BANACH GELFAND TRIPLES FOR GABOR ANALYSIS

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Abstract. It is the purpose of this survey note to show the relevance of a Gelfand triple which is closely connected with time-frequency analysis and Gabor analysis. The Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ and its dual can be shown to be - for a large variety of concrete cases - a convenient substitute for the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ and it's dual, the space of tempered distributions $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$. This concrete pair of Banach spaces is actually a Gelfand triple, which allows to describe in a very intuitive way the properties of the classical Fourier transform and other unitary operators arising in the treatment of various mathematical questions, e.g. multipliers in harmonic analysis. We will demonstrate the usefulness of the Banach Gelfand triple $\big(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)\big)$ within time-frequency analysis, with a special emphasis on questions from *time-frequency analysis* and *Gabor analysis*.

1. Introduction

Gabor analysis is often described as "the part of analysis" which in one way or the other makes use of the family of so-called *time-frequency shift operators*. Here we have to mention first the short-time Fourier transform or sliding-window Fourier transform (in short, STFT), or Dennis Gabor's claim of 1946, that "every function" can be written as a (double) series of time-frequency shifted copies (with suitable complex amplitudes). Even if one wants to discuss the subtleties, these operations on different natural function spaces such as the standard space $\boldsymbol{L}^2(\mathbb{R}^d)$ of "signals of finite energy" (i.e., square integrable functions) the question of convergence of the Gabor series expansions, or the stable reconstruction of a signal from a densely sampled STFT cannot be answered without various extra conditions (typically conditions on the smoothness and decay of the window that is used in forming the STFT).

The correct class of function spaces for time-frequency analysis and Gabor analysis are the *modulation spaces*, since they possess an intrinsic description in terms of STFT or Gabor frames. The smallest member of this class is *Feichtinger's algebra* $S_0(\mathbb{R}^d)$. Consequently, the dual space $S_0'(\mathbb{R}^d)$ of Feichtinger's algebra serves as the largest class of functions and distributions for the discussion of operators and their properties. In between of $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$ sits the Hilbert space $L^2(\mathbb{R}^d)$, actually this triple of Banach spaces forms a Gelfand triple. The notion of *Gelfand triples* allows to express mapping properties of operators (such as the Fourier transform, Gabor frame operators, etc.) in a convenient way. An important consequence is the description of the mapping properties of a linear operator at three levels: at the

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inner level such operators may often be described via integrals (or transformations applied to ordinary functions); at the intermediate Hilbert space level one can describe unitarity properties, while to outest level one can describe the mapping at the level of distributions.

In section 2 we recall basic facts about Bessel sequences, frames and Riesz basis in Hilbert spaces. In section 3 we discuss Gabor frames and their main features in the setting of $L^2(\mathbb{R}^d)$. In section 4 we briefly describe time-frequency representations, especially the short-time Fourier transform. After these preparations we are in the position to introduce in section 5 the key players of our presentation Gelfand triples, and concretely the Gelfand triple $(S_0(\mathbb{R}^d), L^2(\mathbb{R}^d), S_0'(\mathbb{R}^d))$. In section 6 we give an overview of the main results of Feichtinger and Kozek on time-frequency quantization, pseudo-differential operators and their spreading function. Additionally we give the reader a flavor of the usefulness of Banach Gelfand triples for Gabor frame operators. In section 7 we conclude our survey note with some results about Gabor multipliers and its relation to localization operators. The topic developed in this survey note can be found mainly in [18, 20, 21].

Notation. We define $t^2 = t \cdot t$, for $t \in \mathbb{R}^d$, and $xy = x \cdot y$ is the scalar product on $\mathbb{R}^d$.

The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We use the brackets $\langle f, g \rangle$ to denote the extension to $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ of the inner product $\langle f, g \rangle = \int f(t)\overline{g(t)}dt$ on $\boldsymbol{L}^2(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)e^{-2\pi i t\omega}dt$. We recall the space of p-summable sequences

$$\ell^p(J) = \left\{ (a_n)_{n\in J} \,:\, \|(a_n)_{n\in J}\|_{\ell^p} := \left(\sum_{n\in J} |a_n|^p \right)^{1/p} < \infty \right\}.$$

2. Preliminaries

Let $\{g_k\}_{k\in J}$ be a family of an infinite dimensional (separable) Hilbert space $\mathscr{H}$ with inner product $\langle \cdot, \cdot \rangle$. The classical examples are the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ of (equivalence classes of measurable) functions of finite energy ($\boldsymbol{L}^2(\mathbb{R}^d)$-norm) and the sequence space $\ell^2(\mathbb{Z}^d)$ consisting of square summable complex-valued sequences. Similar to the finite dimensional case, we want to represent a signal f in $\mathscr{H}$ as a (now possibly infinite) linear combination of the form

$$f \;\sim\; \sum_{k\in J} c_k g_k \,.$$

At this point there a several questions that arise naturally when considering an infinite sum. First, by convention, we want the sum to converge in the (prescribed) Hilbert-norm, i.e., $\lim_{K\to\infty} \|f - \sum_{k=1}^{K} c_k g_k\| \longrightarrow 0$. Secondly, the sum should converge to the same limit (preferably f) regardless of the summation order we choose (known as *unconditional convergence*). A more subtle point is that we would like to have a continuous linear dependency between the signal f and the coefficients c_k in order to avoid pathological cases in which small alterations in the signal result in uncontrollable changes in the corresponding coefficient sequence and vice-versa. This technical detail accounts for numerical stability.

Obviously, all these requirements are trivially fulfilled in the finite dimensional case. For an infinite family $\{g_k\}$, however, these assumptions have to be ensured before dealing with decomposition and reconstruction issues. Fortunately, there exists concepts in functional analysis that do exactly fit these kind of requirements. For a precise description we need the following definitions.

Definition 1. A family $\{g_k\}$ of a Hilbert space $\mathscr{H}$ is complete in $\mathscr{H}$ if the set of finite linear combination of $\{g_k\}$, write $\mathrm{span}(g_k)$, is dense in $\mathscr{H}$, i.e., every f in $\mathscr{H}$ can be arbitrarily well approximated by elements in $\mathrm{span}(g_k)$ with respect to the $\mathscr{H}$-norm.

In the mathematical literature complete systems are often called "total". The definition makes no claim about the "cost" of approximation. In other words, it is allowed to use more and more complicated coefficient sequences as the approximation quality is increased. In particular, total families do not necessarily allow a series expansion of arbitrary elements from the given Hilbert space.

Definition 2. The family $\{g_k\}_{k\in J}$ of a Hilbert space $\mathscr{H}$ is a basis for $\mathscr{H}$ if for all $f \in \mathscr{H}$ there exists unique scalars $c_k(f)$ such that

$$f \;=\; \sum_{k\in J} c_k(f) g_k \,.$$

Definition 3. A family $\{g_k\}_{k\in J}$ of a Hilbert space $\mathscr{H}$ is a Riesz sequence if there exist bounds $A, B > 0$ such that

$$A\|c\|^2_{\ell^2(J)} \leq \Big\| \sum_{k\in J} c_k g_k \Big\|^2 \leq B\|c\|^2_{\ell^2(J)}\,, \qquad c \in \ell^2(J)\,.$$

A Riesz sequence which generates all $\mathscr{H}$ is called a *Riesz basis* for $\mathscr{H}$.

Riesz bases are somehow "distorted" orthonormal bases as described in the following lemma which reveals all useful properties of a Riesz basis [31].

Lemma 1. *Let $\{g_k\}$ be a sequence in a Hilbert space $\mathscr{H}$. The following are equivalent.*

(1) *$\{g_k\}_{k\in J}$ is a Riesz basis for $\mathscr{H}$.*
(2) *$\{g_k\}_{k\in J}$ is an unconditional basis for $\mathscr{H}$ and g_k are uniformly bounded.*
(3) *$\{g_k\}_{k\in J}$ is a basis for $\mathscr{H}$, and $\sum_{k\in J} c_k g_k$ converges if and only if $\sum_{k\in J} |c_k|^2$ converges.*
(4) *There is an equivalent inner product on $\mathscr{H}$ for which $\{g_k\}_{k\in J}$ is an orthonormal basis for $\mathscr{H}$.*
(5) *$\{g_k\}_{k\in J}$ is a complete Bessel sequence and possesses a bi-orthogonal system $\{h_k\}_{k\in J}$ that is also a complete Bessel sequence.*

The last item of the lemma says that there exists a unique sequence $\{h_k\}_{k\in J}$ such that $\langle g_k, h_j\rangle = \delta_{kj}$ which, combined with the second statement, induces the representation

$$f \;=\; \sum_{k\in J} \langle f, h_k\rangle g_k \;=\; \sum_{k\in J} \langle f, g_k\rangle h_k\,, \qquad f \in \mathscr{H}\,.$$

Hence, Riesz bases are potential candidates for our purpose of signal representation. We point out that the coefficient sequence is always square summable which is an important stability criterion.

A basis allows only unique expansions with respect to the coefficients. In applications it is sometimes more useful to weaken this property. This can be obtained by looking for overcomplete (linearly dependent) sets which is implemented in the concept of frames introduced by Duffin and Schaeffer in 1952 [14].

Definition 4. The sequence $\{g_k\}_{k\in J}$ in a Hilbert space $\mathscr{H}$ is called a Bessel sequence if

$$\sum_{k\in J} |\langle f, g_k\rangle|^2 < \infty\,, \qquad f \in \mathscr{H}\,.$$

Definition 5. A family $\{g_k\}_{k\in J}$ of a Hilbert space $\mathscr{H}$ is a frame of $\mathscr{H}$ if there exist bounds $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{k\in J} |\langle f, g_k\rangle|^2 \le B\|f\|^2\,, \qquad f \in \mathscr{H}\,. \tag{1}$$

If $A = B$, then $\{g_k\}_{k\in J}$ is called a tight frame.

The synthesis map $D : \ell^2(J) \to \mathscr{H}$ of a frame $\{g_k\}_{k\in J}$ is defined by

$$D : (c_k) \to \sum_{k\in J} c_k g_k\,.$$

Its adjoint D^* is the analysis operator $D^*f = (\langle f, g_k\rangle)$. The *frame operator* S is defined by

$$Sf = DD^*f = \sum_{k\in J} \langle f, g_k\rangle g_k\,, \qquad f \in \mathscr{H}\,.$$

By (1), the frame operator satisfies

$$A\langle f, f\rangle \le \langle Sf, f\rangle \le B\langle f, f\rangle\,, \qquad f \in \mathscr{H}\,,$$

and is, therefore, bounded, positive, and invertible. The inverse operator S^{-1} is obviously also positive and has therefore a square root $S^{-1/2}$ (self-adjoint), [42]. The sequence $\{S^{-1/2}g_k\}$ is a tight frame with $A = B = 1$. Indeed,

$$\begin{aligned}\sum_{k\in J}\langle f, S^{-\frac12}g_k\rangle S^{-\frac12}g_k &= S^{-\frac12}\sum_{k\in J}\langle f, S^{-\frac12}g_k\rangle g_k = S^{-\frac12}S(S^{-\frac12}f)\\ &= S^{-\frac12}S^{-\frac12}Sf = If, \quad \forall\, f \in \mathscr{H}.\end{aligned}$$

Every orthonormal basis of $\mathscr{H}$ is a Riesz basis of $\mathscr{H}$ and every Riesz basis of $\mathscr{H}$ is also a frame. The important difference between a Riesz basis and a frame is that the null space $\mathcal{N}(G)$ of the synthesis map D of a frame $\{g_k\}_{k\in J}$ is in general nontrivial which is equivalent to the statement that the range of the analysis map D^* is a (closed) proper subspace of $\ell^2(J)$.

