

Gabor multipliers with varying lattices

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Abstract

In the early days of Gabor analysis it was a common to say that Gabor expansions of signals are interesting due to the natural interpretation of Gabor coefficients, but unfortunately the computation of Gabor coefficients is costly. Nowadays a large variety of efficient numerical algorithms exists (see [28] for a survey) and it has been recognized that stable and robust Gabor expansions can be achieved at low redundancy, e.g., by using a Gaussian atom and any time-frequency lattice of the form $a\mathbf{Z}^d \times b\mathbf{Z}^d \triangleleft \mathbb{R}^{2d}$ with $ab < 1$. Consequently Gabor multipliers, i.e., linear operators obtained by applying a pointwise multiplication of the Gabor coefficients, become an important class of time-variant filters.

It is the purpose of this paper to describe the fact that - provided one uses Gabor atoms from a suitable subspace $S_0(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$ - one has the expected continuous dependence of Gabor multipliers on the ingredients. In particular, we will provide new results which show that a small change of lattice parameters implies only a small change of the corresponding Gabor multiplier (e.g., in the Hilbert-Schmidt norm).

Key Words and Phrases: Gabor frames, Gabor Riesz bases, Gabor multipliers, dual Gabor atom

1 Introduction and Notations

It is the purpose of this paper to demonstrate among others that the use of Gabor atoms from the modulation space $M^1(\mathbb{R}^d)$ (also known as $S_0(\mathbb{R}^d)$, or Feichtinger's algebra) makes Gabor analysis robust in many ways. In particular, we will use the fact that the canonical (= least squares) Gabor coefficients depend continuously on the lattice constants (a, b) , in order to derive, as an example, that Gabor multipliers derived from a smooth and square integrable function over the time-frequency (TF) plane with similar TF-lattice will be close to each other in the sense of the Hilbert Schmidt norm.

In order to make this paper more accessible we do not make use of the full power of modulation spaces, cf. [8] or [19, Chap. 11 and Chap. 12]. We restrict our attention to the modulation spaces of the form $M_g^p(\mathbb{R}^d)$, because they have

a relatively simple description. Furthermore, the spaces $M_s^1(\mathbb{R}^d)$ provide ideal classes of Gabor atoms, and the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ coincides with the intersection of those spaces, i.e., $\mathcal{S}(\mathbb{R}^d) = \bigcap_{s \geq 0} M_s^1(\mathbb{R}^d) = \bigcap_{s > 0} M_s^p(\mathbb{R}^d)$ for any $p \in [1, \infty]$ (cf. [19], p.254). These spaces are defined by the behavior of the short-time Fourier transform $V_g f$ of their elements with respect to an arbitrary non-zero Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$ as described below.

First let us recall some notational conventions: In the sequel, \mathbb{R}^d is regarded as the time domain and \mathbb{R}^{2d} is the time-frequency domain. Making use of the time-frequency shifts $\pi(\lambda)$, $\lambda = (x, \omega) \in \mathbb{R}^{2d}$, defined by $\pi(\lambda)g(t) = \exp(2\pi i \omega t) \cdot g(t-x)$ we define the *short time Fourier transform* for the signal f with window g , both in $L^2(\mathbb{R}^d)$ or f a tempered distribution in $\mathcal{S}'(\mathbb{R}^d)$ an $g \in \mathcal{S}(\mathbb{R}^d)$ a Schwartz function by, cf. [19, p. 41] for technical details,

$$V_g f(\lambda) = \langle f, \pi(\lambda)g \rangle = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \omega t} dt. \quad (1)$$

It is convenient to write for $z = (t, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}^d}$ the symbols $\langle z \rangle := (1 + |t|^2 + |\omega|^2)^{1/2}$ and $\langle t \rangle := (1 + |t|^2)^{1/2}$ resp. $\langle \omega \rangle := (1 + |\omega|^2)^{1/2}$, and $v_s = \langle z \rangle^s$.

Definition 1.1 For $s \in \mathbb{R}$ and $p \in [1, \infty]$ we define

$$M_s^p(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d), V_g f(z) \cdot v_s \in L^p(\mathbb{R}^{2d})\}, \quad (2)$$

with natural norm

$$\|f\|_{M_s^p} := \|V_g f\|_{p, v_s} = \|V_g f \cdot v_s\|_p. \quad (3)$$

In [19] these spaces are denoted by $M_{v_s}^p$. A summary on properties (e.g. inclusion results) of these spaces is given in [12]. In particular the space $M^1(\mathbb{R}^d) := M_0^1(\mathbb{R}^d)$ arising as the special case $p = 1, s = 0$, also known as Feichtinger's algebra (denoted in this context usually by $S_0(\mathbb{R}^d)$, see [19] for details, or [5] for an elementary introduction) will be of relevance for us. For sufficient conditions (expressed in terms of weighted L^p -conditions on the function f and its Fourier transform \hat{f} see [18], or [19, Prop.12.1.6]). For $d = 1$ it is sufficient to have $f, f', f'' \in L^1(\mathbb{R})$ in order to ensure that $f \in S_0(\mathbb{R})$.

