Spectral Properties of a Class of Generalized Landau Operators

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Abstract

We define and study some properties of operators formally given by

$$\widetilde{A} = a\left(\tfrac{1}{2}x + i\partial_y, \tfrac{1}{2}y - i\partial_x\right).$$

In particular we show that when $\widetilde{A}$ is essentially self-adjoint and the symbol a belongs to a certain Shubin class then the spectrum of $\widetilde{A}$ is discrete and coincides with that of the usual Weyl operator $A = a^w(x, -i\partial_x)$. In addition, the eigenvalues of $\widetilde{A}$ are in this case infinitely degenerate and the corresponding eigenfunctions can be obtained from those of A using a family of isometries $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$ indexed by the eigenfunctions of A.

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1 Introduction

This article is devoted to a study of the spectral properties of a class of pseudodifferential operators $\widetilde{A} : \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$ which can be formally written

$$\widetilde{A} = a\left(\tfrac{1}{2}x + i\partial_y, \tfrac{1}{2}y - i\partial_x\right) \tag{1}$$

when a belongs to some conveniently defined symbol class on $\mathbb{R}^{2n}$. These operators generalize the "Landau Hamiltonian"

$$\widetilde{H} = -\left(\partial_x^2 + \partial_y^2\right) + i\left(x\partial_y - y\partial_x\right) + \tfrac{1}{4}(x^2 + y^2) \tag{2}$$

describing the motion of charged particles in a uniform magnetic field which corresponds to the choice $a(x,\xi) = x^2 + \xi^2$ in (1) (Landau and Lifschitz [5]). We therefore find it appropriate to call $\widetilde{A}$ a "Landau operator".

We will prove the following properties of the Landau operators:

- The operator $\widetilde{A}$ and the standard Weyl operator $A = a^w(x, -i\partial_x)$ are intertwined by the elements of a family $(W_\phi)_{\phi \in S}$ of isometries of $L^2(\mathbb{R}^n)$ onto a closed subspace of $L^2(\mathbb{R}^{2n})$, the index set S consisting of all Schwartz functions with unit L^2 norm;
- The operators $\widetilde{A}$ and $A = a^w(x, -i\partial_x)$ have the same eigenvalues, and if u is an eigenfunction of A then each $W_\phi u$ is an eigenfunction of $\widetilde{A}$ for the same eigenvalue; this property already exhibits the fact (well-known for the operator (2) that the eigenvalues of $\widetilde{A}$ are very degenerate;
- We finally state a very precise result (Theorem 6) when a is real and belongs to a Shubin class $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ with $m_0 > 0$; in this case the spectrum of $\widetilde{A}$ is discrete and we are able to describe explicitly all the eigenfunctions in terms of those of the operator A.

Notation 1 *The generic point of $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ is denoted by $z = (x, \xi)$ and that of $T^*\mathbb{R}^{2n} = \mathbb{R}^{4n}$ by (z, ζ). We denote by $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$; its dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions). The scalar product of two functions $u, v \in L^2(\mathbb{R}^n)$ is denoted by $(u|v)$ and that of $U, V \in L^2(\mathbb{R}^{2n})$ by $((U|V))$. The corresponding norms are $||u||$ and $|||U|||$. The space $\mathbb{R}^{2n}$ is equipped with the standard symplectic form σ; in coordinates $\sigma(z, z') = Jz \cdot z'$ where $J(x, \xi) = (\xi, -x)$ ($\cdot$ always denotes the usual Euclidean scalar product on $\mathbb{R}^n$ or $\mathbb{R}^{2n}$), that is $\sigma(z, z') = \xi \cdot .x' - \xi' \cdot x$. The standard symplectic form on $T^*\mathbb{R}^{2n} = \mathbb{R}^{4n}$ is defined by $\omega(z, \zeta; z', \zeta') = \zeta \cdot z' - \zeta' \cdot z$.*

2 The Landau Operators $\widetilde{A}$

We assume that the reader is reasonably familiar with the standard Weyl pseudodifferential calculus as exposed in, for instance, Folland [1], Chapter 2, or Hörmander [4], §18.5.

