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UNIFORM EIGENVALUE ESTIMATES FOR TIME-FREQUENCY LOCALIZATION OPERATORS

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ABSTRACT

Time-variant filters based on Calderón and Gabor reproducing formulas are important tools in time-frequency analysis. The paper studies the behavior of the eigenvalues of these filters. Optimal two-sided estimates of the number of eigenvalues contained in the interval (δ_1, δ_2), where $0 < \delta_1 < \delta_2 < 1$, are obtained. The estimates cover large classes of localization domains and generating functions.

1. *Introduction and statements of the results*

Calderón–Toeplitz and Gabor–Toeplitz operators arise naturally in two contexts:

(i) Toeplitz operators on Fock and Bergman spaces of holomorphic functions;

(ii) time-variant filters based on Calderón and Gabor reproducing formulas.

This paper is concerned with the eigenvalues of a subclass of Calderón–Toeplitz and Gabor–Toeplitz operators which have characteristic functions of bounded domains as symbols. Operators of this class are called time-frequency localization operators. The basic idea of functional calculus is that the operators resemble the main algebraic features of their symbols. We consider symbols that are idempotent with respect to pointwise multiplication, so it is natural to expect that the corresponding operators are at least approximately idempotent. It is easy to verify that time-frequency localization operators are compact, self-adjoint and bounded by 1. In view of these facts and the above-mentioned correspondence principle, one is inclined to think that localization operators should resemble finite dimensional orthogonal projections. We show that this expectation is correct for Gabor–Toeplitz operators and that it is false for Calderón–Toeplitz operators. We identify the basic geometric features responsible for these two different behaviors. Our principal results are two-sided estimates of the number of eigenvalues inside the plunge region corresponding to δ_1, δ_2, where $0 < \delta_1 < \delta_2 < 1$. The plunge region consists of the set of indices of the eigenvalues contained inside the open interval (δ_1, δ_2). The eigenvalues are ordered non-increasingly. Our work generalizes and improves previous results of Daubechies, Paul, Ramanathan and Topiwala [**6**, **8**, **21**].

We now state and discuss our results. Denote by $\lambda_k(\Omega, g)$ the eigenvalues of the time-frequency localization operator $T_{\Omega,g}$ with domain of localization Ω and generating function g (see Section 2 for definitions). Let $M(\delta_1, \delta_2, \Omega, g)$ denote the size of the plunge region corresponding to δ_1, δ_2, that is the number of eigenvalues $\lambda_k(\Omega, g)$ contained inside the open interval (δ_1, δ_2).

The plunge region is a fundamental notion in the theory of filtering. It is the

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region where the transition of the eigenvalues from being close to 1 to being close to 0 occurs. The relative size of the plunge region measures the extent to which the operator fails to be an orthogonal projection. Our estimates of $M(\delta_1, \delta_2, \Omega, g)$ are contained in Theorems 3.1 and 4.1. Theorem 3.1 treats the case of Calderón–Toeplitz operators. We prove that

$$c_1|\Omega| \leqslant M(\delta_1, \delta_2, \Omega, g) \leqslant c_2|\Omega|,$$

where $|\Omega|$ is the hyperbolic volume of the domain of localization and c_1, c_2 are positive constants. The constants c_1, c_2 do not depend on Ω and g as long as conditions (12)–(16) are satisfied (see Section 3). Theorem 4.1 deals with Gabor–Toeplitz operators. We obtain the estimates

$$c_1|(\partial\Omega)^R| \leqslant M(\delta_1, \delta_2, \Omega, g) \leqslant c_2|(\partial\Omega)^R|,$$

where $(\partial\Omega)^R = \{\gamma \in \mathbb{R}^{2n} : d(\gamma, \partial\Omega) < R\}$ is the Euclidean strip of size R around the boundary of the domain Ω, and $|(\partial\Omega)^R|$ is its Euclidean volume. The constants c_1, c_2 do not depend on Ω and g if conditions (25)–(29) hold (see Section 4). In both cases the main point of interest is the rate of growth of the number of eigenvalues inside the plunge region, which is related to expansions of the domain of localization. A typical example in the context of Calderón–Toeplitz operators is the family of hyperbolic balls with a fixed center and with radii bigger than some positive constant. The assumptions of Theorem 3.1 are satisfied in this case and the size of the plunge region is comparable to the volume of the domain of localization. Let us mention two examples related to Gabor–Toeplitz operators. Recall that if the assumptions of Theorem 4.1 are satisfied then the size of the plunge region is comparable to the volume of a fixed strip around the boundary of the domain of localization. In the first example we take a bounded domain with a smooth boundary and we subject it to dilations by a factor greater than some positive constant. The second example is produced by dilating the sides of a unit cube by powers of a given factor, where the powers depend on the sides. As in the first example, we require the dilation factor to be larger than a given positive constant. In both examples the rate of growth of the plunge region size is explicitly computable. We refer the reader to the comments at the end of Sections 3 and 4 for a more detailed discussion.

Let us assume for a moment that the assumptions of Theorems 3.1 and 4.1 hold. For both Calderón–Toplitz and Gabor–Toeplitz operators the number of eigenvalues that are close to 1 is comparable to the volume of the domain of localization. Theorem 3.1 shows that the size of the plunge region where the eigenvalues change from δ_2 to δ_1 is large compared with the size of the region where they are close to 1. Therefore Calderón–Toeplitz localization operators fail to be similar to finite dimensional projections. Theorem 4.1 shows that the size of the plunge region is small compared with the size of the region where the eigenvalues are close to 1. It follows that Gabor–Toeplitz localization operators much better resemble finite dimensional projections.

