CANONICAL SUBGROUPS OF $\mathbb{H}_1 \rtimes SL(2,\mathbb{R})$

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Abstract. We classify, up to inner conjugation, all subgroups of the semidirect products $\mathbb{H}_1 \rtimes SL(2,\mathbb{R})$ and $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$. Our methods can be applied to all Lie groups locally isomorphic to them.

Contents

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1. Introduction

Let $\mathbb{H}_1$ denote the three–dimensional Heisenberg group. The groups $\mathbb{H}_1 \rtimes SL(2,\mathbb{R})$, $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ and their double coverings play an important role in time–frequency analysis, that is in phase space analysis in dimension one (see [F]). Despite their basic relevance, no reference containing detailed information about the structure of their subgroups is available in the literature. This paper fills this gap, providing convenient tables of all connected Lie subgroups of $\mathbb{H}_1 \rtimes SL(2,\mathbb{R})$ and $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ up to inner conjugation: for each conjugacy class we exhibit a natural representative (that is a "canonical" subgroup) together with its normalizer and centralizer.

The main step in the problem at hand is clearly to derive an explicit description of the conjugacy classes of all Lie subalgebras of $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$ under the adjoint actions of the corresponding connected Lie groups. The result concerning $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$ has also an interesting interpretation in terms of Poisson polynomial algebras. Indeed, it is well-known that $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$ is canonicallly isomorphic, as a Lie algebra, to the algebra $\mathcal{P}_2$ of polynomials in two indeterminates and degree ≤ 2 equipped with the usual Poisson bracket $\{f,g\}$. Under the isomorphism, the adjoint action corresponds to affine coordinate changes. Thus, we classify Poisson subalgebras of $\mathcal{P}_2$ up to affine coordinate changes.

Our approach can be applied to derive analogous classifications for *all* connected Lie groups that are locally isomorphic to either $\mathbb{H}_1 \rtimes SL(2,\mathbb{R})$ or $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$. The groups locally isomorphic to $\mathbb{H}_1 \rtimes SL(2,\mathbb{R})$ are of the form $\mathbb{H}_1 \rtimes SL^{(m)}$ or $\mathbb{H}_1^{\text{red}} \rtimes SL^{(m)}$, where $\mathbb{H}_1^{\text{red}} = \mathbb{H}_1/\mathbb{Z}$ is the reduced Heisenberg group and $SL^{(m)}$ is the m–sheet covering of $SL(2,\mathbb{R})$, the case of countably many sheets corresponding to the universal covering. The groups locally isomorphic to $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ are its coverings $\mathbb{R}^2 \rtimes SL^{(m)}$.

Our primary interest in these classification results comes from the issues addressed in [DN], where complete lists are needed for different purposes. On the one hand, we want to describe all possible reproducing formulas – for functions in $L^2(\mathbb{R})$ – that arise by restricting the extended metaplectic (projective) representation of the double cover of $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ to its subgroups. A reproducing formula of this sort reflects, on the level of functions, the properties of those affine transformations of the time–frequency plane that are encoded in the given subgroup H. Moreover, the appropriate notion of equivalence of two such formulas – in a sense made precise in [DN] – may be translated into conjugacy of the corresponding groups. Thus, the normalizers $N(H)$ provide further useful information. In the same paper we also analyze the commutative operator algebras consisting of bounded functions of P^w, the Weyl pseudodifferential operator defined by a real polynomial P of degree ≤ 2. The sensible reduction procedure, in this case, is provided by the classification up to inner conjugation of the one–parameter subgroups of $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$, in the sense that any such operator algebra is conjugate to one of the five canonical algebras corresponding to the five non–conjugate one–parameter subgroups.

The paper is organized as follows. In Section 2 we introduce the groups and algebras we shall be concerned with. In Section 3 we present our classification results together with all the explicit parametrizations. In Section 4 we prove the main theorems. Finally, in Section 5 we briefly discuss Poisson polynomial algebras and covering groups.

2. Preliminaries and notation

The group $SL(2,\mathbb{R})$ is the group of 2×2 real matrices with determinant equal to one and its Lie algebra is identified with $\mathfrak{sl}(2,\mathbb{R})$, the space of 2×2 real, traceless matrices with commutator as bracket. The adjoint action of $SL(2,\mathbb{R})$ on $\mathfrak{sl}(2,\mathbb{R})$ is the usual matrix conjugation $\mathrm{Ad}\, gX = gXg^{-1}$. The linear action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$ gives rise to the semidirect product $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$, where the multiplication is defined by

$$\Big(\left[\begin{smallmatrix} q_1 \\ p_1 \end{smallmatrix}\right], g_1\Big)\Big(\left[\begin{smallmatrix} q_2 \\ p_2 \end{smallmatrix}\right], g_2\Big) = \Big(\left[\begin{smallmatrix} q_1 \\ p_1 \end{smallmatrix}\right] + g_1\left[\begin{smallmatrix} q_2 \\ p_2 \end{smallmatrix}\right], g_1g_2\Big).$$

Consequently, the bracket of $(X_1,A_1),(X_2,A_2)\in\mathbb{R}^2\rtimes\mathfrak{sl}(2,\mathbb{R})$ is

$$[(X_1,A_1),(X_2,A_2)] = (A_1X_2 - A_2X_1, [A_1,A_2]).$$

Observe that $\mathbb{R}^2\rtimes SL(2,\mathbb{R})$ acts naturally on the time–frequency plane $\mathbb{R}^2$ by

$$\Big(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], g\Big)\cdot\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right] = g\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right] + \left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]. \tag{2.1}$$

The exponential mapping $\exp : \mathbb{R}^2\rtimes SL(2,\mathbb{R}) \to \mathbb{R}^2\rtimes\mathfrak{sl}(2,\mathbb{R})$ takes the form

$$\exp t(X,A) = \Big(\int_0^t e^{\tau A}X\, d\tau, e^{tA}\Big),$$

whereas the adjoint action of $(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], g)\in\mathbb{R}^2\rtimes SL(2,\mathbb{R})$ on $(X,A)\in\mathbb{R}^2\rtimes\mathfrak{sl}(2,\mathbb{R})$ is

$$\mathrm{Ad}(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], g)(X,A) = (gX - gAg^{-1}\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], gAg^{-1}).$$

Denote by $\mathbb{H}_1$ the three–dimensional Heisenberg group, that is $\mathbb{R}^2\times\mathbb{R}$ with product:

$$(q_1,p_1,t_1)\cdot(q_2,p_2,t_2) = (q_1+q_2, p_1+p_2, t_1+t_2 - \frac{1}{2}(q_1p_2 - p_1q_2)).$$

Sometimes it is more convenient to write $x = \left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]\in\mathbb{R}^2$ and express the product in terms of the simplectic form ω. Thus if $x_1,x_2\in\mathbb{R}^2$ and

$$\omega(x_1,x_2) = {}^t x_1 J x_2, \quad J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right],$$

we have $(x_1,t_1)\cdot(x_2,t_2) = (x_1+x_2, t_1+t_2-\frac{1}{2}\omega(x_1,x_2))$. The Lie algebra $\mathfrak{h}_1$ of $\mathbb{H}_1$ may be identified with $\mathbb{R}^2\times\mathbb{R}$ with bracket

$$[(X_1,t_1),(X_2,t_2)] = (0, -\omega(X_1,X_2)).$$

The action of $SL(2,\mathbb{R})$ on $\mathbb{H}_1$ given by the automorphisms $A\cdot(x,t) = (Ax,t)$ gives rise to the semidirect product $\mathbb{H}_1\rtimes SL(2,\mathbb{R})$, where the multiplication is given by

$$((x_1,t_1);g_1)((x_2,t_2);g_2) = ((x_1,t_1)\cdot(g_1x_2,t_2);g_1g_2).$$

so that its Lie algebra $\mathfrak{g} = \mathfrak{h}_1\rtimes\mathfrak{sl}(2,\mathbb{R})$ of $\mathbb{H}_1\rtimes SL(2,\mathbb{R})$ has bracket

$$[(X_1,t_1);A_1),((X_2,t_2);A_2)] = ((A_1X_2 - A_2X, -\omega(X_1,X_2)); [A_1,A_2]).$$

The exponential mapping can be shown to be

$$\exp t((X,u);A) = \Big(\int_0^t e^{\tau A}X\, d\tau, tu - \frac{1}{2}\int_0^t \omega(X, \int_0^v e^{\tau A}X\, d\tau)\, dv); e^{tA})\Big),$$

while the adjoint action of $((y,s);g) \in \mathbb{H}_1 \rtimes SL(2,\mathbb{R})$ on $((X,z);A) \in \mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$ is

$$\Big(\big(gX - gAg^{-1}y, z - \omega(y, gX - \frac{1}{2}gAg^{-1}y)\big); gAg^{-1}\Big). \tag{2.2}$$

3. Classification results

3.1. Canonical subalgebras of $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$. As agreed, $\mathfrak{sl}(2,\mathbb{R})$ is the Lie algebra of 2×2 traceless matrices with commutator as bracket. The elements

$$H = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \quad J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right], \quad U = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$$

satisfy $[H,U] = 2U$ and generate the algebras

$$\begin{aligned} \mathfrak{k} &= \big\{tJ : t \in \mathbb{R}\big\}; \\ \mathfrak{a} &= \big\{tH : t \in \mathbb{R}\big\}; \\ \mathfrak{n} &= \big\{tU : t \in \mathbb{R}\big\}; \\ \mathfrak{n} \rtimes \mathfrak{a} &= \big\{tH + uU : t,u \in \mathbb{R}\big\}. \end{aligned}$$

By means of the immersion $X \mapsto (\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], X)$ of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$ we shall identify $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{n} \rtimes \mathfrak{a}$ and $\mathfrak{sl}(2,\mathbb{R})$ itself as subalgebras of $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$. Similarly, by taking semidirect products with $\mathbb{R}^2$ we write

$$\begin{aligned} \mathbb{R}^2 \rtimes \mathfrak{k} &= \big\{(\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right], tJ) : r,s,t \in \mathbb{R}\big\}; \\ \mathbb{R}^2 \rtimes \mathfrak{a} &= \big\{(\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right], tH) : r,s,t \in \mathbb{R}\big\}; \\ \mathbb{R}^2 \rtimes \mathfrak{n} &= \big\{(\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right], tU) : r,s,t \in \mathbb{R}\big\}; \\ \mathbb{R}^2 \rtimes (\mathfrak{n} \rtimes \mathfrak{a}) &= \big\{(\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right], tH + uU) : r,s,t,u \in \mathbb{R}\big\}. \end{aligned}$$

Next, let

$$\mathbb{R}_q = \big\{\left[\begin{smallmatrix} s \\ 0 \end{smallmatrix}\right] : s \in \mathbb{R}\big\} \subset \mathbb{R}^2, \quad \mathbb{R}_p = \big\{\left[\begin{smallmatrix} 0 \\ r \end{smallmatrix}\right] : r \in \mathbb{R}\big\} \subset \mathbb{R}^2. \tag{3.1}$$

The subscripts q and p come from thinking of the plane as phase–space, with position q and momentum p. By means of the immersion $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right] \mapsto (\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right], 0)$ of $\mathbb{R}^2$ in $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$ we shall identify $\mathbb{R}_q$ as a subalgebra of $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$. Thus, we may consider

$$\begin{aligned} \mathbb{R}_q \rtimes \mathfrak{a} &= \big\{(\left[\begin{smallmatrix} s \\ 0 \end{smallmatrix}\right], tH) : s,t \in \mathbb{R}\big\}; \\ \mathbb{R}_q \times \mathfrak{n} &= \big\{(\left[\begin{smallmatrix} s \\ 0 \end{smallmatrix}\right], tU) : s,t \in \mathbb{R}\big\}; \\ \mathbb{R}_q \rtimes (\mathfrak{n} \rtimes \mathfrak{a}) &= \big\{(\left[\begin{smallmatrix} s \\ 0 \end{smallmatrix}\right], tH + uU) : s,t,u \in \mathbb{R}\big\}. \end{aligned}$$

It should be observed that $\mathfrak{n}$ acts on $\mathbb{R}_q$ by zero, so that their product is actually direct, whereas $\mathfrak{k}$ does not act on $\mathbb{R}_q$. Finally, consider the diagonal of $\mathbb{R}_p \times \mathfrak{n}$, written

$$\mathfrak{p} = \big\{(\left[\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right], tU) : t \in \mathbb{R}\big\}$$

because the orbits of the corresponding group in the time–frequency plane are parabolas (see Theorem 3.3). Its vector space direct sum with $\mathbb{R}_q$ leads to the abelian Lie algebra

$$\mathbb{R}_q \oplus \mathfrak{p} = \big\{(\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right], rU) : r \in \mathbb{R}\big\}.$$

The Lie algebras we have introduced, hereafter referred to as canonical, exhaust a list of representatives for the conjugacy classes of subalgebras of $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$.