2.1 Definition and Main Properties

Let $A = a^w(x, D)$ be the Weyl operator with symbol $a \in \mathcal{S}'(\mathbb{R}^n)$:

$$Au(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(\tfrac{1}{2}(x+y), \xi) u(y) dy d\xi$$

for $u \in \mathcal{S}(\mathbb{R}^n)$ the integral being interpreted in some suitable way (oscillatory integral, for instance). We can rewrite this definition as

$$Au = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \mathcal{F}_\sigma a(z_0) T(z_0) u dz_0$$

where $\mathcal{F}_\sigma a(z_0) = \mathcal{F}a(Jz_0)$ is the symplectic Fourier transform of a, and $T(z_0)$ is the Heisenberg–Weyl operator:

$$\widehat{T}(z_0)u(x) = e^{i(\xi_0 \cdot x - \frac{1}{2}\xi_0 \cdot x_0)} u(x - x_0) \tag{3}$$

if $z_0 = (x_0, \xi_0)$. We will also need the family of mappings $W_\phi : \mathcal{S}'(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$ defined, for $\phi \in \mathcal{S}(\mathbb{R}^{2n})$ with $||\phi|| = 1$, by

$$W_\phi u(z) = \left(\tfrac{\pi}{2}\right)^{n/2} W(u,\phi)(\tfrac{1}{2}z). \tag{4}$$

where $W(u,\phi)$ is the cross-Wigner transform of the pair (u,ϕ) (it is the Weyl symbol of the operator with kernel $(2\pi)^{-n} u \otimes \overline{\phi}$). Explicitly:

$$W_\phi u(z) = (2\pi)^{-n/2} e^{\frac{i}{2}\xi\cdot x} \int_{\mathbb{R}^n} e^{-i\xi\cdot y} u(y)\overline{\phi}(x-y)dy.$$

It follows from the usual properties of the cross-Wigner transform that $W_\phi : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$; it extends by continuity and density into a linear mapping $W_\phi : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$. Using the relation $(W_\phi^* U|v) = ((U|W_\phi))$ for $U \in \mathcal{S}(\mathbb{R}^{2n})$, $v \in \mathcal{S}(\mathbb{R}^n)$, one readily verifies that the adjoint $W_\phi^* : L^2(\mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R}^n)$ is given by the formula

$$W_\phi^* U(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^{2n}} e^{i\xi\cdot(y-\frac{1}{2}x)} \phi(x-y)U(x,\xi)dxd\xi. \tag{5}$$

The main result of this section is the following proposition, the second part of which was already proven in [3], Chapter 10. Formula (6) below justifies a posteriori the notation (7).

Proposition 2 *Let $a \in C^\infty(\mathbb{R}^{2n})$ and define $\widetilde{a} \in C^\infty(\mathbb{R}^{4n})$ by*

$$\widetilde{a}(z,\zeta) = a\left(\tfrac{1}{2}z - J\zeta\right). \tag{6}$$

(i) The operator $\widetilde{A} = \widetilde{a}^w(z, -i\partial_z)$ is given by

$$\widetilde{A}U(z) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \mathcal{F}_\sigma a(z_0)\widetilde{T}(z_0)U(z)dz_0 \tag{7}$$

where $U \in \mathcal{S}(\mathbb{R}^{2n})$ and

$$\widetilde{T}(z_0)U(z) = e^{-\frac{i}{2}\sigma(z,z_0)}U(z-z_0). \tag{8}$$

(ii) Each W_ϕ is a linear isometry of $L^2(\mathbb{R}^n)$ onto a closed subspace $\mathcal{H}_\phi$ of $L^2(\mathbb{R}^{2n})$ and we have

$$\widetilde{T}(z_0)W_\phi = W_\phi T(z_0) \quad , \quad \widetilde{A}W_\phi = W_\phi A \tag{9}$$

and

$$W_\phi^*\widetilde{T}(z_0) = W_\phi^* T(z_0) \quad , \quad W_\phi^*\widetilde{A} = \widehat{A}W_\phi^*. \tag{10}$$