Calderón–Toeplitz and Gabor–Toeplitz operators are important tools in time-variant filtering. They provide ways of analyzing signals by describing their frequency content as it varies over time. Their study has involved many people, such as Daubechies, Flandrin, Heil, Paul, Rochberg, Ramanathan, Topiwala and others [**7**, **8**, **12**, **14**, **21**, **22**]. Recently the subject has received increasing attention in the context of time-frequency analysis, a large research area including time-variant filtering (see [**1**, **11**, **13**, **20**, **23**]).

Toeplitz operators on Fock and Bergman spaces constitute a rich class of extensively studied operators. Pioneering work was done by Berezin. Major development took place in the late 1980s by Axler, Berger, Coburn, Luecking, Zheng, Zhu and others (see [**25**] and references therein), and there is still very active ongoing research on the topic [**2**–**5**, **15**]. Calderón–Toeplitz and Gabor–Toeplitz operators include Toeplitz operators on classical Fock spaces and weighted Bergman spaces of the unit disk, thus our results apply in this important context as well. Moreover, it is possible to extend our methods to cover Bergman spaces on bounded symmetric domains [**9**].

Our analysis of eigenvalue distributions follows to a large extent the approach taken by Widom in [**24**]. Many of our computational techniques are based on ideas of Axler, Berger, Coburn and Zhu developed in the context of Toeplitz and Hankel operators defined on Bergman and Fock spaces. In several cases we follow the methods developed by Landau and Widom [**16**] in the context of classical time-frequency localization. One should also remark that our results may be interpreted as two-sided estimates of the second term in the asymptotic expansion of the eigenvalue distribution.

2. *Definitions and basic facts*

The *Calderón reproducing formula* has the form

$$f = \int_0^\infty \int_{\mathbb{R}^n} \langle f, \psi_{u,s}\rangle \psi_{u,s} \frac{du\,ds}{s^{n+1}}, \tag{1}$$

where $f, \psi \in L^2(\mathbb{R}^n)$, $\hat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(x) e^{-2\pi i x\xi} dx$ satisfies $\int_0^\infty |\hat{\psi}(s\xi)|^2 (ds/s) = 1$ for almost every $\xi \in \mathbb{R}^n$, and $\psi_{u,s}(x) = s^{-n/2}\psi((x-u)/s)$, $u \in \mathbb{R}^n$, $s > 0$. The *Gabor reproducing formula* is defined by

$$f = \int_{\mathbb{R}^{2n}} \int_{R^{2n}} \langle f, \phi_{q,p}\rangle \phi_{q,p}\, dq\, dp, \tag{2}$$

where $f, \phi \in L^2(\mathbb{R}^n)$, $||\phi||_{L^2(\mathbb{R}^n)} = 1$, $\phi_{q,p}(x) = e^{2\pi i p x}\phi(x-q)$, $q, p \in \mathbb{R}^n$. For simplicity, we introduce a unified notation, which covers both cases (1) and (2). We will write

$$f = \int_\Pi \langle f, g_\gamma\rangle g_\gamma d\gamma, \tag{3}$$

where Π denotes either $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$ or $\mathbb{R}^{2n}$. In the first case $\gamma = (u, s)$, $u \in \mathbb{R}^n$, $s > 0$, $g_\gamma(x) = s^{-n/2} g((x-u)/s)$ is the action of $\mathbb{R}^{n+1}_+$ on $L^2(\mathbb{R}^n)$, and $d\gamma = du\,ds/s^{n+1}$. In the second case $\gamma = (q, p)$, $q, p \in \mathbb{R}^n$, $g_\gamma(x) = e^{2\pi i p x} g(x-q)$ is the action of $\mathbb{R}^{2n}$ on $L^2(\mathbb{R}^n)$, and $d\gamma = dq\,dp$. The function g is called the *generating function* of the reproducing formula (3). For a bounded function b defined on Π the *Toeplitz operator* $T_{b,g}$, acting on $L^2(\mathbb{R}^n)$, is defined as

$$T_{b,g} f = \int_\Pi b(\gamma)\langle f, g_\gamma\rangle g_\gamma d\gamma. \tag{4}$$

The function b is called the *symbol* of $T_{b,g}$. In case (1) the operator $T_{b,g}$ is called the *Calderón–Toeplitz operator*, and in case (2) the *Gabor–Toeplitz operator*. If b equals χ_Ω, the characteristic function of a bounded domain $\Omega \subset \Pi$, then we denote the corresponding Toeplitz operator by $T_{\Omega,g}$ and we call it the *time-frequency localization operator*. The set Ω is called the *domain of localization*.

It is straightforward to check that the operator norm of $T_{b,g}$ satisfies

$$||T_{b,g}|| \leqslant ||b||_{L^\infty(\Pi, d\gamma)}, \tag{5}$$

that $T_{b,g}$ is trace class if b is integrable, with

$$\mathrm{tr}\, T_{b,g} = ||g||^2_{L^2(\mathbb{R}^n)} \int_\Pi b(\gamma)\, d\gamma, \tag{6}$$

and that for non-negative b

$$T_{b,g} \geqslant 0. \tag{7}$$

It follows from (5), (6) and (7) that time-frequency localization operators $T_{\Omega,g}$ are non-negative and trace class, and that their eigenvalues $\lambda_k(\Omega, g)$ satisfy

$$0 \leqslant \lambda_k(\Omega, g) \leqslant 1, \quad \sum_{k=0}^{\infty} \lambda_k(\Omega, g) = ||g||^2_{L^2(\mathbb{R}^n)} |\Omega|, \tag{8}$$

where $|\Omega|$ denotes the hyperbolic volume of Ω in case (1), and the Euclidean volume of Ω in case (2).

