

A Szegö-Type Theorem for Gabor–Toeplitz Localization Operators

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1. Introduction and Preliminaries

Linear time-frequency methods such as Gabor expansions or the short-time Fourier transform provide ways of analyzing signals by describing their frequency content as it varies over time. In contrast to quadratic methods, Gabor expansions allow one to manipulate signals in a linear way. The resulting time-variant filtering procedures allow one to isolate partial signals or suppress the noise. Gabor expansions have a long history and have been applied by engineers for decades. Despite this fact, their mathematical properties still hide many intricate features. Their study has recently been substantially intensified ([AT; FS; QC; Te] are some recent books that deal with current developments).

The Gabor reproducing formula defines a natural context for time-frequency localization, which is one of the methods for time-variant filtering. By modifying by the weight function, the basic projection operators that build the reproducing formula, we emphasize those regions of the time-frequency plane that correspond to large values of the weight and disregard those where the weight is small. The resulting operator, assigning to the original signal its time-frequency modification, shares many features with classical Toeplitz operators and is called a *Gabor–Toeplitz localization operator*. The weight function is called a *symbol*.

One of the main lines of investigation of time-frequency localization operators is the study of the behavior of their eigenvalues as the domain of localization is expanded by dilations (see [Da2; LW; RT]). The distribution function, indicating how many eigenvalues are bigger than ε ($1 > \varepsilon > 0$), is the principal object in this study. The distribution of the eigenvalues of Gabor–Toeplitz operators was first studied by Daubechies in [Da1]. She observed that if the symbol is a characteristic function of a disk centered at the origin and if the function defining the reproducing formula (i.e., the *window*) is a standard Gaussian, then the corresponding Gabor–Toeplitz localization operator is diagonalized by Hermite functions. She obtained a formula for the eigenvalues in terms of the restricted Gamma function, and this allowed her to analyze the behavior of the distribution of the eigenvalues. Later, a different approach (based on trace formulas) was taken by Ramanathan and Topiwala in [RT]. This approach enabled them to generalize several of the Daubechies results to the context of general bounded domains and general windows. In this

paper we examine the case where the symbols are no longer characteristic functions of bounded domains but instead are general nonnegative, bounded, integrable functions. Our main result is an analog of the classical Szegő theorem. (A nice overview of Szegő-type theorems is contained in the book by Widom [W].) From our result we derive corollaries that describe the behavior of the distribution of the eigenvalues and the size of the plunge region. The first corollary shows that, on the level of order $2n$ (where n denotes dimension), the distribution of the eigenvalues mimics precisely the behavior of the distribution function of the symbol. The second corollary shows that a similar phenomenon occurs for the plunge region of the eigenvalues. The main idea of the proof of our main result comes from the formula linking Toeplitz and Hankel operators.

There are essentially three types of time-frequency localization operators in use:

- (i) compositions of timepass and bandpass filters;
- (ii) restrictions of the reproducing formulas based on coherent state expansions;
- (iii) Weyl pseudodifferential operators with symbols having compact support.

The first class was investigated by Landau, Slepian, Pollak, and Widom (see [L; LW; Sl]). Daubechies, Paul, Ramanathan, and Topiwala (see [Da2; RT]) studied these operators of the second class, which are based on Calderón and Gabor reproducing formulas. The general case was investigated by Dall'Ara, De Mari, and Mauceri in [DDM]. The work in the third direction was done by Flandrin, Heil, Ramanathan, and Topiwala [Fl; HRT]. Still another method of time-frequency localization was developed by Hlawatsch, Kozek, and Krattenthaler in [HKK]. Much useful information on time-frequency localization can be found in a survey paper by Folland and Sitaram [FoS]. For general aspects of time-frequency analysis, we refer the reader to the book of Folland [Fo]. Several new results dealing with function spaces related to Gabor expansions were obtained recently by Feichtinger and Gröchenig in [FG].