We view Gabor analysis as the branch of time-frequency analysis which is concerned with the use of discretely parameterized *coherent families*. Such families are obtained by applying a set of time-frequency shifts, parameterized by the points of a discrete subgroup of $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$, to a given 'Gabor atom' g resp. γ . One is typically asking under which conditions on the pair (γ, g) arbitrary signals from $L^2(\mathbb{R}^d)$ can be expanded in the form

$$f = \sum_{(n,k) \in \mathbf{Z}^{2d}} c_{n,k} M_{nb} T_{ka} g, \quad \text{with } c_{n,k} = \langle f, M_{nb} T_{ka} \gamma \rangle. \quad (4)$$

For a given atom g a corresponding dual function γ can be found if and only if $(g_{n,k}) = (M_{nb} T_{ka} g)_{\mathbf{Z}^d \times \mathbf{Z}^d}$ is a Gabor frame, or equivalently, if the operator

$f \mapsto Sf = \sum_{(n,k) \in \mathbb{Z}^{2d}} \langle f, g_{n,k} \rangle g_{n,k}$ is invertible on $L^2(\mathbb{R}^d)$, with the *canonical* choice $\gamma = \tilde{g} = S^{-1}g$. This particular choice is called *the* dual Gabor atom for g (with respect to (a, b)), and can be characterized among all the functions γ fulfilling (4) in several different ways. For example, it provides the coefficients in $\ell^2(\mathbb{Z}^{2d})$ with minimal norm, resp., is the function γ with the smallest angle (or alternatively distance) to g , or the one closest to g in the L^2 -sense.

In the situation described above, one also has as a consequence of general frame theory (cf. [3]) (for the fixed lattice constants (a, b)) stability of the coefficient mapping $C_\gamma : f \mapsto \langle f, M_{nb}T_{ka}\gamma \rangle$ from $L^2(\mathbb{R}^d)$ into $\ell^2(\mathbb{Z}^{2d})$, as well as of the synthesis operator $R_g : \mathbf{c} \mapsto \sum_{\mathbf{z}^d \times \mathbf{z}^d} c_{n,k} M_{nb}T_{ka}g$, and furthermore it is possible to interchange the order of γ and g (as a consequence of the identity $S^{-1} \circ S = Id = S \circ S^{-1}$).

Unfortunately, despite the fact that the continuous inversion formula for the short-time Fourier transform is valid for arbitrary L^2 -functions g little can be said about its discretized form. There are all kinds of problems if this level of generality is allowed for the windows g : it may happen that there is no finite upper frame bound for all certain lattices (including the very small ones), while one has frames for nearby lattice constants (cf. [11]). It may also happen that one has even an orthonormal basis for some value of (a_0, b_0) (such as $a = 1, b = 1$, with g being the box function, i.e., the indicator function for $[0, 1]$ on the real line), but not for values (a, b) very close to (a_0, b_0) .

2 Stability with respect to lattice parameters

In this section we summarize essentially several explicit statements which follow immediately from more general results provided in a recent paper by Feichtinger and Kaiblinger ([12]) resp. from results valid for more general modulation spaces (see [8, 19]).

Theorem 2.1 *Assume that $g \in M_s^1(\mathbb{R}^d)$ for some $s \geq 0$, and that the triple (g, a_0, b_0) generates a Gabor frame. Then there exists $\delta > 0$ such that for all pairs of lattice constants (a, b) with $|(a, b) - (a_0, b_0)| \leq \delta$ one has: The triple (g, a, b) generates a Gabor frame and the dual Gabor atom $\tilde{g} = \tilde{g}^{(a,b)}$ belongs to $M_s^1(\mathbb{R}^d)$. Moreover, the mapping $(a, b) \mapsto \tilde{g}^{(a,b)}$ is continuous from that δ -ball around (a_0, b_0) into $M_s^1(\mathbb{R}^d)$. In particular, for every $\varepsilon > 0$ there exists $\eta > 0$ such that $\|\tilde{g}^{(a,b)} - \tilde{g}\|_{M_s^1} < \varepsilon$ if only $|(a, b) - (a_0, b_0)| < \eta$.*

A result of this form has a number of interesting consequences. Assume that we are interested in the expansion of functions in the (generalized = fractional) L^2 -Sobolev spaces $H^r(\mathbb{R}^d)$, for some real $r \in [-s, s]$. Then, in the above situation it is true that one characterizes within $\mathcal{S}'(\mathbb{R}^d)$ (or $L^2(\mathbb{R}^d)$, if $s \geq 0$) the membership of $f \in H^r(\mathbb{R}^d)$ by looking either at the STFT $V_g f$ for some $g \in M_s^1(\mathbb{R}^d)$, or, alternatively, consider the (canonical) Gabor coefficients. Recall that we have (by definition, cf. [27])

$$H^r(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d), \hat{f}(\omega) \cdot (1 + |\omega|^2)^{r/2} \in L^2(\mathbb{R}^d)\}.$$