Proof. (i) Let $\widetilde{B}$ be the operator defined by

$$\widetilde{B}U(z) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \mathcal{F}_\sigma a(z_0)\widetilde{T}(z_0)U(z)dz_0.$$

We are going to show that $\widetilde{A}=\widetilde{B}$. The kernel of $\widetilde{B}$ is

$$K(z,u)=\left(\tfrac{1}{2\pi}\right)^{\frac{n}{2}}e^{\frac{i}{2}\sigma(z,u)}\mathcal{F}_\sigma a(z-u) \tag{11}$$

as is easily seen by performing the change of variables $u=z-z_0$ and noting that $\sigma(z,z-u)=-\sigma(z,u)$. The Weyl symbol of $\widetilde{B}$ is thus

$$\widetilde{b}(z,\zeta)=\int_{\mathbb{R}^{2n}}e^{-i\zeta\cdot\eta}K(z+\tfrac{1}{2}\eta,z-\tfrac{1}{2}\eta)d\eta$$

hence, using the identity $\sigma(z+\frac{1}{2}\eta,z-\frac{1}{2}\eta)=-\sigma(z,\eta)$,

$$\widetilde{b}(z,\zeta)=\left(\tfrac{1}{2\pi}\right)^{n}\int_{\mathbb{R}^{2n}}e^{-i\zeta\cdot\eta}e^{-\frac{i}{2}\sigma(z,\eta)}\mathcal{F}_\sigma a(\eta)d\eta. \tag{12}$$

By definition of the symplectic Fourier transform we have

$$e^{-i\zeta\cdot\eta}\mathcal{F}_\sigma a(\eta)=\left(\tfrac{1}{2\pi}\right)^{n}\int_{\mathbb{R}^{2n}}e^{-i\zeta\cdot\eta}e^{-i\sigma(\eta,z)}a(z)dz$$

hence, observing that $\sigma(\eta,z)+\zeta\cdot\eta=\sigma(\eta,z+J\zeta)$,

$$e^{-i\zeta\cdot\eta}\mathcal{F}_\sigma a(\eta)=\left(\tfrac{1}{2\pi}\right)^{n}\int_{\mathbb{R}^{2n}}e^{-i\sigma(\eta,z)}t_{J\zeta}a(z)dz=\mathcal{F}_\sigma(t_{J\zeta}a)(\eta)$$

where $t_{J\zeta}a(z)=a(z-J\zeta)$. Formula (12) can thus be rewritten as

$$\widetilde{b}(2z,\zeta)=\left(\tfrac{1}{2\pi}\right)^{n}\int_{\mathbb{R}^{2n}}e^{-i\sigma(z,\eta)}\mathcal{F}_\sigma(t_{J\zeta}a)(\eta)d\eta;$$

the symplectic Fourier transform being involutive we have $\widetilde{b}(2z,\zeta)=t_{J\zeta}a(z)$. The identity $\widetilde{A}=\widetilde{B}$ follows. (ii) Recalling Moyal's identity