Throughout this paper we will work with the Gabor transform of functions $f \in L^2(\mathbb{R}^n)$ with respect to a fixed square integrable window $\phi \in L^2(\mathbb{R}^n)$. For the sake of convenience we assume that the function ϕ is normalized in $L^2(\mathbb{R}^n)$, that is, $\|\phi\|_{L^2(\mathbb{R}^n)} = 1$. By $\phi_{q,p}$ we denote the *phase-space shift* of ϕ by (q, p) ,

$$\phi_{q,p}(x) = e^{2\pi i p x} \phi(x - q).$$

The coordinates q, p are interpreted as position and momentum (time and frequency in the 1-dimensional setting), and the space \mathbb{R}^{2n} is called the *phase space*. The *Gabor transform* F of f with respect to ϕ is given by

$$F(q, p) = \langle f, \phi_{q,p} \rangle.$$

The functions $\phi_{q,p}$ give rise to the *Gabor reproducing formula*: For any $f \in L^2(\mathbb{R}^n)$ we have

$$f = \int_{\mathbb{R}^{2n}} F(q, p) \phi_{q,p} dq dp,$$

where the integral is understood in a weak sense. Clearly we have $\|F\|_{L^2(\mathbb{R}^{2n})} = \|f\|_{L^2(\mathbb{R}^n)}$. By multiplying the amplitudes $F(q, p)$ by some weight $b(q, p)$, we

can emphasize or eliminate certain parts of the phase-space representation of f . The *Gabor–Toeplitz localization operator* is defined by

$$T_b f = \int_{\mathbb{R}^{2n}} b(q, p) F(q, p) \phi_{q,p}(x) dq dp,$$

where b is assumed to be a nonnegative, bounded, and integrable function defined on the phase space. The function b is called the *symbol* of T_b , or its *Gabor multiplier*.

It is not hard to check that Gabor–Toeplitz localization operators satisfy

$$0 \leq T_b \leq \|b\|_{L^\infty}, \quad (1)$$

that they are trace class, and that

$$\text{tr } T_b = \int_{\mathbb{R}^{2n}} b(\eta) d\eta. \quad (2)$$

Our aim is to study the asymptotic behavior of the eigenvalues of Gabor–Toeplitz localization operators as the symbol is dilated. We define L^∞ normalized dilation by $b_R(\zeta) = b(\zeta/R)$, $R > 0$. We are interested in the behavior of the eigenvalues $\lambda_i(R)$ of T_{b_R} as $R \rightarrow \infty$. It follows from (1) and (2) that

$$0 \leq \lambda_i(R) \leq 1 \quad \text{and} \quad \sum_{i=0}^{\infty} \lambda_i(R) = R^{2n} \int_{\mathbb{R}^{2n}} b(\eta) d\eta.$$

For more background and further details, see [Da2] and [RT].

2. The Main Result and its Consequences

THEOREM 2.1. *Let $\phi \in L^2(\mathbb{R}^n)$, $\|\phi\|_{L^2(\mathbb{R}^n)} = 1$, and $b \in L^1(\mathbb{R}^{2n})$ with $0 \leq b \leq 1$. Then, for any continuous function h defined on the closed interval $[0, 1]$, the following asymptotic formula holds:*

$$\lim_{R \rightarrow \infty} \frac{\text{tr}(T_{b_R} h(T_{b_R}))}{R^{2n}} = \int_{\mathbb{R}^{2n}} b(\eta) h(b(\eta)) d\eta. \quad (3)$$

REMARK 1. The operator T_{b_R} is a nonnegative trace class operator because its symbol is nonnegative and integrable. Its operator norm is bounded by $\mathbf{1}$ because its symbol is bounded by 1. The functional calculus of self-adjoint operators allows all bounded functions defined on its spectrum to act on it. In particular, all continuous functions defined on $[0, 1]$ may be applied to T_{b_R} . Each continuous function h provides a bounded operator $h(T_{b_R})$. The operator norm of $h(T_{b_R})$ is bounded by the maximum of the function h . The operator $T_{b_R} h(T_{b_R})$ is trace class, and it makes sense to compute its trace.

