PRECURSORS IN MATHEMATICS: EARLY WAVELET BASES

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The plain fact that wavelet families are very interesting orthonormal systems for $L^2(\mathbf{R})$ makes it natural to view them as an important contribution to the field of orthogonal expansions of functions. This classical field of mathematical analysis was particularly flourishing in the first 30 years of the 20th century, when detailed discussions of the convergence of orthogonal series, in particular of trigonometric series, were undertaken.

Alfred Haar describes the situation in his 1910 paper in Math. Annalen appropriately as follows: for any given (family of) orthonormal system(s) of functions on the unit interval $[0,1]$ one has to ask the following questions:

- *convergence theory* (sufficient conditions that a series is convergent);
- *divergence theory* (in contrast to convergence theory it exhibits examples of relatively “decent” functions for which nevertheless no good convergence, e.g., at that time mostly in the pointwise or uniform sense, takes place);
- *summability theory* (to which extent can summation methods help to overcome the problems of divergence);
- *uniqueness theory* (under which circumstances can one be assured that in case of convergence of the series of partial sums of an orthogonal expansion of a function, its limit equals the original function).

Alfred Haar’s 31-page paper is mainly concerned with the properties of the partial sum operators, by studying the integral kernel of the corresponding projection operators and deriving corresponding properties from it, not only for the case of the classical trigonometric functions, but also for orthonormal systems related to Sturm-Liouville differential equations. Recall that only two years earlier Féjer (also published in Math. Annalen) had shown that Cesaro summability (also denoted as C1-summability) was a way to overcome the problem of divergence of a classical Fourier series for the case of continuous functions, while in the years before obviously inherent problems with the questions of pointwise convergence of the partial sums of Fourier series had been revealed. In fact, at the time of the writing of Haar’s paper it was not at all clear whether the divergence problem even for continuous functions is shared by general orthonormal systems of functions or only by those considered up to that time.

In this sense the construction of what is nowadays called Haar’s orthonormal basis for $L^2([0,1])$ in Chap. 3 of his paper provides an answer to this very question and had probably no additional relevance to the author at that time. Indeed, only the last 9 pages of the paper are concerned with the description of the Haar system, verifying the orthogonality and completeness relations, and above all the uniform convergence of the partial sums of the Haar series to f, for any continuous function f on $[0,1]$. More specifically, he shows that one has pointwise convergence of the Haar partial sums at all the Lebesgue points of f, i.e., at those points where the pointwise derivative of the antiderivate of f exists. It is well-known that this is true for almost all points in $[0,1]$ for a given Lebesgue integrable function on the interval. Moreover, the limit of the Haar partial sums is exactly this real number.

It is certainly of great value to the community that the effort of compiling this volume also brought the translation of Haar's article from German into English, so that readers can now check what Haar had really done in his paper. Obviously he did not see the structure of a wavelet basis (at least not as something worth commenting on), as he was working over the interval $[0,1]$ only. So it was left to Yves Meyer to point out (what is nowadays almost the usual way of explaining the basics of wavelet orthonormal systems) that one can see the Haar system (naturally extended to the whole real line) as the first orthonormal system of this kind, with the disadvantage that it consists of discontinuous functions only.

Obviously Haar's example did not exclude the possibility that uniform convergence for continuous functions could only arise for orthonormal systems of functions which are discontinuous. Hence it was left as an open problem whether one could also have an orthonormal system of *continuous* functions on $[0,1]$ with the same properties. The answer was provided 18 years later in volume 100 of Math. Annalen, in the year 1928, by Philip Franklin, a professor at the MIT (Cambridge, Massachusetts), in a concise 8 page paper.[1]

Franklin's paper provides the description of an orthonormal basis (again over $[0,1]$), which is nowadays called *Franklin's system*, consisting of appropriately defined continuous, piecewise linear functions, with a node sequence at points of the form $k/2^n$, successively inserted within $[0,1]$. He proves that this series expansion is uniformly convergent for continuous functions on the interval, and in the quadratic mean for square-integrable, measurable functions in the Lebesgue sense. In a final remark Franklin also mentions that one may obtain the Haar basis by starting from his basis, taking derivatives and then orthonormalizing the resulting system. He also indicates that several of the properties of the Haar system could be derived easily in this way.

The next paper — in chronological order — in this section is due to J. O. Strömberg, entitled: "A modified Franklin system and higher-order spline systems on $\mathbf{R}^n$ as unconditional bases for Hardy spaces". It appeared in Vol. II of the Conference on Harmonic Analysis in honour of A. Zygmund's 80-th birthday, which took place more than 50 years after the publication of Franklin's paper, in March 1981 (the volume was published in 1983), i.e., long before Yves Meyer's construction was carried out (this fact was recognized by him on several occasions later on). Peter Jones, one of the editors of that volume, has described the situation at a recent wavelet conference in Hong Kong with these words: "here we were all sitting, listening to his talk, without realizing the relevance of the construction he had given, or putting it into the appropriate context."