The sequence $\{\tilde g_{k\in J}\}$ with $\tilde g_k = S^{-1}g_k$, is also a frame with frame bounds $1/B$ and $1/A$. It is a dual frame for $\{g_k\}$ in the sense that

$$f \;=\; \sum_{k\in J}\langle f, \tilde g_k\rangle g_k \;=\; \sum_{k\in J}\langle f, g_k\rangle \tilde g_k\,, \qquad f \in \mathscr{H}\,.$$

Again we see, that frames do indeed fit our purpose for signal analysis and signal recovery. In contrast to Riesz bases, frames have, in general, no bi-orthogonal relation. Moreover, the dual frame is not unique. The canonical dual $\{S^{-1}g_k\}$ is the one that is producing minimal ℓ^2 coefficients as already shown in [14]. For alternative dual frames there exist constructive approaches that rely on the canonical dual. In [6, 36], it is shown that any dual frame of $\{g_k\}$ can be written as

$$S^{-1}g_k + h_k - \sum_{j\in J} \langle S^{-1}g_k, g_j\rangle h_j \,, \tag{2}$$

where $\{h_k\}$ is a Bessel sequence.

The lack of uniqueness has the advantage that if one coefficient is missing out of the sequence $\langle f, g_k\rangle$, the whole signal can still be completely recovered as long as $\{g_k\}$ is a frame but no Riesz basis. Similarly, any frame that is not a Riesz basis is still a frame when discarding single frame elements. Studies about the conservation of the frame property when discarding frame elements are known as *excesses of frames* [1, 2].

3. Gabor analysis on $\boldsymbol{L}^2$

We define the Fourier transform of an integrable function by $\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t)e^{-2\pi i t\omega}dt$. The translation operator T_x and the modulation operator M_ω are given by

$$T_x f = f(\cdot - x), \qquad M_\omega f = e^{2\pi i \omega \cdot} f(\cdot), \quad x, \omega \in \mathbb{R}^d. \tag{3}$$

Combined together they give rise to the so-called time-frequency shift $\pi(\lambda)$:

$$\pi(\lambda) = M_\omega T_x, \quad (x, \omega) \in \mathbb{R}^{2d}. \tag{4}$$

Note that

$$\pi(\lambda_2)\pi(\lambda_1) = e^{2\pi i (x_1\omega_2 - x_2\omega_1)}\pi(\lambda_1)\pi(\lambda_2)$$

for $\lambda_1 = (x_1, \omega_1), \lambda_2 = (x_2, \omega_2) \in \mathbb{R}^{2d}$.

A time-frequency lattice Λ is a discrete subgroup of $\mathbb{R}^{2d}$ ($= \mathbb{R}^d \times \hat{\mathbb{R}}^d$) with compact quotient. Its redundancy $|\Lambda|$ is the reciprocal value of the measure of a fundamental domain for the quotient $\mathbb{R}^{2d}/\Lambda$.

For a lattice Λ in $\mathbb{R}^{2d}$ and a so-called *Gabor atom* $g \in \boldsymbol{L}^2$ we define the associated Gabor family by

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g\}_{\lambda\in\Lambda} \,.$$

If $\mathcal{G}(g, \Lambda)$ is a frame for $\boldsymbol{L}^2$, we call it a *Gabor frame*. Since Λ has a group structure, the frame operator

$$Sf = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g$$

has the property that it commutes with all time-frequency shifts of the form $\pi(\lambda)$ for $\lambda \in \Lambda$. Therefore, the canonical dual frame of $\mathcal{G}(g, \Lambda)$ is simply given by $\mathcal{G}(h, \Lambda)$ with $h = S^{-1}g$. The fact, that a canonical dual frame of a Gabor frame is again a Gabor frame, i.e., generated by a single function, is the key property in many applications. It reduces computational issues to solving the linear system $Sh = g$.

A special and widely studied case are separable lattices of the form $\alpha\mathbb{Z} \times \beta\mathbb{Z}$ for some positive lattice parameters α and β, whose redundancy is simply $(\alpha\beta)^{-1}$. The prototype of a function generating Gabor frames for such separable lattices is the Gaussian

$$\psi(x) = e^{-\pi x^2 \sigma^2}\,. \tag{5}$$

for some real $\sigma > 0$. The Gaussian generates a Gabor frame if and only if $\alpha\beta < 1$ [39, 43]. We emphasize that for $\alpha\beta = 1$ the Gaussian generates a *unstable* generating system for $\boldsymbol{L}^2(\mathbb{R}^d)$, i.e., the resulting Gabor family is complete but coefficient sequences must not be bounded. In this context we mention a central result, the so-called *density theorem* and refer to [29] for detailed discussions. An elegant elementary proof of the density theorem has been provided by Janssen [33].

Theorem 1. *Assume that $\mathcal{G}(g,\alpha,\beta)$ is a frame. Then, $\alpha\beta \le 1$. Moreover, $\mathcal{G}(g,\alpha,\beta)$ is a Riesz basis for $\boldsymbol{L}^2(\mathbb{R}^d)$ if and only if $\alpha\beta = 1$.*

In his seminal paper [24], Gabor chose the integer lattice $a = b = 1$ in $\mathbb{R}^2$ and used the Gaussian in order to define a Gabor system with maximal time-frequency localization. However, as mentioned above, this system is no longer stable though complete, and, indeed, the celebrated Balian-Low Theorem [3, 38] states that good time-frequency localization and Gabor Riesz bases are not compatible:

Theorem 2. (Balian-Low) *If $\mathcal{G}(g,1,1)$ constitutes a Riesz basis for $\boldsymbol{L}^2(\mathbb{R})$, then*

$$\int_{\mathbb{R}} |g(t)|^2 t^2 dt \int_{\mathbb{R}} |\hat{g}(\omega)|^2 \omega^2 d\omega = \infty\,.$$

The Balian-Low Theorem reveals a form of uncertainty principle and has inspired fundamental research, see [29] and references therein.

In the sequel we state some fundamental results of Gabor frames and the Gabor frame operator (Gabor frame-type operator). To this end we need the notion of the *adjoint lattice* Λ° of Λ which is the set of all elements in $\mathbb{R}^{2d}$ that satisfy the commutation property

$$\pi(\lambda^\circ)\pi(\lambda) = \pi(\lambda)\pi(\lambda^\circ) \quad \text{for all} \quad \lambda \in \Lambda\,.$$

Note that Λ° is again a lattice of $\mathbb{R}^{2d}$ (and that $\Lambda^{\circ\circ} = \Lambda$). Instead of the frame operator we will use the more general notion of a frame-type operator $S_{g,\gamma,\Lambda}$ associated to the pair (g,γ), where γ takes the role of an "analyzing" and g the role of a "synthesizing" window:

$$S_{g,\gamma,\Lambda} f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle\, \pi(\lambda) g\,, \qquad f \in \boldsymbol{L}^2(\mathbb{R}^d)\,.$$

This sum converges in $\boldsymbol{L}^2(\mathbb{R}^d)$ for all $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ as long both functions g,γ are Bessel atoms for Λ, that is, $\mathcal{G}(g,\Lambda)$ and $\mathcal{G}(\gamma,\Lambda)$ are Gabor Bessel families. For the fundamental results to hold with respect to norm convergence we need a little bit more than Bessel sequences. It is that both atoms g (and analogously γ) satisfy

$$\sum_{\lambda^\circ\in\Lambda^\circ} |\langle g, g_{\lambda^\circ}\rangle| < \infty\,, \tag{A'}$$

also known as the *Tolimieri-Orr's condition.* This somewhat technical property is used for controlling convergence problems (by altering the convergence definition Condition A' can be weakened). Condition A' is in general not easy to verify. In particular, if Condition A' holds for one lattice, there is, in general, no guarantee that it holds also for a different lattice. This problem, however, is overcome by the Feichtinger algebra $\boldsymbol{S}_0$ (Section 5) which defines a class of functions for which Condition A' is satisfied for any lattice in $\mathbb{R}^{2d}$.

We summarize the fundamental results of Gabor analysis in the following theorem that is given in [22] in a slightly more general context. The statements go back to the seminal papers [12, 32, 45]. They are, however, all consequences of the fundamental identity of Gabor analysis extensively studied in [19].

Theorem 3. *Let Λ be a lattice in $\mathbb{R}^{2d}$ with adjoint lattice Λ°. Then, for g, h satisfying A', the following holds.*

(1) *(Fundamental Identity of Gabor Analysis)*

$$\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)\gamma\rangle\langle\pi(\lambda)g,h\rangle = |\Lambda| \sum_{\lambda^\circ\in\Lambda^\circ}\langle g,\pi(\lambda^\circ)\gamma\rangle\,\langle\pi(\lambda^\circ)f,h\rangle \tag{6}$$

for all $f, h \in \boldsymbol{L}^2(\mathbb{R}^d)$, where both sides converge absolutely.

(2) *(Wexler-Raz Identity)*

$$S_{g,\gamma,\Lambda}f = |\Lambda|\cdot S_{f,\gamma,\Lambda^\circ}g \tag{7}$$

for all $f \in \boldsymbol{L}^2(\mathbb{R}^d)$.

(3) *(Janssen Representation)*

$$S_{g,\gamma,\Lambda} = |\Lambda| \sum_{\lambda^\circ\in\Lambda^\circ}\langle\gamma,\pi(\lambda^\circ)g\rangle\,\pi(\lambda^\circ) \tag{8}$$

where the series converges unconditionally in the strong operator sense.

In Section 6 we explicitly derive the Janssen representation of the Gabor frame operator from advanced concepts in harmonic analysis and provide a much deeper insight into this topic.

Another important result is the Ron-Shen Duality Principle which is often referred to [41] although it appeared already in [32] and [12].

Theorem 4. *Let $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ and Λ be a lattice in $\mathbb{R}^{2d}$ with adjoint Λ°. Then the Gabor system $\mathcal{G}(g,\Lambda)$ is a frame for $\boldsymbol{L}^2(\mathbb{R}^d)$ if and only if $\mathcal{G}(g,\Lambda^\circ)$ is a Riesz basis for its closed linear span. In this case, the quotient of the two frame bounds and quotient of the Riesz bounds (alternatively the condition number of the corresponding frame operator and the Gramian matrix, respectively), coincide.*

The last important identity in Gabor Analysis that we want to present in this section is the Wexler-Raz Biorthogonality Relation which basically says that g and γ are dual Gabor windows if and only if $S_{g,\gamma,\Lambda} = Id$. That is, according to Janssen Representation, exactly the case when

$$\langle\gamma,\pi(\lambda^\circ)g\rangle \;=\; |\Lambda|^{-1}\,\delta_{0,\lambda^\circ}\,.$$

Alternatively this relation can be described by what is a true biorthogonality (using again Kronecker's Delta):

$$\langle\pi(\lambda^{\circ\prime})\gamma,\pi(\lambda^\circ)g\rangle \;=\; |\Lambda|^{-1}\,\delta_{\lambda^{\circ\prime},\lambda^\circ}\,.$$

In the next section we describe basic and more advanced studies in harmonic analysis that contribute to a better understanding of the Gabor frame operator.

4. Time-frequency representations

Traditionally we extract the frequency information of a signal f by means of the Fourier transform $\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t)e^{-2\pi it\omega}dt$. If we know $\hat{f}(\omega)$ for all frequencies ω, then our signal f can be reconstructed by the inversion formula $f(t) = \int_{\widehat{\mathbb{R}^d}} \hat{f}(\omega)e^{2\pi it\omega}d\omega$ (valid pointwise or in the quadratic mean).