REMARK 2. The mapping

$$h \rightarrow \frac{1}{R^{2n}} \text{tr}(T_{b_R} h(T_{b_R}))$$

defines a bounded functional on $C([0, 1])$. The representing measure equals

$$\mu_R = \frac{1}{R^{2n}} \sum_{k=0}^{\infty} \lambda_k(R) \delta_{\lambda_k(R)}.$$

Formulas (1) and (2) imply that the total variation of μ_R equals

$$\|\mu_R\| = \frac{1}{R^{2n}} \sum_{k=0}^{\infty} \lambda_k(R) = \int_{\mathbb{R}^{2n}} b(\eta) d\eta.$$

Let us define a measure μ by the integral formula

$$\int_0^1 h(t) d\mu(t) = \int_{\mathbb{R}^{2n}} b(\eta) h(b(\eta)) d\eta.$$

With this interpretation, formula (3) is equivalent to the statement that the measures μ_R converge weakly to μ as $R \rightarrow \infty$. A standard approximation argument implies that it is enough to prove (3) for $h(t) = t^n$, $n = 0, 1, 2, \dots$

Formula (3) directly leads to descriptions of the asymptotic behavior of the distribution of the eigenvalues and the size of the plunge region. Let us denote the distribution of the sequence $\{\lambda_i(R)\}_{i=0}^{\infty}$ by $N(\delta, R) = |\{i : \lambda_i(R) > \delta\}|$ with $0 < \delta < 1$ and the size of the plunge region by $M(\delta_1, \delta_2, R) = |\{k : \delta_1 < \lambda_k(R) < \delta_2\}|$ with $0 < \delta_1 < \delta_2 < 1$. In the following corollaries we assume that the window ϕ and the symbol b satisfy the same requirements as in Theorem 2.1.

COROLLARY 2.2. *Let $0 < \delta < 1$. If $|\{\eta : b(\eta) = \delta\}| = 0$, then*

$$\lim_{R \rightarrow \infty} \frac{N(\delta, R)}{R^{2n}} = |\{\eta : b(\eta) > \delta\}|. \quad (4)$$

COROLLARY 2.3. *Let $0 < \delta_1 < \delta_2 < 1$. If $|\{\eta : b(\eta) = \delta_i\}| = 0$ for $i = 1, 2$, then*

$$\lim_{R \rightarrow \infty} \frac{M(\delta_1, \delta_2, R)}{R^{2n}} = |\{\eta : \delta_1 < b(\eta) < \delta_2\}|. \quad (5)$$

COMMENTS. (i) Formulas (4) and (5) show that, asymptotically, the eigenvalues $\lambda_i(R)$ mimic the behavior of the symbol function. This phenomenon is related to the fact that, asymptotically, the calculus of Gabor–Toeplitz localization operators becomes a perfect calculus. Such a perfect calculus property is a desired feature of quantization procedures. In our case, the quantization is the assignment of the operator T_b to its symbol b . The perfect calculus property can be expressed (in a trace class sense) as follows: For any two nonnegative, bounded, integrable symbols b^1 and b^2 , we have

$$\lim_{R \rightarrow \infty} \frac{1}{R^{2n}} \|T_{b_R^1 b_R^2} - T_{b_R^1} T_{b_R^2}\|_{S^1} = 0 \quad (6)$$

and

$$\lim_{R \rightarrow \infty} \frac{1}{R^{2n}} \|T_{b_R^1} T_{b_R^2}\|_{S^1} = \lim_{R \rightarrow \infty} \frac{1}{R^{2n}} \|T_{b_R^1 b_R^2}\|_{S^1} = \int_{\mathbb{R}^{2n}} b^1(\eta) b^2(\eta) d\eta.$$

The perfect calculus property (6) holds only in the asymptotic sense. Even for characteristic functions of bounded sets, the map $b \rightarrow T_b$ fails to be a homomorphism.