Looking at the paper nowadays, at a time when the standard facts about wavelets are well-known to a wide community, one easily recognizes that Strömberg is really describing an orthonormal wavelet system in the "classical sense" for the Hilbert space $L^2(\mathbf{R})$ or even $L^2(\mathbf{R}^n)$, $n \geq 1$, inspired by Franklin's basis and the concept of higher-order spline bases. Again, the motivation for the paper was a very specific question: can one have unconditional bases for the (real) Hardy spaces $H^p(\mathbf{R}^n)$

[1] For the historically interested reader we note that at this time along with David Hilbert, Otto Blumenthal, and Constantin Caratheodory, also Albert Einstein was an editor of this journal.

(even for $p < 1$)? As a matter of fact, Strömberg's approach did overcome some difficulties in an earlier construction given by L. Carleson [3], whose result in turn followed the non-constructive existence proof given by Maurey.

Strömberg's paper itself proceeds from a description of his new wavelet system to results stating that it is not only an orthonormal basis for $L^2(\mathbf{R})$, but also an unconditional basis for $H^p(\mathbf{R})$ (for $p > 1/(m + 5/2)$, where m is the degree of the splines used). Moreover it is shown that the H^p-norm is equivalent to a certain solid sequence space norm (i.e., a norm on the space of coefficients which has the property that coefficients which are smaller in absolute value yield functions with smaller H^p-norm).

When Y. Meyer got into contact with J. Morlet and his system of functions (generated by the "Mexican hat function", the second derivative of a Gauss function), which according to numerical experiments worked almost like an orthonormal system, he was aware of the fact that according to Balian's result of 1981 [1] one could not have an orthonormal basis of Weyl-Heisenberg form with good joint time-frequency concentration, and therefore, according to his own words, he was going to verify that it was also impossible to have a wavelet orthonormal system, as we know it today. To his surprise his first conjecture turned out to be false, because he himself was able to produce a system of band-limited Schwartz functions (in other words, the Fourier transforms of what are nowadays called *Meyer wavelets* have compact support and are infinitely smooth), which is an orthonormal basis for $L^2(\mathbf{R})$. In fact, from the group-theoretical perspective (which had been already brought into view by Alex Grossmann, who was the intermediator between Morlet and Meyer) the analogy between the two cases was quite tempting. In each case a certain irreducible, (square) integrable group representation providing a continuous — abstract — wavelet transform, played a key role, also providing a continuous reproduction formula due to an abstract version of Calderon's reproducing formula, which was then also the basis for other reconstruction principles (such as the atomic decomposition methods for coorbit spaces developed by Feichtinger and Gröchenig [6]). Meyer's wavelet construction shows that there are indeed decisive differences between the case of the non-unimodular $ax + b$-group and the Heisenberg group, which among other properties is nilpotent and even possesses compact neighborhoods of the identity which are invariant under inner automorphisms.

Yves Meyer's first paper — although certainly widely circulated — was actually not published in one of the standard mathematical journals, but rather in the prestigious "Séminaire Bourbaki", 38eme annee, 1985/86, in the issue of February 1986 (published 1987 in Asterisque 145–146). Nevertheless it is fair to say that it marks the beginning of "modern wavelet theory" and that due to his enthusiasm for the possible far-reaching consequences of this discovery wavelets became a hot topic within a short time. Reportedly the construction was carried out in the summer of 1985, and presented in the seminar in October of the same year. When I met Yves Meyer in Paris in February of 1986 he told me that he himself, and also P. G. Lemarié, had found two constructions of orthonormal wavelet systems, but that they did not yet have a general scheme of constructing wavelets, so it was not yet clear at that time how one could systematically construct wavelets with certain additional properties.

Aside from the construction itself it is quite remarkable to observe, reading this paper more than 18 years after its appearance, how clear the relevance of the new wavelet system is already explained, based on the fact that it is a universal (in some sense) family of Schwartz functions which provides an unconditional basis for quite a wide range of Banach spaces of functions or tempered distributions, including Sobolev spaces, but also H_1 and BMO (in the w*-sense). Also a bit surprisingly, no direct reference to the pioneering work of J. Peetre or H. Triebel is made, two mathematicians who had developed many details concerning the relevant family of Banach spaces of functions resp. distributions (nowadays called Besov–Triebel–Lizorkin spaces), using a Fourier approach which in turn makes essential use of Littlewood–Paley theory.