However, in many situations it is of relevance to know, *how long* each frequency appears in the signal f, e.g., for a pianist playing a piece of music. Mathematically this leads to the study of functions $S(f)(t,\omega)$ of the signal f, which describe the time-frequency content of f over "time" t. In the following we mention the most prominent time-frequency representations.

In the last century researchers such as E. Wigner, Kirkwood, and Rihaczek had invented different time-frequency representations, [46, 35]. The work of Wigner and Kirkwood was motivated by the description of a particle in quantum mechanics by a joint probability distribution of position and momentum of the particle. More concretely, in 1932 Wigner introduced the first time-frequency representation of a function $f \in L^2(\mathbb{R}^d)$ by

$$W(f)(x,\omega) = \int_{\mathbb{R}^d} f(x+\frac{t}{2})\overline{f(x-\frac{t}{2})}e^{-2\pi i\omega t}dt, \tag{9}$$

the so called *Wigner distribution* of f. Later Kirkwood proposed another time-frequency representation, which was in a different context rediscovered by Rihazcek. Both researchers associated to a function $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ the following expression

$$R(f)(x,\omega) = f(x)\overline{\hat{f}(\omega)}e^{-2\pi ix\omega},$$

the *Kirkwood-Rihaczek distribution* of f.

Nowadays, the *short-time Fourier transform* (STFT) has become the standard tool for (linear) time-frequency analysis. It is used as a measure of the time-frequency content of a signal f (energy distribution), but it also establishes a connection to the Heisenberg group.

The STFT provides information about local (smoothness) properties of the signal f. This is achieved by localization of f near t through multiplication with some *window function* g and a subsequent Fourier transform providing information about the frequency content of f in this segment. Typically g is concentrated around the origin. If g is compactly supported only a segment of f in some interval or ball around t is relevant, but g can be any non-zero Schwartz function such as the Gaussian. Overall we have:

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi it\omega}dt, \text{ for } (x,\omega) \in \mathbb{R}^{2d}, \tag{10}$$

In 1927 Weyl pointed out that the translation and modulation operator satisfy the following commutation relation

$$T_x M_\omega = e^{-2\pi ix\omega} M_\omega T_x, \ (x,\omega) \in \mathbb{R}^{2d}. \tag{11}$$

$\{T_x : x \in \mathbb{R}^d\}$ and $\{M_\omega : \omega \in \mathbb{R}^d\}$ are Abelian groups of unitary operators, with the infinitesimal generators given by differentiation and multiplication operator, respectively. Therefore the commutation relation (11) is the analogue of Heisenberg's commutation relation for the differentiation and multiplication operator.

The time-frequency shifts $M_\omega T_x$ for $(x, \omega) \in \mathbb{R}^{2d}$ satisfy the following composition law:

$$\pi(x,\omega)\pi(y,\eta) = e^{-2\pi i x\cdot\eta}\pi(x+y,\omega+\eta), \tag{12}$$

for $(x, \omega), (y, \eta)$ in the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$. i.e. the mapping $(x, \omega) \mapsto \pi(x, \omega)$ defines (only) a *projective representation* of the time-frequency plane (viewed as an Abelian group) $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$. By adding a toral component, i.e. $\tau \in \mathbb{C}$ with $|\tau| = 1$ one can augment the phase space $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ to the so-called *Heisenberg group* $\mathbb{R}^d \times \widehat{\mathbb{R}^d} \times T$ and the mapping $(x, \omega, \tau) \mapsto \tau M_\omega T_x$ defines a (true) unitary representation of the Heisenberg group [23], the so-called *Schrödinger representation.* From this point of view the definition of $V_g f$ can be interpreted as representation coefficients:

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle, \qquad f,g \in \boldsymbol{L}^2(\mathbb{R}^d).$$

The STFT is linear in f and conjugate linear in g. The choice of the window function g influences the properties of the STFT remarkably. One example of a good window class is the Schwartz space of rapidly decreasing functions. Later we will discuss another function space, which is perfectly suited as a good class of windows, *Feichtinger's algebra.*

Furthermore, for $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$ the STFT $V_g f$ is uniformly continuous on $\mathbb{R}^{2d}$, i.e., we can sample the $V_g f$ without problem. This fact is of great relevance in the discussion of Gabor frames.

By Parseval's theorem and an application of the commutation relations (11) we derive the following relation

$$V_g f(x,\omega) = e^{-2\pi i x\omega} V_{\hat{g}}\hat{f}(\omega,-x), \tag{13}$$

which is sometimes called the *fundamental identity of time-frequency analysis* [29]. The equation (13) expresses the fact that the STFT is a joint time-frequency representation and that the Fourier transform amounts to a rotation of the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ by an angle of $\frac{\pi}{2}$ whenever the window g is Fourier invariant. Another important consequence of the definition of STFT (10) and the commutation relations (11) is the *covariance property* of the STFT:

$$V_g(T_u M_\eta f)(x,\omega) = e^{-2\pi i u\omega} V_g f(x-u,\omega-\eta). \tag{14}$$

Later we will draw an important conclusion of the basic identity of time-frequency analysis (13) and the covariance property of the STFT (14): isometric Fourier invariance and the invariance under TF-shifts of Feichtinger's algebra.

As for the Fourier transform there is also a Parseval's equation for the STFT which is referred to as *Moyal's formula.*

Lemma 2. *(Moyal's Formula) Let $f_1, f_2, g_1, g_2 \in \boldsymbol{L}^2(\mathbb{R}^d)$ then $V_{g_1}f_1$ and $V_{g_2}f_2$ are in $\boldsymbol{L}^2(\mathbb{R}^{2d})$ and the following identity holds:*

$$\langle V_{g_1}f_1, V_{g_2}f_2\rangle_{\boldsymbol{L}^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}\,. \tag{15}$$

Moyal's formula implies that orthogonality of windows g_1, g_2 resp. of signals f_1, f_2 implies orthogonality of their STFT's. Most importantly we observe that one has for normalized $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ (i.e. with $\|g\|_2 = 1$):

$$\|V_g f\|_{\boldsymbol{L}^2(\mathbb{R}^{2d})} = \|f\|_{\boldsymbol{L}^2(\mathbb{R}^d)},$$

for all $f \in \boldsymbol{L}^2(\mathbb{R}^d)$, i.e., the STFT is an isometry from $\boldsymbol{L}^2(\mathbb{R}^d)$ to $\boldsymbol{L}^2(\mathbb{R}^{2d})$.

Another consequence of Moyal's formula is an inversion formula for the STFT. Assume that the analysis window $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ and the synthesis window $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$ satisfy $\langle g, \gamma\rangle \neq 0$. Then for $f \in \boldsymbol{L}^2(\mathbb{R}^d)$

$$f = \frac{1}{\langle g, \gamma\rangle} \iint_{\mathbb{R}^{2d}} \langle f, \pi(x,\omega)g\rangle\, \pi(x,\omega)\gamma\, dx d\gamma. \tag{16}$$

We observe that in contrast to the Fourier inversion the building blocks of the STFT inversion formula are just time-frequency shifts of a square- integrable function. Therefore also the Riemannian sums corresponding to this inversion integral are functions in $\boldsymbol{L}^2(\mathbb{R}^d)$ and are even norm convergent in $\boldsymbol{L}^2(\mathbb{R}^d)$ for nice windows (from Feichtinger's algebra, see later).

5. The Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$

Since Feichtinger's discovery of the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ in 1979 many results have shown that $\boldsymbol{S}_0(\mathbb{R}^d)$ is a good substitute of Schwartz's space $\mathcal{S}(\mathbb{R}^d)$ of test functions (except if one is interested in a discussion of partial differential equations). Furthermore $\boldsymbol{S}_0(\mathbb{R}^d)$ has turned out as the appropriate setting for the treatment of questions in harmonic analysis on $\mathbb{R}^d$ (actually on a general locally compact Abelian group G, even without using their structure theory). In this section we recall well-known properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ which we will need later in our discussion of Gabor frame operators. Nowadays the space $\boldsymbol{S}_0(\mathbb{R}^d)$ is called *Feichtinger's algebra* since it is a Banach algebra with respect to pointwise multiplication and convolution.

Definition 6. A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if, for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$,

$$\|f\|_{S_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable windows. Often it is convenient to use the Gaussian as a window.

The above definition of $\boldsymbol{S}_0(\mathbb{R}^d)$ [29](different from the original one [15]) allows for an easy derivation of the basic properties of Feichtinger's algebra in the following lemma.

Lemma 3. *Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:*

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.
(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

Proof:

(1) For $z = (u,\eta)$ in the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ one has:
$$\|\pi(u,\eta)f\|_{S_0} = \iint_{\mathbb{R}^{2d}} |V_g f(x-u,\omega-\eta)|\, dxd\omega =$$
$$= \iint_{\mathbb{R}^{2d}} |V_g f(x,\omega)|\, dxd\omega = C\|f\|_{S_0}.$$
(2) The key of the argument is an application of the fundamental identity of time-frequency analysis (13) to a Fourier invariant window g and the independence of the definition of $\boldsymbol{S}_0(\mathbb{R}^d)$ for $g \in \mathcal{S}(\mathbb{R}^d)$. For simplicity we choose g the (Fourier invariant) Gaussian $g_0(x) = 2^{d/4}e^{-\pi x^2}$:
$$\|\hat{f}\|_{S_0} = \iint_{\mathbb{R}^{2d}} |V_{g_0}\hat{f}(x,\omega)|\, dxd\omega = \iint_{\mathbb{R}^{2d}} |V_{\widehat{g_0}}\hat{f}(x,\omega)|\, dxd\omega =$$
$$\iint_{\mathbb{R}^{2d}} |V_{g_0}f(-\omega,x)|\, dxd\omega = \iint_{\mathbb{R}^{2d}} |V_{g_0}f(x,\omega)|\, dxd\omega = \|f\|_{S_0}.$$

□

Later we will need that $\boldsymbol{S}_0(\mathbb{R}^d)$ is *dense* and *continuously embedded* into $\boldsymbol{L}^p(\mathbb{R}^d)$ for any $p \in [1,\infty)$. The original motivation for Feichtinger's introduction of $\boldsymbol{S}_0(\mathbb{R}^d)$ was the search for a *smallest* member in the family of all time-frequency homogenous Banach spaces. For a proof of all these assertions we refer the reader to the original paper of Feichtinger or Gröchenig's book on time-frequency analysis [29].

Another reason for usefulness of $\boldsymbol{S}_0(\mathbb{R}^d)$ is the fact that $\boldsymbol{S}_0(\mathbb{R}^d)$ is a natural domain for the application of Poisson summation [28].

Lemma 4. *Let Λ be a lattice in $\mathbb{R}^d$ and $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ then*
$$\sum_{\lambda\in\Lambda} f(\lambda) = |\Lambda|^{-1} \sum_{\lambda^\perp\in\Lambda^\perp} \hat{f}(\lambda^\perp) \tag{17}$$
holds pointwise and with absolute convergence.

Here $\Lambda^\perp$ is the orthogonal lattice for Λ, e.g. $\Lambda^\perp = (A^{-1})^t\mathbb{Z}^d$ for $\Lambda = A\mathbb{Z}^d$, where A is a non-singular matrix describing Λ.

In 1958 I. M. Gelfand and A G. Kostyuchenko introduced Gelfand triples in their study of the spectral theory of self-adjoint operators [25]. They were motivated by the work of Dirac on the foundations of quantum mechanics and Schwartz's theory of distributions.