Theorem 3.1. *Any proper Lie subalgebra* $\mathfrak{h}$ *of* $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$ *is conjugate to one of the following non–conjugate canonical Lie algebras, listed together with* $\mathfrak{h}^{(1)} = [\mathfrak{h},\mathfrak{h}]$, $\mathfrak{h}^{(2)} = [\mathfrak{h}^{(1)},\mathfrak{h}^{(1)}]$, $\mathfrak{h}_{(2)} = [\mathfrak{h},\mathfrak{h}_{(1)}]$ *and their algebraic structure.*

dim/n.	$\mathfrak{h}$	$\mathfrak{h}^{(1)}$	$\mathfrak{h}^{(2)}$	$\mathfrak{h}_{(2)}$	*structure*
(1.*i*)	$\mathfrak{k}$	0	0	0	*abelian*
(1.*ii*)	$\mathfrak{a}$	0	0	0	*abelian*
(1.*iii*)	$\mathfrak{n}$	0	0	0	*abelian*
(1.*iv*)	$\mathfrak{p}$	0	0	0	*abelian*
(1.*v*)	$\mathbb{R}_q$	0	0	0	*abelian*
(2.*i*)	$\mathfrak{n} \rtimes \mathfrak{a}$	$\mathfrak{n}$	0	$\mathfrak{n}$	*solvable*
(2.*ii*)	$\mathbb{R}_q \rtimes \mathfrak{a}$	$\mathbb{R}_q$	0	$\mathbb{R}_q$	*solvable*
(2.*iii*)	$\mathbb{R}_q \times \mathfrak{n}$	0	0	0	*abelian*
(2.*iv*)	$\mathbb{R}_q \oplus \mathfrak{p}$	0	0	0	*abelian*
(2.*v*)	$\mathbb{R}^2$	0	0	0	*abelian*
(3.*i*)	$\mathbb{R}_q \rtimes (\mathfrak{n} \rtimes \mathfrak{a})$	$\mathbb{R}_q \times \mathfrak{n}$	0	$\mathbb{R}_q \times \mathfrak{n}$	*solvable*
(3.*ii*)	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})$	*semisimple*
(3.*iii*)	$\mathbb{R}^2 \rtimes \mathfrak{k}$	$\mathbb{R}^2$	0	$\mathbb{R}^2$	*solvable*
(3.*iv*)	$\mathbb{R}^2 \rtimes \mathfrak{a}$	$\mathbb{R}^2$	0	$\mathbb{R}^2$	*solvable*
(3.*v*)	$\mathbb{R}^2 \rtimes \mathfrak{n}$	$\mathbb{R}_q$	0	0	*solvable*
(4.*i*)	$\mathbb{R}^2 \rtimes (\mathfrak{n} \rtimes \mathfrak{a})$	$\mathbb{R}^2 \rtimes \mathfrak{n}$	$\mathbb{R}_q$	$\mathbb{R}^2 \rtimes \mathfrak{n}$	*solvable*

In the first column, the first index denotes dimension.

3.2. **Canonical subgroups of** $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$**.** By means of the exponential mapping it is easy to compute the connected subgroups that correspond to the canonical Lie algebras described in Theorem 3.1. For the reader's convenience, we give explicit parametrizations of such groups. Most notations are self–explanatory. First of all, put

$$k_\theta = \exp\theta J = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix};$$
$$a_t = \exp tH = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, \text{ and } d_s = a_{\log s} = \begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix},\ s>0;$$
$$u_t = \exp tU = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

The above generate the canonical subgroups of $SL(2,\mathbb{R})$, namely

$$K = \left\{k_\theta : \theta \in \mathbb{R}\right\} = SO(2,\mathbb{R});$$
$$A = \left\{d_s : s > 0\right\};$$
$$N = \left\{u_t : t \in \mathbb{R}\right\};$$
$$NA = \left\{u_t d_{s^{1/2}} : t \in \mathbb{R}, s > 0\right\}.$$

Along with the above groups, we shall consider the following variants

$$\pm N = \left\{\begin{bmatrix} \varepsilon & t \\ 0 & \varepsilon \end{bmatrix} : \varepsilon = \pm 1, t \in \mathbb{R}\right\};$$
$$\pm NA = \left\{\begin{bmatrix} a & t \\ 0 & a^{-1} \end{bmatrix} : a \neq 0, t \in \mathbb{R}\right\};$$
$$\Delta = \pm A = \left\{\operatorname{diag}(a, a^{-1}) : a \neq 0\right\};$$
$$\Delta^0 = \langle \Delta, J\rangle \text{ (group generated by } \Delta \text{ and } J).$$

The normalizer $N(\cdot)$ and centralizer $Z(\cdot)$ in $SL(2,\mathbb{R})$ of the canonical groups K, A and N of $SL(2,\mathbb{R})$ and of the canonical algebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ of $\mathfrak{sl}(2,\mathbb{R})$ are

$$\begin{aligned}
&N(K) = Z(K) = K = N(\mathfrak{k}) = Z(\mathfrak{k});\\
&N(A) = N(\mathfrak{a}) = \Delta^0,\ Z(A) = Z(\mathfrak{a}) = \Delta;\\
&N(N) = N(\mathfrak{n}) = \pm NA,\ Z(N) = Z(\mathfrak{n}) = \pm N,
\end{aligned}$$

as one checks by direct computation. Finally,

$$\begin{aligned}
&\mathbb{R}^2 \rtimes K = \left\{\left(\left[\begin{smallmatrix} q\\ p\end{smallmatrix}\right], k_\theta\right) : p, q, \theta \in \mathbb{R}\right\};\\
&\mathbb{R}^2 \rtimes A = \left\{\left(\left[\begin{smallmatrix} q\\ p\end{smallmatrix}\right], d_s\right) : p, q \in \mathbb{R}, s > 0\right\};\\
&\mathbb{R}^2 \times N = \left\{\left(\left[\begin{smallmatrix} q\\ p\end{smallmatrix}\right], u_t\right) : p, q, t \in \mathbb{R}\right\};\\
&\mathbb{R}^2 \rtimes NA = \left\{\left(\left[\begin{smallmatrix} q\\ p\end{smallmatrix}\right], u_t d_{s^{1/2}}\right) : p, q, t \in \mathbb{R}, s > 0\right\}.\\
&\mathbb{R}_q \rtimes A = \left\{\left(\left[\begin{smallmatrix} q\\ 0\end{smallmatrix}\right], d_s\right) : q \in \mathbb{R}, s > 0\right\};\\
&\mathbb{R}_q \times N = \left\{\left(\left[\begin{smallmatrix} q\\ 0\end{smallmatrix}\right], u_t\right) : q, t \in \mathbb{R}\right\};\\
&\mathbb{R}_q \rtimes NA = \left\{\left(\left[\begin{smallmatrix} q\\ 0\end{smallmatrix}\right], u_t d_{s^{1/2}}\right) : q, t \in \mathbb{R}, s > 0\right\}.\\
&P = \left\{\left(\left[\begin{smallmatrix} \frac{t^2}{2}\\ t\end{smallmatrix}\right], u_t\right) : t \in \mathbb{R}\right\}\\
&\mathbb{R}_q \cdot P = \left\{\left(\left[\begin{smallmatrix} q\\ t\end{smallmatrix}\right], u_t\right) : r \in \mathbb{R}\right\}.
\end{aligned}$$

Theorem 3.2. *Any connected Lie subgroup of $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ is conjugate to one of the following non–conjugate subgroups, listed together with their normalizers $N(\cdot)$ and centralizers $Z(\cdot)$.*

dim/n.	*group*	$N(\cdot)$	$Z(\cdot)$
(1.*i*)	K	K	K
(1.*ii*)	A	Δ^0	Δ
(1.*iii*)	N	$\mathbb{R}_q \rtimes \pm NA$	$\mathbb{R}_q \rtimes \pm N$
(1.*iv*)	P	P	P
(1.*v*)	$\mathbb{R}_q$	$\mathbb{R}_q \rtimes \pm NA$	$\mathbb{R}_q \times N$
(2.*i*)	NA	$\pm NA$	$\pm I$
(2.*ii*)	$\mathbb{R}_q \rtimes A$	$\mathbb{R}_q \rtimes \Delta$	I
(2.*iii*)	$\mathbb{R}_q \times N$	$\mathbb{R}^2 \times \pm NA$	$\mathbb{R}_q \times N$
(2.*iv*)	$\mathbb{R}_q \cdot P$	$\mathbb{R}^2 \times N$	$\mathbb{R}_q \cdot P$
(2.*v*)	$\mathbb{R}^2$	G	I
(3.*i*)	$\mathbb{R}_q \rtimes NA$	$\mathbb{R}_q \rtimes \pm NA$	I
(3.*ii*)	$SL(2,\mathbb{R})$	$SL(2,\mathbb{R})$	$\pm I$
(3.*iii*)	$\mathbb{R}^2 \rtimes K$	$\mathbb{R}^2 \rtimes K$	I
(3.*iv*)	$\mathbb{R}^2 \rtimes A$	$\mathbb{R}^2 \rtimes \Delta^0$	I
(3.*v*)	$\mathbb{R}^2 \times N$	$\mathbb{R}^2 \rtimes \pm NA$	I
(4.*i*)	$\mathbb{R}^2 \rtimes NA$	$\mathbb{R}^2 \rtimes \pm NA$	I

In the first column, the first index denotes dimension.

For some purposes (see [DN]) it is of interest to describe the geometric nature of the orbits of the various canonical groups in the time–frequency plane. We shall also use such structures to show that the canonical groups are mutually non–conjugate.

