GABOR-TYPE MATRICES AND DISCRETE HUGE GABOR TRANSFORMS

Sigang Qiu
Department of Mathematics
University of Connecticut
U-9, Storrs
CT 06269-3009, USA
sigang@math.uconn.edu

Hans. G. Feichtinger
Department of Mathematics
University of Vienna
Strudlhofgasse 4, A-1090
Vienna, Austria
fei@tyche.mat.univie.ac.at

ABSTRACT
We introduce a class of Gabor-type matrices and show that the product of two Gabor-type matrices is still a Gabor-type matrix possessing the same type. The key point is based on the fact that the multiplication of Gabor-type matrices can be converted into the “multiplication” of small block matrices. We propose an efficient algorithm, the specially devised block-multiplication. As an interesting consequence we show that Gabor operators corresponding to Gabor triples \((g_k, a, b)\) for any Gabor signals \(g_k(k = 1, 2)\) are commutative provided that \(ab\) divides the signal length.

1. INTRODUCTION
It is by now well known, that the non-orthogonality of a Gabor family, obtained from a Gabor atom (or Gabor window, or building block) by time-frequency shifts along some discrete TF-lattice, is not such a big problem for the determination of coefficients, if the so-called dual Gabor window is given. Indeed, it is then sufficient to calculate sampling values of a STFT of the signal that has to be expanded, with the (conjugate of) the dual Gabor window as STFT-window. In a number of papers \([3, 6, 7, 2, 8, 9, 10]\) of the literature, the authors have described several practical approaches to calculate the dual Gabor atom for the discrete and periodic (= finite) setting. Most of those methods however are very much restricted by the size of the window (or signal) length, due to limitations of the size of matrices that can be handled or inverted on a given computer. In order to eliminate the continuous Gabor transform it is natural to sample the Gabor window sufficiently dense and obtain an approximate dual Gabor window by means of discrete methods. Even if the support of the window itself is not large a high sampling rate will lead to a discrete Gabor transform with “huge” window size \(N\). The aim of this note is to propose a new way of performing the multiplication for a special class of matrices which are called Gabor-type matrices by substituting it by a small special block matrix “multiplication”. Based on this special multiplication we can calculate the dual Gabor window, and the inverse of the Gabor frame matrix for “huge” Gabor window. For much more detailed discussions we refer to \([5]\).

We will use some relevant notations introduced in \([3]\). Specifically,

1. The signal is always viewed as a \(N\)-periodic row vector in \(C^N\). \(a\) and \(b\) denote lattice constants which divide \(N\). We call \(a = \frac{N}{m}\) and \(b = \frac{N}{k}\) the associated dual lattice constants.

2. The rotation operator with rotation number \(a\) acting on vectors is as follows:

\[\text{rot}(x, a) := (x_{N-a}, x_{N-a+1}, ..., x_{N-1}, x_0, ..., x_{N-a-1}).\]

Similarly we can define the rotation operator \(\text{rotm}\) on column vectors.

The matrix rotation on matrix \(B\) with rotation number \(a\) can be obtained by rotating all the column vectors of \(B\). That is,

\[\text{rotm}(B, a) := (\text{rot}(B_1, a), \text{rot}(B_2, a), ..., \text{rot}(B_m, a)).\]

where \(B_i\) is the \(i\)-th column vector of \(B\), \(l = 1, 2, ..., m\).

Besides, we introduce the following definitions.

Definition 1 (Gabor-type Block Matrix) We call an \(N \times a\) matrix \(B\) a Gabor\((a, b)\)-type block matrix if all the nonzero entries of \(B\) are distributed only in the \(k\)-th (for \(k = 0, \pm h, \pm 2h, ..., \pm (b-1)h\)) subdiagonal, which is equivalently to say that, \(B\) can be written as

\[
B = \begin{pmatrix}
    c_{1,1} & 0 & \ldots & 0 \\
    0 & c_{2,2} & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & c_{a,a} \\
    c_{b+1,1} & 0 & \ldots & 0 \\
    0 & c_{b+2,2} & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & c_{b+a,a} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{(b-1)b+1,1} & 0 & \ldots & 0 \\
    0 & c_{(b-1)b+2,2} & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & c_{(b-1)b+a,a}
\end{pmatrix}
\]
where $c_{k,l} \in \mathbb{C}$.