Now we indicate the relationship between operators $T_{b,g}$ and Toeplitz operators acting on Hilbert spaces with reproducing kernels. It follows directly from the reproducing formula (3) that the transform $W_g : L^2(\mathbb{R}^n) \longrightarrow L^2(\Pi, d\gamma)$ given by

$$W_g f(\gamma) = \langle f, g_\gamma \rangle$$

is an isometry and that the integral operator $P_g : L^2(\Pi, d\gamma) \longrightarrow L^2(\Pi, d\gamma)$

$$P_g H(\zeta) = \int_\Pi H(\eta) \langle g_\eta, g_\zeta \rangle d\eta$$

is an orthogonal projection onto $W_g(L^2(\mathbb{R}^n))$. Thus, if M_b denotes multiplication by b on $L^2(\Pi, d\gamma)$, the operator $P_g M_b P_g$ is a Toeplitz operator acting on the Hilbert space $W_g(L^2(\mathbb{R}^n))$ and it has the matrix representation

$$\begin{bmatrix} W_g T_{b,g} W_g^* & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to the decomposition $L^2(\Pi, d\gamma) = W_g(L^2(\mathbb{R}^n)) \oplus W_g(L^2(\mathbb{R}^n))^\perp$. The above representation shows that as far as nonzero eigenvalues are concerned we may substitute $T_{b,g}$ by $P_g M_b P_g$.

The main tools in our analysis of localization operators are *Hankel operators*

$$H_{b,g} = (I - P_g) M_b P_g$$

and *commutators*

$$C_{b,g} = [M_b, P_g] = M_b P_g - P_g M_b.$$

They are defined for bounded symbols b. Later in this paper we will use the following formulas relating commutators, Hankel and Toeplitz operators

$$P_g M_{\chi_\Omega} P_g - (P_g M_{\chi_\Omega} P_g)^2 = H^*_{\Omega,g} H_{\Omega,g}, \tag{9}$$

$$C_{\Omega,g} = H_{\Omega,g} - H^*_{\Omega,g}, \tag{10}$$

$$H_{\Omega,g} = C_{\Omega,g} P_g. \tag{11}$$

It is easy to check the above formulas by direct computation.

3. *Calderón–Toeplitz localization operators*

The upper half space $\mathbb{R}_+^{n+1}$ is equipped with the hyperbolic metric d, given by

$$\cosh(d(\zeta,\eta)) = 1 + \frac{|\zeta-\eta|^2}{2st},$$

where $\zeta = (u,s)$, $\eta = (v,t)$, $u,v \in \mathbb{R}^n$, $s,t > 0$, and with the hyperbolic measure

$$d\zeta = \frac{du\,ds}{s^{n+1}}.$$

By $B(\gamma,r)$ we denote the hyperbolic ball with center γ and radius r, and by $|\Omega|$ the hyperbolic volume of a set $\Omega \subset \mathbb{R}^{n+1}$, that is, its hyperbolic measure. The *averaging operator* $A_r : L^2(\mathbb{R}_+^{n+1}, d\zeta) \longrightarrow L^2(\mathbb{R}_+^{n+1}, d\zeta)$ is defined as

$$A_r b(\eta) = |B(\eta,r)|^{-1} \int_{B(\eta,r)} b(\gamma)\,d\gamma.$$

For all $r > 0$ the operator norm a_r of A_r, satisfies $a_r < 1$ [**19**, Lemma 5.2]. For $M > 0$ let

$$I(M) = \int_{\mathbb{R}_+^{n+1}} e^{-Md(\zeta,e)}\,d\zeta,$$

where $e = (0,\ldots,0,1)$. It is a standard fact that $I(M) < \infty$ for $M > n$ and $I(M) = \infty$ for $M \leqslant n$ (see for example [**18**]). For $R > 0$ let $L(R)$ denote the supremum of the set consisting of those numbers $r > 0$ for which there exists a sequence $\{\gamma_i\}_{i=1}^{\infty} \subset \mathbb{R}_+^{n+1}$ such that the balls $B(\gamma_i, R)$ cover $\mathbb{R}_+^{n+1}$, and the balls $B(\gamma_i, r)$ are pairwise disjoint. It is easy to see that $L(R) \geqslant R/2$ for every $R > 0$.

Let Ψ be a family of functions $\psi \in L^2(\mathbb{R}^n)$ satisfying the following conditions:

$$\|\psi\|^2_{L^2(\mathbb{R}^n)} \leqslant N_\Psi, \tag{12}$$

$$\int_0^\infty |\hat{\psi}(s\xi)|^2 \frac{ds}{s} = 1 \quad \text{for almost everywhere } \xi \in \mathbb{R}^n, \tag{13}$$

$$|\langle \psi_\eta, \psi_\zeta\rangle|^2 \geqslant \frac{c_{r_\Psi}}{|B(e,r_\Psi)|^2} \int_{\mathbb{R}_+^{n+1}} \chi_{B(\gamma,r_\Psi)}(\eta)\chi_{B(\gamma,r_\Psi)}(\zeta)\,d\gamma, \tag{14}$$

$$|\langle \psi_\eta, \psi_\zeta\rangle|^2 \leqslant C_\Psi e^{-M_\Psi d(\zeta,\eta)}, \tag{15}$$

where N_Ψ, r_Ψ, c_{r_Ψ}, C_Ψ are positive constants and $M_\Psi > n$.

Let $\mathscr{C}$ be a family of bounded domains Ω contained in $\mathbb{R}_+^{n+1}$ satisfying the following condition:

$$|\Omega^{R_\mathscr{C}}| \leqslant C_\mathscr{C} |\Omega|, \tag{16}$$

where $R_\mathscr{C}$, $C_\mathscr{C}$ are positive constants, and

$$\Omega^R = \{\gamma : d(\gamma,\Omega) < R\}.$$

Recall that for $0 < \delta_1 < \delta_2 < 1$ the size of the plunge region of the eigenvalues $\lambda_k(\Omega,\psi)$ is defined as

$$M(\delta_1,\delta_2,\Omega,\psi) = |\{k : \delta_1 < \lambda_k(\Omega,\psi) < \delta_2\}|.$$

Now we state the main result of this section.