An important result of linear algebra is the theorem on the existence of eigenvectors for any self-adjoint linear operator A on $\mathbb{R}^d$. The situation changes drastically when one passes from the finite to the infinite-dimensional case, since it can happen that a unitary operators does not have any (non-zero) eigenvector. Particular examples of such operators are the translation operator T_x and the modulation operator M_ω on $\boldsymbol{L}^2(\mathbb{R}^d)$. Let us present an easy argument showing that the translation operator $T_x, x \neq 0$, has no eigenvectors in $\boldsymbol{L}^2(\mathbb{R}^d)$. Assume that $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ satisfies

$$T_x f(t) = af(t), \tag{18}$$

which by the Fourier transform is equivalent to

$$M_{-x}\hat{f}(\omega) = a\hat{f}(\omega) \quad \text{a.e.}. \tag{19}$$

But this is only possible if the function $\hat{f}$ equals zero a.e., up to the points with $e^{2\pi i\omega x} \neq a$, i.e., it differs from zero only on a set of measure zero, hence $\hat{f} = 0$ and finally $f = 0 \in \boldsymbol{L}^2(\mathbb{R}^d)$. In other words, the translation operator T_x does not have eigenvectors in the space $\boldsymbol{L}^2(\mathbb{R}^d)$. On the other hand we are not too far off with the claim that T_x has the eigenvectors $e^{-2\pi it\omega}$ corresponding to the eigenvalue $e^{2\pi ix\omega}$, and the claim that any function f in $\boldsymbol{L}^2(\mathbb{R}^d)$ can be (kind of) expanded in terms of the eigenvectors $e^{-2\pi it\omega}$, by suitable interpretation of the inversion formula for the Fourier transform (valid pointwise for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$):

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega)e^{2\pi it\omega}d\omega. \tag{20}$$

Furthermore, the action of the translation operator is given by

$$T_x f(t) = \int_{\mathbb{R}^d} e^{2\pi ix\omega}\hat{f}(\omega)e^{2\pi it\omega}d\omega,$$

which is a continuous analog of the spectral decomposition of a self-adjoint operator in $\mathbb{R}^d$.

More concretely, the system of eigenfunctions $\{e^{-2\pi it\omega} : \omega \in \widehat{\mathbb{R}^d}\}$ is complete in the sense that for any function f in $L^2(\mathbb{R}^d)$ Parseval's equality holds

$$\int_{\mathbb{R}^d} |f(t)|^2 dt = \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 d\omega.$$

The obvious problem is the fact that $\boldsymbol{L}^2(\mathbb{R}^d)$ does not contain the system of eigenvectors of the translation operator T_x. But they can be considered as linear functionals on $\boldsymbol{S}_0(\mathbb{R}^d)$. This as well es several similar observations suggests to study operators on a Hilbert space via a dense subspace and its associated dual space. In our example it is actually possible to start from $\boldsymbol{S}_0(\mathbb{R}^d)$ and construct $\boldsymbol{L}^2(\mathbb{R}^d)$ as completion of $\boldsymbol{S}_0(\mathbb{R}^d)$ with respect to norm corresponding to the usual scalar product $\langle f, g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}dt$.

In this context it turns out that $\boldsymbol{S}_0(\mathbb{R}^d)$ has the important additional property that both δ-distributions and the pure frequencies $\chi_\omega(x) = e^{-2\pi ix\omega}$ (for all $\omega \in \mathbb{R}^d$) are in a natural way elements of $\boldsymbol{S}_0'(\mathbb{R}^d)$, i.e. define bounded linear functionals on $\boldsymbol{S}_0(\mathbb{R}^d)$. This dual space can be defined via STFT as follows [29]:

$$\boldsymbol{S}_0'(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \; : \|f\|_{\boldsymbol{S}_0'(\mathbb{R}^d)} = \|V_g f\|_{L^\infty} = \sup_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} |V_g f(x,\omega)| < \infty. \right\}$$

It is now easy to verify that $\delta \in \boldsymbol{S}_0'(\mathbb{R}^d)$. Indeed, for $g \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\sup_{(x,\omega)\in\mathbb{R}^d\times\widehat{\mathbb{R}^d}} |V_g\delta(x,\omega))| = \sup_{(x,\omega)\in\mathbb{R}^d\times\widehat{\mathbb{R}^d}} |\langle \delta, M_\omega T_x g\rangle| = \sup_{(x,\omega)\in\mathbb{R}^d\times\widehat{\mathbb{R}^d}} |g(-x)| = \|g\|_{L^\infty} < \infty.$$

We are now in a situation similar to the one inspiring Gelfand to introduce what is nowadays called a Gelfand triple. The main idea being the observation, that a triple of spaces – consisting of the Hilbert space itself, a small (topological vector) space contained in the Hilbert space, and its dual – allows a much better description of the situation. The advantage in our case is the fact that we can even take a Banach space, namely $\boldsymbol{S}_0(\mathbb{R}^d)$. Hence we can work with the following formal definition:

Definition 7. A (Banach) *Gelfand triple* consists of some Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ which is continuously and densely embedded into some Hilbert space $\mathscr{H}$, which in turn is w^*-continuously and densely embedded into the dual Banach space $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$.

We shall use the symbol $(\boldsymbol{B}, \mathscr{H}, \boldsymbol{B}')$ for such a triple of spaces. In this setting the inner product on $\mathscr{H}$ extends in a natural way to a pairing between B and B' (producing an anti-linear functional of the same norm).

As another consequence we mention an extension of an eigenvector of a bounded operator on a Hilbert space $\mathscr{H}$. Let A be a linear operator on a Banach space B then a linear functional F is a *generalized eigenvector* of A to the eigenvalue λ if

$$F(Af) = \lambda F(f), \qquad \text{for all } f \in B.$$

This notion allows to interpret the characters $\chi_\omega(x) = e^{-2\pi i\omega x}$ as generalized eigenvectors for the translation operator T_x on $\boldsymbol{S}_0(\mathbb{R}^d)$. Furthermore the set of generalized eigenvectors $\{\chi_\omega : \omega \in \mathbb{R}^d\}$ is complete by Plancherel's theorem, i.e., if $\hat{f}(\omega) = \langle \chi_\omega, f\rangle = 0$ for all $\omega \in \mathbb{R}^d$ implies $f \equiv 0$. This suggests to think of the Fourier transform of f at frequency ω as the evaluation of the linear functional $\langle \chi_\omega, f\rangle$.

The treatment of the translation operator T_x on $\boldsymbol{L}^2(\mathbb{R}^d)$ is a particular case of a general theorem by Gelfand that for any self-adjoint operator A on a Hilbert space $\mathscr{H}$ there exists a nuclear space and a complete system of generalized eigenvectors, see [26]. The advantage of the approach presented here is that instead of a (maybe complicated) nuclear topological vector space a relatively simple-minded Banach space can be used.

The introduction of Gelfand triples does not only offer a better description of a self-adjoint operator but it also allows to a simplification of proofs. For example, in the discussion of the Fourier transform $\mathcal{F}$ we consider it as an object on $\boldsymbol{S}_0(\mathbb{R}^d)$ where everything is well-defined and Parseval's formula and taking the inverse Fourier transform is justified by the nice properties of $\boldsymbol{S}_0(\mathbb{R}^d)$. By a density argument we get all properties of the Fourier transform on the level of $\boldsymbol{L}^2(\mathbb{R}^d)$. And we obtain an extension of the Fourier transform to $\boldsymbol{S}_0'(\mathbb{R}^d)$ by duality, the so-called *generalized Fourier transform.*

The preceding discussion suggests the following lemma which says that assertions for an operator on the $\boldsymbol{S}_0$-level are actually statements for $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{S}_0'$, respectively.

Lemma 5. *The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:*

(1) *$\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}^d})$,*
(2) *$\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}^d})$,*
(3) *$\mathcal{F}$ is a weak* (as well as a norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\widehat{\mathbb{R}^d})$.*

Furthermore we have that Parseval's formula

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle \tag{21}$$

is valid for $(f,g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$ and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

The properties of Fourier transform are expressed by the *Gelfand bracket*

$$\langle f, g\rangle_{(\boldsymbol{S}_0,\boldsymbol{L}^2,\boldsymbol{S}_0')(\mathbb{R}^d)} = \langle \hat{f}, \hat{g}\rangle_{(\boldsymbol{S}_0,\boldsymbol{L}^2,\boldsymbol{S}_0')(\widehat{\mathbb{R}}^d)} \tag{22}$$

which combines the functional brackets of Banach spaces and that of the inner-product for the Hilbert space.

The Fourier transform is a prototype for the notion of a Gelfand triple isomorphism.

Definition 8. If $(\boldsymbol{B}_1, \mathscr{H}_1, \boldsymbol{B'}_1)$ and $(\boldsymbol{B}_2, \mathscr{H}_2, \boldsymbol{B'}_2)$ are Gelfand triples then an operator A is called a *[unitary] Gelfand triple isomorphism* if

(1) A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
(2) A is a [unitary operator resp.] isomorphism from $\mathscr{H}_1$ to $\mathscr{H}_2$.
(3) A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B'}_1$ and $\boldsymbol{B'}_2$.

In this terminology the Fourier transform is a unitary Gelfand triple isomorphism of the Gelfand triple automorphism on $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ (isomorphism with itself). In the following lemma we give conditions for the extension of a linear mapping given on $\boldsymbol{S}_0(\mathbb{R}^d)$ to a unitary mapping on $\boldsymbol{L}^2(\mathbb{R}^d)$.

Lemma 6. *(cf. [22]) Let U be a unitary mapping from $\boldsymbol{L}^2(\mathbb{R}^d)$ to $\boldsymbol{L}^2(\mathbb{R}^d)$. The mapping U extends to a Gelfand triple isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ if and only if the restrictions of U to $\boldsymbol{S}_0(\mathbb{R}^d)$ defines a bounded bijective linear mapping from $\boldsymbol{S}_0(\mathbb{R}^d)$ onto itself.*

Due to this lemma we only have to check the properties of U at the $\boldsymbol{S}_0$-level, i.e., to verify the existence of some $C > 0$ such that

$$\|Uf\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \leq C\|f\|_{\boldsymbol{S}_0(\mathbb{R}^d)}. \tag{23}$$

The discussion of the Fourier transform $\mathcal{F}$ on the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ allows to think of $\mathcal{F}$ as a bounded operator between $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$ with a distributional kernel $k(t,\omega) = e^{-2\pi i t\omega}$. The existence of a distributional kernel holds for any bounded operator between $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$. This important fact is the so-called *kernel theorem* for $\boldsymbol{S}_0(\mathbb{R}^d)$ (cf. [18], Thm. 7.4.2). Before we give a precise description of this important fact we recall the notion of a Wilson basis. With the help of a Wilson basis we can adapt a linear algebra reasoning to the infinite-dimensional setting.

In 1991 Daubechies, Jaffard and Journé [11] followed an idea of Wilson in their construction of an orthonormal basis from a Gabor system $\mathcal{G}(g,\Lambda)$ of $\boldsymbol{L}^2(\mathbb{R}^d)$. Wilson suggested that the building blocks $\pi(x,\omega)g$ of an orthonormal basis of $\boldsymbol{L}^2(\mathbb{R}^d)$ should be symmetric in ω and should be concentrated at ω and $-\omega$.

Definition 9. For $g \in \boldsymbol{L}^2$ the associated Wilson system $\mathcal{W}(g)$ consists of functions

$$\psi_{k,n} = c_n T_{\frac{k}{2}}(M_n + (-1)^{k+n} M_{-n})g, \ (k,n) \in \mathbb{Z}^d \times \mathbb{N}_0,$$

where $c_0 = \frac{1}{2}$ and $c_n = \frac{1}{\sqrt{2}}$ for $n \geq 1$, $\psi_{k,0} = T_k g$ and $\psi_{2k+1,0} = 0$ for $k \in \mathbb{Z}$.