Theorem 3.3. *The orbits of the canonical subgroups of* $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ *on the plane* $\mathbb{R}^2 = \{\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right] : x, \xi \in \mathbb{R}\}$ *relative to the action (2.1) are:*

dim/n.	*group*	*orbits*
(1.*i*)	K	*circles centered at the origin; the origin;*
(1.*ii*)	A	*branches of hyperbola* $x\xi =$ *cost.; the four half–axes; the origin;*
(1.*iii*)	N	*points on the* x*–axis; horizontal lines;*
(1.*iv*)	P	*parabolas of the form* $x = \frac{1}{2}\xi^2 +$ *cost.;*
(1.*v*)	$\mathbb{R}_q$	*the* x *axis;*
(2.*i*)	NA	*the half–planes* $\xi > 0$ *and* $\xi < 0$*; the two half* x *axes; the origin;*
(2.*ii*)	$\mathbb{R}_q \rtimes A$	*the half–planes* $\xi > 0$ *and* $\xi < 0$*; the* x *axis;*
(2.*iii*)	$\mathbb{R}_q \times N$	*horizontal lines;*
(2.*iv*)	$\mathbb{R}_q \cdot P$	*the plane;*
(2.*v*)	$\mathbb{R}^2$	*the plane;*
(3.*i*)	$\mathbb{R}_q \rtimes NA$	*the half–planes* $\xi > 0$ *and* $\xi < 0$*; the* x *axis;*
(3.*ii*)	$SL(2,\mathbb{R})$	*the punctured plane; the origin;*
(3.*iii*)	$\mathbb{R}^2 \rtimes K$	*the plane;*
(3.*iv*)	$\mathbb{R}^2 \rtimes A$	*the plane;*
(3.*v*)	$\mathbb{R}^2 \times N$	*the plane;*
(4.*i*)	$\mathbb{R}^2 \rtimes NA$	*the plane.*

3.3. **Canonical subalgebras of $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$.** As it is clear from the formula expressing the bracket in $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$, its center is

$$\mathfrak{z} = \left\{ \left((\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], z); 0 \right) : z \in \mathbb{R} \right\}.$$

For a real parameter α, we put

$$\begin{aligned}
\mathfrak{k}_\alpha &= \left\{ \left((\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \alpha t); tJ \right) : t \in \mathbb{R} \right\}; \\
\mathfrak{a}_\alpha &= \left\{ \left((\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \alpha t); tH \right) : t \in \mathbb{R} \right\}; \\
\mathfrak{n}_\alpha &= \left\{ \left((\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \alpha t); tU \right) : t \in \mathbb{R} \right\}; \\
\mathfrak{n} \rtimes \mathfrak{a}_\alpha &= \left\{ \left((\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \alpha t); tH + uU \right) : t, u \in \mathbb{R} \right\}.
\end{aligned}$$

If $\alpha = 0$, the corresponding subscript will be omitted. The first three are obviously abelian, while $\mathfrak{n} \rtimes \mathfrak{a}_\alpha$ is solvable, its derived algebra being $\mathfrak{n}$. Next we consider

$$\begin{aligned}
\mathfrak{z} \times \mathfrak{k} &= \left\{ \left((\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], z); tJ \right) : z, t \in \mathbb{R} \right\}; \\
\mathfrak{z} \times \mathfrak{a} &= \left\{ \left((\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], z); tH \right) : z, t \in \mathbb{R} \right\}; \\
\mathfrak{z} \times \mathfrak{n} &= \left\{ \left((\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], z); tU \right) : z, t \in \mathbb{R} \right\}; \\
\mathfrak{z} \times (\mathfrak{n} \rtimes \mathfrak{a}) &= \left\{ \left((\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], z); tH + uU \right) : z, t, u \in \mathbb{R} \right\}; \\
\mathfrak{z} \times \mathfrak{sl}(2,\mathbb{R}) &= \left\{ \left((\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], z); tH + uU + wJ \right) : z, t, u, w \in \mathbb{R} \right\},
\end{aligned}$$

whose algebraic structure is readily seen: the first three are abelian and the fourth is solvable since the central factor does not play any role; the last one is reductive because it stands as the product of its center and its derived algebra, which is semisimple. Other canonical semidirect products are:

$$\begin{aligned}
&\mathfrak{h}_1 \rtimes \mathfrak{k} = \Big\{\Big(([\begin{smallmatrix} s \\ r \end{smallmatrix}], z); tJ\Big) : s, r, z, t \in \mathbb{R}\Big\};\\
&\mathfrak{h}_1 \rtimes \mathfrak{a} = \Big\{\Big(([\begin{smallmatrix} s \\ r \end{smallmatrix}], z); tH\Big) : s, r, z, t \in \mathbb{R}\Big\};\\
&\mathfrak{h}_1 \rtimes \mathfrak{n} = \Big\{\Big(([\begin{smallmatrix} s \\ r \end{smallmatrix}], z); tU\Big) : s, r, z, t \in \mathbb{R}\Big\};\\
&\mathfrak{h}_1 \rtimes (\mathfrak{n} \rtimes \mathfrak{a}) = \Big\{\Big(([\begin{smallmatrix} s \\ r \end{smallmatrix}], z); tH + uU\Big) : s, r, z, t, u \in \mathbb{R}\Big\}
\end{aligned}$$

for which the algebraic structure is less obvious but follows by simple computations and is stated in Theorem 3.4. By means of the immersion $[\begin{smallmatrix} s \\ r \end{smallmatrix}] \mapsto (([\begin{smallmatrix} s \\ r \end{smallmatrix}], 0); 0)$ of $\mathbb{R}^2$ in $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$ we shall identify $\mathbb{R}_q$ as a subalgebras of $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$. Its direct sum with the center will be denoted for short

$$\mathbb{R}_q Z = \Big\{\Big(([\begin{smallmatrix} s \\ 0 \end{smallmatrix}], z); 0\Big) : s, z \in \mathbb{R}\Big\}.$$

By means of $\mathbb{R}_q$ and $\mathbb{R}_q Z$ we define natural semidirect products

$$\begin{aligned}
&\mathbb{R}_q \rtimes \mathfrak{a}_\alpha = \Big\{\Big(([\begin{smallmatrix} s \\ 0 \end{smallmatrix}], \alpha t); tH\Big) : t, s \in \mathbb{R}\Big\},\ \alpha \in \mathbb{R};\\
&\mathbb{R}_q \times \mathfrak{n}_\alpha = \Big\{\Big(([\begin{smallmatrix} s \\ 0 \end{smallmatrix}], \alpha t); tU\Big) : t, s \in \mathbb{R}\Big\},\ \alpha \in \mathbb{R};\\
&\mathbb{R}_q \rtimes (\mathfrak{n} \rtimes \mathfrak{a}_\alpha) = \Big\{\Big(([\begin{smallmatrix} s \\ 0 \end{smallmatrix}], \alpha t); tH\Big) : t, s \in \mathbb{R}\Big\},\ \alpha \in \mathbb{R};\\
&\mathbb{R}_q Z \rtimes \mathfrak{a} = \Big\{\Big(([\begin{smallmatrix} s \\ 0 \end{smallmatrix}], z); tH\Big) : s, z, t \in \mathbb{R}\Big\};\\
&\mathbb{R}_q Z \times \mathfrak{n} = \Big\{\Big(([\begin{smallmatrix} s \\ 0 \end{smallmatrix}], z); tU\Big) : s, z, t \in \mathbb{R}\Big\};\\
&\mathbb{R}_q Z \rtimes (\mathfrak{n} \rtimes \mathfrak{a}) = \Big\{\Big(([\begin{smallmatrix} s \\ 0 \end{smallmatrix}], z); tH + uU\Big) : s, z, t, u \in \mathbb{R}\Big\}.
\end{aligned}$$

Notice that $\mathfrak{n}$ acts on $\mathbb{R}_q Z$ by zero, so that their semidirect product is actually direct. Finally, the set–theoretic injection $(X, A) \mapsto ((X, 0); A)$ of $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$ maps $\mathfrak{p}$ into a subalgebra of the latter, denoted in the same way. Therefore we write

$$\begin{aligned}
&\mathfrak{z} \times \mathfrak{p} = \Big\{\Big(([\begin{smallmatrix} 0 \\ r \end{smallmatrix}], z); rU\Big) : z, r \in \mathbb{R}\Big\};\\
&\mathbb{R}_q Z \oplus \mathfrak{p} = \Big\{\Big(([\begin{smallmatrix} s \\ r \end{smallmatrix}], z); rU\Big) : s, r, z \in \mathbb{R}\Big\}.
\end{aligned}$$

The notation $\mathbb{R}_q Z \oplus \mathfrak{p}$ reflects the fact that this algebra is the vector space direct sum of $\mathbb{R}_q Z$ and $\mathfrak{p}$ and *not* their direct product as Lie algebras. As for the algebraic structure, observe that $\mathfrak{z} \times \mathfrak{p}$ is abelian while $\mathbb{R}_q Z \oplus \mathfrak{p}$ is nilpotent, because $[\mathbb{R}_q Z \oplus \mathfrak{p}, \mathbb{R}_q Z \oplus \mathfrak{p}] = \mathfrak{z}$. It is isomorphic, but not conjugate, to the Heisenberg algebra $\mathfrak{h}_1$.

The Lie algebras we have introduced, hereafter referred to as canonical, exhaust a list of representatives for the conjugacy classes subalgebras of $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$, as stated in the next classification result.

Theorem 3.4. *Any proper Lie subalgebra $\mathfrak{h}$ of $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$ is conjugate to one of the following types of Lie algebras, listed together with $\mathfrak{h}^{(1)} = \mathfrak{h}_{(1)} = [\mathfrak{h}, \mathfrak{h}]$, $\mathfrak{h}^{(2)} = [\mathfrak{h}^{(1)}, \mathfrak{h}^{(1)}]$, $\mathfrak{h}_{(2)} = [\mathfrak{h}, \mathfrak{h}_{(1)}]$ and their algebraic structure.*

dim/N.	$\mathfrak{h}$	$\mathfrak{h}^{(1)}$	$\mathfrak{h}^{(2)}$	$\mathfrak{h}_{(2)}$	*structure*
$(1.i)$	$\mathfrak{z}$	0	0	0	*abelian*
$(1.ii)$	$\mathfrak{k}_\alpha,\ \alpha\in\mathbb{R}$	0	0	0	*abelian*
$(1.iii)$	$\mathfrak{a}_\alpha,\ \alpha\in\mathbb{R}$	0	0	0	*abelian*
$(1.iv)$	$\mathfrak{n}_\alpha,\ \alpha\in\mathbb{R}$	0	0	0	*abelian*
$(1.v)$	$\mathfrak{p}$	0	0	0	*abelian*
$(1.vi)$	$\mathbb{R}_q$	0	0	0	*abelian*
$(2.i)$	$\mathfrak{z}\times\mathfrak{k}$	0	0	0	*abelian*
$(2.ii)$	$\mathfrak{z}\times\mathfrak{a}$	0	0	0	*abelian*
$(2.iii)$	$\mathfrak{z}\times\mathfrak{n}$	0	0	0	*abelian*
$(2.iv)$	$\mathfrak{n}\rtimes\mathfrak{a}_\alpha,\ \alpha\in\mathbb{R}$	$\mathfrak{n}$	0	$\mathfrak{n}$	*solvable*
$(2.v)$	$\mathbb{R}_q\rtimes\mathfrak{a}_\alpha,\ \alpha\in\mathbb{R}$	$\mathbb{R}_q$	0	$\mathbb{R}_q$	*solvable*
$(2.vi)$	$\mathbb{R}_q\times\mathfrak{n}_\eta,\ \eta\in\{0,\pm1\}$	0	0	0	*abelian*
$(2.vii)$	$\mathfrak{z}\times\mathfrak{p}$	0	0	0	*abelian*
$(2.viii)$	$\mathbb{R}_qZ$	0	0	0	*abelian*
$(3.i)$	$\mathbb{R}_q\rtimes(\mathfrak{n}\rtimes\mathfrak{a}_\alpha),\ \alpha\in\mathbb{R}$	$\mathbb{R}_q\times\mathfrak{n}$	0	$\mathbb{R}_q\times\mathfrak{n}$	*solvable*
$(3.ii)$	$\mathfrak{z}\times(\mathfrak{n}\rtimes\mathfrak{a})$	$\mathfrak{n}$	0	$\mathfrak{n}$	*solvable*
$(3.iii)$	$\mathbb{R}_qZ\rtimes\mathfrak{a}$	$\mathbb{R}_q$	0	$\mathbb{R}_q$	*solvable*
$(3.iv)$	$\mathbb{R}_qZ\times\mathfrak{n}$	0	0	0	*abelian*
$(3.v)$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})$	*semisimple*
$(3.vi)$	$\mathbb{R}_qZ\oplus\mathfrak{p}$	$\mathfrak{z}$	0	0	*nilpotent*
$(3.vii)$	$\mathfrak{h}_1$	$\mathfrak{z}$	0	0	*nilpotent*
$(4.i)$	$\mathfrak{h}_1\rtimes\mathfrak{k}$	$\mathfrak{h}_1$	$\mathfrak{z}$	$\mathfrak{h}_1$	*solvable*
$(4.ii)$	$\mathfrak{h}_1\rtimes\mathfrak{a}$	$\mathfrak{h}_1$	$\mathfrak{z}$	$\mathfrak{h}_1$	*solvable*
$(4.iii)$	$\mathfrak{h}_1\rtimes\mathfrak{n}$	$\mathbb{R}_qZ$	0	$\mathfrak{z}$	*solvable*
$(4.iv)$	$\mathbb{R}_qZ\rtimes(\mathfrak{n}\rtimes\mathfrak{a})$	$\mathbb{R}_q\times\mathfrak{n}$	0	$\mathbb{R}_q\times\mathfrak{n}$	*solvable*
$(4.v)$	$\mathfrak{z}\times\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})$	*reductive*
$(5.i)$	$\mathfrak{h}_1\rtimes(\mathfrak{n}\rtimes\mathfrak{a})$	$\mathfrak{h}_1\rtimes\mathfrak{n}$	$\mathbb{R}_qZ$	$\mathfrak{h}_1\rtimes\mathfrak{n}$	*solvable*

In the first column, the first index denotes dimension. Moreover, whenever a parameter appears, algebras corresponding to distinct parameters are not conjugate.