$$((W(u,\phi)|W(v,\psi)))=\left(\tfrac{1}{2\pi}\right)^{n}(u|v)(\phi|\psi)$$

we have $((W_\phi u|W_\phi v))=(u|v)$ hence W_ϕ is an isometry. Let us set $P_\phi=W_\phi W_\phi^*$. We have $P_\phi=P_\phi^*$ and, since $W_\phi^*W_\phi$ is the identity on $L^2(\mathbb{R}^n)$, $P_\phi^2=P_\phi$ hence P_ϕ is the orthogonal projection of $L^2(\mathbb{R}^{2n})$ onto a closed subspace $\mathcal{H}_\phi$; that subspace is precisely the range of W_ϕ. The proof of the intertwining relations $\widetilde{T}(z_0)W_\phi=W_\phi T(z_0)$ and $W_\phi^*\widetilde{T}(z_0)=W_\phi^*T(z_0)$ is purely computational, using the explicit definitions of the involved operators (see [3], Theorem 10.10(i)). The formulae $\widetilde{A}W_\phi=W_\phi A$ and $W_\phi^*\widetilde{A}=\widehat{A}W_\phi^*$ follow taking definition (7) of $\widetilde{A}$ into account. ∎

Remark 3 *The operators $\widetilde{T}(z_0)$ satisfy the same commutation relations as the Heisenberg–Weyl operators $T(z_0)$ and one can prove that they correspond to an irreducible unitary representation of the Heisenberg group on each of the Hilbert spaces $\mathcal{H}_\phi$ (see [3], §10.3.1).*

2.2 Composition and Adjoint

We are going to prove that the product of two Landau operators (when defined) is again a Landau operator, and that the adjoint of a Landau operator is obtained –as in standard Weyl calculus– by taking the complex conjugate of the symbol.

Recall that if A and B have Weyl symbols a and b then the Weyl symbol of the product $C = AB$ is given by

$$c(z) = \left(\tfrac{1}{4\pi}\right)^{2n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{2}\sigma(z',z'')} a(z + \tfrac{1}{2}z') b(z - \tfrac{1}{2}z'') dz' dz''. \tag{13}$$

Proposition 4 *(i) The symbol $\widetilde{c}$ of the product $\widetilde{C} = \widetilde{A}\widetilde{B}$ of two Landau operators is $\widetilde{c}(z, \zeta) = c\left(\frac{1}{2}z - J\zeta\right)$ where c is the Weyl symbol of the product AB. Hence*

$$\widetilde{A}\widetilde{B} = \widetilde{AB}. \tag{14}$$

(ii) The symbol of the adjoint $\widetilde{A}^*$ is the complex conjugate $\overline{\widetilde{a}}$ of the symbol $\widetilde{a}$ of $\widetilde{A}$. Hence $\widetilde{A}^*$ is (essentially) self-adjoint if and only if a is real.

Proof. In view of formula (13) the Weyl symbol of $\widetilde{A}\widetilde{B}$ is given by

$$\widetilde{c}(z,\zeta) = \left(\tfrac{1}{4\pi}\right)^{4n} \int_{\mathbb{R}^{4n}} e^{\frac{i}{2}\omega(z',\zeta';z'',\zeta')} a\left[\tfrac{1}{2}(z + \tfrac{1}{2}z') - J(\zeta + \tfrac{1}{2}\zeta')\right] \times$$
$$b\left[\tfrac{1}{2}(z - \tfrac{1}{2}z'') - J(\zeta - \tfrac{1}{2}\zeta'')\right] dz' dz'' d\zeta' d\zeta''$$

with ω is the symplectic form on $T^*\mathbb{R}^{2n} = \mathbb{R}^{4n}$. Defining new variables $u' = \frac{1}{2}z' - J\zeta'$ and $u'' = \frac{1}{2}z'' - J\zeta''$ this formula becomes

$$\widetilde{c}(z,\zeta) = \left(\tfrac{1}{4\pi}\right)^{4n} \int_{\mathbb{R}^{4n}} I(u, u'') a(\tfrac{1}{2}(z + u') - J\zeta) \times$$
$$b(\tfrac{1}{2}(z - u'') - J\zeta) dz' dz'' du' du''$$

with

$$I(u, u'') = \int_{\mathbb{R}^n} \exp\left[-\frac{i}{2}(\sigma(z', z'' - u'') - \sigma(u', z''))\right] dz' dz''.$$