They proved the following theorem which shows a method for the construction of a Wilson basis from a Gabor system $\mathcal{G}(g, \frac{1}{2}\mathbb{Z} \times \mathbb{Z})$. Later Feichtinger, Gröchenig, and Walnut [16] showed that Wilson systems provide an unconditional basis for $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$ endowed with the w^*-topology. Therefore Wilson systems provide us with a natural class of bases for time-frequency analysis. The existence of an unconditional basis for $\boldsymbol{S}_0(\mathbb{R}^d)$ will be very helpful in our discussion of the kernel theorem for $\boldsymbol{S}_0(\mathbb{R}^d)$ and their construction relies heavily on the functorial properties of $\boldsymbol{S}_0$, cf. [18].

Theorem 5. *Let $\mathcal{G}(g, \frac{1}{2}\mathbb{Z} \times \mathbb{Z})$ be a tight frame for $\boldsymbol{L}^2(\mathbb{R})$ with $\|g\| = 1$ and $g(x) = \overline{g(-x)}$. Then the Wilson system $\mathcal{W}(g)$ is an orthonormal basis of $\boldsymbol{L}^2(\mathbb{R})$.*

As a corollary we get Wilson bases for $\boldsymbol{L}^2(\mathbb{R}^d)$ by taking tensor products.

Corollary 1. *Let $\mathcal{W}(g)$ be a Wilson basis for $\boldsymbol{L}^2(\mathbb{R})$ and define $\boldsymbol{\Psi}_{k,n} = \prod_{j=1}^d \psi_{r_j,s_j}$ for $(r,s) \in \mathbb{Z}^d \times \mathbb{N}_0$. Then $\boldsymbol{\Psi}_{k,n}$ is an orthonormal basis for $\boldsymbol{L}^2(\mathbb{R}^d)$.*

In applications of mathematics one often has to deal with linear systems. In the discrete and finite case each linear system is a linear mapping from the input space $\mathbb{R}^n$ into the output space $\mathbb{R}^m$ of our system and its action is given by matrix multiplication after a choice of bases in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively (similarly from $\mathbb{C}^n$ to $\mathbb{C}^m$ using complex matrices).

A linear system in infinite dimensions may be considered as a continuous analog of matrix multiplication (replacing summation by integration), i.e.,

$$g(x) = Kf(x) = \int_{\mathbb{R}^d} k(x,y) f(y) dy.$$

We can think of the input values $f(y)$ as being listed in an infinite column vector and $k(x,y)$ as an infinite matrix, the so-called *kernel* of K, and the integral $\int_{\mathbb{R}^d} k(x,y) f(y) dy$ providing the entries of the output vector in the expected way. In signal processing, such a model is known as a *linear time-variant system*.

For a wide range of function spaces (covering practically all cases relevant for applications) and by means of the use of generalized functions this analogy can be given a precise mathematical meaning. The natural way of describing this context is via so-called *kernel theorems*. Although only Hilbert Schmidt operators can be described as integral operators with $\boldsymbol{L}^2$-kernels, every bounded linear system A on $\boldsymbol{L}^2(\mathbb{R}^d)$ can be uniquely described by some distributional kernel $K \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$.

Suppose we have an integral operator K with distributional kernel k on $\boldsymbol{S}_0(\mathbb{R}^d)$, i.e., we think of K in a weak sense

$$\langle Kf, g\rangle = \langle k, g \otimes \bar{f}\rangle, \ f, g \in \boldsymbol{S}_0(\mathbb{R}^d),$$

where $g \otimes f$ denotes the tensor product $g(x)f(y)$, then K is a bounded operator between $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$. Since by duality we deduce that

$$|\langle Kf, g\rangle| = |\langle k, g \otimes \bar{f}\rangle| \leq \|k\|_{\boldsymbol{S}_0'} \|g \otimes \bar{f}\|_{\boldsymbol{S}_0} = \|k\|_{\boldsymbol{S}_0'} \|f\|_{\boldsymbol{S}_0} \|g\|_{\boldsymbol{S}_0}$$

is true for all $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, we have that $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$. Therefore the operator K is bounded between $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$ with the following estimate for the operator norm of K:

$$\|K\|_{op} \leq \|k\|_{\boldsymbol{S}_0'} .$$

The non-trivial aspect of the kernel theorem is that the converse is true.

Theorem 6. *If K is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes \bar{f}\rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$.*

We only sketch a proof and refer the interested reader to the book of Gröchenig [29] for the technical details.

We define the infinite matrix $\mathbf{a} = \big(a_{(l,m),(r,s)}\big)$ of the operator K with respect to a multivariate Wilson basis $\mathcal{W}(g)$ by

$$a_{(l,m),(r,s)} = \langle K\boldsymbol{\Psi}_{r,s}, \boldsymbol{\Psi}_{l,m}\rangle. \tag{24}$$

Then the matrix $(\mathbf{a})$ is bounded from $\ell^1(\mathbb{Z}^d \times \mathbb{N}_0)$ to $\ell^\infty(\mathbb{Z}^d \times \mathbb{N}_0^d)$. We therefore can define a kernel k for K as in linear algebra by

$$k = \sum_{l,m,r,s} a_{(l,m),(r,s)} \boldsymbol{\Psi}_{l,m} \otimes \boldsymbol{\Psi}_{r,s}. \tag{25}$$

Now, we know that $\{\boldsymbol{\Psi}_{l,m} \otimes \boldsymbol{\Psi}_{r,s}\}$ is an orthonormal basis for $\boldsymbol{L}^2(\mathbb{R}^{2d})$ which yields that $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ with weak*-convergence of the sum.

An important corollary of the preceding discussion is the following observation.

Corollary 2. *Let $(\boldsymbol{\Psi}_{\mathbf{k},\mathbf{n}})$ be an orthonormal Wilson basis for $\boldsymbol{L}^2(\mathbb{R}^d)$ then the coefficient mapping $D : f \mapsto \langle f, \boldsymbol{\Psi}_{k,n}\rangle$ induces a Gelfand triple isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z}^d \times \mathbb{N}^d)$.*

Proof: Since $(\boldsymbol{\Psi}_{k,n})$ is an orthonormal basis of $\boldsymbol{L}^2(\mathbb{R}^d)$ the analysis operator $f \mapsto \langle f, \boldsymbol{\Psi}_{k,n}\rangle$ is an isomorphism between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\ell^2(\mathbb{Z}^d \times \mathbb{N}^d)$. The Wilson system $(\boldsymbol{\Psi}_{k,n})$ is an unconditional basis for $\boldsymbol{S}_0(\mathbb{R}^d)$ and therefore the analysis operator gives an isomorphism between $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\ell^1(\mathbb{Z}^d \times \mathbb{N}^d)$. By duality we obtain that $\boldsymbol{S}_0'(\mathbb{R}^d)$ is isomorph to $\ell^\infty(\mathbb{Z}^d \times \mathbb{N}^d)$. □

6. The spreading function and pseudo-differential operators

The notion of a Gelfand triple has turned out to be a very fruitful concept for investigations in Gabor analysis, see [18], [10], [13]. In this section we present some results of Feichtinger and Kozek on Gelfand triples for time-frequency analysis. All these results have their origin in the search of a mathematical framework for problems in signal analysis. Many problems in applications are modelled as linear time-variant systems (LTV). In the last section we learned that a LTV is just an integral operator K acting on signals with finite energy,

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy, \qquad f \in \boldsymbol{L}^2(\mathbb{R}^d). \tag{26}$$

The quality of an integral operator K on $\boldsymbol{L}^2(\mathbb{R}^d)$ relies on properties of its kernel k. For example integrability conditions on k yield to classes of nice operators. The most prominent class of operators, the *Hilbert-Schmidt* operators $\mathcal{HS}$ are defined in terms of integrability conditions. Namely, an integral operator K on $\boldsymbol{L}^2(\mathbb{R}^d)$ is a *Hilbert-Schmidt* operator if $k \in \boldsymbol{L}^2(\mathbb{R}^d \times \mathbb{R}^d)$.

The class of Hilbert-Schmidt operators $\mathcal{HS}$ has a natural inner product. Let $K_1, K_2 \in \mathcal{HS}$ with kernels k_1, k_2, respectively. Then

$$\langle K_1, K_2 \rangle_{\mathcal{HS}} := \langle k_1, k_2 \rangle_{\boldsymbol{L}^2(\mathbb{R}^d \times \mathbb{R}^d)} \tag{27}$$

defines an inner product on $\mathcal{HS}$. The associated *Hilbert-Schmidt norm* $\|K\|_{\mathcal{HS}} := \big(\langle K_1, K_2 \rangle_{\mathcal{HS}}\big)^{1/2}$ gives $\mathcal{HS}$ the structure of a Hilbert space [42]. Furthermore we recall that every Hilbert-Schmidt operator on $\mathcal{HS}$ is a compact operator on $\boldsymbol{L}^2(\mathbb{R}^d)$. Recall that a compact operator K on $\boldsymbol{L}^2(\mathbb{R}^d)$ is of Hilbert-Schmidt type if and only if there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ in $\boldsymbol{L}^2(\mathbb{R}^d)$ and a sequence of scalars $(\lambda_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ such that

$$Kf = \sum_{n\in\mathbb{N}} \lambda_n \langle e_n, f \rangle e_n. \tag{28}$$

The sequence of scalars $(\lambda_n)_{n\in\mathbb{N}}$ are actually the eigenvalues of K and $\|K\|_{\mathcal{HS}} = \big(\sum_{n\in\mathbb{N}} |\lambda_n|^2\big)^{1/2}$. The space of Hilbert-Schmidt operators $\mathcal{HS}$ is not closed in the C^*-algebra $\mathcal{K}$ of compact operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ with respect to the operator norm and there exist compact operators which are not of Hilbert-Schmidt type. But $\mathcal{HS}$ is a two-sided ideal in $\mathcal{K}$.

If we choose as orthonormal basis of $\boldsymbol{L}^2(\mathbb{R}^d)$ a Wilson basis $(\boldsymbol{\Psi}_{\mathbf{k},\mathbf{n}})$ then the preceding observations led to an isomorphism between $\mathcal{HS}$ and $\ell\boldsymbol{L}^2(\mathbb{Z}^d \times \mathbb{N}^d)$. Now we can make use of the concept of Gelfand triples, but this time we take the Hilbert-Schmidt operators as Hilbert space of an *Operator Gelfand triple*. We observe that the kernel theorem for $\boldsymbol{S}_0(\mathbb{R}^d)$ provides us with another class of operators with "smooth kernels". We write $\mathcal{L}$ for the space of bounded linear operators on a Banach space B. One finds that $K \in \mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d))$ can be identified with kernels $k \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ and is dense in $\mathcal{HS}$. But the class of Hilbert-Schmidt operators $\mathcal{HS}$ is dense in $\mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))$ and therefore $(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d)))$ is indeed a Gelfand triple. In this setting the kernel theorem can be interpreted as a unitary Gelfand triple isomorphism between this triple and their kernels in $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \mathbb{R}^d)$. There is another Gelfand triple isomorphism that associates the $\mathcal{HS}$ Gelfand triple with the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$: the so-called *spreading symbol* of operators.