3.4. **Canonical subgroups of $\mathbb{H}_1\rtimes SL(2,\mathbb{R})$.** First of all, it is clear that

$$Z=\Big\{\Big((0,z);I\Big):z\in\mathbb{R}\Big\}.$$

is the center of $\mathbb{H}_1\rtimes SL(2,\mathbb{R})$. The groups

$$K_\alpha=\Big\{\Big((0,t\alpha);k_t\Big):t\in\mathbb{R}\Big\},\ \alpha\in\mathbb{R};$$
$$A_\alpha=\Big\{\Big((0,t\alpha);a_t\Big):t\in\mathbb{R}\Big\},\ \alpha\in\mathbb{R};$$
$$N_\alpha=\Big\{\Big((0,t\alpha);u_t\Big):t\in\mathbb{R}\Big\},\ \alpha\in\mathbb{R};$$
$$NA_\alpha=\Big\{\Big((0,s\alpha),u_ta_s\Big):s,t\in\mathbb{R}\Big\},\ \alpha\in\mathbb{R}.$$

correspond to K, A, N and NA for $\alpha=0$, in which case the subscript will be omitted. Some groups arise as direct products of the form $Z\times H$, where H is a subgroup of $SL(2,\mathbb{R})$, like $Z\times K$, $Z\times A$, $Z\times N$, but also, for example $Z\times\Delta$, $Z\times\Delta^0$. Lie groups containing semidirect factors like $\mathbb{R}_q$ or $\mathbb{R}_qZ$ are written in the natural way, namely

$$\mathbb{R}_q \rtimes A_\alpha = \left\{ \left(([\begin{smallmatrix} q \\ 0 \end{smallmatrix}], t\alpha), a_t \right) : q, t \in \mathbb{R} \right\}, \ \alpha \in \mathbb{R};$$
$$\mathbb{R}_q \times N_\alpha = \left\{ \left(([\begin{smallmatrix} q \\ 0 \end{smallmatrix}], t\alpha), u_t \right) : q, t \in \mathbb{R} \right\}, \ \alpha \in \mathbb{R};$$
$$\mathbb{R}_q \rtimes (NA_\alpha) = \left\{ \left(([\begin{smallmatrix} q \\ 0 \end{smallmatrix}], s\alpha), u_t a_s \right) : q, s, t \in \mathbb{R} \right\}, \ \alpha \in \mathbb{R};$$
$$\mathbb{R}_q Z \times N = \left\{ \left(([\begin{smallmatrix} q \\ 0 \end{smallmatrix}], z); u_t \right) : q, t, z \in \mathbb{R} \right\};$$
$$\mathbb{R}_q Z \rtimes NA = \left\{ \left(([\begin{smallmatrix} q \\ 0 \end{smallmatrix}], z); u_t d_{s^{1/2}} \right) : q, t, z \in \mathbb{R}, s > 0 \right\};$$
$$\mathbb{R}_q Z \rtimes \pm N = \left\{ \left(([\begin{smallmatrix} q \\ 0 \end{smallmatrix}], z); n \right) : q, z \in \mathbb{R}, n \in \pm N \right\};$$
$$\mathbb{R}_q Z \rtimes \pm NA = \left\{ \left(([\begin{smallmatrix} q \\ 0 \end{smallmatrix}], z); b \right) : q, z \in \mathbb{R}, b \in \pm NA \right\}.$$

The last groups we need arise by starting from the Lie group P whose Lie algebra is $\mathfrak{p} \subset \mathfrak{h}_1 \rtimes \mathfrak{sl}(2, \mathbb{R})$ (see section 3.3) and successively taking normalizers in $\mathbb{H}_1 \rtimes SL(2, \mathbb{R})$. They correspond to the algebras $\mathfrak{p}$, $\mathfrak{z} \times \mathfrak{p}$ and $\mathbb{R}_q Z \oplus \mathfrak{p}$, respectively, that is

$$P = \left\{ \left((\left[\begin{smallmatrix} \frac{1}{2}t^2 \\ t \end{smallmatrix} \right], \tfrac{1}{12}t^3), u_t \right) : t \in \mathbb{R} \right\};$$
$$Z \times P = \left\{ \left((\left[\begin{smallmatrix} \frac{1}{2}t^2 \\ t \end{smallmatrix} \right], z + \tfrac{1}{12}t^3), u_t \right) : t, z \in \mathbb{R} \right\} = N(P);$$
$$\mathbb{R}_q Z \cdot P = \left\{ \left(([\begin{smallmatrix} q \\ t \end{smallmatrix}], z), u_t \right) : q, t, z \in \mathbb{R} \right\} = N(Z \times P).$$

It should be observed that the one–dimensional group P written above projects onto the subgroup $P \subset \mathbb{R}^2 \rtimes SL(2, \mathbb{R})$ considered in section 3.2 under the quotient map modulo the center Z. Indeed, $\mathbb{H}_1 \rtimes SL(2, \mathbb{R})/Z \simeq \mathbb{R}^2 \rtimes SL(2, \mathbb{R})$. We are now in a position for stating the main classification result at the level of groups.

Theorem 3.5. *Any connected Lie subgroup of $\mathbb{H}_1 \rtimes SL(2, \mathbb{R})$ is conjugate to one of the following non–conjugate subgroups, listed together with their normalizers $N(\cdot)$ and centralizers $Z(\cdot)$.*

dim/n.	*group*	$N(\cdot)$	$Z(\cdot)$
$(1.i)$	Z	G	G
$(1.ii)$	$K_\alpha, \ \alpha \in \mathbb{R}$	$Z \times K$	$Z \times K$
$(1.iii)$	$A_\alpha, \ \alpha \in \mathbb{R} \setminus \{0\}$	$Z \times \Delta$	$Z \times \Delta$
$(1.iv)$	A	$Z \times \Delta^0$	$Z \times \Delta$
$(1.v)$	$N_\alpha, \ \alpha \in \mathbb{R} \setminus \{0\}$	$\mathbb{R}_q Z \rtimes \pm N$	$\mathbb{R}_q Z \rtimes \pm N$
$(1.vi)$	N	$\mathbb{R}_q Z \rtimes \pm NA$	$\mathbb{R}_q Z \rtimes \pm N$
$(1.vii)$	P	$Z \times P$	$Z \times P$
$(1.viii)$	$\mathbb{R}_q$	$\mathbb{R}_q Z \rtimes \pm NA$	$\mathbb{R}_q Z \times N$
$(2.i)$	$Z \times K$	$Z \times K$	$Z \times K$
$(2.ii)$	$Z \times A$	$Z \times \Delta^0$	$Z \times \Delta$
$(2.iii)$	$Z \times N$	$\mathbb{R}_q Z \times \pm NA$	$\mathbb{R}_q Z \times \pm N$
$(2.iv)$	$NA_\alpha, \alpha \in \mathbb{R}$	$Z \times \pm NA$	$Z \times \pm I$
$(2.v)$	$\mathbb{R}_q \rtimes A_\alpha, \ \alpha \in \mathbb{R}$	$\mathbb{R}_q Z \rtimes \Delta$	Z
$(2.vi)$	$\mathbb{R}_q \times N$	$\mathbb{R}_q Z \rtimes \pm NA$	$\mathbb{R}_q Z \times N$
$(2.vii)$	$\mathbb{R}_q \times N_\eta, \ \eta \in \{\pm 1\}$	$\mathbb{R}_q Z \rtimes \pm N$	$\mathbb{R}_q Z \times N$
$(2.viii)$	$Z \times P$	$\mathbb{R}_q Z \cdot P$	$Z \times P$
$(2.ix)$	$\mathbb{R}_q Z$	$\mathbb{H}_1 \rtimes \pm NA$	$\mathbb{R}_q Z \times N$

$(3.i)$	$\mathbb{R}_q \rtimes (NA_\alpha),\ \alpha \in \mathbb{R}$	$\mathbb{R}_q Z \rtimes \pm NA$	Z
$(3.ii)$	$Z \times NA$	$Z \times \pm NA$	$Z \times \pm I$
$(3.iii)$	$\mathbb{R}_q Z \rtimes A$	$\mathbb{R}_q Z \rtimes \Delta$	Z
$(3.iv)$	$\mathbb{R}_q Z \times N$	$\mathbb{H}_1 \rtimes \pm NA$	$\mathbb{R}_q Z \times N$
$(3.v)$	$SL(2,\mathbb{R})$	$Z \times SL(2,\mathbb{R})$	$Z \times \pm I$
$(3.vi)$	$\mathbb{R}_q Z \cdot P$	$\mathbb{H}_1 \rtimes N$	Z
$(3.vii)$	$\mathbb{H}_1$	G	Z
$(4.i)$	$\mathbb{H}_1 \rtimes K$	$\mathbb{H}_1 \rtimes K$	Z
$(4.ii)$	$\mathbb{H}_1 \rtimes A$	$\mathbb{H}_1 \rtimes \Delta^0$	Z
$(4.iii)$	$\mathbb{H}_1 \rtimes N$	$\mathbb{H}_1 \rtimes \pm NA$	Z
$(4.iv)$	$\mathbb{R}_q Z \rtimes NA$	$\mathbb{R}_q Z \rtimes \pm NA$	Z
$(4.v)$	$Z \times SL(2,\mathbb{R})$	$Z \times SL(2,\mathbb{R})$	$Z \times \pm I$
$(5.i)$	$\mathbb{H}_1 \rtimes NA$	$\mathbb{H}_1 \rtimes \pm NA$	Z

In the first column, the first index denotes dimension.

4. Proofs

4.1. **Proof of Theorem 3.1.** In the course of the proof of Theorem 3.1 we shall use Proposition 4.3 below, a variant of the next well–known result.

Proposition 4.1. *Any non–trivial subalgebra of* $\mathfrak{sl}(2,\mathbb{R})$ *is conjugate to either* $\mathfrak{n}$, $\mathfrak{a}$, $\mathfrak{k}$ *or* $\mathfrak{n} \rtimes \mathfrak{a}$.