We call $b \times a$ a matrix $\hat{B}$ the associated “nonzero” block matrix if $\hat{B}$ is constructed from $B$ as follows:

$$
\hat{B} = \begin{pmatrix}
  c_{1,1} & c_{2,2} & \cdots & c_{a,a} \\
  c_{a+1,1} & c_{b+2,2} & \cdots & c_{b+a,a} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{(b-1)a+1,1} & c_{(a-1)b+2,2} & \cdots & c_{(a-1)a+b,a}
\end{pmatrix}
$$

The $N \times a$ Gabor-type block matrix $B$ is always considered as $N$-periodic column-wise and $a$-periodic row-wise. The associated $b \times a$ “nonzero” block matrix $\hat{B}$ is considered as $b$-periodic column-wise and $a$-periodic row-wise. Equivalently, if $B = (t_{i,j})_{N \times a}$ and $\hat{B} = (s_{i,j})_{b \times a}$, then we always assume that $t_{i+N,j+n} = t_{ij}$ and $s_{i+b,j+n} = s_{ij}$ for $i,j \in IN$.

**Definition 2 (Gabor-type Matrix)** We say that an $N \times N$ matrix $G$ is a Gabor $(a,b)$-type matrix if $G$ can be formulated as

$$G = \left[ B, \text{rotm}(B,a), \ldots, \text{rotm}^{a-1}(B,a) \right]$$

where $B$ is an $N \times a$ Gabor $(a,b)$-type block matrix and $\text{rotm} = \text{rotm}(.,a)$ is a rotation operator. We will call $B$ the “Gabor-type block matrix associated to $G$. $B$ is also called the “nonzero” block matrix associated to $G$.

It is obvious that Gabor-type matrix $G$, the associated block matrix $B$ and nonzero block $\hat{B}$ are determined each other completely. Therefore, we only need to work with the small size “nonzero” block matrix $\hat{B}$.

2. MAIN RESULTS

In this section we present some of our main results and propose an algorithm to manipulate the discrete large Gabor transforms. [5].

**Theorem 1** The matrix product of two Gabor $(a,b)$-type matrices is itself a Gabor $(a,b)$-type matrix. Furthermore, a non-singular matrix is of Gabor $(a,b)$-type if and only if its inverse is of the same type.

Since Gabor-type matrices is completely determined by the associated nonzero block matrices, we present the following algorithm for carrying out the small block matrix multiplication instead of the Gabor-type matrix multiplication.

**Theorem 2 (Block-Multiplication)** Under the assumptions of Theorem 1, assume that $\hat{B}_1 = (w_{k,l})_{b \times a}$ and $\hat{B}_2 = (v_{k,l})_{b \times a}$ are two Gabor-type nonzero block matrices associated to Gabor $(a,b)$-type matrices $G_1$ and $G_2$. Let $\hat{B} := (w_{k,l})_{b \times a}$ be the corresponding nonzero block matrix of $G$. Then, for $q = 1,2, \ldots, b$ and $s = 1,2, \ldots, a$, the general $(q,s)$ entry of $\hat{B}$ can be calculated via the formula:

$$w_{q,s} = \sum_{p=1}^{b} r_1(p,q) r_2(p,s) v_{p,s},$$

where $r_1(p,q) = \text{mod}(b + q - p + 1, b)$ and $r_2(p,s) = \text{mod}(s + (p-1)b, a)$.

Using the concepts of matrix algebra [1], we have the following statements.

**Theorem 3** Let

$$G = \{ G \in \mathbb{C}^{N \times N} : G \text{ is Gabor } (a,b)\text{-type} \}$$

Then $G \subseteq \mathbb{C}^{N \times N}$ is matrix algebra of dimension $ab$. Furthermore, if $ab$ divides $N$, then $G$ is an $ab$-dimensional commutative matrix algebra.

We call $G$ a Gabor-type matrix algebra. It is easy to see that if $G \in G$ is nonsingular then $G^{-1} \in G$.

Let $g \in \mathbb{C}^{N}$ be a Gabor atom and $(a,b)$ be lattice constants (i.e., divisors of $N$). Using the notation cited in [2],

$\text{GAB} = \text{GAB}(g,a,b)$ is the $N^2 \times N$ Gabor basic matrix whose row vectors are from the Gabor family $\{ M_{ab} T_{s} g \}$. The Gabor operator $S$ corresponding to $(g,a,b)$ has the following matrix representation:

$$S x = x * (\text{GAB}' * \text{GAB}) \text{ for } x \in \mathbb{C}^{N},$$

where we call $G = \text{GAB}' * \text{GAB}$ the associated Gabor matrix. Since Gabor matrices are Gabor-type matrices [3], we can easily deduce an interesting consequence of Theorem 3.