The second construction, due to Lemarié and Meyer, was then published in 1986 (already submitted in December of 1985) in Revista Matematica Iberoamericana, Vol. 2. It refers to Wojtaszczyk's paper [9] on the Franklin system as an unconditional basis for H_1, but does not mention Strömberg's paper. It provides a new systems of functions, obtained by dyadic dilations and integer translations of a $2^n - 1$ functions ψ_k, such that the overall family is an orthonormal basis for $L^2(\mathbf{R}^n)$. These functions are Schwartz functions, in fact their Fourier transforms are C^∞-functions with compact support, hence they are decaying faster than the inverse of any polynomial. Furthermore they have vanishing moments of all orders, or, equivalently, the Taylor series of $\hat{\psi}_k$ at zero is trivial. Obviously this also implies that the functions $\hat{\psi}_k$ cannot be analytic, and hence one cannot have exponential decay for the members of the family ψ_k. A recent construction of band-limited wavelets with subexponential decay can be found in [5], which in turn is based on a systematic description of Lemarié–Meyer (band-limited) wavelets as given in [2].

One should mention here an alternative construction, due to Lemarié ([7], cited in the paper of G. Battle as a preprint), which has not been included in the present volume. It gives the first construction of exponentially decaying wavelet functions of a given smoothness and any given (finite) number of vanishing moments. The main part of his paper uses spline functions in order to construct for any given natural number m a suitable function ψ of class C^{m-2} for which ψ and its first $m-2$ derivatives are $O(e^{-\varepsilon|t|})$ for a certain $\varepsilon > 0$, and $\int_{\mathbf{R}} t^k \psi(t)\,dt = 0$ for $0 \le k \le m-1$. He also generalizes the result to the multi-dimensional case there.

Recall that it was Ingrid Daubechies who was able to construct compactly supported wavelet systems with similar properties. Her paper [4] appeared in 1988 and marks one of the cornerstones of the field, for at least two reasons: compactly supported wavelets are important for applications, and moreover the so-called cascade algorithm made the calculation of wavelet coefficients numerically efficient.

The practical importance of wavelet systems actually depends very much on the fact already in the included paper by Lemarié and Meyer from 1986: wavelet orthonormal bases of "nice functions" are also unconditional bases for a variety of Banach spaces of functions or distributions on $\mathbf{R}^n$. In fact, once one started to consider compression and denoising methods based on the wavelet expansion, such as the methods of hard or soft thresholding (which simply consists of the idea of discarding the small wavelet coefficients, interpreting them as noise), the fact that they provide unconditional bases for the different spaces of smooth functions automatically implied that the (non-linear) thresholding operators are in fact uniformly bounded on all of those spaces. In fact, if the convergence of the wavelet expansion was conditional (based on a specific order), one could not expect that such a property is granted a priori. That the wavelet system has to consist of "nice functions" is obvious, since it is clear that such a property cannot be valid for the simple Haar system (except for the L^2-context), because a finite approximating partial sum of a wavelet series of a smooth function would of course be just a step function, having no smoothness at all. In contrast, the partial sums of the wavelet expansion for a function in some Sobolev space (to give an example) is convergent in that very space, whenever the function expanded belongs to that (think of a Sobolev or Besov) space, and the wavelet system satisfies sufficiently high moment conditions.

The last paper in this series is due to Guy Battle, and is entitled "A block spin construction of ondelettes, Part I: Lemarié Functions". It appeared in Commun. Math. Physics in 1987, and was motivated by potential applications in physics. In fact, the author refers to an earlier construction (a polynomial generalization of Haar's system to $\mathbf{R}^n$) which he had published together with G. Federbush in 1982 and 1983 under the name of "phase cell cluster expansions". Their early wavelet-like systems (using a finite number of generators) did have good concentration and vanishing moments, but poor regularity. Stimulated by the constructions of Lemarié and Meyer, he provides a cell-cluster approach to the construction of orthonormal wavelet systems. The two main steps are first an orthogonalization at the level of dyadic scales, which is then complemented by the (meanwhile standard) procedure of "symmetric orthonormalization" (preserving the group structure of the system: starting from a Riesz basis of functions, obtained by shifts along a lattice, one obtains an equivalent orthonormal basis, with the same structure, with identical closed linear span within the Hilbert space L^2).

In summary, this section provides access to a selection of papers which play a historical role within mathematical analysis, and which belong certainly to the most cited papers over an extended period of time. Since some of those papers have been available so far only in their original language (German in the case of Haar's paper, translated by G. Zimmermann, and French in the case of Yves Meyer's papers, translated by J. Horváth) it is a valuable service to the community that we are seeing carefully translated English versions in this volume. Recall that, e.g., Haar's basis (extended to all of $\mathbf{R}$), is quite often taken as the most natural and simple example of a wavelet orthonormal system, with all the algebraic properties easily verified, which have played an important role, e.g., in Ingrid Daubechies' construction of compactly supported wavelets of a prescribed smoothness [4].

It is almost characteristic of the constructions provided here that they arose from very specific questions in a functional analytic context, which by themselves would have been considered as interesting mathematical problems by a small group of experts only. However, the resulting constructions turned out to be of high relevance for many applications, and in a number of contexts completely different from the original ones. Hence they provide good examples of "how mathematical research may work," and that the investigation of fundamental mathematical questions often yields surprising developments that benefit the entire scientific community.

At the end let us mention that additional historical information on the early times of wavelet theory can also be found in [8].

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