THEOREM 3.1. *If Ψ and $\mathscr{C}$ are the families of generating functions and localization* *domains satisfying conditions* (12)–(16), *then for any* $\delta_1 > 0$ *sufficiently close to* 0 *and any* $\delta_2 < 1$ *sufficiently close to* 1 *there are positive constants* c_1, c_2 *such that for all* $\psi \in \Psi$, $\Omega \in \mathscr{C}$

$$c_1|\Omega| \leqslant M(\delta_1, \delta_2, \Omega, \psi) \leqslant c_2|\Omega|.$$

The proof is based on the following lemmas.

LEMMA 3.2. *If* $\psi \in L^2(\mathbb{R}^n)$ *satisfies conditions* (13) *and* (14), *then for any bounded domain* $\Omega \subset \mathbb{R}^{n+1}_+$

$$\sum_k \lambda_k(\Omega, \psi)(1 - \lambda_k(\Omega, \psi)) \geqslant c_{r_\Psi} \frac{1 - a^2_{r_\Psi}}{2} |\Omega|.$$

LEMMA 3.3. *If* $\psi \in L^2(\mathbb{R}^n)$ *satisfies conditions* (13) *and* (15), *and* $\Omega \subset \mathbb{R}^{n+1}_+$ *is a bounded domain satisfying condition* (16), *then for any* $0 < p \leqslant 1$

$$\sum_k \lambda_k^p(\Omega, \psi) \leqslant C_\Psi^p e^{pM_\Psi R_{\mathscr{C}}/2} I(pM_\Psi) \frac{\left|B\left(e, \frac{R_{\mathscr{C}}}{2}\right)\right|^p}{\left|B\left(e, L\left(\frac{R_{\mathscr{C}}}{2}\right)\right)\right|} C_{\mathscr{C}}|\Omega|.$$

Proof of Theorem 3.1. For $0 < \delta_1 < \delta_2 < 1$ let

$$I(\Omega, \psi, \delta_2) = \{k : \lambda_k(\Omega, \psi) \geqslant \delta_2\},$$
$$II(\Omega, \psi, \delta_1, \delta_2) = \{k : \delta_1 < \lambda_k(\Omega, \psi) < \delta_2\},$$
$$III(\Omega, \psi, \delta_1) = \{k : \lambda_k(\Omega, \psi) \leqslant \delta_1\}.$$

The upper bound follows immediately from the trace formula (8). Indeed, we have

$$|II| \leqslant \frac{1}{\delta_1(1-\delta_2)} \sum_k \lambda_k(1 - \lambda_k) \leqslant \frac{1}{\delta_1(1-\delta_2)} \sum_k \lambda_k = \frac{||\psi||^2_{L^2}|\Omega|}{\delta_1(1-\delta_2)} \leqslant \frac{N_\Psi|\Omega|}{\delta_1(1-\delta_2)}.$$

By Lemma 3.2 we obtain

$$c_{r_\Psi} \frac{1 - a^2_{r_\Psi}}{2} |\Omega| \leqslant \sum_k \lambda_k(1 - \lambda_k) = \sum_I + \sum_{II} + \sum_{III}. \tag{17}$$

Observe that

$$\sum_I \lambda_k(1 - \lambda_k) \leqslant (1 - \delta_2) N_\Psi |\Omega|. \tag{18}$$

It follows from Lemma 3.3 that for any $0 < p \leqslant 1$

$$\sum_{III} \lambda_k(1 - \lambda_k) \leqslant \sum_{III} \lambda_k \leqslant \delta_1^{1-p} \sum_k \lambda_k^p$$
$$\leqslant \delta_1^{1-p} C_\Psi^p e^{pM_\Psi R_{\mathscr{C}}/2} I(pM_\Psi) \frac{\left|B\left(e, \frac{R_{\mathscr{C}}}{2}\right)\right|^p}{\left|B\left(e, L\left(\frac{R_{\mathscr{C}}}{2}\right)\right)\right|} C_{\mathscr{C}}|\Omega|. \tag{19}$$

Take p close enough to 1 so that $pM_\Psi > n$. Since $I(pM_\Psi) < \infty$ we may conclude from (17), (18) and (19) that for δ_2 sufficiently close to 1 and δ_1 sufficiently close to 0

$$\sum_{II} \lambda_k(1-\lambda_k) \geqslant c_1|\Omega| \tag{20}$$

for some positive constant c_1. Clearly (20) implies the lower bound. □

Proof of Lemma 3.2. From (9) and (10) it follows that

$$\sum_k \lambda_k(1-\lambda_k) = \|H_{\Omega,\psi}\|_{S^2}^2 \geqslant \tfrac{1}{4}\|C_{\Omega,\psi}\|_{S^2}^2, \tag{21}$$

where $\|\cdot\|_{S^2}$ is the Hilbert–Schmidt norm. Since the integral kernel of $C_{\Omega,\psi}$ equals

$$(\chi_\Omega(\zeta)-\chi_\Omega(\eta))\langle\psi_\eta,\psi_\zeta\rangle,$$

by condition (14) we obtain

$$\begin{aligned}\|C_{\Omega,\psi}\|_{S^2}^2 &\geqslant \frac{c_{r_\Psi}}{|B(e,r_\Psi)|^2}\int\int\int(\chi_\Omega(\zeta)-\chi_\Omega(\eta))^2\chi_{B(\gamma,r_\Psi)}(\eta)\chi_{B(\gamma,r_\Psi)}(\zeta)\,d\gamma\,d\eta\,d\zeta\\ &= 2c_{r_\Psi}(|\Omega|-\|A_{r_\Psi}\chi_\Omega\|_{L^2}^2) \geqslant 2c_{r_\Psi}(1-a_{r_\Psi}^2)|\Omega|.\end{aligned} \tag{22}$$