As a motivation we discuss a problem of great practical interest: communication with cellar phones. In modern communication cellar phones play a crucial rule in everyday life. How do engineers solve the problem of transmitting a signal f from a sender A to a receiver B? In the most general situation sender A and receiver B move in different directions with certain velocities which yields to a variation of the path lengths of the transmitted signal f and due to the Doppler effect to a change of frequencies. Therefore B receives a signal of the following form

$$\tilde{f} = \iint_{\mathbb{R}^2} \eta(K)(x,\omega) M_\omega T_x f dx d\omega, \tag{29}$$

where the function $\eta(K)$ models the effect of the channel by the amount of time-frequency shifts arising as just described, applied the the signal f. The receiver B is not interested in the signal $\tilde{f}$ but in the original signal f. From a mathematical point of view $\tilde{f}$ is just the action of an operator K on the signal f, i.e., $\tilde{f} = Kf$. In this picture B has to invert the operator K to get the information contained in the signal f. Operators of this form are called *pseudo-differential operators* and arise naturally in many problems of physics, engineering and mathematics. The function $\eta(K)$ is the so-called *spreading function* of the operator K. In the following we look for conditions on the spreading function $\eta(K)$ which allow an inversion of our pseudo-differential operator K.

First the equation (29) suggests a decomposition of a general operator K on $\boldsymbol{L}^2(\mathbb{R}^d)$ as a continuous superposition of time-frequency shifts.

$$K = \iint_{\mathbb{R}^{2d}} \eta(K)(x,\omega) M_\omega T_x dx d\omega. \tag{30}$$

We already know such a decomposition of the identity operator on $\boldsymbol{L}^2(\mathbb{R}^d)$ since this is the inversion formula for the STFT:

$$I_{\boldsymbol{L}^2(\mathbb{R}^d)} = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^{2d}} V_g f(x,\omega) M_\omega T_x dx d\omega \tag{31}$$

for $g, \gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$ with $\langle g, \gamma \rangle \neq 0$.

The non-commutativity of translation and modulation operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ leads to a twisted convolution of the spreading functions of two operators K and L. Let $K, L \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ and $\eta(K), \eta(L)$ their spreading functions respectively. Then the spreading function of the composition KL is given by *twisted convolution* of $\eta(K)$ and $\eta(L)$:

$$\eta(KL)(x,\omega) = \iint_{\mathbb{R}^2} \eta(K)(x',\omega')\eta(L)(x-x',\omega-\omega') e^{-2\pi i x'(\omega-\omega')} d\omega'. \tag{32}$$

The spreading function of the adjoint operator K^* is given by

$$\eta(K^*)(x,\omega) = \overline{\eta(K)(-x,-\omega)} \cdot e^{-2\pi i x\omega} \tag{33}$$

and therefore leads to a noncommutative involution. Later we will return to this topic in the context of Gröchenig/Leinert's resolution of the "irrational case"-conjecture [30].

The relation between the kernel k of an operator K from the Gelfand triple $(\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0))$ and its spreading function $\eta(K)$ is given by the following mapping from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$

$$\eta(K)(x,\omega) = \int_{\mathbb{R}^d} k(y, y-x) e^{-2\pi i y\omega} dy, \tag{34}$$

which is very useful in the calculation of the spreading function of an operator K. It can be interpreted literally at the lowest level (integrals etc. exist), and extend by continuity to the "upper levels". Moreover it can be described by the fact that it is the unique Gelfand triple isomorphism which maps TF-shift operators onto the corresponding Dirac measures in the TF-plane (hence reproducing exactly the situation we had in the finite case).

The spreading function of an operator K is an object living on the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Therefore a further understanding of its properties should be done according to the structure of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ which is closely related to the structure of the Euclidean plane $\mathbb{R}^d \times \mathbb{R}^d$. Namely, the time-frequency plane is a symplectic manifold, i.e., there exists a non-degenerate 2-form $\Omega(X,Y) = y \cdot \omega - x \cdot \eta$ for two points $X = (x,\omega), Y = (y,\eta)$ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Since Ω is non-degenerate there is a unique invertible skew-symmetric linear operator $\mathcal{J}$ on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ such that the symplectic form Ω and the Euclidian inner product are related as follows: $\Omega(X,Y) = \langle \mathcal{J}X, Y\rangle$ for all $X, Y \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. This implies an important fact about the characters of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Namely, the characters are given by $\{\chi_s(X,Y) = e^{2\pi i \Omega(X,Y)} | X \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$ for a fixed $Y \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Therefore it is natural to analyse a function F on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with the *symplectic Fourier Transform*

$$\mathcal{F}_s F(X) = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(Y) e^{2\pi i \Omega(X,Y)} dY \tag{35}$$

instead of the Fourier transform $\mathcal{F}$ induced by the standard inner-product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d \times \mathbb{R}^d$. From the relation between symplectic form and inner-product we obtain that the symplectic Fourier transform $\mathcal{F}_s$ is just a Fourier transform followed by a rotation by $\frac{\pi}{2}$ since $\mathcal{J}$ describes a rotation by $\frac{\pi}{2}$ around the origin of $\mathbb{R}^d \times \mathbb{R}^d$. This fact allows us to derive similiar statements for the symplectic Fourier transform as for the Euclidian Fourier transform.

(1) $\mathcal{F}_s$ is a unitary mapping from $\boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ onto $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d \times \mathbb{R}^d)$.
(2) $\mathcal{F}_s^{-1} = \mathcal{F}_s$ (involutive property);
(3) $\mathcal{F}_s \left(\boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d) \right) = \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

By duality we obtain that

Proposition 1. *The symplectic Fourier transform $\mathcal{F}_s$ defines a unitary Gelfand triple automorphism on $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.*

Another reason for our choice of $\boldsymbol{S}_0(\mathbb{R}^{2d})$ as space of test functions is that the Poisson summation formula for symplectic Fourier transform holds pointwise and with absolute convergence. Recently, we have shown that the Fundamental Identity of Gabor Analysis can be dervied by an application of Poisson summation to a product of two STFT's:

Theorem 7. *Let Λ a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with adjoint lattice Λ° and $F \in \boldsymbol{S}_0(\mathbb{R}^{2d})$. Then*

$$\sum_{\lambda \in \Lambda} F(\lambda) = \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{F}_s F(\lambda^\circ) \tag{36}$$

holds pointwise and with absolute convergence on both sides.

The spreading function is an important tool for the description of (slowly) time-variant channels in communication theory, but it is not the only symbol of associated with a linear operator. In the theory of pseudo- differential operators the *Kohn-Nirenberg symbol* (KN), denoted by $\sigma(K)$, is used for an operator $K \in (\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \mathbb{R}^d)$. It is defined as the symplectic Fourier transform of the spreading function $\eta(K)$:

$$\sigma(x,\omega) = \mathcal{F}_s\eta(K) = \iint_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \eta(K)e^{2\pi i(y\cdot\omega - x\cdot\eta)}dyd\eta,\ (x,\omega)\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d. \tag{37}$$

If $Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$ then $\sigma(K) = \int_{\mathbb{R}^d} k(x,x-y)e^{-2\pi i y\cdot\omega}dy$. In signal analysis $\sigma(K)$ was introduced by Zadeh and is called the *time-varying transfer function* of a system modelled by K. As an example we mention the KN symbol of a rank-one operator $f \otimes \overline{g}$, which describes the mapping $h \mapsto \langle h, g\rangle f$, is equal to

$$\sigma(f\otimes\overline{g})(x,\omega) = f(x)\overline{\hat{g}(\omega)}e^{-2\pi i x\cdot\omega}, \quad (x,\omega)\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d, \tag{38}$$

the Rihaczek distribution of f against g. For $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$ we have that the KN-symbol $\sigma(f\otimes\overline{g}) \in \boldsymbol{S}_0(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$ which in turn implies (using the last equation) that $(x,\omega)\mapsto e^{2\pi i x\cdot\omega}$ is a pointwise multiplier on $\boldsymbol{S}_0(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$.

After these preparations we can state one of our main results:

Theorem 8. *The spreading function $K \mapsto \eta(K)$ is a unitary Gelfand triple isomorphism from $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ to $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$.*

Corollary 3. *The KN symbol of K induces a unitary Gelfand triple isomorphism between $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$.*

Another consequence of the preceding theorem is the following Gelfand-bracket identities for $K_1, K_2 \in (\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0))$:

$$\langle K_1, K_2\rangle_{(\mathcal{B},\mathcal{HS},\mathcal{B}')} = \langle \eta(k_1), \eta(k_2)\rangle_{(\boldsymbol{S}_0,\boldsymbol{L}^2,\boldsymbol{S}_0')(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \tag{39}$$

$$= \langle \sigma(k_1), \sigma(k_2)\rangle_{(\boldsymbol{S}_0,\boldsymbol{L}^2,\boldsymbol{S}_0')(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}, \tag{40}$$

with $\mathcal{B} = \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ and $\mathcal{B}' = \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0)$ respectively.

The KN symbol of a rank-one operator $f \otimes \overline{g}$, which is the mapping $h \mapsto \langle h, g\rangle f$, is the Rihaczek distribution and by an application of the (inverse) symplectic Fourier transform we get another time-frequency distribution: the STFT!

Lemma 7. *For $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$ the rank-one operator $f \otimes \overline{g}$ has a kernel in $\boldsymbol{S}_0(\mathbb{R}^d)$. Moreover the corresponding spreading function is*

$$\eta(f\otimes\overline{g})(x,\omega) = \int_{\mathbb{R}^d} f(x)\overline{g(y-x)}e^{-2\pi i y\cdot\omega}dy \tag{41}$$

and hence coincides with $V_g f \in \boldsymbol{S}_0(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$.

In the light of this result the inversion formula for the STFT is a superposition of time-frequency shifts with the spreading function of the rank-one operator $g \otimes \overline{\gamma}$ for

$g, \gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$ with $\langle g, \gamma \rangle \neq 0$:

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(f \otimes \overline{g})(x, \omega) T_x M_\omega \gamma \, dx d\omega. \tag{42}$$

Recall that in analogy with the characters $\{\chi_\omega : \omega \in \widehat{\mathbb{R}^d}\}$ the time-frequency shifts $\{\pi(X) : X = (x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$ would be an orthonormal set with respect to the Hilbert-Schmidt inner product $\langle \cdot, \cdot \rangle_{\mathcal{HS}}$ and $\eta(f \otimes \overline{g})(x, \omega) = \langle f \otimes \overline{g}, \pi(x, \omega) \rangle_{\mathcal{HS}}$ but as in the case of Fourier transform the building blocks $\pi(X)$ for $X \in \mathbb{R}^d$ of our orthonormal system $\{\pi(X) : X = (x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$ are not Hilbert- Schmidt. As in our treatment of the Fourier transform it is not so important that the building blocks are elements of our Hilbert space but that they allow us to get expressions as they would be an orthonormal set of elements in our Hilbert space.

As a first example we state a generalization of the inversion formula for the STFT from $\boldsymbol{L}^2(\mathbb{R}^d)$ to the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, where for $f \in \boldsymbol{S}_0'(\mathbb{R}^d)$ the formula is interpreted in a weak sense.

Proposition 2. *Let $g, \gamma \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $\langle g, \gamma \rangle \neq 0$. Then*

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(f \otimes \overline{g})(x, \omega) T_x M_\omega \gamma \; dx d\omega. \tag{43}$$

holds for $f \in (\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

That is a special case of a general statement about the spreading function.

Theorem 9. *Any $K \in (\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \mathcal{H}S, \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0))$ has a representation*

$$K = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle K, \pi(x, \omega) \rangle_{\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')} \, \pi(x, \omega) \, dx d\omega \tag{44}$$

convergent in the strong resp. weak*-sense. The (complex-valued) amplitude function arising in this context, i.e. $\eta(K)(x, \omega) = \langle K, \pi(x, \omega) \rangle_{\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')}$, is called the spreading distribution of the operator K.