Theorem 2.2 Let (g, \tilde{g}, a, b) be as above, i.e., with $g \in M_s^1(\mathbb{R}^d)$ such that (g, a, b) generates a Gabor frame. Then the following properties are equivalent for $f \in \mathcal{S}'(\mathbb{R}^d)$ and for any pair (a, b) close enough to (a_0, b_0) :

- i) $f \in H^r(\mathbb{R}^d)$, with $\|f\|_{H^r(\mathbb{R}^d)} = \|\hat{f}(\omega)(1 + |\omega|^2)^{r/2}\|_{L^2}$,
- ii) $V_g f(t, \omega) \cdot (1 + |\omega|^2)^{r/2} \in L^2(\mathbb{R}^{2d})$, hence $\|V_g f(z)\langle \omega \rangle^r\|_{L^2(\mathbb{R}^{2d})} < \infty$,
- iii) $\left(\sum_{(n,m) \in \mathbf{Z}^{2d}} |V_g f(na, mb)|^2 (1 + |m|^2)^r \right)^{1/2} < \infty$
- iv) $\left(\sum_{(n,m) \in \mathbf{Z}^{2d}} |V_{\tilde{g}(a,b)} f(na, mb)|^2 (1 + |m|^2)^r \right)^{1/2} < \infty$

The continuity of $(a, b) \rightarrow \tilde{g}^{(a,b)}$ (valid for the $M_s^1(\mathbb{R}^d)$ -norm) yields the continuity result given below. It shows that the relative error which occurs when in the reconstruction formula with respect to the lattice $a_0 \mathbf{Z}^d \times b_0 \mathbf{Z}^d$ a dual Gabor atom $\tilde{g}^{(a,b)}$ is used instead of the “correct one” which would be $\tilde{g}^{(a_0, b_0)}$.

Theorem 2.3 Assume that (g, a_0, b_0) generates a Gabor frame, for some $g \in M_s^1(\mathbb{R}^d)$. Then for any $\varepsilon > 0$ there exists some $\eta > 0$ such that the use of the canonical dual $\tilde{g}^{(a,b)}$ instead of $\tilde{g}^{(a_0, b_0)}$ results in a (uniformly) small relative error in the expansion of arbitrary elements of $H^r(\mathbb{R}^d)$, $0 \leq |r| \leq s$, whenever $|(a, b) - (a_0, b_0)| < \eta$, i.e., one has for all $f \in H^r(\mathbb{R}^d)$:

$$\left\| \sum_{(k,n) \in \mathbf{Z}^{2d}} [V_{\tilde{g}^{(a_0, b_0)}} f(ka_0, nb_0) - V_{\tilde{g}^{(a,b)}} f(ka_0, nb_0)] M_{nb_0} T_{ka_0} g \right\|_{H^r} < \varepsilon \cdot \|f\|_{H^r}.$$

The above result follows from the boundedness properties (uniform with respect to the range of values $r \in [-s, s]$) from $H^s(\mathbb{R}^d)$ into the corresponding weighted ℓ^2 -space over \mathbf{Z}^{2d} (for atoms in $M^r(\mathbb{R}^d)$) in combination with analogous bounded properties for the synthesis mapping, which maps sequences over \mathbf{Z}^{2d} back into infinite Gabor sums (which are unconditionally norm convergent in all the H^r -spaces). The continuity statement for the analysis mapping is the more interesting one and is therefore stated explicitly:

Theorem 2.4 Assume that (g, a_0, b_0) generates a Gabor frame, for some $g \in M_s^1(\mathbb{R}^d)$. Then for any $\beta > 0$ there exists some $\eta > 0$ such that the following is true: Whenever $|(a, b) - (a_0, b_0)| < \eta$, then for all $f \in H^r(\mathbb{R}^d)$

$$\left(\sum_{(k,n) \in \mathbf{Z}^{2d}} [V_{\tilde{g}^{(a_0, b_0)}} f(ka_0, nb_0) - V_{\tilde{g}^{(a,b)}} f(ka_0, nb_0)]^2 (1 + |nb_0|^2)^s \right)^{1/2} < \beta \cdot \|f\|_{H^r}.$$

Remark: (i) Note that the above two theorems include - as special cases - the case $s = 0$, i.e., $L^2(\mathbb{R}^d) = H^0(\mathbb{R}^d)$, in which case instead of $M_s^1(\mathbb{R}^d)$ one just has to take the Segal algebra $S_0(\mathbb{R}^d)$.