Combining (21) and (22) we achieve the desired estimate. □

Proof of Lemma 3.3. Observe that if $\{\gamma_i\}_{i=1}^\infty \subset \mathbb{R}_+^{n+1}$ is such a sequence that the balls $B(\gamma_i,R)$ cover $\mathbb{R}_+^{n+1}$, and b is a non-negative, bounded, integrable function defined on $\mathbb{R}_+^{n+1}$, then

$$\sum_k \lambda_k^p(b,\psi) \leqslant C_\Psi^p e^{pM_\Psi R} I(pM_\Psi)\sum_i\left(\int_{B(\gamma_i,R)} b(\zeta)\,d\zeta\right)^p. \tag{23}$$

Indeed, we have

$$\begin{aligned}\sum_k \lambda_k^p(b,\psi) &= \int_{\mathbb{R}_+^{n+1}}\langle T_{b,\psi}^p\psi_\eta,\psi_\eta\rangle\,d\eta\\ &\leqslant \int_{\mathbb{R}_+^{n+1}}\langle T_{b,\psi}\psi_\eta,\psi_\eta\rangle^p\,d\eta\\ &= \int_{\mathbb{R}_+^{n+1}}\left(\int_{\mathbb{R}_+^{n+1}} b(\zeta)|\langle\psi_\zeta,\psi_\eta\rangle|^2 d\zeta\right)^p d\eta\\ &\leqslant \int_{\mathbb{R}_+^{n+1}}\left(\int_{\mathbb{R}_+^{n+1}}\sum_i\chi_{B(\gamma_i,R)}(\zeta)b(\zeta)|\langle\psi_\zeta,\psi_\eta\rangle|^2 d\zeta\right)^p d\eta\\ &\leqslant \int_{\mathbb{R}_+^{n+1}}\sum_i\left(\int_{B(\gamma_i,R)} b(\zeta)C_\Psi e^{-M_\Psi d(\zeta,\eta)}d\zeta\right)^p d\eta\\ &\leqslant C_\Psi^p\sum_i\int_{\mathbb{R}_+^{n+1}}\left(\int_{B(\gamma_i,R)} b(\zeta)e^{-M_\Psi(d(\gamma_i,\eta)-d(\gamma_i,\zeta))}d\zeta\right)^p d\eta\\ &\leqslant C_\Psi^p e^{pM_\Psi R}\sum_i\int_{\mathbb{R}_+^{n+1}} e^{-pM_\Psi d(\gamma_i,\eta)}d\eta\left(\int_{B(\gamma_i,R)} b(\zeta)\,d\zeta\right)^p\\ &= C_\Psi^p e^{pM_\Psi R} I(pM_\Psi)\sum_i\left(\int_{B(\gamma_i,R)} b(\zeta)\,d\zeta\right)^p.\end{aligned}$$

Let us take $\{\gamma_i\}_{i=1}^{\infty} \subset \mathbb{R}_+^{n+1}$ such that the balls $B(\gamma_i, R_\epsilon/2)$ cover $\mathbb{R}_+^{n+1}$, and that the balls $B(\gamma_i, r)$ are pairwise disjoint. We obtain

$$\sum_i \left(\int_{B(\gamma_i, R_\epsilon/2)} \chi_\Omega(\zeta)\, d\zeta\right)^p \leqslant \left|B\left(e, \frac{R_\epsilon}{2}\right)\right|^p \left|\left\{i : B\left(\gamma_i, \frac{R_\epsilon}{2}\right) \cap \Omega \neq \varnothing\right\}\right|$$
$$\leqslant \frac{\left|B\left(e, \frac{R_\epsilon}{2}\right)\right|^p}{|B(e,r)|} |\{\eta : d(\Omega, \eta) < R_\epsilon\}|. \tag{24}$$

It is enough to combine (23), (24) and (16) to complete the proof. □

Our estimates carry over to weighted Bergman spaces on the upper half plane. The reason for this is the fact that the reproducing kernels corresponding to Bergman wavelets ψ_α satisfy appropriate bounds (see **[17]**). For a family Ψ we may take any of the sets $\{\psi_\alpha : 0 < \epsilon_\Psi \leqslant \alpha \leqslant M_\Psi\}$ for any pair of positive constants ϵ_Ψ, M_Ψ.

Let us discuss the conditions imposed on generating functions.

(i) Both conditions (14) and (15) have radial character with respect to the hyperbolic distance and they are nicely adjusted to the natural geometry of $\mathbb{R}_+^{n+1}$.

(ii) Condition (14) expresses the fact that there is a uniform control from below of the square of the absolute value of the reproducing kernel by the convolutional square of the characteristic function of a hyperbolic ball with fixed radius. We normalize characteristic functions in L^1.

(iii) Condition (15) is the standard almost diagonality condition. It can be expressed in terms of certain regularity properties of generating functions.

Now we discuss condition (16), imposed on the domains of localization.

(iv) One can directly check that the family of all hyperbolic balls with radii bigger than some $R_0 > 0$ satisfies condition (16).

(v) Condition (16) may fail for the following reason. If Ω consists of isolated thin parts, then the volume of Ω^ϵ is substantially larger than the volume of Ω. If the domains Ω do not contain thin parts then condition (16) holds. Indeed, one may prove that if there are positive constants ϵ, δ such that for all $\gamma \in \Omega$

$$|B(\gamma, \epsilon) \cap \Omega| \geqslant \delta,$$

then

$$|\Omega^\epsilon| \leqslant C_{\epsilon,\delta}|\Omega|,$$

where $C_{\epsilon,\delta}$ is a positive constant which depends on ϵ and δ, but not on Ω.