Recall that the adjoint action of $(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], g) \in \mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ on $(X, A) \in \mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$ is given by

$$\mathrm{Ad}(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], g)(X, A) = (gX - gAg^{-1}\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], gAg^{-1}).$$

Proposition 4.2. *Any subalgebra* $\mathfrak{h}$ *of* $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$ *is conjugate to a subalgebra* $\bar{\mathfrak{h}}$ *for which* $\pi_2(\bar{\mathfrak{h}}) \in \Big\{\{0\}, \mathfrak{n}, \mathfrak{a}, \mathfrak{k}, \mathfrak{n} \rtimes \mathfrak{a}, \mathfrak{sl}(2,\mathbb{R})\Big\}$, *where* π_2 *is the projection* $\pi_2(X, A) = A$.

Proof. Observe that π_2 is a Lie algebra homomorphism intertwining the adjoint actions of $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ and $SL(2,\mathbb{R})$, and apply Proposition 4.1. □

We shall prove that any subalgebra $\mathfrak{h}$ of $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$ is conjugate to one of the listed ones. The fact that they are mutually non–conjugate follows from Theorem 3.2, where it is proved that the same property holds for the corresponding connected subgroups by inspecting the structures of their orbits in the time–frequency plane. We shall not write the explicit calculations leading to the description of $\mathfrak{h}^{(1)} = \mathfrak{h}_{(1)} = [\mathfrak{h}, \mathfrak{h}]$, $\mathfrak{h}^{(2)} = [\mathfrak{h}^{(1)}, \mathfrak{h}^{(1)}]$ and $\mathfrak{h}_{(2)} = [\mathfrak{h}, \mathfrak{h}_{(1)}]$, nor shall we prove the assertions concerning the algebraic structures because these are all straightforward matters.

CASE 1: $\dim \mathfrak{h} = 1$. In view of Proposition 4.3 we may assume that

$$\pi_2(\mathfrak{h}) \in \Big\{\{0\}, \mathfrak{n}, \mathfrak{a}, \mathfrak{k}\Big\}.$$

• If $\pi_2(\mathfrak{h}) = \{0\}$, then $\mathfrak{h} \subset \Big\{(\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right], 0) : r, s \in \mathbb{R}\Big\}$ is a one–dimensional subspace of $\mathbb{R}^2$ so that $\mathfrak{h} = \mathbb{R}_q$ up to the linear action of $SL(2,\mathbb{R})$, that is up to conjugation.

• Assume next $\pi_2(\mathfrak{h}) \in \{\mathfrak{n}, \mathfrak{a}, \mathfrak{k}\}$, and let A be either H, U or J. Then there exists $(\alpha,\beta) \in \mathbb{R}^2$ such that

$$\mathfrak{h} = \left\{ \left(\left[\begin{smallmatrix} \alpha t \\ \beta t \end{smallmatrix} \right], tA \right) : t \in \mathbb{R} \right\}.$$

◇ If $\alpha = \beta = 0$ then $\mathfrak{h}$ is either $\mathfrak{a}$, $\mathfrak{k}$ or $\mathfrak{n}$.

◇ Suppose next $\alpha = 0$ and $\beta \neq 0$. If $A = U$, the group element

$$g = \left(\left[\begin{smallmatrix} 0 \\ \beta^{1/3} \end{smallmatrix} \right], \left[\begin{smallmatrix} \beta^{1/3} & 1 \\ 0 & \beta^{-1/3} \end{smallmatrix} \right] \right)$$

conjugates $\mathfrak{h}$ to $\mathfrak{p}$. If $A = H$, then $g = (\left[\begin{smallmatrix} 0 \\ \beta \end{smallmatrix} \right], I)$ conjugates $\mathfrak{h}$ to $\mathfrak{a}$. If $A = J$, then $g = (\left[\begin{smallmatrix} -\beta \\ 0 \end{smallmatrix} \right], I)$ conjugates $\mathfrak{h}$ to $\mathfrak{k}$.

◇ Finally, suppose $\alpha \neq 0$ and $\beta = 0$. If $A = U$, then $g = (\left[\begin{smallmatrix} 0 \\ \alpha \end{smallmatrix} \right], I)$ conjugates $\mathfrak{h}$ to $\mathfrak{n}$. If $A = H$, $g = (\left[\begin{smallmatrix} \alpha \\ 0 \end{smallmatrix} \right], I)$ conjugates $\mathfrak{h}$ to $\mathfrak{a}$. If $A = J$, $g = (\left[\begin{smallmatrix} 0 \\ \alpha \end{smallmatrix} \right], I)$ conjugates $\mathfrak{h}$ to $\mathfrak{k}$.

<u>CASE 2:</u> $\dim \mathfrak{h} = 2$. In view of Proposition 4.3 we may assume that

$$\pi_2(\mathfrak{h}) \in \left\{ \{0\}, \mathfrak{n}, \mathfrak{a}, \mathfrak{k}, \mathfrak{n} \rtimes \mathfrak{a} \right\}.$$

• The case $\pi_2(\mathfrak{h}) = \{0\}$ is trivial and yields $\mathfrak{h} = \mathbb{R}^2$.

• Assume next $\pi_2(\mathfrak{h}) \in \{\mathfrak{n}, \mathfrak{a}, \mathfrak{k}\}$ and let A be either H, U or J. Here two cases arise:

(a) $\mathfrak{h}$ contains the line $\{(0, tA) : t \in \mathbb{R}\}$;

(b) $\mathfrak{h}$ does not contain the line $\{(0, tA) : t \in \mathbb{R}\}$.

In case (a) there exists a non–zero $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\mathfrak{h} = \left\{ \left(u \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right], tA \right) : t, u \in \mathbb{R} \right\}.$$

The bracket of two elements in $\mathfrak{h}$ takes the form

$$[\left(u \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right], tA \right), \left(u' \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right], t'A \right)] = \left((tu' - t'u) A \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right], 0 \right)$$

and the set $\left\{ (u \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right], tA) : t, u \in \mathbb{R} \right\}$ is a Lie algebra if and only if (α, β) is an eigenvector of A. If $A = U$, then $(\alpha, \beta) = \lambda(1, 0)$ for some $\lambda \neq 0$ and we obtain $\mathbb{R}_q \times \mathfrak{n}$. If $A = H$ then either $\mathfrak{h} = \{(\left[\begin{smallmatrix} s \\ 0 \end{smallmatrix} \right], tH) : t, s \in \mathbb{R}\}$ or $\mathfrak{h} = \left\{(\left[\begin{smallmatrix} 0 \\ s \end{smallmatrix} \right], tH) : t, s \in \mathbb{R}\right\}$. The first case is $\mathbb{R}_q \rtimes \mathfrak{a}$ and the second reduces to $\mathbb{R}_q \rtimes \mathfrak{a}$ by conjugating with $g = (0, J)$. The case $A = J$ cannot occur because J has no non–zero eigenvectors.

In case (b), for some linear functional $\tau(s, r)$ we have

$$\mathfrak{h} = \left\{ \left(\left[\begin{smallmatrix} s \\ r \end{smallmatrix} \right], \tau(s, r) A \right) : s, r \in \mathbb{R} \right\}.$$

The bracket of two elements in $\mathfrak{h}$ takes the form

$$[\left(\left[\begin{smallmatrix} s \\ r \end{smallmatrix} \right], \tau(s, r) A \right), \left(\left[\begin{smallmatrix} s' \\ r' \end{smallmatrix} \right], \tau(s', r') A \right)] = \left(\tau(s, r) A \left[\begin{smallmatrix} s' \\ r' \end{smallmatrix} \right] - \tau(s', r') A \left[\begin{smallmatrix} s \\ r \end{smallmatrix} \right], 0 \right)$$

and the set $\{(\left[\begin{smallmatrix} s \\ r \end{smallmatrix} \right], \tau(s, r) A) : s, r \in \mathbb{R}\}$ is a Lie algebra if and only if

$$\tau(s, r) \tau(A \left[\begin{smallmatrix} s' \\ r' \end{smallmatrix} \right]) - \tau(s', r') \tau(A \left[\begin{smallmatrix} s \\ r \end{smallmatrix} \right]) = 0$$

for all $s, r, s', r' \in \mathbb{R}$. This condition is equivalent to

$$v_\tau \otimes {}^t A v_\tau = {}^t A v_\tau \otimes v_t,$$

where v_τ is the vector representing the functional $\tau(\cdot,\cdot)$, that is $\tau(s,r) = \langle [\begin{smallmatrix} s \\ r \end{smallmatrix}], v_\tau \rangle$. The latter condition, in turn, holds if and only if v_τ is an eigenvector of ${}^tA$. If $A = U$, then $\mathfrak{h} = \{([\begin{smallmatrix} s \\ r \end{smallmatrix}], \alpha r U) : s, r \in \mathbb{R}\}$ for some $\alpha \neq 0$, and $g = (0, \left[\begin{smallmatrix} \alpha^{-1/3} & 0 \\ 0 & \alpha^{1/3} \end{smallmatrix}\right])$ conjugates $\mathfrak{h}$ to $\mathbb{R}_q \oplus \mathfrak{p}$. If $A = H$, then either $\mathfrak{h} = \{([\begin{smallmatrix} s \\ r \end{smallmatrix}], \alpha r H) : s, r \in \mathbb{R}\}$ or $\mathfrak{h} = \{([\begin{smallmatrix} s \\ r \end{smallmatrix}], \alpha s H) : s, r \in \mathbb{R}\}$ for some $\alpha \neq 0$.Take $g = (\left[\begin{smallmatrix} 0 \\ -\alpha^{-1} \end{smallmatrix}\right], I)$ in the first case and $g = (\left[\begin{smallmatrix} 0 \\ -\alpha^{-1} \end{smallmatrix}\right], J)$ in the second case to conjugate $\mathfrak{h}$ to $\mathbb{R}_q \rtimes \mathfrak{a}$. Again, the case $A = J$ cannot occur.

- Now we consider the case $\pi_2(\mathfrak{h}) = \mathfrak{n} \rtimes \mathfrak{a}$. We can represent $\mathfrak{h}$ in the form

$$\mathfrak{h} = \left\{ \left(\left[\begin{smallmatrix} \sigma(t,u) \\ \rho(t,u) \end{smallmatrix}\right], tH + uU \right) : t, u \in \mathbb{R} \right\},$$

where $\sigma(t,u)$ and $\rho(t,u)$ are linear functionals on $\mathbb{R}^2$. The bracket of two elements in $\mathfrak{h}$ takes the form

$$\begin{aligned} &\left[\left(\left[\begin{smallmatrix} \sigma(t,u) \\ \rho(t,u) \end{smallmatrix}\right], tH + uU \right), \left(\left[\begin{smallmatrix} \sigma(t',u') \\ \rho(t',u') \end{smallmatrix}\right], t'H + u'U \right)\right] \\ &= \left((tH + uU) \left[\begin{smallmatrix} \sigma(t',u') \\ \rho(t',u') \end{smallmatrix}\right] - (t'H + u'U) \left[\begin{smallmatrix} \sigma(t,u) \\ \rho(t,u) \end{smallmatrix}\right], 2(tu' - t'u)U \right), \end{aligned}$$

so that the condition

$$(tH + uU) \begin{bmatrix} \sigma(t',u') \\ \rho(t',u') \end{bmatrix} - (t'H + u'U) \begin{bmatrix} \sigma(t,u) \\ \rho(t,u) \end{bmatrix} = \begin{bmatrix} \sigma(0, 2(tu' - t'u)) \\ \rho(0, 2(tu' - t'u)) \end{bmatrix} \tag{4.1}$$

must be satisfied for all $t, u, t', u' \in \mathbb{R}$. Let $v_\rho = (v_1, v_2)$ denote the vector representing ρ; equating the second coordinates in (4.1) forces $v_2 = 0$. Thus

$$\mathfrak{h} = \left\{ \left(\left[\begin{smallmatrix} \sigma(t,u) \\ v_1 t \end{smallmatrix}\right], tH + uU \right) : t, u \in \mathbb{R} \right\}.$$

Conjugating with $g = ([\begin{smallmatrix} 0 \\ -v_1 \end{smallmatrix}], I)$ leads to

$$\left\{ \left(\left[\begin{smallmatrix} \tilde\sigma(t,u) \\ 0 \end{smallmatrix}\right], tH + uU \right) : t, u \in \mathbb{R} \right\}$$

with a new functional $\tilde\sigma$. Let $v_{\tilde\sigma} = (s_1, s_2)$ denote the vector representing $\tilde\sigma$. Equating now the first coordinates in (4.1) forces $s_2 = 0$. Conjugating with $g = ([\begin{smallmatrix} s_1 \\ 0 \end{smallmatrix}], I)$ the algebra $\{([\begin{smallmatrix} s_1 t \\ 0 \end{smallmatrix}], tH + uU) : t, u \in \mathbb{R}\}$ leads to $\mathfrak{n} \rtimes \mathfrak{a}$.