(ii) In the above statement we have only discussed the continuous dependence

on the lattice constant. It is, however, also true that one may modify the Gabor atom as well (however only in the M_s^1 -sense). For example, it would be justified to replace a Schwartz function g by a compactly supported version, obtained by localizing it in a decent way.

(iii) There are a few results in the literature which show that some changes of the sampling points (e.g., in the sense of jitter error, which is supposed to be uniformly small). Although in some of these papers (e.g. [2]) slightly weaker assumptions on the Gabor atoms are made (e.g. it may suffice to assume that g and \hat{g} are in Wiener's algebra) usually no continuous dependence of (the canonical) Gabor coefficients can be derived in that context.

(iv) Even for the case $s = 0$ the use of Gabor atoms in $S_0(\mathbb{R}^d) = M^1(\mathbb{R}^d)$ (as opposed to simply assuming the L^2 -frame condition by itself) has the advantage of guaranteeing (cf. [17]) that one can make stronger statements for functions $f \in M^1(\mathbb{R}^d)$, where the corresponding sequence space is $\ell^1(\mathbb{Z}^{2d})$ instead of $\ell^2(\mathbb{Z}^{2d})$. On the other hand one knows for the elements from the larger space $M^\infty(\mathbb{R}^d) = S'_0(\mathbb{R}^d)$, and hence for functions from a general L^p -space (for the full range $p \in [1, \infty]$), that the error for the coefficients (by using $\tilde{g}^{(a,b)}$ instead of $\tilde{g}^{(a_0,b_0)}$) is at least small in the sense of the sup-norm and that consequently the reconstruction error is small in the $S'_0(\mathbb{R}^d)$ -sense.

3 Gabor Multipliers and their continuity

There is a large variety of possible results concerning Gabor multipliers and Anti-Wick operators which arise by applying a pointwise multiplication operation (with some function or distribution) to the short-time Fourier transform before applying its inverse. Typically good mapping properties can be described in terms of modulation spaces, see [14] for a first survey. For the present paper we choose the more standard context of operators on $L^2(\mathbb{R}^d)$, but go for the case that the resulting Gabor multipliers are Hilbert Schmidt operators. Note, that in the terminology of these Anti-Wick calculus one has the following result (see [1, Theorem 3.1]).

Theorem 3.1 *For $\gamma, g \in L^2(\mathbb{R}^d)$ and a pointwise multiplier $\mathbf{m} \in L^p(\mathbb{R}^{2d})$ one has that the corresponding Anti-Wick operator $A_u^{\gamma,g}$ belong to the Schatten class S^p . In particular, L^2 -multipliers \mathbf{m} yield Hilbert Schmidt operators (and up to some constant their \mathcal{HS} norm is dominated by the L^2 -norm of their symbol \mathbf{m}).*

The main result of this paper describes the fact that operators which arise as Gabor multipliers over different (but close) lattices, using as pointwise multipliers for the canonical Gabor coefficients (these are also called *upper symbols*) the sampling values of a continuous and square integrable function over those lattices will be close in the sense of the Hilbert Schmidt norm, provided that involved atoms are in $S_0(\mathbb{R}^d)$. For fixed atoms g_1 and $g_2 \in S_0(\mathbb{R}^d)$ the result has already been announced as Theorem. 5.6.1 in [16]. Here we go for a more general situation, which actually leads to faster convergence rates (at least ac-

ording to our numerical experiments). The above mentioned theorem reads as follows:

Theorem 3.2 *Let g_1, g_2 be atoms in $S_0(\mathbb{R}^d)$, and for some $s > d$ let $\mathbf{m} \in H^s(\mathbb{R}^{2d})$ be given, which is then both a continuous and square integrable function. Furthermore, let (a_k, b_k) be a sequence of lattice constants satisfying $(a_k, b_k) \rightarrow (a_0, b_0)$ for $k \rightarrow \infty$, for some pair (a_0, b_0) of positive lattice constants. Write G_k for the Gabor multipliers, with windows g_1 and g_2 , the TF-lattices $a_k \mathbf{Z}^d \times b_k \mathbf{Z}^d$, and corresponding multiplier sequences $\mathbf{m}_k = (\mathbf{m}(na_k, lb_k))_{n,l \in \mathbf{Z}^d}$. Then the operators G_k converge to G_0 in the \mathcal{HS} -norm.*

We have to skip the proof of this statement here, as it is lengthy and will require the use of Kohn–Nirenberg symbols of the corresponding operators (cf. [7]). An alternative variant with stronger assumptions on the symbol, but also with a stronger conclusion, follows. Its proof (or rather an outline of arguments) can be well described in the present context.