The basic tool in the proof is the fact that the spreading representation maps a time-frequency shift $\pi(x, \omega)$ for $(x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ on the Dirac measure δ_X at $X = (x, \omega)$, i.e., $\eta(\pi(X)) = \delta_X$ and the relation between the spreading function and the kernel of an operator K.

The preceding theorem is the mathematical justification of a widely used statement that the spreading function of an operator K is a measure for the time-frequency content of K.

In our intuition we move an operator K over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and want there is a simply relation between the original symbol of K and the symbol after a movement to $(x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. The KN-symbol of an operator K is shifted by $T_{x,\omega}$ in the time-frequency plane.

Lemma 8. *Let K belong to one of the spaces $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{H}S, \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0))$, then $\pi(x, \omega) K \pi(x, \omega)^*$, the conjugation of K by $\pi(x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ corresponds to translation of the KN symbol $\sigma(K)$,*

$$\sigma(\pi(x, \omega) K \pi(x, \omega)^*) = T_{(x,\omega)}\big(\sigma(K)\big). \tag{45}$$

This property of the KN symbol is of central importance in our study of the Gabor frame operator to which we devote the final part of this section. Let $\mathcal{G} = (g, \Lambda)$ be a Gabor system for a lattice $\Lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then the Gabor frame operator $S_{g,\Lambda}$ commutes with all time-frequency shifts of the lattice Λ, i.e.

$$\pi(\lambda) S_{g,\Lambda} \pi(\lambda)^* = S_{g,\Lambda}, \text{ for all } \lambda \in \Lambda. \tag{46}$$

This fact was the motivation for Feichtinger and Kozek to introduce the class of Λ-invariant operators [18].

Definition 10. Let Λ a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and K an operator concentrated on Λ. Then K is called Λ-*invariant* if $\pi(\lambda)K = K\pi(\lambda)$ for all $\lambda \in \Lambda$.

In the following we want to find the support of the spreading function $\eta(K)$ of an Λ-invariant operator $K \in (\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0))$. As a first step towards this result we study spreading representations of K on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

Lemma 9. *Let $K_1, K_2 \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ with spreading function $\eta(K_1), \eta(K_2)$, respectively. Then*

(1) $\eta(K_1K_2)(\lambda) = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(K_1)(\mu)\eta(K_2)(\lambda-\mu)\rho(\lambda-\mu,\mu)d\mu$ *with* $\rho(X,Y) = e^{2\pi i(y\cdot\omega - x\cdot\eta)}$ *for* $X = (x,\omega), Y = (y,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.
(2) $supp(\eta(K_1)\eta(K_2)) \subset supp(K_1) + supp(K_2)$.
(3) $|\eta(K_1K_2)| = |\eta(K_1)| * |\eta(K_2)|$ *for* $\eta(K_1), \eta(K_2) \in L^1_{loc}(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

The proof of (i) is a consequence of the commutation relation for time-frequency shifts and the fact that for $K_1 \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ and $K_2 \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ also $K_1K_2 \in \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0)$. Now each operator K in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ has an absolutely convergent spreading representation and therefore our result holds pointwise. The support condition follows from the analogous result for the ordinary convolution.

By abstract reasons each Λ-invariant operator K has a representation in the set of all operators concentrated on $\Lambda^\circ = \{\lambda^\circ \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d | \pi(\lambda)\pi(\lambda^\circ)\} = \pi(\lambda^\circ)\pi(\lambda\}\}$ since K lies in the commutant of the (C^*, von Neumann) algebra generated by $\{\pi(\lambda) : \lambda \in \Lambda\}$. The set Λ° is the so-called *adjoint lattice* since it is the annihilator subgroup of Λ for the symplectic Fourier transform $\mathcal{F}_s$ and if $\Lambda^\perp$ is the annihilator subgroup of Λ with respect to $\mathcal{F}$ then $\Lambda^\circ = \mathcal{J}\Lambda^\perp$.

The time-frequency invariance of $\boldsymbol{S}_0(\mathbb{R}^d)$ implies that K and $\pi(\lambda)K$ are in the Gelfand triple $(\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0))$, too. Therefore, the Λ-invariance of T translates into a periodicity condition for the symbol $\sigma(K)$

$$\sigma(K) = T_\lambda(\sigma(K)), \quad \lambda \in \Lambda\,. \tag{47}$$

This periodicity condition corresponds to a support condition for the spreading function since $\eta(K)(\lambda) = \eta(K)(\lambda)e^{-2\pi\Omega(\lambda,\mu)}$. But $\{e^{-2\pi\Omega(\lambda,\mu)} | \lambda \in \Lambda\}$ for a fixed $\mu \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is a group of characters on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ yields that

$$\text{supp}(\eta(K)) \subset \mathcal{J}\Lambda^\perp = \Lambda^\circ. \tag{48}$$

The fact that distributions in $\boldsymbol{S}_0'(\mathbb{R}^d)$ with support in a discrete subgroup is a sum of Dirac measures with a bounded sequence of coefficients implies that for some bounded sequence (c_{λ°) over Λ°

$$\eta(K) = \sum_{\lambda^\circ \in \Lambda^\circ} c_{\lambda^\circ} \delta_{\lambda^\circ} \tag{49}$$

with $c_{\lambda^\circ} = (K)_{\lambda^\circ} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d/\Lambda^\circ} \sigma(K)(\mu) e^{2\pi i \Omega(\lambda,\mu)} d\mu$.

Returning to the description in the operator domain we arrive at the following characterization

Theorem 10. *Let $K \in (\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'), \mathcal{H}S, \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0))$ and $\sigma(K)$ the KN symbol. Then $\sigma(K)$ is a Λ-periodic distribution with a symplectic Fourier transform supported on Λ^0. Furthermore*

$$K = \sum_{\lambda^\circ \in \Lambda^\circ} (K)_{\lambda^\circ} \pi(\lambda^\circ). \tag{50}$$

Corollary 4. *The mapping $\sigma(K) \mapsto (K)_{\lambda^\circ}$ is a unitary Gelfand triple isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d/\Lambda)$ and $(\ell^1, \ell^2, \ell^\infty)(\Lambda^\circ)$.*

Note that the time-frequency invariance of $\boldsymbol{S}_0(\mathbb{R}^d)$ implies the boundedness of K on $\boldsymbol{S}_0(\mathbb{R}^d)$ since

$$\|K\|_{\mathcal{L}(S_0)} = \| \sum_{\lambda^\circ \in \Lambda^\circ} (K)_{\lambda^\circ} \pi(\lambda^\circ) \|_{\mathcal{L}(S_0)} \le \sum_{\lambda^\circ \in \Lambda^\circ} |(K)_{\lambda^\circ}|. \tag{51}$$

The next theorem shows that for any Λ-invariant operator K with $\sigma(K) \in \boldsymbol{S}_0'((\mathbb{R}^d \times \widehat{\mathbb{R}}^d)/\Lambda)$ there exists a prototype operator $P \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ such that periodization of P in the time-frequency plane is corresponds to sampling of the spreading function $\eta(P)$ on Λ^0.

Theorem 11. *Let K be a Λ-invariant operator with $\sigma(K) \in \boldsymbol{S}_0'((\mathbb{R}^d \times \widehat{\mathbb{R}}^d)/\Lambda)$. Then there exists some $P \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ such that its periodization is exactly K*

$$K = \sum_{\lambda \in \Lambda} \pi(\lambda)\, P\, \pi(\lambda)^* = \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} \langle P, \pi(\lambda^\circ) \rangle_{\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')}\ \pi(\lambda^\circ). \tag{52}$$

Remark 1. The preceding result is a discrete analog of our spreading representation for operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ which in the context of Gabor analysis leads to the so-called *Janssen representation* of the Gabor frame operator.

The proof of the theorem is based on two important features of the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

(1) $\{U \mapsto \pi(\lambda)\, U\, \pi(\lambda)^* | \lambda \in \Lambda\}$ defines unitary representation of Λ which gives the Λ-invariance of K.
(2) An application of the Poisson summation formula for the symplectic Fourier transform to $\sigma(P)$ with respect to the lattice Λ maps the periodization of

$$\sigma(K) = \sum_{\lambda \in \Lambda} T_\lambda(\sigma(P)) \tag{53}$$

to the sampling of the spreading function $\eta(P)$ on the lattice Λ°.

As an application we state that the Gabor frame operator $S_{g,\Lambda}$ of a Gabor system $\mathcal{G}(g,\Lambda)$ with $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ is generated by shifting a rank-one operator along the lattice Λ. In addition we use the fact the spreading function of a rank-one operator is the STFT. Altogether we therefore have

$$S_{g,\Lambda} = \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ)\gamma\rangle \pi(\lambda^\circ) \tag{54}$$

with $\gamma \in \boldsymbol{S}_0(\mathbb{R}^d)$. The last equation (54) is the so-called Jannsen representation of $S_{g,\Lambda}$ which decomposes $S_{g,\Lambda}$ into an *absolutely convergent* series of time-frequency shifts. In (54) we used implicitely another pleasant property of $\boldsymbol{S}_0(\mathbb{R}^d)$.

Lemma 10. *Let $g, \gamma \in \boldsymbol{S}_0(\mathbb{R}^d)$ and Λ a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then (g,γ) satisfy Tolimieri-Orr's condition (A'):*

$$\sum_{\lambda \in \Lambda} |\langle g, \gamma_\lambda\rangle| < \infty, \qquad (A').$$

This stability of Condition (A') for $g, \gamma \in \boldsymbol{S}_0(\mathbb{R}^d)$ with respect to lattice changes makes Feichtinger's algebra such an important object in Gabor analysis. In a recent work Feichtinger and Kaiblinger have drawn some deep consequences from this fact. Roughly speaking, they proved that the set of functions in $\boldsymbol{S}_0(\mathbb{R}^d)$ which generate a Gabor frame is *"open"* [17].

We close our discussion of the Gabor frame operator with a striking result of Gröchenig/Leinert on the quality of the canonical dual of a Gabor system $\mathcal{G}(g,\Lambda)$ generated by a window $g \in \boldsymbol{S}_0(\mathbb{R}^d)$.

Theorem 12. *Let $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and $\mathcal{G}(g,\Lambda)$ a Gabor frame of $\boldsymbol{L}^2(\mathbb{R}^d)$. Then $\gamma_0 = S_{g,\Lambda}^{-1} g$ is in $\boldsymbol{S}_0(\mathbb{R}^d)$.*

Their proof is based on a noncommutative version of Wiener's lemma for the Banach algebra $\ell^1(\Lambda)$ with twisted convolution $\natural$ as product and noncommutative involution $*$ as described above for the spreading function of a product of two operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ and the spreading function of the adjoint of an operator in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$. A special case of their main result is that $(\ell^1, \natural, *)$ is a *symmetric* Banach algebra. In this context their Wiener lemma is expressed as the inverse-closedness of the Banach algebra

$$\mathcal{A}(\Lambda) = \{A \in \mathcal{B}(\boldsymbol{L}^2(\mathbb{R}^d)) \,|\, A = \sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda),\ (a_\lambda) \in \ell^1(\Lambda)\}$$

of absolutely convergent time-frequency series in the C^*-algebra $C^*(\Lambda)$ generated by the time-frequency shifts $\{\pi(\lambda) : \lambda \in \Lambda\}$. In other words, the argument is based on the highly non-trivial fact that a element of $\mathcal{A}(\Lambda)$ which is invertible in $C^*(\Lambda)$ has its inverse already in $\mathcal{A}(\Lambda)$.