4. *Gabor–Toeplitz localization operators*

In this section d denotes the Euclidean metric on $\mathbb{R}^{2n}$, $d(\eta,\zeta) = |\eta - \zeta|$, $B(\gamma, r)$ stands for the Euclidean ball with center γ and radius r, and $|\Omega|$ is the Lebesgue measure of Ω.

Let Φ be a family of functions $\phi \in L^2(\mathbb{R}^n)$ satisfying the following conditions:

$$\|\phi\|_{L^2(\mathbb{R}^n)} = 1, \tag{25}$$

$$|\langle\phi_\eta, \phi_\zeta\rangle|^2 \geqslant \frac{c_{r_\Phi}}{|B(0, r_\Phi)|^2} \int_{\mathbb{R}^{2n}} \chi_{B(\gamma, r_\Phi)}(\eta)\chi_{B(\gamma, r_\Phi)}(\zeta)\, d\gamma, \tag{26}$$

$$|\langle\phi_\eta, \phi_\zeta\rangle| \leqslant C_\Phi(1 + |\eta - \zeta|)^{-N_\Phi}, \tag{27}$$

where r_Ψ, c_{r_Ψ}, C_Ψ are positive constants and $N_\Phi > 2n + 1$.

As in the previous section, given a subset $\Omega \subset \mathbb{R}^{2n}$ and $R > 0$, we put

$$\Omega^R = \{\eta \in \mathbb{R}^{2n} : d(\eta, \Omega) < R\}.$$

Let $\mathscr{G}$ be a family of bounded domains $\Omega \subset \mathbb{R}^{2n}$ satisfying the following conditions:

$$|\{\gamma : |B(\gamma, r_\Phi) \cap \Omega| > \delta_{\mathscr{G}}, |B(\gamma, r_\Phi) \cap \Omega^c| > \delta_{\mathscr{G}}\}| \geqslant c_{\mathscr{G}}|(\partial\Omega)^{R_{\mathscr{G}}}|, \tag{28}$$

$$|(S_\Omega^k)^{n^{1/2}}| \leqslant C_{\mathscr{G}}|(\partial\Omega)^{R_{\mathscr{G}}}|, \tag{29}$$

where $R_{\mathscr{G}}$, $\delta_{\mathscr{G}}$, $c_{\mathscr{G}}$, $C_{\mathscr{G}}$ are positive constants, and

$$S_\Omega^k = \{\eta \in \Omega : k + 1 > d(\eta, \partial\Omega) \geqslant k\}.$$

The next theorem is the main result of this section.

THEOREM 4.1. *If Φ and $\mathscr{G}$ are families of generating functions and localization domains satisfying conditions (25)–(29), then for any $\delta_1 > 0$ sufficiently close to 0 and any $\delta_2 < 1$ sufficiently close to 1 there are positive constants c_1, c_2 such that for all $\phi \in \Phi$, $\Omega \in \mathscr{G}$*

$$c_1|(\partial\Omega)^{R_{\mathscr{G}}}| \leqslant M(\delta_1, \delta_2, \Omega, \phi) \leqslant c_2|(\partial\Omega)^{R_{\mathscr{G}}}|.$$

The proof of Theorem 4.1 is based on the following facts involving the singular values $\mu_k(\Omega, \phi)$ of the commutators $C_{\Omega,\phi}$.

PROPOSITION 4.2. *If $\phi \in L^2(\mathbb{R}^n)$ satisfies (25) and $\Omega \subset \mathbb{R}^{2n}$ is a bounded domain, then for all $p \geqslant 1/2$*

$$2^{-2p}\sum_k \mu_k^{2p}(\Omega, \phi) \leqslant \sum_k (\lambda_k(\Omega, \phi)(1 - \lambda_k(\Omega, \phi)))^p \leqslant \sum_k \mu_k^{2p}(\Omega, \phi).$$

LEMMA 4.3. *If $\phi \in L^2(\mathbb{R}^n)$ satisfies (25) and (26) and a bounded domain $\Omega \subset \mathbb{R}^{2n}$ satisfies (28), then*

$$\sum_k \mu_k^2(\Omega, \phi) \geqslant \frac{2c_{r_\Phi} c_{\mathscr{G}} \delta_{\mathscr{G}}^2}{|B(0, r_\Phi)|^2}|(\partial\Omega)^{R_{\mathscr{G}}}|.$$

LEMMA 4.4. *If $\phi \in L^2(\mathbb{R}^n)$ satisfies (25) and (27) and a bounded domain $\Omega \subset \mathbb{R}^{2n}$ satisfies (29), then*

$$\sum_k \mu_k(\Omega, \phi) \leqslant 2C_\Phi C_{\mathscr{G}}(A_{N_\Phi} + B_{N_\Phi})|(\partial\Omega)^{R_{\mathscr{G}}}|,$$

where

$$A_{N_\Phi} = \left(1 + \frac{n^{1/2}}{2}\right)^{N_\Phi} \sum_{k\geqslant 0} \int_{|\gamma| \geqslant k-(n^{1/2}/2)} (1 + |\gamma|)^{-N_\Phi} d\gamma,$$

$$B_{N_\Phi} = \sum_{k\geqslant 0}\left(\int_{|\gamma|\geqslant k} (1 + |\gamma|)^{-2N_\Phi} d\gamma\right)^{1/2}.$$

Proof of Theorem 4.1. In this proof c will denote a constant which may change its value from place to place. As in the proof of Theorem 3.1 we define

$$I(\Omega, \phi, \delta_2) = \{k : \lambda_k(\Omega, \phi) \geqslant \delta_2\},$$
$$II(\Omega, \phi, \delta_1, \delta_2) = \{k : \delta_1 < \lambda_k(\Omega, \phi) < \delta_2\},$$
$$III(\Omega, \phi, \delta_1) = \{k : \lambda_k(\Omega, \phi) \leqslant \delta_1\}.$$