CASE 3: $\dim \mathfrak{h} = 3$. In view of Proposition 4.3 we may assume that

$$\pi_2(\mathfrak{h}) \in \left\{ \mathfrak{n}, \mathfrak{a}, \mathfrak{k}, \mathfrak{n} \rtimes \mathfrak{a}, \mathfrak{sl}(2, \mathbb{R}) \right\}.$$

- The cases $\pi_2(\mathfrak{h}) \in \{\mathfrak{n}, \mathfrak{a}, \mathfrak{k}\}$ are trivial and lead to $\mathbb{R}^2 \rtimes \mathfrak{n}$, $\mathbb{R}^2 \rtimes \mathfrak{a}$ and $\mathbb{R}^2 \rtimes \mathfrak{n}$, respectively.

- Assume next $\pi_2(\mathfrak{h}) = \mathfrak{n} \rtimes \mathfrak{a}$. Here two cases arise:
  (a) $\mathfrak{h}$ contains the line $\{(0, tH) : t \in \mathbb{R}\}$;
  (b) $\mathfrak{h}$ does not contain the line $\{(0, tH) : t \in \mathbb{R}\}$.

In case (a), either

$$\mathfrak{h} = \left\{ \left(w \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right], tH + uU \right) : t, u, w \in \mathbb{R} \right\}$$

for some non–zero vector $[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}] \in \mathbb{R}^2$ or

$$\mathfrak{h} = \left\{ \left([\begin{smallmatrix} s \\ r \end{smallmatrix}], tH + \omega(s,r)U \right) : r, s, t \in \mathbb{R} \right\}$$

for some non-zero linear functional ω on $\mathbb{R}^2$. The set $\{(w\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right], tH+uU) : t,u,w\in\mathbb{R}\}$ is a Lie algebra if and only if the vector $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ is a simultaneous eigenvector of the matrices $\left[\begin{smallmatrix}t & u\\ 0 & -t\end{smallmatrix}\right]$ for all $t,u\in\mathbb{R}$. This implies $\beta=0$ and $\mathfrak{h}=\mathbb{R}_q\rtimes(\mathfrak{n}\rtimes\mathfrak{a})$. On the other hand, the set $\{(\left[\begin{smallmatrix}s\\ r\end{smallmatrix}\right], tH+\omega(s,r)U) : r,s,t\in\mathbb{R}\}$ is a Lie algebra if and only if

$$\omega\Big((tH+\omega(s,r)U)\left[\begin{smallmatrix}s'\\ r'\end{smallmatrix}\right]-(tH+\omega(s',r')U)\left[\begin{smallmatrix}s\\ r\end{smallmatrix}\right]\Big)=2\Big(t\omega(s',r')-t'\omega(s,r)\Big)$$

for all $r,s,t,r',s',t'\in\mathbb{R}$. Taking $t=t'=0$ we obtain

$$v_\omega\otimes{}^tUv_\omega={}^tUv_\omega\otimes v_\omega,$$

where v_ω represents ω. Hence $v_\omega=0$, so that this case does not occur.

In case (b), for some linear functional τ on $\mathbb{R}^3$

$$\mathfrak{h}=\Big\{\Big(\left[\begin{smallmatrix}s\\ r\end{smallmatrix}\right],\tau(s,r,u)H+uU\Big) : r,s,u\in\mathbb{R}\Big\}$$

and this set is a Lie algebra if and only if for all $r,s,u,r',s',u'\in\mathbb{R}$

$$\tau\Big((\tau(s,r,u)H+uU)\left[\begin{smallmatrix}s'\\ r'\end{smallmatrix}\right]-(\tau(s',r',u')H+u'U)\left[\begin{smallmatrix}s\\ r\end{smallmatrix}\right],2(\tau(s,r,u)u'-\tau(s',r',u')u)\Big)=0.$$

Let $v_\tau=(t_1,t_2,t_3)\in\mathbb{R}^3$ represent τ. For $u=u'=0$ the above relation reduces to

$$v_{\tilde\tau}\otimes{}^tHv_{\tilde\tau}={}^tHv_{\tilde\tau}\otimes v_{\tilde\tau},$$

where $\tilde\tau(s,r)=\tau(s,r,0)$. Thus $v_{\tilde\tau}=(t_1,t_2)$ is an eigenvector of ${}^tH=H$ and either $t_2=0$ or $t_1=0$. A direct computation shows that if $t_2=0$ then also $t_1=0$. This is not possible because $\dim\pi_2(\mathfrak{h})=2$. Next, one shows that if $t_1=0$, then $t_2t_3=0$. But since t_2 cannot vanish, necessarily $t_3=0$. Hence

$$\mathfrak{h}=\Big\{\Big(\left[\begin{smallmatrix}s\\ r\end{smallmatrix}\right],t_2rH+uU\Big) : r,s,u\in\mathbb{R}\Big\}$$

and by conjugating with dilations we may assume $t_2=1$. A final conjugation with $g=(\left[\begin{smallmatrix}0\\ -1\end{smallmatrix}\right],I)$ gives $\mathfrak{h}=\mathbb{R}_q\rtimes(\mathfrak{n}\rtimes\mathfrak{a})$.

- Finally, assume $\pi_2(\mathfrak{h})=\mathfrak{sl}(2,\mathbb{R})$. Then, for some linear functionals σ and ρ on $\mathbb{R}^3$

$$\mathfrak{h}=\Big\{\Big(\left[\begin{smallmatrix}\sigma(t,u,w)\\ \rho(t,u,w)\end{smallmatrix}\right],tH+uU+wJ\Big) : t,u,w\in\mathbb{R}\Big\}.$$

Now, since $\mathfrak{n}\rtimes\mathfrak{a}$ is a subalgebra of $\mathfrak{sl}(2,\mathbb{R})$, necessarily

$$\Big\{\Big(\left[\begin{smallmatrix}\sigma(t,u,0)\\ \rho(t,u,0)\end{smallmatrix}\right],tH+uU\Big) : t,u\in\mathbb{R}\Big\}$$

is a subalgebra of $\mathbb{R}^2\rtimes\mathfrak{sl}(2,\mathbb{R})$. But this is precisely what we had in the case $\dim\mathfrak{h}=2$ and $\pi_2(\mathfrak{h})=\mathfrak{n}\rtimes\mathfrak{a}$, Therefore we may assume that both $(t,u)\mapsto\sigma(t,u,0)$ and $(t,u)\mapsto\rho(t,u,0)$ are the zero functional and there must exist $(\alpha,\beta)\in\mathbb{R}^2$ such that

$$\mathfrak{h}=\Big\{\Big(\left[\begin{smallmatrix}\alpha w\\ \beta w\end{smallmatrix}\right],tH+uU+wJ\Big) : t,u,w\in\mathbb{R}\Big\}.$$

As easily seen, the above set is closed under taking brackets if and only if

$$(tH+uU+wJ)\left[\begin{smallmatrix}\alpha w'\\ \beta w'\end{smallmatrix}\right]-(t'H+u'U+w'J)\left[\begin{smallmatrix}\alpha w\\ \beta w\end{smallmatrix}\right]=\left[\begin{smallmatrix}-2\alpha(tw'-t'w)\\ -2\beta(tw'-t'w)\end{smallmatrix}\right]$$

for all $t,u,w,t',u',w'\in\mathbb{R}$. It is now straightforward to check that this may happen if and only if $\alpha=\beta=0$. Hence $\mathfrak{h}=\mathfrak{sl}(2,\mathbb{R})$.

CASE 4: $\dim \mathfrak{h} = 4$. In view of Proposition 4.3 we may assume that

$$\pi_2(\mathfrak{h}) \in \Big\{\mathfrak{n} \rtimes \mathfrak{a}, \mathfrak{sl}(2,\mathbb{R})\Big\}.$$

- The case $\pi_2(\mathfrak{h}) = \mathfrak{n} \rtimes \mathfrak{a}$ is trivial and leads to $\mathbb{R}^2 \rtimes (\mathfrak{n} \rtimes \mathfrak{a})$.
- Assume $\pi_2(\mathfrak{h}) = \mathfrak{sl}(2,\mathbb{R})$. Then, for some linear functionals σ and ρ on $\mathbb{R}^4$

$$\mathfrak{h} = \Big\{\Big(\Big[\begin{smallmatrix} \sigma(s,t,u,w) \\ \rho(s,t,u,w) \end{smallmatrix}\Big], tH + uU + wJ\Big) : s,t,u,w \in \mathbb{R}\Big\}.$$

Arguing as in the previous case, the set

$$\Big\{\Big(\Big[\begin{smallmatrix} \sigma(s,t,u,0) \\ \rho(s,t,u,0) \end{smallmatrix}\Big], tH + uU\Big) : s,t,u \in \mathbb{R}\Big\}$$

must also be a subalgebra of $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$. This algebra is three dimensional and projects under π_2 onto $\mathfrak{n} \rtimes \mathfrak{a}$. Therefore, arguing as we did for this case, we may assume that for some real number ρ_0 and some functional $\tilde{\sigma}$ on $\mathbb{R}^2$

$$\mathfrak{h} = \Big\{\Big(\Big[\begin{smallmatrix} \tilde{\sigma}(s,w) \\ \rho_0 w \end{smallmatrix}\Big], tH + uU + wJ\Big) : s,t,u,w \in \mathbb{R}\Big\}.$$

Conjugating with $g = \big(\big[\begin{smallmatrix} -\rho_0 \\ 0 \end{smallmatrix}\big], I\big)$ we obtain

$$\mathfrak{h} = \Big\{\Big(\Big[\begin{smallmatrix} \sigma^\dagger(s,t,w) \\ 0 \end{smallmatrix}\Big], tH + uU + wJ\Big) : s,t,u,w \in \mathbb{R}\Big\}$$

for some new functional $\sigma^\dagger$ on $\mathbb{R}^3$. Finally,

$$\Big[\Big(\Big[\begin{smallmatrix} \sigma^\dagger(s,t,w) \\ 0 \end{smallmatrix}\Big], tH + uU + wJ\Big), \Big(\Big[\begin{smallmatrix} \sigma^\dagger(s',t',w') \\ 0 \end{smallmatrix}\Big], t'H + u'U + w'J\Big)\Big] = \\ \Big(\Big[\begin{smallmatrix} t\sigma^\dagger(s',t',w') - t'\sigma^\dagger(s,t,w) \\ w'\sigma^\dagger(s,t,w) - w\sigma^\dagger(s',t',w') \end{smallmatrix}\Big], [tH + uU + wJ, t'H + u'U + w'J]\Big)$$

shows that $w'\sigma^\dagger(s,t,w) - w\sigma^\dagger(s',t',w') = 0$ for all $s,t,w,s',t',w' \in \mathbb{R}$. Choosing $w = w' = 1$ implies that $(s,t) \mapsto \sigma^\dagger(s,t,1)$ is the zero functional and therefore $\sigma^\dagger$ depends only on w. But then $\mathfrak{h}$ would be three dimensional, a contradiction. Hence this case does not occur.

