the condition (4) is not satisfied. (5) is not satisfied either.

Remark: It is well known that G being bounded below is a necessary condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $L^2(R)$, cf. [3]. Theorem 2.1 shows that this condition is not necessary for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame sequence. However, it is implicit in (7) that G has to be bounded below on $R - N_G$ in order for Theorem 2.1 to work, and an easy modification of the proof in [3] shows that this is actually a necessary condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame for $L^2(R - N_G)$. We shall later give examples of frame sequences for which G is not bounded below on $R - N_G$.

In case g has support in an interval of length $\frac{1}{b}$ an equivalent condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame sequence can be given. First, observe that by (3) this condition on g implies that for all continuous functions f with compact support, we have

$$\sum_{m,n\in Z} |< f, E_{mb}T_{na}g >|^2 = \frac{1}{b}\int |f(x)|^2 G(x)dx.$$

It is not hard to show that this actually holds for all $f \in L^2(R)$, cf [6].

Corollary 2.2: *Suppose that $g \in L^2(R)$ has compact support in an interval I of length $|I| \leq 1/b$. Then $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame sequence with bounds A, B if and only if*

$$0 < bA \leq \sum_{n\in Z} |g(x-na)|^2 \leq bB, \quad \textit{for a.e. } x \in R - N_G.$$

In that case $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is actually a frame for $L^2(R - N_G)$.

Proof: Suppose that g has support in an interval I of length $|I| \leq \frac{1}{b}$. If $0 < bA \leq G(x) \leq bB$ for a.e. $x \in R - N_G$, it follows from Theorem 2.1 that $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame sequence with the desired bounds. Now suppose that $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame sequence with bounds A, B. Then, for every interval I of length $|I| = 1/b$ and every function $f \in L^2(I)$,

$$\sum_{m,n} | < f, E_{mb}T_{na}g > |^2 = \frac{1}{b}\int_R |f(x)|^2 G(x)dx \leq B\|f\|^2.$$

But this is clearly equivalent to

$$G(x) = \sum_{n\in Z} |g(x-na)|^2 \leq Bb \ a.e..$$

To prove the lower bound for G we proceed by way of contradiction. Suppose that for some $\epsilon > 0$ we have $0 < G(x) \leq (1-\epsilon)Ab$ on a set of positive measure. In this case there is a set Δ of positive measure and supported in an interval of length $\leq \frac{1}{b}$ so that $0 < G(x) \leq (1-\epsilon)Ab$ on Δ. Then, for any function $f \in L^2(R)$ supported on Δ, we have

$$\sum_{m,n} | < f, E_{mb}T_{na}g > |^2 = \frac{1}{b}\int_R |f(x)|^2 G(x)dx$$

$$\leq \frac{(1-\epsilon)Ab}{b}\int_R |f(x)|^2 dx = (1-\epsilon)A\|f\|^2.$$

Since $G(x) > 0$ on Δ, there is a $k \in Z$ so that $\chi_\Delta T_{ka}g$ is not the zero function. With $\Delta' := \Delta \cap Supp(T_{ka}g)$ we have

$$f := \chi_{\Delta'}T_{ka}g \in \overline{span}\{E_{mb}T_{ka}g\}_{m\in Z} \subseteq \overline{span}\{E_{mb}T_{na}g\}_{m,n\in Z},$$

so the above calculation shows that the lower bound for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is at most $(1-\epsilon)A$, which is a contradiction. Thus

$$G(x) \geq bA \text{ for } a.e. \ x \in R - N_G.$$

In case the condition in Cor. 2.2 is satisfied, it follows from Theorem 2.1 that $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame for $L^2(R - N_G)$.
Q.E.D.

For functions g with the property that the translates $T_{na}g, n \in Z$, have disjoint support we can give an equivalent condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame sequence. Define the function

$$\tilde{G}(x) : R \to [0,\infty], \quad \tilde{G}(x) = \sum_{m\in Z} |g(x + \frac{m}{b})|^2.$$

Proposition 2.3: *Let $g \in L^2(R), a, b > 0$ and suppose that*

$$(9) \quad supp(g) \cap supp(T_{na}g) = \emptyset, \ \forall n \in Z - \{0\}.$$

Then $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame sequence with bounds A, B if and only if there exist $A, B > 0$ such that

$$bA \leq \sum_{m\in Z} |g(x + \frac{m}{b})|^2 \leq bB \text{ for a.e. } x \in R - N_{\tilde{G}}.$$

In that case, $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a Riesz sequence iff $N_{\tilde{G}}$ has measure zero.

Proof: Because of the support condition (9), it is clear that $\{E_{mb}g\}_{m\in Z}$ is a frame sequence iff $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame sequence, in which case the sequences have the same frame bounds. But by Corollary 1.3 $\{E_{mb}g\}_{m\in Z}$ is a frame sequence with bounds A, B iff

$$bA \leq \sum_{m\in Z} |g(x + \frac{m}{b})|^2 \leq bB \text{ for a.e. } x \in R - N_{\tilde{G}}.$$

Also, $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a Riesz sequence iff $\{E_{mb}g\}_{m\in Z}$ is a Riesz sequence, which, by Cor. 1.3, is the case iff $N_{\tilde{G}}$ has measure zero. **Q.E.D.**

We are now ready to show that G being bounded below on $R - N_G$ (by a positive number) is not a necessary condition for $\{E_{mb}T_{na}g\}_{m,n\in Z}$ to be a frame sequence.