(ii) Our results indicate that there is some kind of instability in the behavior of the plunge region with respect to perturbations of the symbol. Let us start with a symbol that is a characteristic function of a bounded domain with smooth boundary. Let us also assume that the function defining the reproducing formula is regular. In this case the plunge region grows no faster than R^{2n-1} (see [RT]). If we perturb the symbol and make it attain values that are smaller than 1 on a set of positive measure but keep it bounded by 1, then the plunge region changes its behavior. After the perturbation, it grows asymptotically with the rate R^{2n} .

(iii) If the symbol b is a characteristic function of a bounded domain Ω , then its distribution function is constant on $[0, 1)$ and its value coincides with the volume of Ω , denoted by $|\Omega|$. In this case it is relatively straightforward to derive an estimate of the second term of the asymptotic expansion of $N(\delta, R)$ —that is, an estimate of the error made by substituting $R^{2n}|\Omega|$ for $N(\delta, R)$:

$$N(\delta, R) - R^{2n}|\Omega| \leq c_\delta R^{2n} \left(|\Omega| - \iint R^{2n} \Phi(R(\zeta - \eta)) \chi_\Omega(\zeta) \chi_\Omega(\eta) d\eta d\zeta \right), \quad (7)$$

where $c_\delta = \max(\frac{1}{\delta}, \frac{1}{1-\delta})$ and $\Phi(\zeta) = |\langle \phi, \phi_\zeta \rangle|^2$. If the domain is regular enough and the function Φ has sufficient decay at infinity, then the right-hand side of (7) may be controlled by cR^{2n-1} (see [RT]).

(iv) The operators T_b with $b = \chi_\Omega$, where Ω is a bounded domain, resemble the operators investigated in [W]. The main difference is that twisted convolution is used in our context, whereas Widom makes use of the standard convolution.

(v) Our results show that it is possible to replace the \liminf in Theorem 1 of [RT] by a true limit without making the additional assumptions formulated in Corollary 1 of [RT].

The proofs of (6) and (7) will be presented at the end of Section 3.

3. Proofs

Since $\|\phi\|_{L^2(\mathbb{R}^n)} = 1$, it follows from the Gabor reproducing formula that the Gabor transform $W_\phi: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^{2n})$ given by the formula

$$W_\phi f(q, p) = \langle f, \phi_{q,p} \rangle = F(q, p)$$

is an isometry and that the integral operator $P_\phi: L^2(\mathbb{R}^{2n}) \rightarrow L^2(\mathbb{R}^{2n})$,

$$P_\phi H(q, p) = \int_{\mathbb{R}^{2n}} H(s, r) \langle \phi_{s,r}, \phi_{q,p} \rangle ds dr,$$

is an orthogonal projection onto $W_\phi(L^2(\mathbb{R}^n))$. It is easy to check that the operator $P_\phi M_b P_\phi$ has the matrix representation

$$\begin{bmatrix} W_\phi T_b W_\phi^* & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to the decomposition $L^2(\mathbb{R}^{2n}) = W_\phi(L^2(\mathbb{R}^n)) \oplus W_\phi(L^2(\mathbb{R}^n))^\perp$. This matrix representation shows that, as far as our results are concerned, we may identify the operators T_b and $P_\phi M_b P_\phi$.

By $H_b = (I - P_\phi)M_b P_\phi$ we denote the Hankel operator with the symbol b . The proof of our main theorem is based on the following lemma.

LEMMA 3.1. *If b is a nonnegative, bounded, integrable function defined on \mathbb{R}^{2n} , then*

$$\lim_{R \rightarrow \infty} \frac{\|H_{b_R}\|_{S^2}}{R^n} = 0, \quad (8)$$

where $\|\cdot\|_{S^2}$ denotes the Hilbert–Schmidt norm.