In order to formulate a more general result we will make use of the Wiener amalgam spaces $W^2(\mathbb{R}^{2d}) := W(C^0, \ell^2)(\mathbb{R}^{2d})$ which consists of all bounded and continuous functions for which the sequence $c_n := \max_{y \in Q+n} |f(x)|$, with $Q = [0, 1]^{2d}$ being the unit cube and $n \in \mathbf{Z}^{2d}$, is square summable, i.e., belong to $\ell^2(\mathbf{Z}^{2d})$. It is not difficult to show that this condition is equivalent to the assumption that a local maximal function $f^\#$, given by $f^\#(t) = \max_{|y| \leq 1} |f(t+y)|$ belongs to $L^2(\mathbb{R}^d)$, and $\|f\|_{W^2} \simeq \|f^\#\|_2$.

There are various sufficient conditions for f to belong to $W^2(\mathbb{R}^{2d})$: for example, if f is in a classical fractional L^2 -Sobolev space of order $s > d$, because then H^s is of the form $\mathcal{F}(L_w^2)$ for some weight function over \mathbb{R}^{2d} which satisfies $1/w \in L^2(\mathbb{R}^{2d})$. As a consequence $H^s(\mathbb{R}^{2d}) \subseteq \mathcal{F}[W(L^2, \ell^1)](\mathbb{R}^{2d}) \subseteq W(\mathcal{FL}^1, \ell^2)(\mathbb{R}^{2d}) \subseteq W(C_0, \ell^2)(\mathbb{R}^{2d})$ by the Hausdorff-Young inequality for Wiener amalgam spaces (as described in [6]). An even weaker sufficient condition is provided in Lemma 3.4 given at the end of this paper. The main result of this paper now reads as follows:

Theorem 3.3 *Let $\mathbf{m} \in W^2(\mathbb{R}^{2d})$ be given, and let (g_k) and (γ_k) be two sequences which converge in $S_0(\mathbb{R}^d)$, with limits g_0 and γ_0 respectively. Furthermore let (a_k, b_k) be a sequence of lattice constants satisfying $(a_k, b_k) \rightarrow (a_0, b_0)$ for $k \rightarrow \infty$, for some pair (a_0, b_0) of positive lattice constants. Write G_k for the Gabor multipliers, with analysis window γ_k , synthesis window g_k , using the TF-lattice $a_k \mathbf{Z}^d \times b_k \mathbf{Z}^d$, and corresponding multiplier sequences (also called upper symbols), $\mathbf{m}_k = (\mathbf{m}(na_k, lb_k))_{n,l \in \mathbf{Z}^d}$, i.e., explicitly, we have for $k \geq 0$:*

$$G_k(f) = \sum_{(n,l) \in \mathbf{Z}^{2d}} \mathbf{m}(na_k, lb_k) \langle f, M_{lb_k} T_{na_k} \gamma_k \rangle M_{lb_k} T_{na_k} g_k. \quad (5)$$

Then the operators G_k converge to G_0 in the Hilbert-Schmidt norm.

Proof: Let us collect a few basic facts first:

1. Let us make at the beginning some easy simplifications: Since the sequences (g_k) and (γ_k) are convergent in $S_0(\mathbb{R}^d)$ we may suppose that both of them are bounded in $S_0(\mathbb{R}^d)$, by some constant $C_1 > 0$.

We may assume as well without loss of generality that $a_k > a_0/2 > 0$ and $b_k > b_0/2 > 0$ for all k , i.e., the density of all the lattices involved is bounded above. Equivalently, the discrete measures $\mu_k = \sum_{n,l} \delta_{(na_k, lb_k)}$ are uniformly (with respect to k) bounded in the Wiener amalgam space $W(M, \ell^\infty)(\mathbb{R}^{2d})$, which amounts to the claim that for any ball $B_R(x)$ of radius R the number of points from the lattice $a_k \mathbf{Z}^d \times b_k \mathbf{Z}^d$ is uniformly bounded (with respect to x).

We also note that $W^2(\mathbb{R}^{2d})$ is a subspace of the continuous and bounded functions, therefore the multiplier sequences $(\mathbf{m}(na_k, lb_k))$ are uniformly bounded. Since the functions g_k and γ_k in $S_0(\mathbb{R}^d)$ are Bessel sequences for arbitrary lattices in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ we are assured that the coefficients (taken with respect to γ_k) are in $\ell^2(\mathbb{Z}^{2d})$, and that synthesis is also well convergent in $L^2(\mathbb{R}^d)$ (cf. [17] for the technical details), i.e., (5) is well defined.