Since there are no other cases, the proof is complete. □

4.2. **Proof of Theorems 3.2 and 3.3.** Theorem 3.3 follows from direct computations, all of which are elementary. As for Theorem 3.2, the list follows from Theorem 3.1 by taking exponentials, and the structure of normalizers and centralizers may be established by straightforward calculations. It only remains to be proved that all canonical subgroups are mutually non–conjugate, where conjugation in the group, namely $i_z y = zyz^{-1}$, $z, y \in \mathbb{R}^2 \rtimes SL(2,\mathbb{R})$, is given by

$$i_{([\begin{smallmatrix} q \\ p \end{smallmatrix}], g)}\big(\big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big], h\big) = \big((I - ghg^{-1})\big[\begin{smallmatrix} q \\ p \end{smallmatrix}\big] + g\big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big], ghg^{-1}\big).$$

In particular, the $SL(2,\mathbb{R})$–component is given by the usual matrix conjugation. Let $z \in \mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ and denote by $\alpha_z : \mathbb{R}^2 \to \mathbb{R}^2$ the action (2.1). If $\big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] \in \mathbb{R}^2$ and H is a subgroup of $\mathbb{R}^2 \rtimes SL(2,\mathbb{R})$, denote by $\mathcal{O}^H_{[\begin{smallmatrix} x \\ \xi \end{smallmatrix}]}$ the orbit

$$\mathcal{O}^H_{[\begin{smallmatrix} x \\ \xi \end{smallmatrix}]} = \big\{\alpha_h\big(\big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big]\big) : h \in H\big\}.$$

If H and H' are conjugate subgroups, that is $H' = i_z(H)$ for some $z \in \mathbb{R}^2 \rtimes SL(2,\mathbb{R})$, then the following obvious relation holds:

$$\alpha_z(\mathcal{O}^H_{\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]}) = \mathcal{O}^{H'}_{\alpha_z(\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right])}.$$

Thus conjugate groups have the same sets of orbits. This fact, together with the observation that conjugate groups must share dimension and algebraic structure, and must have conjugate $SL(2,\mathbb{R})$–components, shows, by simple inspection, that all canonical subgroups are mutually non–conjugate.

4.3. **Proof of Theorem 3.4.** The projection $\pi : \mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R}) \to \mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$ given by $((X,z);A) \mapsto (X;A)$ shows that $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})/\mathfrak{z}$ is isomorphic to $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$. This is the key observation for the proof, in the form stated in following simple result.

Proposition 4.3. *Any subalgebra $\mathfrak{h}$ of $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$ is conjugate to a subalgebra $\bar{\mathfrak{h}}$ for which $\pi(\bar{\mathfrak{h}})$ is a canonical subalgebra of $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$. Two subalgebras of $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$ are conjugate only if they project onto conjugate subalgebras in $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$.*

Proof. Let $P : \mathbb{H}_1 \rtimes SL(2,\mathbb{R}) \to \mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ denote the projection $((Y,s);g) \mapsto (Y;g)$. Since $dP = \pi$, or simply from (2.2), it follows that for $X \in \mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$ and $g \in \mathbb{H}_1 \rtimes SL(2,\mathbb{R})$ we have $\mathrm{Ad}(Pg)(\pi(X)) = \pi(\mathrm{Ad}\, gX)$. Let now Pg be the group element that conjugates $\pi(\mathfrak{h})$ to a canonical subalgebra of $\mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$. Then $\bar{\mathfrak{h}} = \mathrm{Ad}\, g(\mathfrak{h})$ satisfies the first assertion. If $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are two conjugate subalgebras in $\mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$, that is $\mathrm{Ad}\, g(\mathfrak{q}_1) = \mathfrak{q}_2$ for some $g \in \mathbb{H}_1 \rtimes SL(2,\mathbb{R})$, then $\mathrm{Ad}(Pg)(\pi(\mathfrak{q}_1)) = \pi(\mathfrak{q}_2)$, whence the second assertion □

Thus we may assume that $\pi(\mathfrak{h})$ is canonical. In the course of the proof we see that at most two canonical subalgebras project to the same canonical algebra. In any such case the two algebras have different dimensions and cannot be conjugate. This establishes the fact that all algebras appearing in the list are mutually non–conjugate.

CASE 1: $\pi(\mathfrak{h}) = \{0\}$. Clearly $\mathfrak{h} = \mathfrak{z}$.

CASE 2: $\pi(\mathfrak{h}) = \mathfrak{k}$. There exist $\alpha, \beta \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \left(\left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \alpha t + \beta z\right); tJ\right)$ for some $t, z \in \mathbb{R}$. If $\beta = 0$, then $\mathfrak{h} = \mathfrak{k}_\alpha$. Otherwise put $\alpha t + \beta z = w$ and $\mathfrak{h} = \mathfrak{z} \times \mathfrak{k}$.

CASE 3: $\pi(\mathfrak{h}) = \mathfrak{a}$. There exist $\alpha, \beta \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \left(\left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \alpha t + \beta z\right); tH\right)$ for some $t, z \in \mathbb{R}$. If $\beta = 0$, then $\mathfrak{h} = \mathfrak{a}_\alpha$. Otherwise put $\alpha t + \beta z = w$ and $\mathfrak{h} = \mathfrak{z} \times \mathfrak{a}$.

CASE 4: $\pi(\mathfrak{h}) = \mathfrak{n}$. There exist $\alpha, \beta \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \left(\left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \alpha t + \beta z\right); tU\right)$ for some $t, z \in \mathbb{R}$. If $\beta = 0$, then $\mathfrak{h} = \mathfrak{n}_\alpha$. Otherwise put $\alpha t + \beta z = w$ and $\mathfrak{h} = \mathfrak{z} \times \mathfrak{n}$.

CASE 5: $\pi(\mathfrak{h}) = \mathfrak{p}$. There exist $\alpha, \beta \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \left(\left(\left[\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right], \alpha t + \beta z\right); tU\right)$ for some $t, z \in \mathbb{R}$. Conjugating with $\left(\left(\left[\begin{smallmatrix} \alpha \\ 0 \end{smallmatrix}\right], 0\right); I\right)$ this becomes $\left(\left(\left[\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right], \beta z\right); tU\right)$. If $\beta = 0$, then $\mathfrak{h} = \mathfrak{p}$. Otherwise put $\beta z = w$ and $\mathfrak{h} = \mathfrak{z} \times \mathfrak{p}$.

CASE 6: $\pi(\mathfrak{h}) = \mathbb{R}_q$. There exist $\alpha, \beta \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \big(([\begin{smallmatrix} s \\ 0 \end{smallmatrix}], \alpha s + \beta z); 0\big)$ for some $s, z \in \mathbb{R}$. If $\beta \neq 0$, put $\alpha s + \beta z = w$ and $\mathfrak{h} = \mathbb{R}_q Z$. Otherwise conjugate with $\big(([\begin{smallmatrix} 0 \\ -\alpha \end{smallmatrix}], 0); I\big)$ to get $\mathfrak{h} = \mathbb{R}_q$.

CASE 7: $\pi(\mathfrak{h}) = \mathfrak{n} \rtimes \mathfrak{a}$. There exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \big(([\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}], \alpha t + \beta u + \gamma z); tH + uU\big)$ for some $t, u, z \in \mathbb{R}$. If $\gamma \neq 0$, then put $\alpha t + \beta u + \gamma z = w$ and $\mathfrak{h} = \mathfrak{z} \times (\mathfrak{n} \rtimes \mathfrak{a})$. If $\gamma = 0$, it easily seen that for X and X' in the above parametrization, $[X, X'] \in \mathfrak{h}$ only if $\beta = 0$. Thus $\mathfrak{h} = \mathfrak{n} \rtimes \mathfrak{a}_\alpha$.

CASE 8: $\pi(\mathfrak{h}) = \mathbb{R}_q \rtimes \mathfrak{a}$. There exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \big(([\begin{smallmatrix} s \\ 0 \end{smallmatrix}], \alpha t + \beta s + \gamma z); tH\big)$ for some $s, t, z \in \mathbb{R}$. If $\gamma \neq 0$, then put $\alpha t + \beta s + \gamma z = w$ and $\mathfrak{h} = \mathbb{R}_q Z \rtimes \mathfrak{a}$. If $\gamma = 0$, it easily seen that for X and X' in the above parametrization, it can be $[X, X'] \in \mathfrak{h}$ only if $\beta = 0$. Thus $\mathfrak{h} = \mathbb{R}_q \rtimes \mathfrak{a}_\alpha$.

CASE 9: $\pi(\mathfrak{h}) = \mathbb{R}_q \times \mathfrak{n}$. There exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \big(([\begin{smallmatrix} s \\ 0 \end{smallmatrix}], \alpha s + \beta u + \gamma z); uU\big)$ for some $s, u, z \in \mathbb{R}$. If $\gamma \neq 0$, then put $\alpha t + \beta s + \gamma z = w$ and $\mathfrak{h} = \mathbb{R}_q Z \times \mathfrak{n}$. If $\gamma = 0$, conjugate with $(\big(\big[\begin{smallmatrix} 0 \\ -\alpha/t \end{smallmatrix}\big], 0); a_t\big)$ with $t \neq 0$ to get

$$\Big(\Big(\Big[\begin{smallmatrix} ts + \frac{\alpha}{t}(t^2 u) \\ 0 \end{smallmatrix}\Big], \big(\tfrac{\beta}{t^2} - \tfrac{1}{2}\tfrac{\alpha^2}{t^2}\big)(t^2 u)\Big); (t^2 u) U\Big).$$

Put $v = (t^2 u)$ and $w = ts + \frac{\alpha}{t}(t^2 u)$, obtaining

$$\Big(\Big(\begin{bmatrix} w \\ 0 \end{bmatrix}, [\frac{2\beta - \alpha^2}{2t^2}]v\Big); vU\Big).$$

By suitably choosing t we may let $(2\beta - \alpha^2/2t^2) \in \{0, \pm 1\}$. Thus $\mathfrak{h} = \mathbb{R}_q \times \mathfrak{n}_\eta$.

CASE 10: $\pi(\mathfrak{h}) = \mathbb{R}_q \oplus \mathfrak{p}$. There exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \big(([\begin{smallmatrix} q \\ t \end{smallmatrix}], \alpha q + \beta t + \gamma z); tU\big)$ for some $q, t, z \in \mathbb{R}$. It easily seen that for X and X' in the above parametrization,

$$[X, X'] = \Big(\big([\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}], q't - qt'\big); 0\Big),$$

so that necessarily $q't - qt' = \alpha \cdot 0 + \beta \cdot 0 + \gamma z$ for all $q, t, q', t' \in \mathbb{R}$ and some z. This can only happen if $\gamma \neq 0$. Put then $\alpha t + \beta s + \gamma z = w$ and $\mathfrak{h} = \mathbb{R}_q Z \oplus \mathfrak{p}$.

CASE 11: $\pi(\mathfrak{h}) = \mathbb{R}^2$. There exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \big(([\begin{smallmatrix} q \\ p \end{smallmatrix}], \alpha q + \beta p + \gamma z); 0\big)$ for some $p, q, z \in \mathbb{R}$. It easily seen that for X and X' in the above parametrization,

$$[X, X'] = \Big(\big([\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}], q'p - qp'\big); 0\Big),$$

so that necessarily $q'p - qp' = \alpha \cdot 0 + \beta \cdot 0 + \gamma z$ for all $q, p, q', p' \in \mathbb{R}$ and some z. This can only happen if $\gamma \neq 0$. Put then $\alpha t + \beta s + \gamma z = w$ and $\mathfrak{h} = \mathfrak{h}_1$.