Using the properties of the Fourier transform $I(u, u'')$ is easily calculated and one finds that it is equal to $(4\pi)^{2n} e^{\frac{i}{2}\sigma(u',u')}$; formula (14) follows. Part (ii) of the proposition follows from formula (6) and the fact that $\widetilde{a}$ is the Weyl symbol of $\widetilde{A}$ viewed as an operator $\mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$. ■

3 Shubin Classes and Landau Operators

3.1 The Shubin classes $H\Gamma_\rho^{m_1,m_0}$ and $HG_\rho^{m_1,m_0}$

We recall the following definitions and results from Shubin [6] (Chapter 4). Let m_0, m_1, and ρ be real numbers such that $m_0 \leq m_1$ and $0 < \rho \leq 1$. The symbol class $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ consists of all functions $a \in C^\infty(\mathbb{R}^{2n})$ such that for $|z|$ sufficiently large the following properties hold:

$$C_0|z|^{m_0} \leq |a(z)| \leq C_1|z|^{m_1} \tag{15}$$

for some $C_0, C_1 \geq 0$ and, for every $\alpha \in \mathbb{N}^n$ there exists $C_\alpha \geq 0$ such that

$$|\partial_z^\alpha a(z)| \leq C_\alpha |a(z)||z|^{-\rho|\alpha|}. \tag{16}$$

We denote by $HG_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ the class of operators A with τ-symbols a_τ belonging to $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$; this means that for every $\tau \in \mathbb{R}$ there exists $a_\tau \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ such that

$$Au(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a_\tau((1-\tau)x + \tau y, \xi)u(y)dyd\xi;$$

choosing $\tau = \frac{1}{2}$ this means, in particular, that every Weyl operator with $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ is in $HG_\rho^{m_1,m_0}(\mathbb{R}^{2n})$. It turns out that the condition $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ is also sufficient, because if $a_\tau \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ is true for some τ then it is true for all τ.

The main appeal of Shubin's classes comes from the following property, which is essential for the proof of the main result (Theorem 6):

(Sh) Let $A \in HG_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ with $m_0 > 0$. If A is formally self-adjoint, that is if $(Au|v)_{L^2} = (u|Av)_{L^2}$ for all $u, v \in C_0^\infty(\mathbb{R}^n)$, then A is essentially self-adjoint and has discrete spectrum in $L^2(\mathbb{R}^n)$. Moreover there exists an orthonormal basis of eigenfunctions $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ $(j = 1, 2, ...)$ with eigenvalues $\lambda_j \in \mathbb{R}$ such that $\lim_{j\to\infty} |\lambda_j| = \infty$.

3.2 The main theorem

The key to the relationship between the spectral properties of operators $A \in HG_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ and the corresponding Landau operators is following property of the isometries W_ϕ:

Proposition 5 *Assume that $(\phi_j)_j$ is an orthonormal basis of $L^2(\mathbb{R}^n)$. Then the vectors $\Phi_{j,k} = W_{\phi_j}\phi_k$ form an orthonormal basis of $L^2(\mathbb{R}^{2n})$ and we have $\Phi_{j,k} \in \mathcal{H}_j \cap \mathcal{H}_k$.*

Proof. Since the W_{ϕ_j} are isometries the vectors $\Phi_{j,k}$ form an orthonormal system. It is thus sufficient to show that if $U \in L^2(\mathbb{R}^{2n})$ is orthogonal to the family $(\Phi_{j,k})_{j,k}$ (and hence to all the spaces $\mathcal{H}_{\phi_j}$) then $U = 0$. Assume that $((U|\Phi_{jk})) = 0$ for all j, k. Since