Proof. Let $c(\eta) = b^{1/2}(\eta)$. Observe that the integral kernel of $M_{c_R} P_\phi M_{c_R}$ equals

$$c_R(\zeta) \langle \phi_\eta, \phi_\zeta \rangle c_R(\eta).$$

We have

$$\begin{aligned} \operatorname{tr}(P_\phi M_{b_R} P_\phi)^2 &= \operatorname{tr}(M_{c_R} P_\phi M_{c_R})^2 \\ &= R^{2n} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} R^{2n} \Phi(R(\zeta - \eta)) b(\eta) b(\zeta) d\eta d\zeta, \end{aligned} \quad (9)$$

where $\Phi(\eta) = |\langle \phi, \phi_\eta \rangle|^2$. Since $\int_{\mathbb{R}^{2n}} \Phi(\eta) d\eta = 1$, $\Phi \geq 0$, and $b \in L^2(\mathbb{R}^{2n})$, we obtain

$$\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} R^{2n} \Phi(R(\zeta - \eta)) b(\eta) b(\zeta) d\eta d\zeta \rightarrow \|b\|_{L^2(\mathbb{R}^{2n})}^2 \quad (10)$$

as $R \rightarrow \infty$. It follows from (9) and (10) that

$$\lim_{R \rightarrow \infty} \frac{\operatorname{tr}(P_\phi M_{b_R} P_\phi)^2}{R^{2n}} = \|b\|_{L^2}^2.$$

The proof of (8) is completed by observing that

$$P_\phi M_{b_R}^2 P_\phi - (P_\phi M_{b_R} P_\phi)^2 = P_\phi M_{b_R} (I - P_\phi) M_{b_R} P_\phi$$

and that

$$\operatorname{tr}(P_\phi M_{b_R}^2 P_\phi) = R^{2n} \|b\|_{L^2(\mathbb{R}^{2n})}^2. \quad \square$$

REMARK. The preceding statement remains true for all nonnegative $b \in L^2(\mathbb{R}^{2n})$; one need only adjust properly the definition of Hankel operators.

Proof of Theorem 2.1. In view of Remark 2, it is enough to prove that

$$\lim_{R \rightarrow \infty} \frac{\operatorname{tr}(P_\phi M_{b_R} P_\phi)^m}{R^{2n}} = \int_{\mathbb{R}^{2n}} b^m(\eta) d\eta \quad (11)$$

for $m = 1, 2, 3, \dots$. Let $Q_\phi = I - P_\phi$. We observe that

$$\begin{aligned} P_\phi M_{b_R}^m P_\phi &= P_\phi M_{b_R} (P_\phi + Q_\phi) M_{b_R} \dots M_{b_R} P_\phi \\ &= (P_\phi M_{b_R} P_\phi)^m + \text{remaining terms}, \end{aligned} \quad (12)$$

and each of the remaining terms contains a block of the form

$$\dots P_\phi M_{b_R} Q_\phi M_{b_R} Q_\phi \dots Q_\phi M_{b_R} P_\phi \dots,$$

with Q_ϕ occurring at least once. In the second and all remaining occurrences of Q_ϕ we replace Q_ϕ with $I - P_\phi$ and expand the resulting expression into a sum of products. We thus obtain a sum of terms

$$\dots P_\phi M_{b_R} Q_\phi M_{b_R}^k P_\phi \dots$$

We conclude that

$$|\text{tr}(\text{remaining terms})|$$

can be estimated from above by a sum of expressions of the form

$$\|P_\phi M_{b_R} Q_\phi M_{b_R}^k P_\phi\|_{S^1},$$

where S^1 denotes the trace class. By the Cauchy–Schwartz inequality we obtain

$$\|P_\phi M_{b_R} Q_\phi M_{b_R}^k P_\phi\|_{S^1} \leq \|P_\phi M_{b_R} Q_\phi\|_{S^2} \|Q_\phi M_{b_R}^k P_\phi\|_{S^2}. \quad (13)$$

Now we may combine (12), (13), and Lemma 3.1 to obtain (11):

$$\lim_{R \rightarrow \infty} \frac{\text{tr}(P_\phi M_{b_R} P_\phi)^m}{R^{2n}} = \lim_{R \rightarrow \infty} \frac{\text{tr}(P_\phi M_{b_R}^m P_\phi)}{R^{2n}} = \int_{\mathbb{R}^{2n}} b^m(\eta) d\eta. \quad \square$$