2. Since the compactly supported functions are dense in $W^2(\mathbb{R}^{2d})$ for every $\varepsilon' > 0$ there exists a compactly supported plateau-function p with values $p(x) \in [0, 1]$ such that $\|\mathbf{m} \cdot p - \mathbf{m}\|_{W^2} < \varepsilon'$.
3. Using the pointwise multiplier property of Wiener amalgam spaces in the form that $W(C^0, \ell^2)(\mathbb{R}^{2d}) \cdot W(M, \ell^\infty)(\mathbb{R}^{2d}) \subseteq W(M, \ell^2)(\mathbb{R}^{2d})$ we obtain, for some fixed constant $C_2 > 0$:

$$\|\mathbf{m} \cdot \mu_k - \mathbf{m} \cdot p \cdot \mu_k\|_{W(M, \ell^2)} < C \|\mu_k\|_{W(M, \ell^\infty)} \|\mathbf{m} \cdot p - \mathbf{m}\|_{W^2} < C_2 \varepsilon'$$

or more explicitly

$$\left\| \sum_{(n,l) \in \mathbf{Z}^{2d}} (1 - p(na_k, lb_k)) \mathbf{m}(na_k, lb_k) \delta_{(na_k, lb_k)} \right\|_{W(M, \ell^\infty)} < C_2 \cdot \varepsilon' \quad (6)$$

4. Since the local measure norm of a sum of (discretely distributed) Dirac -measures being just the sum of absolute amplitudes over balls of a fixed radius is getting smaller if their amplitudes are made smaller or replaced by zero we can conclude from the above estimate that for every $\varepsilon > 0$ there exists a finite subset $F \subseteq \mathbf{Z}^d \times \mathbf{Z}^d$ such that for all k

$$\left\| \sum_{(n,l) \notin F} \mathbf{m}(na_k, lb_k) \delta_{(na_k, lb_k)} \right\|_{W(M, \ell^\infty)} \leq C_2 \cdot \varepsilon'. \quad (7)$$

This is possible since the lattice constants are bounded away from zero (uniformly) and therefore it can be assured that the points with labels in $(\mathbf{Z}^d \times \mathbf{Z}^d) \setminus F$ will correspond to points in phase space which are outside the support of p , and are therefore represented within the estimate (6) with amplitude 1.

5. In order to derive the claimed convergence statement it now makes sense to describe both G_k and G_0 in terms of a (double) sum of rank one operators (cf. [13]), which we can then split into two parts, the sum over F and $(\mathbf{Z}^d \times \mathbf{Z}^d) \setminus F$. As in [13] we write $P^{(k)} = g_k \otimes \gamma_k^*$ for the kernel of the rank one operator $f \mapsto \langle f, \gamma_k \rangle g_k$. Conjugation of any operator P by a TF-shift $\pi(\lambda)$, i.e., $P \mapsto P_\lambda := \pi(\lambda)' \circ P \circ \pi(\lambda)$ is described by the symbol $(\pi \otimes \pi^*)$. Using the symbol $(\pi \otimes \pi^*)(\lambda)(P) = P_\lambda$, we can rewrite G_k as

$$G_k = \sum_{(n,l) \in \mathbf{Z}^d \times \mathbf{Z}^d} \mathbf{m}(na_k, lb_k) [(\pi \otimes \pi^*)(na_k, lb_k)P^{(k)}] \quad (8)$$

with $P^{(k)} = g_k \otimes \gamma_k^* \in S_0(\mathbb{R}^d) \otimes S_0(\mathbb{R}^d) \subseteq S_0(\mathbb{R}^{2d}) \subseteq W(C_0, \ell^1)(\mathbb{R}^{2d})$, and $\|P^{(k)}\|_{W(C_0, \ell^1)(\mathbb{R}^{2d})} \leq \|P^{(k)}\|_{S_0(\mathbb{R}^{2d})} \leq \|g_k\|_{S_0(\mathbb{R}^d)} \|\gamma_k\|_{S_0(\mathbb{R}^d)} \leq C_1^2$.

6. Recall furthermore that according to the kernel theorem for $S_0(\mathbb{R}^d)$ one has for every bounded linear operator T from $S_0(\mathbb{R}^d)$ into $S'_0(\mathbb{R}^d)$, and due to the embeddings $S_0(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow S'_0(\mathbb{R}^d)$ also for any bounded linear mapping from $L^2(\mathbb{R}^d)$ into itself a uniquely determined distributional kernel $K_T \in S'_0(\mathbb{R}^{2d})$, which in turn has a Kohn-Nirenberg symbol $\sigma(K) \in S'_0(\mathbb{R}^{2d})$. This relationship is an extension of the unitary mapping between the Hilbert space of all Hilbert Schmidt operators (with the standard scalar product $\langle T, S \rangle_{\mathcal{HS}} = \text{trace}(TX^*)$) and the Hilbert space $L^2(\mathbb{R}^{2d})$, which can be described for Schwartz functions (and indeed for functions in $S_0(\mathbb{R}^{2d})$ using integrals involving a partial Fourier transforms. (see [19, (14.6), p.304] or [13, section 7.5] for details.) In view of these facts it is sufficient to show that the Kohn-Nirenberg symbols of our operators are convergent in $L^2(\mathbb{R}^{2d})$.
7. One very useful property in this context is the *shift covariance* property of the Kohn-Nirenberg symbol, i.e., the fact (cf. formula (7.5.14) in [13])