By Proposition 4.2 and Lemma 4.4 we obtain

$$|II| \leqslant \frac{1}{\delta_1(1-\delta_2)}\sum_k \lambda_k(1-\lambda_k) \leqslant \frac{1}{\delta_1(1-\delta_2)}\sum_k \mu_k^2$$
$$\leqslant \frac{2}{\delta_1(1-\delta_2)}\sum_k \mu_k \leqslant \frac{c}{\delta_1(1-\delta_2)}|(\partial\Omega)^{R_{\mathscr{G}}}|,$$

and the upper bound follows. Proposition 4.2 and Lemma 4.3 give

$$\left(\sum_I + \sum_{II} + \sum_{III}\right) \lambda_k(1-\lambda_k) \geqslant \frac{1}{4}\sum_k \mu_k^2 \geqslant c|(\partial\Omega)^{R_{\mathscr{G}}}|. \tag{30}$$

By Proposition 4.2 and Lemma 4.4 we get

$$\sum_I \lambda_k(1-\lambda_k) \leqslant (1-\delta_2)^{1/2}\sum_k \lambda_k^{1/2}(1-\lambda_k)^{1/2} \leqslant c(1-\delta_2)^{1/2}|(\partial\Omega)^{R_{\mathscr{G}}}| \tag{31}$$

and

$$\sum_{III} \lambda_k(1-\lambda_k) \leqslant \delta_1^{1/2}\sum_k \lambda_k^{1/2}(1-\lambda_k)^{1/2} \leqslant c\delta_1^{1/2}|(\partial\Omega)^{R_{\mathscr{G}}}|. \tag{32}$$

Taking δ_1 close enough to 0 and δ_2 close enough to 1, from (30), (31) and (32) we obtain

$$\sum_{II} \lambda_k(1-\lambda_k) \geqslant c|(\partial\Omega)^{R_{\mathscr{G}}}|,$$

where c is some positive constant. This proves the estimate from below. □

Proof of Proposition 4.2. Since λ_k are the eigenvalues of $P_\phi M_{\chi_\Omega} P_\phi$, the eigenvalues of $P_\phi M_{\chi_\Omega} P_\phi - (P_\phi M_{\chi_\Omega} P_\phi)^2$ are $\lambda_k - \lambda_k^2$. The proof follows easily from (9), (10) and (11). □

Proof of Lemma 4.3. The integral kernel of $C_{\Omega,\phi}$ equals

$$(\chi_\Omega(\zeta) - \chi_\Omega(\eta))\langle \phi_\eta, \phi_\zeta\rangle.$$

Condition (26) implies that

$$\sum_k \mu_k^2(\Omega,\phi) \geqslant \frac{c_{r_\Phi}}{|B(0,r_\Phi)|^2}\int_{\mathbb{R}^{2n}}\int_{B(\gamma,r_\Phi)}\int_{B(\gamma,r_\Phi)} (\chi_\Omega(\zeta)-\chi_\Omega(\eta))^2\, d\eta\, d\zeta\, d\gamma$$
$$\geqslant \frac{2c_{r_\Phi}\delta_{\mathscr{G}}^2}{|B(0,r_\Phi)|^2}|\{\gamma : |B(\gamma,r_\Phi)\cap\Omega| > \delta_{\mathscr{G}}, |B(\gamma,r_\Phi)\cap\Omega^c| > \delta_{\mathscr{G}}\}|$$
$$\geqslant \frac{2c_{r_\Phi}c_{\mathscr{G}}\delta_{\mathscr{G}}^2}{|B(0,r_\Phi)|^2}|(\partial\Omega)^{R_{\mathscr{G}}}|$$

and this finishes the proof. □

Proof of Lemma 4.4. Observe that

$$(\chi_\Omega(\zeta)-\chi_\Omega(\eta))\langle\phi_\eta,\phi_\zeta\rangle = \chi_\Omega(\zeta)\chi_{\Omega^c}(\eta)\langle\phi_\eta,\phi_\zeta\rangle - \chi_{\Omega^c}(\zeta)\chi_\Omega(\eta)\langle\phi_\eta,\phi_\zeta\rangle. \tag{33}$$

Denote now by $\|\cdot\|_{S^1}$ the trace class norm. Formulas (2) and (33) imply that

$$\begin{aligned}\|C_{\Omega,\phi}\|_{S^1} &\leqslant \int_{\mathbb{R}^{2n}} \|\chi_\Omega(\zeta)\chi_{\Omega^c}(\eta)\langle\phi_\eta,\phi_\gamma\rangle\langle\phi_\gamma,\phi_\zeta\rangle\|_{S^1}\,d\gamma \\ &\quad + \int_{\mathbb{R}^{2n}} \|\chi_{\Omega^c}(\zeta)\chi_\Omega(\eta)\langle\phi_\eta,\phi_\gamma\rangle\langle\phi_\gamma,\phi_\zeta\rangle\|_{S^1}\,d\gamma \\ &= 2\int_{\mathbb{R}^{2n}} \|\chi_\Omega(\zeta)\langle\phi_\zeta,\phi_\gamma\rangle\|_{L^2}\|\chi_{\Omega^c}(\eta)\langle\phi_\eta,\phi_\gamma\rangle\|_{L^2}\,d\gamma \\ &= 2\left(\int_\Omega + \int_{\Omega^c}\right)\|\chi_\Omega(\zeta)\langle\phi_\zeta,\phi_\gamma\rangle\|_{L^2}\|\chi_{\Omega^c}(\eta)\langle\phi_\eta,\phi_\gamma\rangle\|_{L^2}\,d\gamma. \qquad (34)\end{aligned}$$