We end up this section by recalling another way of representing an integral operator: the Weyl form of a pseudo-differential operator. First, the Wigner distribution defined in (9) can be generalized to a pair of functions f, g as follows:

$$W(f,g)(x,\omega) = \int f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-2\pi i \omega t}\, dt. \tag{55}$$

Then the Weyl operator L_{σ^w} of symbol $\sigma^w \in \mathcal{S}'(\mathbb{R}^{2d})$ is defined by

$$\langle L_{\sigma^w} f, g\rangle = \langle \sigma^w, W(g,f)\rangle, \qquad f,g \in \mathcal{S}(\mathbb{R}^d). \tag{56}$$

An easy computation shows that

$$L_{\sigma^w} = \iint_{\mathbb{R}^{2d}} \widehat{\sigma^w}(\omega,-x) M_\omega T_x dx d\omega,$$

i.e., L_{σ^w} is the operator K defined in (30), with symbol $\eta(K)$ given by

$$\eta(K)(x,\omega) = \widehat{\sigma^w}(\omega,-x).$$

As a consequence,

Corollary 5. *The Weyl symbol σ^w of L_{σ^w} induces a unitary Gelfand triple isomorphism between $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.*

7. Gabor Multipliers

In this section we study the interplay between *Gabor multipliers* and suitable Gelfand triples. A number of basic results can be obtained as a combination of known facts about both the *analysis* and the *synthesis* mapping associated with a Gabor or Weyl–Heisenberg family, and the standard properties of multiplication operators, acting between Banach sequence spaces, based for example, on Hölder's inequality. For a detailed treatement of this subject we refer the reader to [21].

Since the atoms used to build Gabor multipliers should generate Bessel families with respect to general TF-lattices Λ, windows g will be most often taken from the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$. In particular, such windows will generate Bessel families for all the lattices $a\mathbb{Z}^d \times b\mathbb{Z}^d \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d, a > 0, b > 0$.

Definition 11. Let g_1, g_2 be two $\boldsymbol{L}^2$-functions, Λ a TF-lattice for $\mathbb{R}^d$, i.e., a discrete subgroup of the phase space $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Furthermore let $\mathbf{m} = (\mathbf{m}(\lambda))_{\lambda\in\Lambda}$ be a complex-valued sequence on Λ. Then the *Gabor multiplier* associated to the triple (g_1, g_2, Λ) with (*strong* or) *upper symbol* $\mathbf{m}$ is given by

$$G_{\mathbf{m}}(f) = G_{g_1,g_2,\Lambda,\mathbf{m}}(f) = \sum_{\lambda\in\Lambda} m(\lambda)\langle f, \pi(\lambda)g_1\rangle\, \pi(\lambda)g_2.$$

We simply write $G_{g,\Lambda,\mathbf{m}}$ for the case $g_1 = g = g_2$.

It is obvious from this definition that Gabor multipliers are essentially (infinite) linear combinations of rank-one operators $f \mapsto \langle f, \pi(\lambda)g_1\rangle\ \pi(\lambda)g_2$, with coefficients m_λ. Whenever $g_1 = g = g_2$ and $\|g\|_2 = 1$ these building blocks are just the orthogonal projections onto the $1D$-subspaces of $\boldsymbol{L}^2$ generated by the elements of the WH-family $(\pi(\lambda)g)_{\lambda\in\Lambda}$. Depending on the properties of the *analysis window* g_1, the *synthesis window* g_2 and the *multiplier sequence* $\mathbf{m} = (m_\lambda)_{\lambda\in\Lambda}$ the overall operator $G_{g_1,g_2,\Lambda,\mathbf{m}}$ is bounded between various spaces. Typically one would require that both g_1 and g_2 are *Bessel atoms* with respect to the given lattice Λ, and that $\mathbf{m}$ is bounded. In this case the coefficient mapping using g_1, mapping f to the sequence of sampling values of the STFT $V_{g_1}f(\Lambda)$ maps $\boldsymbol{L}^2(\mathbb{R}^d)$ into $\ell^2(\Lambda)$ (by definition), and also the synthesis mapping $\mathbf{c} \mapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g_2$ is bounded from $\ell^2(\Lambda)$ to $\boldsymbol{L}^2(\mathbb{R}^d)$, and thus the overall operator is bounded on $\boldsymbol{L}^2(\mathbb{R}^d)$.

There are many good reasons to assume that the windows g_1 and g_2 should be chosen from $\boldsymbol{S}_0(\mathbb{R}^d)$. Among them notice that $\boldsymbol{S}_0(\mathbb{R}^d)$ is much larger than the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, used often in such a context *just for convenience*. On the other hand, $\boldsymbol{L}^2(\mathbb{R}^d)$ is a too large reservoir, since some of the more interesting results described below are not valid for all windows in $\boldsymbol{L}^2(\mathbb{R}^d)$.

In order to concentrate on the essential properties we shall state some of our results only for the case $g_1 = g_2 = g$, assuming that (g, Λ) generates a *tight* Gabor frame. In this particular case a minimal *symbolic calculus* is valid, in the sense that the constant multiplier $\mathbf{m} \equiv 1$ yields a multiple of the identity operator. Summarizing these basic facts we have:

Theorem 13. *Assume that $g \in \boldsymbol{S}_0(\mathbb{R}^d)$. Then one has:*

(i) *If $\mathbf{m} \in \ell^\infty(\Lambda)$, then $G_{\mathbf{m}} = G_{g,\Lambda,\mathbf{m}}$ defines a bounded operator on $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$, and the operator norm of $G_{\mathbf{m}}$ can be estimated (up to some constant) by $\|\mathbf{m}\|_\infty$.*
(ii) *The Gabor multiplier generated by $\mathbf{m}(\lambda) \equiv 1$ is a multiple of the identity operator if and only if (g, Λ) generates a tight Gabor frame.*
(iii) *$G_{\mathbf{m}}$ is a compact operator on $\boldsymbol{L}^2(\mathbb{R}^d)$ and on $\boldsymbol{S}_0(\mathbb{R}^d)$, if $\mathbf{m} \in c_o(\Lambda)$, i.e., if $m(\lambda) \to 0$ for $\lambda \to \infty$ (in the sense of Λ)*
(iv) *If $\mathbf{m} \in \ell^2(\Lambda)$, then $G_{\mathbf{m}}$: $\boldsymbol{S}_0'(\mathbb{R}^d) \to \boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\mathbb{R}^d) \to \boldsymbol{S}_0(\mathbb{R}^d)$.*
(v) *For $\mathbf{m} \in \ell^1(\Lambda)$ the operator $G_{\mathbf{m}}$ operator on $\boldsymbol{L}^2(\mathbb{R}^d)$, maps $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$.*

Proof: These statements follow from the boundedness properties of the coefficient resp. synthesis mappings (for fixed lattice Λ), as described in some detail in Section 3.3.3 of [22]. □

Of course it would be possible to make similar statements for other classes of windows. For example, any $g \in \boldsymbol{S}_0'(\mathbb{R}^d)$ in combination with an ℓ^1 multiplier sequence yields still a (compact) linear operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ into $\boldsymbol{S}_0'(\mathbb{R}^d)$, to mention a rather extreme possible variant. A more traditional approach to TF-analysis making use of Schwartz functions and tempered distributions would probably make use of $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ and $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ (instead of $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$) in the above context.

For general pairs (g_1, g_2) from $\boldsymbol{S}_0(\mathbb{R}^d)$ an even more compact formulation of the above theorem using the terminology of *Gelfand triples* can be given:

Theorem 14. *For every pair (g_1, g_2) in $\boldsymbol{S}_0(\mathbb{R}^d)$, and any TF-lattice Λ, the mapping from the strong symbol (multiplier) $(\mathbf{m}(\lambda))_{\lambda\in\Lambda}$ to the corresponding Gabor multiplier $G_{g_1,g_2,\Lambda,\mathbf{m}}$ maps the Gelfand triple $(\ell^1(\Lambda), \ell^2(\Lambda), \ell^\infty(\Lambda))$ into the bounded operators with kernel in the corresponding Gelfand triple*
$\Big(\boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), S_0'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)\Big)$, i.e., into $(\mathcal{B}, \mathcal{HS}, \mathcal{B}')$.

In the last part of this section we summarize the mapping properties between the space of symbols and the membership of the resulting Gabor multiplier in one of the typical operator ideals within the bounded operators on the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$. Again we fix a pair (g_1, g_2) in $\boldsymbol{S}_0(\mathbb{R}^d)$, and the TF-lattice Λ.

Theorem 15. *Assume that g, g_1, g_2 are in $\boldsymbol{S}_0(\mathbb{R}^d)$. Then one has:*

(i) *If $\mathbf{m}$ is bounded, then $G_{g_1,g_2,\Lambda,\mathbf{m}}$ is a bounded operator on $\boldsymbol{L}^2(\mathbb{R}^d)$.*
(ii) *If $\mathbf{m}$ is real-valued, then $G_{g,\Lambda,\mathbf{m}}$ is a self-adjoint operator on $\boldsymbol{L}^2(\mathbb{R}^d)$.*
(iii) *If $\mathbf{m} \in c_o(\Lambda)$, then $G_{g_1,g_2,\Lambda,\mathbf{m}}$ is a compact operator on $\boldsymbol{L}^2(\mathbb{R}^d)$.*

(iv) *If* $\mathbf{m} \in \ell^2(\Lambda)$, *then* $G_{g_1,g_2,\Lambda,\mathbf{m}}$ *is a Hilbert–Schmidt operator on* $\boldsymbol{L}^2(\mathbb{R}^d)$.
(v) *If* $\mathbf{m} \in \ell^1(\Lambda)$, *then* $G_{g_1,g_2,\Lambda,\mathbf{m}}$ *is a trace-class operator on* $\boldsymbol{L}^2(\mathbb{R}^d)$.

Proof: Most of these statements follow from general facts about operator ideal properties of linear operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ with kernels in the Gelfand triple $(\mathcal{B}, \mathcal{HS}, \mathcal{B}')$. Obviously $\boldsymbol{L}^2$-kernels correspond (exactly) to Hilbert–Schmidt operators. On the other hand the operators in $\mathcal{B}$, i.e., with S_0-kernels, are absolutely convergent sums of rank-one operators, and hence they are trace-class. Since the sequences with a finite number of non-zero coefficients generate finite rank operators, the density of such sequences in $c_o(\Lambda)$ implies (iii). Relation (ii) is easily verified directly and the main application of the symmetry assumption between analysis and synthesis, i.e., the choice $g_1 = g_2 = g$, is the investigation of the eigenvalue behaviour of operators with real symbols. □

Remark 2. The main statements of the above theorem can be summarized in the terminology of Gelfand triples by saying that for atoms $g_1, g_2 \in \boldsymbol{S}_0(\mathbb{R}^d)$ the mapping $(m_\lambda)_{\lambda\in\Lambda} \mapsto G_{g_1,g_2,\Lambda,\mathbf{m}}$ maps the Gelfand triple of sequence spaces $(\ell^1(\Lambda), \ell^2(\Lambda), \ell^\infty(\Lambda))$ into the Gelfand triple of operator ideals, consisting of trace-class operators, $\mathcal{HS}$ and the class of all bounded linear operators on $\boldsymbol{L}^2(\mathbb{R}^d)$.

Remark 3. The Gabor multipliers are special cases of the so-called *localization operators*. They have been studied by many authors, we refer the reader to [7, 8, 9] and references therein.

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