CASE 12: $\pi(\mathfrak{h}) = \mathbb{R}_q \rtimes (\mathfrak{n} \rtimes \mathfrak{a})$. There exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \big(([\begin{smallmatrix} q \\ 0 \end{smallmatrix}], \alpha t + \beta u + \gamma q + \delta z); tH + uU\big)$ for some $q, t, u, z \in \mathbb{R}$. If $\delta \neq 0$

then put $\alpha t + \beta u + \gamma q + \delta z = w$ and $\mathfrak{h} = \mathbb{R}_q Z \rtimes (\mathfrak{n} \rtimes \mathfrak{a})$. If $\delta = 0$, it easily seen that for X and X' in the above parametrization,

$$[X, X'] = \Big(\Big(\left[\begin{smallmatrix} tq'-t'q \\ 0 \end{smallmatrix}\right], 0\Big); 2(tu' - t'u)U\Big),$$

so that necessarily $0 = \alpha \cdot 0 + 2\beta(tu' - t'u) + \gamma(tq' - t'q)$ for all $q, t, u, q', t',' u \in \mathbb{R}$ and some z. This can only happen if $\beta = \gamma = 0$. Thus $\mathfrak{h} = \mathbb{R}_q \rtimes (\mathfrak{n} \rtimes \mathfrak{a}_\alpha)$.

<u>CASE 13:</u> $\pi(\mathfrak{h}) = \mathfrak{sl}(2, \mathbb{R})$. There exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \Big(\big(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \alpha a + \beta b + \gamma c + \delta z\big); \big(\left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right]\big)\Big)$ for some $a, b, c, z \in \mathbb{R}$. If $\delta \neq 0$ then put $\alpha a + \beta b + \gamma c + \delta z = w$ and $\mathfrak{h} = \mathfrak{z} \times \mathfrak{sl}(2, \mathbb{R})$. If $\delta = 0$, it easily seen that for X and X' in the above parametrization, $[X, X'] \in \mathfrak{h}$ only if

$$0 = \alpha(bc' - b'c) + 2\beta(ab' - a'b) + 2\gamma(ca' - c'a)$$

for all $a, b, c, a', b', c' \in \mathbb{R}$, so that necessarily $0 = \alpha = \beta = \gamma$. Thus $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{R})$.

<u>CASE 14:</u> $\pi(\mathfrak{h}) = \mathbb{R}^2 \rtimes \mathfrak{k}$. There exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \Big(\big(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], \alpha q + \beta p + \gamma t + \delta z\big); tJ\Big)$ for some $q, p, t, z \in \mathbb{R}$. If $\delta \neq 0$ then put $\alpha q + \beta p + \gamma t + \delta z = w$ and $\mathfrak{h} = \mathfrak{h}_1 \rtimes \mathfrak{k}$. If $\delta = 0$, it easily seen that for X and X' in the above parametrization, $[X, X'] \in \mathfrak{h}$ only if

$$pq' - qp' = \alpha(tp' - t'p) + \beta(qt' - q't)$$

for all $p, q, t, p', q', t' \in \mathbb{R}$, which is impossible for all $\alpha, \beta \in \mathbb{R}$.

<u>CASE 15:</u> $\pi(\mathfrak{h}) = \mathbb{R}^2 \rtimes \mathfrak{a}$. There exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \Big(\big(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], \alpha q + \beta p + \gamma t + \delta z\big); tH\Big)$ for some $q, p, t, z \in \mathbb{R}$. If $\delta \neq 0$ then put $\alpha q + \beta p + \gamma t + \delta z = w$ and $\mathfrak{h} = \mathfrak{h}_1 \rtimes \mathfrak{a}$. If $\delta = 0$, it easily seen that for X and X' in the above parametrization, $[X, X'] \in \mathfrak{h}$ only if

$$pq' - qp' = \alpha(tq' - t'q) + \beta(pt' - qp't)$$

for all $p, q, t, p', q', t' \in \mathbb{R}$, which is impossible for all $\alpha, \beta \in \mathbb{R}$.

<u>CASE 16:</u> $\pi(\mathfrak{h}) = \mathbb{R}^2 \rtimes \mathfrak{n}$. There exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \Big(\big(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], \alpha q + \beta p + \gamma u + \delta z\big); uU\Big)$ for some $q, p, u, z \in \mathbb{R}$. If $\delta \neq 0$ then put $\alpha q + \beta p + \gamma t + \delta z = w$ and $\mathfrak{h} = \mathfrak{h}_1 \rtimes \mathfrak{n}$. If $\delta = 0$, it easily seen that for X and X' in the above parametrization, $[X, X'] \in \mathfrak{h}$ only if

$$pq' - qp' = \alpha(tp' - t'p)$$

for all $p, q, t, p', q', t' \in \mathbb{R}$, which is impossible for all $\alpha \in \mathbb{R}$.

<u>CASE 17:</u> $\pi(\mathfrak{h}) = \mathbb{R}^2 \rtimes (\mathfrak{n} \rtimes \mathfrak{a})$. There exist $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \Big(\big(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], \alpha t + \beta u + \gamma q + \delta p + \varepsilon z\big); tH + uU\Big)$ for some $q, p, t, u, z \in \mathbb{R}$. If $\varepsilon \neq 0$ then put $\alpha t + \beta u + \gamma q + \delta p + \varepsilon z = w$ and $\mathfrak{h} = \mathfrak{h}_1 \rtimes (\mathfrak{n} \rtimes \mathfrak{a})$. If $\varepsilon = 0$, it easily seen that for X and X' in the above parametrization, $[X, X'] \in \mathfrak{h}$ only if

$$pq' - qp' = \beta(tu' - t'u) + \gamma(tq' - t'q + up' - u'p) + \delta(pt' - p't)$$

for all $p, q, t, u, p', q', t', u' \in \mathbb{R}$, which is impossible for all $\beta, \gamma, \delta \in \mathbb{R}$.

CASE 18: $\pi(\mathfrak{h}) = \mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$. There exist $\alpha, \beta, \gamma, \delta, \varepsilon, \omega \in \mathbb{R}$ such that any $X \in \mathfrak{h}$ can be written $X = \left(\left(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], \alpha a + \beta b + \gamma c + \delta q + \varepsilon p + \omega z\right); \left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right]\right)$ for some $q, p, a, b, c, z \in \mathbb{R}$. If $\omega \neq 0$ then put $\alpha a + \beta b + \gamma c + \delta q + \varepsilon p + \omega z = w$ and $\mathfrak{h} = \mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$. If $\omega = 0$, it easily seen that for X and X' in the above parametrization, $[X, X'] \in \mathfrak{h}$ only if

$$\begin{aligned} pq' - qp' &= \alpha(bc' - b'c) + 2\beta(ab' - a'b) + 2\gamma(ca' - c'a) \\ &\quad + \delta(aq' - a'q + bp' - b'p) + \varepsilon(cq' - c'q + pa' - p'a) \end{aligned}$$

for all $p, q, a, b, c, p', q', a', b', c' \in \mathbb{R}$, which is impossible for all $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{R}$.

Since there are no othere cases the proof is complete.

5. Related issues

5.1. **Polynomial algebras.** The classification of Theorem 3.4 may be expressed in terms of (Poisson) polynomial algebras. Let $\mathcal{P}_2$ denote the Lie algebra consisting of all polynomials of degree ≤ 2 in two indeterminates, equipped with the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial \xi} - \frac{\partial f}{\partial \xi}\frac{\partial g}{\partial x}, \qquad f(x,\xi),\ g(x,\xi) \in \mathcal{P}_2.$$

Let $\Phi : \mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R}) \to \mathcal{P}_2$ be defined by

$$\Phi\Big(\big(\big(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], t\big); A\big)\Big) = -\frac{1}{2}\begin{bmatrix} x & \xi \end{bmatrix} JA \begin{bmatrix} x \\ \xi \end{bmatrix} - \begin{bmatrix} x & \xi \end{bmatrix} J \begin{bmatrix} q \\ p \end{bmatrix} - t \tag{5.1}$$

Explicitly, if $\left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right] \in \mathfrak{sl}(2,\mathbb{R})$, then

$$\Phi\Big(\big(\big(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], t\big); A\big)\Big)(x,\xi) = -\frac{c}{2}\, x^2 + ax\xi + \frac{b}{2}\, \xi^2 - px + q\xi - t.$$

As easily checked, Φ is a Lie algebra isomorphism. Observe that $\mathfrak{h}_1$ is mapped by Φ onto $\mathcal{P}_1$, whereas $\mathfrak{sl}(2,\mathbb{R})$ corresponds to homogeneous polynomials of degree 2. The adjoint action on $\mathcal{P}_2$ is given by affine coordinate change. In other words, $g = ((y,s); B) \in G$ acts on $\mathbb{R}^2$ by $((y,s);B) \cdot {}^t[x,\xi] = B\, {}^t[x,\xi] + y$ and $\Phi(\operatorname{Ad} gH) = \Phi(H) \circ g^{-1}$. Thus we identify Poisson subalgebras of $\mathcal{P}_2$ up to affine coordinate changes. The explicit list is omitted.

The isomorphism (5.1) plays an important role in the harmonic analysis in phase space. Let ε denote the extended metaplectic representation of $\mathbb{H}_1 \rtimes SL(2,\mathbb{R})$ (see e.g. [F] for more details) and let σ^w denote the Weyl pseudodifferential operator with symbol the tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^2)$ like, for example, a polynomial. The following formula holds:

$$\varepsilon(\exp X) = e^{2\pi i (\Phi(X))^w}, \qquad X \in \mathfrak{g}.$$

Thus $\mathcal{P}_2$ is a natural model for ε via the Weyl calculus. This correspondence has been used in [DN] in connection with the problem of describing the commutative algebras generated by the restriction of ε to one–parameter subgroups of $\mathbb{H}_1 \rtimes SL(2,\mathbb{R})$.

5.2. **Coverings.** Our results could be extended to all Lie groups locally isomorphic to either $G_1 = \mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ or $G_2 = \mathbb{H}_1 \rtimes SL(2,\mathbb{R})$, because their Lie algebras are either $\mathfrak{g}_1 = \mathbb{R}^2 \rtimes \mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{g}_2 = \mathfrak{h}_1 \rtimes \mathfrak{sl}(2,\mathbb{R})$. The matter essentially reduces to the following steps:

(a) Identify the universal coverings of G_1 and G_2, compute their discrete central subgroups and then form the quotients of G_1 and G_2 modulo their discrete central subgroups. Such quotients exhaust the class of all Lie groups locally isomorphic to either G_1 or G_2. This is easily done: for G_2 one gets the groups $\mathbb{H}_1 \rtimes SL^{(m)}$ or $\mathbb{H}_1^{\mathrm{red}} \rtimes SL^{(m)}$, where $\mathbb{H}_1^{\mathrm{red}} = \mathbb{H}_1/\mathbb{Z}$ is the reduced Heisenberg group and $SL^{(m)}$ is the m–sheet covering of $SL(2,\mathbb{R})$. The case of countably many sheets corresponds to the universal covering SL^{∞}. The groups locally isomorphic to G_1 are its coverings $\mathbb{R}^2 \rtimes SL^{(m)}$. The explicit construction of the universal covering SL^{∞} may be found for example in [LV], while the explicit construction of all other groups together with many useful formulae and observations may be found in [A].

(b) Given a group H_i locally isomorphic to G_i, $i = 1, 2$, compute the exponential mapping $\exp : \mathfrak{g}_i \to H_i^m$ and write the canonical subgroups corresponding to the canonical subalgebras of $\mathfrak{g}_i$.

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