We start by estimating $\int_{\Omega^c}$. Let $Q_l = l+[-1/2,1/2]^{2n}$, where l has integer coordinates. An application of (27) and (29) gives

$$\begin{aligned}\int_{\Omega^c} \|\chi_\Omega(\zeta)\langle\phi_\zeta,\phi_\gamma\rangle\|_{L^2}\|\chi_{\Omega^c}(\eta)\langle\phi_\eta,\phi_\gamma\rangle\|_{L^2}\,d\gamma &\leqslant \int_{\Omega^c}\left(\int_\Omega |\langle\phi_\zeta,\phi_\gamma\rangle|^2\,d\zeta\right)^{1/2} d\gamma \\ &\leqslant \sum_k \int_{\Omega^c}\left(\int_{S_\Omega^k} |\langle\phi_\zeta,\phi_\gamma\rangle|^2\,d\zeta\right)^{1/2} d\gamma \\ &\leqslant C_\Phi \sum_k \int_{\Omega^c}\left(\int_{S_\Omega^k} (1+|\zeta-\gamma|)^{-2N_\Phi}\,d\zeta\right)^{1/2} d\gamma \\ &\leqslant C_\Phi \sum_k\sum_l \int_{\Omega^c}\left(\int_{Q_l\cap S_\Omega^k} (1+|\zeta-\gamma|)^{-2N_\Phi}\,d\zeta\right)^{1/2} d\gamma \\ &\leqslant C_\Phi\left(1+\frac{n^{1/2}}{2}\right)^{N_\Phi}\sum_k\sum_l\int_{\Omega^c}(1+|l-\gamma|)^{-N_\Phi}\,d\gamma\,|Q_l\cap S_\Omega^k|^{1/2} \\ &\leqslant C_\Phi\left(1+\frac{n^{1/2}}{2}\right)^{N_\Phi}\sum_k\int_{|\gamma|\geqslant k-(n^{1/2}/2)}(1+|\gamma|)^{-N_\Phi}\,d\gamma\,|(S_\Omega^k)^{n^{1/2}}| \\ &\leqslant C_\Phi C_{\mathcal{G}}\left(1+\frac{n^{1/2}}{2}\right)^{N_\Phi}\sum_k\int_{|\gamma|\geqslant k-(n^{1/2}/2)}(1+|\gamma|)^{-N_\Phi}\,d\gamma\,|(\partial\Omega)^{R_{\mathcal{G}}}|.\end{aligned}$$

Now we estimate $\int_\Omega$. We apply again (27) and (29) and we obtain

$$\begin{aligned}\int_\Omega \|\chi_\Omega(\zeta)\langle\phi_\zeta,\phi_\gamma\rangle\|_{L^2}\|\chi_{\Omega^c}(\eta)\langle\phi_\eta,\phi_\gamma\rangle\|_{L^2}\,d\gamma &\leqslant \int_\Omega\left(\int_{\Omega^c}|\langle\phi_\zeta,\phi_\gamma\rangle|^2\,d\zeta\right)^{1/2} d\gamma \\ &\leqslant C_\Phi\sum_k\int_{S_\Omega^k}\left(\int_{\Omega^c}(1+|\zeta-\gamma|)^{-2N_\Phi}\,d\zeta\right)^{1/2} d\gamma \\ &\leqslant C_\Phi\sum_k |S_\Omega^k|\left(\int_{|\zeta|\geqslant k}(1+|\zeta|)^{-2N_\Phi}\,d\zeta\right)^{1/2} \\ &\leqslant C_\Phi C_{\mathcal{G}}\sum_k\left(\int_{|\zeta|\geqslant k}(1+|\zeta|)^{-2N_\Phi}\,d\zeta\right)^{1/2}|(\partial\Omega)^{R_{\mathcal{G}}}|.\end{aligned}$$

The proof follows by combining (33), (34) and the above estimates of $\int_{\Omega^c}$ and $\int_\Omega$. □

The estimates of this section apply to the classical Fock space defined on $\mathbb{C}^n$. It is enough to take a properly normalized Gaussian for ϕ and to apply the Bargmann transform.

The geometric meaning of the conditions imposed on reproducing kernels is similar to that discussed in Section 3. One needs to replace the hyperbolic metric by the Euclidean distance.

(i) Both (26) and (27) have radial character.

(ii) Condition (26) is similar to (14). One needs only to use standard convolution on $\mathbb{R}^{2n}$ instead of '$ax + b$' convolution.

(iii) There are however differences on the level of generating functions. Modulation spaces **[13]** give now the right function scale to express minimal regularity necessary for (27) to hold.

Two examples of domains satisfying conditions (28) and (29) are as follows.

(a) Let $W \subset \mathbb{R}^{2n}$ be a bounded domain with a smooth boundary. Define the family $\mathcal{G} = \{\lambda W : \lambda \geqslant 1\}$. One may compute the asymptotic of the size of the plunge region. It is λ^{2n-1}.

(b) Let Q_1, Q_2 be some cubes contained in $\mathbb{R}^n$. Consider the family $\mathcal{G} = \{\lambda^\alpha Q_1 \times \lambda^\beta Q_2 : \lambda \geqslant 1\}$. Now the asymptotic of the size of the plunge region is $\lambda^{\max\{\alpha,\beta\}+(n-1)\min\{\alpha,\beta\}/n}$.

It is possible to prove Theorem 4.1 without the strip condition (29). However, in this case condition (27) has to be strengthened by taking a larger constant N_Φ.

It is possible to prove a Szegö-type formula for Gabor–Toeplitz operators with continuous symbols with compact support (see **[10]**).

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