Proof of Corollary 2.2. Let $g_\delta(t) = \chi_{(\delta, 1]}(t)/t$ and let $g_{\delta, \varepsilon}^+$, $g_{\delta, \varepsilon}^-$ be continuous approximations of g_δ satisfying the following conditions:

- (i) $0 \leq g_{\delta, \varepsilon}^- \leq g_\delta \leq g_{\delta, \varepsilon}^+ \leq 1/\delta$,
- (ii) $g_{\delta, \varepsilon}^-$ and $g_{\delta, \varepsilon}^+$ coincide with g_δ outside the interval $[\delta - \varepsilon, \delta + \varepsilon]$.

Observe that

$$N(\delta, R) = \text{tr}(T_{b_R} g_\delta(T_{b_R})),$$

$$\text{tr}(T_{b_R} g_{\delta, \varepsilon}^-(T_{b_R})) \leq N(\delta, R) \leq \text{tr}(T_{b_R} g_{\delta, \varepsilon}^+(T_{b_R})), \quad (14)$$

and

$$\begin{aligned} \frac{\text{tr}(T_{b_R} g_{\delta, \varepsilon}^+(T_{b_R})) - \text{tr}(T_{b_R} g_{\delta, \varepsilon}^-(T_{b_R}))}{R^{2n}} &= \frac{\text{tr}(T_{b_R} (g_{\delta, \varepsilon}^+ - g_{\delta, \varepsilon}^-)(T_{b_R}))}{R^{2n}} \\ &\rightarrow \int_{\mathbb{R}^{2n}} b(\eta) (g_{\delta, \varepsilon}^+ - g_{\delta, \varepsilon}^-)(b(\eta)) d\eta \leq \frac{1}{\delta} |\{\eta : \delta - \varepsilon \leq b(\eta) \leq \delta + \varepsilon\}|. \end{aligned} \quad (15)$$

Formulas (14) and (15), together with the fact that

$$|\{\eta : \delta - \varepsilon \leq b(\eta) \leq \delta + \varepsilon\}| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

imply that $N(\delta, R)/R^{2n}$ converges as $R \rightarrow \infty$. Clearly the limit equals $|\{\eta : b(\eta) > \delta\}|$. \square

Proof of Corollary 2.3. The proof follows directly from Corollary 2.2. \square

Proof of (6). It is enough to apply the Cauchy–Schwartz inequality and Lemma 3.1 to the identity

$$P_\phi M_{b_R^1 b_R^2} P_\phi - P_\phi M_{b_R^1} P_\phi M_{b_R^2} P_\phi = H_{b_R^1}^* H_{b_R^2}. \quad \square$$

Proof of (7). Let $g_1(t) = t\chi_{[0,\delta]}(t)$ and $g_2(t) = (1-t)\chi_{(\delta,1]}(t)$. Clearly,

$$N(\delta, R) - R^{2n}|\Omega| = \text{tr}(\chi_{(\delta,1]}T_{R\Omega} - T_{R\Omega}) = -\text{tr} g_1(T_{R\Omega}) + \text{tr} g_2(T_{R\Omega}).$$

We obtain

$$\begin{aligned} & |N(\delta, R) - R^{2n}|\Omega|| \\ & \leq \text{tr}(g_1 + g_2)(T_{R\Omega}) \leq \max\left(\frac{1}{\delta}, \frac{1}{1-\delta}\right) \text{tr}(T_{R\Omega} - T_{R\Omega}^2) \\ & = \max\left(\frac{1}{\delta}, \frac{1}{1-\delta}\right) R^{2n} \left(|\Omega| - \iint R^{2n} \Phi(R(\zeta - \eta)) \chi_{\Omega}(\zeta) \chi_{\Omega}(\eta) d\eta d\zeta \right). \end{aligned}$$

The last equality follows from the proof of Lemma 3.1. \square

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