$$\sigma[(\pi \otimes \pi^*)(\lambda)K] = T_\lambda[\sigma(K)],$$

where $T_\lambda F(x) = F(x - \lambda)$ is the ordinary translation of functions or distributions on $\mathbb{R}^d \times \mathbb{R}^d$. As a consequence we can describe $\sigma(G_k)$ as

$$\sum_{(n,l) \in \mathbf{Z}^d \times \mathbf{Z}^d} \mathbf{m}(na_k, lb_k) [T_{(na_k, lb_k)}\sigma(P^{(k)})] = (\mu_k \cdot \mathbf{m}) * \sigma(P^{(k)}) \quad (9)$$

Note that for $g, \gamma \in S_0(\mathbb{R}^d)$ one has (cf. [13, (7.5.9)]) the resulting KN-symbol $\sigma(P^{(k)})$ coincides with the continuous function $g_k(x)\widehat{\gamma}_k(\xi)\overline{\exp(ix\xi)}$ which belongs to $S_0(\mathbb{R}^{2d})$ (because $S_0(\mathbb{R}^d) \otimes S_0(\mathbb{R}^d) \subseteq S_0(\mathbb{R}^{2d})$ and $(x, \xi) \mapsto \exp(ix\xi)$ is a pointwise multiplier of $S_0(\mathbb{R}^{2d})$). Alternatively (using the above estimate for $\|P^{(k)}\|_{S_0(\mathbb{R}^{2d})}$) we may use the fact that the unitary relation between kernels P of Hilbert Schmidt operators and their symbols $\sigma(P)$ restricts to an isomorphism between $S_0(\mathbb{R}^{2d})$ and $S_0(\mathbb{R}^d \times \widehat{\mathbb{R}^d})$.

8. Clearly we can now split this sum (for all k , including $k = 0$) into a finite sum over F and an infinite sum over $(\mathbf{Z}^d \times \mathbf{Z}^d) \setminus F$ which contributes over- all very little. Indeed, using the convolution relations for Wiener amalgams ([4],[22]) of the form $W(M, \ell^\infty)(\mathbb{R}^{2d}) * W(C_0, \ell^1)(\mathbb{R}^{2d}) \subseteq W(C_0, \ell^2)(\mathbb{R}^{2d})$ and the interpretation of the ‘‘infinite part’’ as the convolution product

$$\sum_{(n,l) \notin F} \mathbf{m}(na_k, lb_k) T_{(na_k, lb_k)} \sigma(P^{(k)}) = \left[\sum_{(n,l) \notin F} \mathbf{m}(na_k, lb_k) \delta(na_k, lb_k) \right] * \sigma(P^{(k)})$$

which allows us to derive the following estimate:

$$\left\| \sum_{(n,l) \notin F} \mathbf{m}(na_k, lb_k) T_{(na_k, lb_k)} \sigma(P^{(k)}) \right\|_{W^2(\mathbb{R}^{2d})} \leq$$

$$C \cdot \left\| \sum_{(n,l) \notin F} \mathbf{m}(na_k, lb_k) \delta_{(na_k, lb_k)} \right\|_{W(M, \ell^\infty)} \|P^{(k)}\|_{W(C_0, \ell^1)} \leq C(C_2 \varepsilon') C_1^2 < \varepsilon/2,$$

if ε' is chosen appropriately.

9. For the remaining finite sum over F we observe that for each of the individual terms $\mathbf{m}(na_k, lb_k) \mapsto \mathbf{m}(na_0, lb_0)$ as $(a_k, b_k) \rightarrow (a_0, b_0)$.
10. Finally we have to observe that, of course, also

$$\|(\pi \otimes \pi^*)(\lambda_k) P^{(k)} - (\pi \otimes \pi^*)(\lambda_0) P^{(0)}\|_{\mathcal{HS}} \rightarrow 0$$

as $\lambda_k \rightarrow \lambda_0$, because we can split the sum as follows:

$$\begin{aligned} & \|(\pi \otimes \pi^*)(\lambda_k)(P^{(k)} - P^0)\|_{\mathcal{HS}} + \|[(\pi \otimes \pi^*)(\lambda_k) - (\pi \otimes \pi^*)(\lambda_0)]P^0\|_{\mathcal{HS}} \leq \\ & \|P^{(k)} - P^0\|_{\mathcal{HS}} + \|T_{(\lambda_k - \lambda_0)} \sigma(P^0) - \sigma(P^0)\|_{L^2(\mathbb{R}^{2d})} \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

11. Altogether we find that it is possible, by choosing ε' sufficiently small and (a_k, b_k) close enough to (a_0, b_0) that we can obtain for any given $\varepsilon > 0$

$$\|G_k - G_0\|_{\mathcal{HS}} = \|\sigma(G_k - G_0)\|_{L^2(\mathbb{R}^{2d})} < \varepsilon \quad (10)$$

by choosing sufficiently large k , q.e.d..