$$((U|\Phi_{jk})) = ((U|W_{\phi_j}\phi_k)) = (W_{\phi_j}^* U|\phi_k)$$

it follows that $W_{\phi_j}^* U = 0$ for all j since $(\phi_j)_j$ is a basis; using the anti-linearity of W_ϕ in ϕ we have in fact $W_\phi^* U = 0$ for all $\phi \in L^2(\mathbb{R}^n)$. Let us show that this implies that $U = 0$. In view of formula (5) we have

$$\begin{aligned} W_\phi^* U(y) &= (2\pi)^{-n/2} \int_{\mathbb{R}^{2n}} e^{i\xi\cdot(y-\frac{1}{2}x)} \phi(x-y) U(x,\xi) dx d\xi \\ &= \int_{\mathbb{R}^n} F_{(2)}^{-1} U(x, y - \tfrac{1}{2}x) \phi(x-y) dx \end{aligned}$$

where $F_{(2)}^{-1}$ is the inverse partial Fourier transform of U in the second set of variables. The condition $W_\phi^* U = 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ implies that $F_{(2)} U = 0$ and hence $U = 0$. That $\Phi_{j,k} \in \mathcal{H}_j \cap \mathcal{H}_k$ is clear by definition of the Hilbert spaces $\mathcal{H}_j$. ∎

We now have all the elements needed to prove the main result of this article.

Theorem 6 *Let $A \in HG_\rho^{m_1,m_0}(\mathbb{R}^{2n})$. (i) The operators A and $\widetilde{A}$ have same eigenvalues; if u is an eigenfunction of A corresponding to the eigenvalue λ then $U_\phi = W_\phi u$ is an eigenfunction of $\widetilde{A}$ corresponding to λ, for every ϕ, and we have $U_\phi \in \mathcal{S}(\mathbb{R}^{2n})$. (ii) Assume in addition that $m_0 > 0$ and that A is formally self-adjoint. Then $\widetilde{A}$ has discrete spectrum $(\lambda_j)_{j\in\mathbb{N}}$ and $\lim_{j\to\infty} |\lambda_j| = \infty$; (iii) The eigenfunctions of $\widetilde{A}$ are in this case given by $\Phi_{jk} = W_{\phi_j}\phi_k$ where the ϕ_j are the eigenfunctions of A; (iv) We have $\Phi_{jk} \in \mathcal{S}(\mathbb{R}^{2n})$ and the Φ_{jk} form an orthonormal basis of $L^2(\mathbb{R}^{2n})$.*

Proof. (i) That every eigenvalue of A also is an eigenvalue of $\widetilde{A}$ is clear: if $Au = \lambda u$ for some $u \neq 0$ then

$$\widetilde{A}(W_\phi u) = W_\phi A u = \lambda(W_\phi u)$$

and $W_\phi u \neq 0$ because W_ϕ is injective; this proves at the same time that $W_\phi u$ is an eigenfunction of $\widetilde{A}$. Assume conversely that $\widetilde{A}U = \lambda U$ for $U \neq 0$. For every ϕ we have, using the equality $W_\phi^* \widetilde{A} = A W_\phi^*$,

$$A W_\phi^* U = W_\phi^* \widetilde{A} U = \lambda W_\phi^* U$$

hence λ is an eigenvalue of A and $W_\phi^* U$ will be an an eigenfunction of A if it is different from zero. Let us prove this is indeed the case. Recall that $W_\phi W_\phi^* = P_\phi$ is the orthogonal projection on $\mathcal{H}_\phi$. Assume that $W_\phi^* U = 0$; then $P_\phi U = 0$ for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, and hence $U = 0$ in view of Proposition 5; but this is not possible since U is an eigenfunction. That we have $U_\phi \in \mathcal{S}(\mathbb{R}^{2n})$ is clear since $W_\phi : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$. Properties (ii)–(iv) immediately follows from (i) using property (**Sh**) of the Shubin classes $HG_\rho^{m_1,m_0}(\mathbb{R}^{2n})$. ∎

The symbol $a(z) = x^2 + \xi^2$ of the Landau Hamiltonian (2) is in $H\Gamma_1^{2,2}(\mathbb{R}^2)$, hence Theorem 6 applies. The eigenvalues of $\widetilde{H}$ are those of the Hermite operator $-\partial_x^2 + x^2$ and are thus the numbers $\lambda_j = 2j + 1$ $(j = 0, 1, 2, ...)$ and the corresponding eigenfunctions ϕ_j are conveniently rescaled Hermite functions.