**Corollary 1** Let $(g_k,a,b)$ be two Gabor triples for $k = 1,2$, the associated Gabor operations $S_k$ are defined as

$$S_k x = \sum_{n=0}^{N} \sum_{m=0}^{N} \langle x, M_{ab} T_{s=0} g_k \rangle M_{ab} T_{s=0} g_k$$

where $x \in \mathbb{C}^{N}$. Assume that $ab$ divides $N$ (especially, the critical sampling case where $ab = N$), then the Gabor operators $S_k$ for $k = 1,2$ are commutative, i.e.,

$$S_1 S_2 = S_2 S_1.$$

Applying the special block-multiplication established in Theorem 2, we obtain the following algorithm, which is called the block-frame (BKFR) method. It can be used to compute the dual Gabor atom and is workable even for huge Gabor atom.

**Algorithm 1 (BKFR-method)** Let $(g,a,b)$ be any Gabor triple which generates frame and $G$ be the associated Gabor frame matrix. Then there exists a positive constant $r(0 < r < 1)$ such that $Q := rG$ satisfies $\| I - Q \| \leq 1 - r \| G \| := \gamma < 1$ where $Q := rG$ and the dual Gabor atom $\hat{g}$ can be obtained iteratively. Setting $\hat{g}_0 = r g$ and for $n \geq 0$,

$$\hat{g}_{n+1} = \hat{g}_n * \left( I + (I - Q)^n \right),$$

one obtains

$$\lim_{n \to \infty} \hat{g}_n = \hat{g}$$

with the error estimate after $n$ iterations

$$\| \hat{g} - \hat{g}_n \| \leq \gamma^n \| \hat{g} \|.$$
Remarks

- Since $G$ is a Gabor $(a, b)$-type matrix [3], $I - Q$ is also a Gabor $(a, b)$-type matrix. $(I - Q)^n$ for $n = 0, 1, \ldots$ in Eq. (1) can be iteratively calculated via the special $8 \times 8$ block-matrix multiplication Theorem 2 efficiently. This yields that Algorithm 1 can be performed very fast.

- Since the (best) Gabor frame bounds are usually not easy to be determined, we usually get the Gabor frame upper bound $U_b$ by the following easy way [4]:

$$U_b = \|G\|_1 = \|\tilde{B}\|_1$$

where $\tilde{B}$ is the nonzero block matrix corresponding to $G$. Then take

$$r = \frac{2}{U_b + \frac{1}{U_b}}$$

and $Q = rG$

in the BKFR-Algorithm. In this case, we can easily check that $\gamma = |U_b - \frac{1}{U_b}|/\left(U_b + \frac{1}{U_b}\right)$. The specially devised Algorithm for computing the best Gabor upper bound is proposed in [5].

- If the Gabor atom $g$ is well concentrated, the associated Gabor matrix $S$ is usually diagonal dominant. In this case, we perform the Algorithm 1 with $S_d = D^{-1}d S$ instead of $S$, where $D$ is a diagonal matrix with the same diagonal elements as $S$. We call this special algorithm modified BKFR-method.

- Algorithm 1 does also gives a new way of computing the inverse of Gabor frame operator.

3. NUMERICAL RESULTS

In this section we present some of our numerical results. The experiments shows that the Algorithm presented here has the big advantage of dealing with arbitrary Gabor signals even with huge signal length efficiently. All the numerical experiments were carried out using MATLAB 4.0 on a SUN-Station.

Figure 1 shows the comparison of convergent rates to compute the corresponding dual Gabor to the illustrated Gabor atom between the BKFR-method and CG-method presented in [4].

Figure 2 illustrates the Gabor atom and the associated dual Gabor atom; the original Chirp-signal and the reconstructed one. The signal length $N = 5120$ and the lattice constants $(a, b) = (32, 10)$. The reconstruction error in this case is in order of $10^{-13}$, which can be considered as an error-free reconstruction practically.

Figure 3 shows the comparison of convergent rates of modified BKFR-method, BKFR-method and CG-method ???. The signal length $N = 2400$ and the lattice constants $(a, b)$ are $(40, 25)$. The reconstruction error is around the order of $O(10^{-15})$.

4. CONCLUSION

We introduce a class of Gabor-type matrices which proved to be closed under the usual matrix multiplication. We propose an algorithm to perform the Gabor-type matrix multiplication with the simple block-multiplication. As an application to discrete Gabor transforms we present the BKFR-algorithm to compute the Gabor analysis window function iteratively.

5. ACKNOWLEDGEMENTS

The first author would like to thank the hospitality of the Department of Mathematics of the University of Vienna, where the main part of the paper was prepared.

6. REFERENCES


Figure 1: Convergent rates comparison between CG-method and BKFR-method.

Figure 2: Gabor atom, the associated dual Gabor atom; the original Chirp signal and the reconstructed Chirp. The signal length $N = 5120$ and the lattice constants $(a, b) = (32, 10)$.

Figure 3: Convergent rates comparison between CG-method, BKFR-method and modified BKFR-method.