Remarks: (1) The above result can be used in many ways. For example one can start with a fixed pair $g_k = g_1$ and $\gamma_k = g_2$ for all k , and will thus have a proof for Theorem.5.6.1 in [16] (it was restated above).

(2) Even more interesting is the situation if one adapts either the synthesis or the analysis window to the other one, in order to ensure that the constant symbol $\mathbf{m} \equiv 1$ yields the identity operator as Gabor multiplier. For example, we may fix $g_k = g$ and choose $\gamma_k = \tilde{g}^{(k)}$, which is determined as the inverse of the frame operator associated with the triple $(g, (a_k, b_k))$. Indeed, in this case

one can make use of the continuity of the mapping $\tilde{g}^{(k)} \rightarrow \tilde{g}^{(0)}$ in the sense of $S_0(\mathbb{R}^d)$ for $k \rightarrow \infty$, as a consequence of the main result of [12].

(3) From a practical point of view we want report the following: Since one has a trivial form of convergence for $\mathbf{m} \equiv 1$ in this case, and all the ingredients are local (in a TF-sense) it is plausible that the Gabor multipliers obtained by this construction are faster convergent towards the limit G_0 than for the case that one just fixes a pair of functions, such as $(g, \tilde{g}^{(0)})$ (the dual with respect to the limiting lattice generated by (a_0, b_0)). This fact has been verified numerically in a number of typical cases, but so far we do not yet have a detailed theoretical argument why (and when) this phenomenon will occur.

(4) Of course, one could generalize Theorem 3.3 even further by allowing a sequence of continuous functions \mathbf{m}_k being convergent to \mathbf{m}_0 in the sense of $W^2(\mathbb{R}^{2d})$.

(5) Another variant would be to use an upper symbol \mathbf{m} which is continuous and vanishing at infinity, i.e., from $C^0(\mathbb{R}^{2d})$. In that case the arguments of the proof provided above would yield convergence in the sense of the operator norm (instead of the \mathcal{HS} -norm).

(6) If however \mathbf{m} is just a bounded and continuous function (even if it is equicontinuous or $\equiv 1$) one cannot have convergence of G_k towards G_0 in the generality stated above. What is true, however, is strong operator norm convergence for a large class of modulation spaces. Hence, for example, it is not hard to derive (uniformly for compact subset of $M_r^p(\mathbb{R}^d)$, with $r \in [-s, s]$) that

$$\|G_k(f) - G_0(f)\|_{M_s^p(\mathbb{R}^d)} \rightarrow 0 \text{ for each } f \in M_s^p(\mathbb{R}^d) \text{ as } k \rightarrow \infty. \quad (11)$$

(7) With the ingredients as in Theorem. 3.3, but with $\mathbf{m} \in W(C_0, \ell^1)(\mathbb{R}^{2d})$ one can show that the operators G_k converge to G_0 in the trace class norm. Under slightly stronger assumptions ($\mathbf{m} \in S_0(\mathbb{R}^{2d})$) this result has been stated as Theorem. 5.6.2 in [16]. Using Theorem. 5.3.3., one can even show that convergence takes place in the operator norm for operators from $S'_0(\mathbb{R}^d)$ to $S_0(\mathbb{R}^d)$.

We conclude this paper with a simple sufficient condition for a potential multiplier function on \mathbb{R}^2 to belong to $W(C_0, \ell^2)$.

Lemma 3.4 *Assume that a bounded, continuous function F on \mathbb{R}^2 has the property that F , but also $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial^2 F}{\partial x \partial y}$ belong to $L^2(\mathbb{R}^2)$. Then F belongs to $W(C_0, \ell^2)(\mathbb{R}^2)$.*

Proof: The proof can be based on the fact that via the Fourier transform the assumptions imply that $\hat{f} \cdot w \in L^2(\mathbb{R}^2)$, for $w(x, y) = (1+|x|)(1+|y|)$. Since clearly $1/w \in L^2(\mathbb{R}^2)$ we can apply the Cauchy Schwartz inequality to $\hat{f} = \hat{f}w \cdot \frac{1}{w}$ in order to recognize that $L_w^2 \hookrightarrow L^1(\mathbb{R}^2)$. Consequently $\hat{f} \in W(\mathcal{FL}^2, \ell^1)(\mathbb{R}^2)$ and the Hausdorff Young principle for amalgams can be applied (cf. [6]), implying that $f \in W(\mathcal{FL}^1, \ell^2)(\mathbb{R}^2) \subseteq W(C_0, \ell^2)(\mathbb{R}^2)$.

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