Using well-known formulae expressing the cross-Wigner transforms of pairs of Hermite functions in terms of Laguerre polynomial $\mathcal{L}_j^k$ of degree j and order k (see e.g. Folland [1]) one recovers the usual expressions

$$\Phi_{j+k,k}(z) = (-1)^j \frac{1}{\sqrt{2\pi}} \left(\frac{j!}{(j+k)!}\right)^{\frac{1}{2}} 2^{-\frac{k}{2}} z^k \mathcal{L}_j^k(\tfrac{1}{2}|z|^2) e^{-\frac{|z|^2}{4}}$$

and $\Phi_{j,j+k} = \overline{\Phi_{j+k,k}}$ for $k = 0, 1, 2, ...$ for the eigenfunctions of $\widetilde{H}$ found in the physics literature (see e.g. Landau and Lifschitz [5]).

4 Conclusions and Comments

We have shown in this article that the study of the spectral properties of operators obtained from a symbol by using the "quantization rules"

$$x_j \longrightarrow \widetilde{X_j} = \tfrac{1}{2} x_j + i\partial_{y_j} \quad , \quad y_j \longrightarrow \widetilde{Y_j} = \tfrac{1}{2} y_j - i\partial_{x_j}$$

is equivalent to that of the usual Weyl quantization. One can actually show that we obtain the same spectra with any choice of quantization satisfying the canonical commutation relation $\left[\widetilde{X_j}, \widetilde{Y_k}\right] = i\delta_{jk} \frac{1}{2} x + i\partial_y, \frac{1}{2} y - i\partial_x$. In particular, this is true for the alternative choice

$$x_j \longrightarrow \widehat{X_j} = x_j + \tfrac{1}{2} i\partial_{y_j} \quad , \quad y_j \longrightarrow \widehat{Y_j} = y_j - \tfrac{1}{2} i\partial_{x_j}$$

("Bopp shifts") which is of great interest in deformation quantization. In fact, denoting by $\widehat{A}$ the operator $a(\widehat{X}, \widehat{Y})$ one can show that for any function $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$ we have $\widehat{A}\Psi = a \star \Psi$ where $\star$ is the Moyal star-product. The first part of Theorem 6 can then be restated by saying that the study of the "star-genvalue equation" $a \star \Psi = \lambda \Psi$ is reduced to that of the eigenvalue problem $A\psi = \lambda\psi$; the corresponding eigenfunctions are then obtained by using the modified transform W'_ϕ defined by $W'_\phi \psi = (2\pi\hbar)^{n/2} W(\psi, \phi)$. We will come back to this relationship in a forthcoming paper.

One of the reasons for which Shubin introduced the classes $H\Gamma_\rho^{m_1,m_0}$ and $HG_\rho^{m_1,m_0}$ we used in this article was to study global hypoellipticity. In fact, he showed that if $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ then the conditions $u \in \mathcal{S}'(\mathbb{R}^n)$ and $Au \in \mathcal{S}(\mathbb{R}^n)$ imply $u \in \mathcal{S}(\mathbb{R}^n)$. Although it is not true that the condition $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ implies that $\widetilde{A} \in HG_\rho^{m_1,m_0}(\mathbb{R}^{4n})$ it is possible to show that the global hypoellipticity property carries over to Landau operators; see de Gosson[2]. In particular one recovers the global hypoellipticity of the Landau Hamiltonian (2) which was proven by Wong [7] using very different methods.

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