Example: Let $a, b > 0$ and suppose that $\frac{1}{ab} \notin N$. Chose $\epsilon > o$ such that

$$[0,\epsilon] + na \cap [\frac{1}{b}, \frac{1}{b} + \epsilon] = \emptyset, \ \forall n \in Z.$$

This implies that $\epsilon < min(a, \frac{1}{b})$. Define

$$g(x) := \begin{cases} x & \text{if } x \in [0,\epsilon] \\ \sqrt{1 - (x - \frac{1}{b})^2} & \text{if } x \in [\frac{1}{b}, \frac{1}{b} + \epsilon] \\ 0 & \text{otherwise} \end{cases}$$

Then the condition (9) in Proposition 2.3 is satisfied. Also, for $x \in [0, \epsilon]$,

$$\tilde{G}(x) = \sum\nolimits_{m \in Z} |g(x + \frac{m}{b})|^2 = g(x)^2 + g(x+1)^2 = 1$$

and for $x \in]\epsilon, \frac{1}{b}]$, we have $\tilde{G}(x) = 0$. Thus, by Proposition 2.3 $\{E_{mb}T_{na}g\}_{m,n \in Z}$ is a frame sequence. But for $x \in [0, \epsilon]$,

$$G(x) = \sum_{n \in Z} |g(x - na)|^2 = x^2.$$

Thus G is not bounded below by a positive number on $R - N_G$. By the remark after Theorem 2.1 this implies that $\overline{span}\{E_{mb}T_{na}g\}_{m,n \in Z} \neq L^2(R - N_G)$.

For $ab > 1$ it is even possible to construct an orthonormal sequence having all the features of the above example. For example, let $a = 2, b = 1$ and

$$g(x) := \begin{cases} x & \text{if } x \in [0,1] \\ \sqrt{2x - x^2} & \text{if x } \in]1,2] \\ 0 & \text{otherwise} \end{cases}$$

Since

$$\sum\nolimits_{m \in Z} |g(x + \frac{m}{b})|^2 = 1, \ \forall x,$$

it follows by Proposition 2.3 that $\{E_m T_{2n} g\}_{m,n \in Z}$ is a Riesz sequence with bounds $A = B = 1$, which implies that $\{E_m T_{2n} g\}_{m,n \in Z}$ is an orthonormal sequence. But $G(x) = \sum_{n \in Z} |g(x - 2n)|^2$ is not bounded below on $R - N_G$.

G being bounded above is still a necessary condition for $\{E_{mb}T_{na}g\}_{m,n \in Z}$ to be a frame sequence (repeat the argument in Cor. 2.2). $\tilde{G}$ also has to be bounded above:

Proposition 2.4: *If* $\{E_{mb}T_{na}g\}_{m,n \in Z}$ *is a frame sequence with upper bound* B*, then*

$$\sum_{m \in Z} |g(x + \frac{m}{b})|^2 \leq B \quad a.e.$$

Proof: If $\{E_{mb}T_{na}g\}_{m,n \in Z}$ is a frame sequence then $\{\mathcal{F}^{-1} E_{mb}T_{na}g\}_{m,n \in Z} = \{T_{mb}E_{na}\check{g}\}_{m,n \in Z}$ is a frame sequence with the same bounds. In particular the sequence $\{T_{mb}\check{g}\}_{m,n\in Z}$ has the upper frame bound B. By Theorem 1.2 (or, more precisely, the proof of it in [2]) it follows that

$$\sum_{m\in Z}|g(\frac{x+m}{b})|^2 \leq B \text{ for } a.e.\ x$$

It follows that $\sum_{m\in Z}|g(x+\frac{m}{b})|^2 \leq B\ \ a.e.$ **Q.E.D.**

Remark: Recall that a wavelet frame for $L^2(R)$ has the form

$$\{\frac{1}{a^{n/2}}g(\frac{x}{a^n}-mb)\}_{m,n\in Z},$$

where $a>1, b>0$ and $g\in L^2(R)$ are fixed.

As well as Weyl-Heisenberg frames, wavelet frames play a very important role in applications. The theory for the two types of frames was developed at the same time, with the main contribution due to Daubechies. Several results for Weyl-Heisenberg frames has counterparts for wavelet frames. For example, Theorem 5.1.6 in [6] gives sufficient conditions for $\{\frac{1}{a^{n/2}}g(\frac{x}{a^n}-mb)\}_{m,n\in Z}$ to be a frame based on a calculation similar to (3).

Also our results for Weyl-Heisenberg frames has counterparts for wavelet frames. The ideas in the proof of Theorem 2.1 can be used to modify [6], Theorem 5.1.6, which leads to the following:

Theorem 2.5: *Let $a>1, b>0$ and $g\in L^2(R)$ be given. Let $N:=\{\gamma\in[1,a]|\ \sum_{n\in Z}|\hat{g}(a^n\gamma)|^2=0\}$ and suppose that*

$$A := \inf_{|\gamma|\in[1,a]-N}\left[\sum_{n\in Z}|\hat{g}(a^n\gamma)|^2-\sum_{k\neq 0}\sum_{n\in Z}|\hat{g}(a^n\gamma)\hat{g}(a^n\gamma+k/b)|\right]>0,$$

$$B := \sup_{|\gamma|\in[0,a]}\sum_{k,n\in Z}|\hat{g}(a^n\gamma)\hat{g}(a^n\gamma+k/b)|<\infty.$$

Then $\{\frac{1}{a^{n/2}}g(\frac{x}{a^n}-mb)\}_{m,n\in Z}$ is a frame sequence with bounds $\frac{A}{b},\frac{B}{b}$.

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