ORTHONORMAL SAMPLING FUNCTIONS

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Abstract. We investigate functions $\phi(x)$ whose translates $\{\phi(x-k)\}$, where k runs through the integer lattice $\mathbb{Z}$, provide a system of orthonormal sampling functions. The cardinal sine, whose important role in the sampling theory of bandlimited functions is well documented, is the classic example. For the bandlimited case we provide a complete characterization of such functions ϕ and give examples with rapid decay including a construction which is symmetric. We also analyze the general case of arbitrary sampling rate, $a > 0$, which leads to some unexpected observations.

1. Introduction

We introduce the relevant notions for our study by starting from a basic model for encoding discrete-time data into continuous-time functions. It is a standard technique in the framework of shift-invariant systems. Given a generator function ϕ in the space $L^2(\mathbb{R})$ of square-integrable functions, the data of complex numbers c_k is transformed into a function f on $\mathbb{R}$ by the synthesis mapping

$$(c_k)_{k\in\mathbb{Z}} \quad \mapsto \quad f(x) = \sum_{k\in\mathbb{Z}} c_k\, \phi(x-k), \quad x \in \mathbb{R}.$$

It is well known if ϕ is the cardinal sine,

$$\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}, \qquad x \in \mathbb{R},$$

then the encoding has the following useful features:

(i) The synthesis mapping is an isometry from $\ell^2(\mathbb{Z})$ into $L^2(\mathbb{R})$.
(ii) The data can be reconstructed in a convenient way by sampling

$$f \quad \mapsto \quad (f(k))_{k\in\mathbb{Z}}\,.$$

(iii) The range of the synthesis mapping consists of bandlimited functions, i.e, functions with compactly supported Fourier transform.

These properties correspond to the following conditions on the generator $\phi \in L^2(\mathbb{R})$. By $\delta_{k,0}$ we denote the Kronecker delta.

2000 *Mathematics Subject Classification.* Primary 42C15; Secondary 41A05, 42C05, 42C40, 47A15.
Key words and phrases. Shift-orthonormal, sampling function, interpolation, cardinal sine, shift-invariant space, wavelets, multiresolution analysis, scaling function.
The first author was supported by the Austrian Science Fund FWF grant P-15605.

(i) ϕ is shift-orthonormal, i.e.,

$$\int_{-\infty}^{\infty} \phi(x)\overline{\phi(x-k)}dx = \delta_{k,0}, \qquad k \in \mathbb{Z}.$$

(ii) ϕ is a sampling function (fundamental function of interpolation, cardinal function), i.e., it is continuous and

$$\phi(k) = \delta_{k,0}, \qquad k \in \mathbb{Z}.$$

(iii) ϕ is bandlimited.

The cardinal sine is in many ways the prototypical generator. However, the poor decay properties of the sinc are often disadvantageous. One of our objectives, inspired by a query posed by H. G. Feichtinger, is to obtain generators with better decay, thus providing functions that may replace the sinc-function in various applications. It turns out that constraints on the bandwidth impose limitations to such properties as rapid decay or symmetry. These limitations together with what is possible are summarized in Section 2 as Theorem 1. Section 2 also contains a characterization of a class of bandlimited shift-orthonormal sampling functions together with examples that indicate what is possible regarding decay.

The case of more general sampling rates, i.e., replacing the integer lattice $\mathbb{Z}$ by a scaled lattice $a\mathbb{Z}$ with some positive number a other than one, is usually dealt with by simply scaling the generator; but not without introducing an additional normalizing factor. Thus the existence and nature of generators ϕ which are both $a\mathbb{Z}$-sampling and $a\mathbb{Z}$-shift-orthonormal with any given a different from one is not immediately transparent. See the introductory paragraphs of Section 3. Nevertheless we can give a complete description of the situation in this general case, see Theorem 2 in Section 3. Section 4 is devoted to the proof of this theorem, which requires several technical lemmas.

The conditions (i), (ii), (iii) on a function ϕ arise naturally in wavelet and sampling theory, for example see [5, 6] for relatively recently published texts on these subjects which also contain further references. Indeed, generators of shift-invariant systems, wavelets, and scaling functions with combinations of these features have been analyzed to some extent in the literature, see, e.g., [1–4, 7–10]. Some of our observations are direct consequences of established machinery while others may have been noted earlier. In the text we indicate which observations, to our knowledge, have been recorded by other authors and provide explicit references.

2. Shift-orthonormal sampling functions

It is useful to study the notions introduced above in the Fourier domain. We use the following normalization of the Fourier transform, for integrable ϕ,

$$\widehat{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x)e^{-2\pi i\xi x}dx, \qquad \xi \in \mathbb{R}.$$

As is well-known, for example see [5, Lemma 7.5], a function $\phi \in L^2(\mathbb{R})$ is shift-orthonormal if and only if

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi - k)|^2 = 1, \qquad \text{for a.e. } \xi \in \mathbb{R}. \tag{1}$$

A function $\phi \in L^2(\mathbb{R})$ with integrable Fourier transform is a sampling function if and only if

$$\sum_{k\in\mathbb{Z}} \widehat{\phi}(\xi - k) = 1, \qquad \text{for a.e. } \xi \in \mathbb{R}. \tag{2}$$

We note that the assumption that $\widehat{\phi}$ be integrable is always satisfied for bandlimited $\phi \in L^2(\mathbb{R})$.

Our first main result is the following classification, including the examples below, for bandlimited shift-orthonormal sampling functions. A continuous function ϕ is said to have rapid decay if $\lim_{|x|\to\infty} x^N \phi(x) = 0$ for all $N = 1, 2, \dots$.

Theorem 1. *Suppose $\phi \in L^2(\mathbb{R})$ is a shift-orthonormal sampling function such that*

$$\operatorname{supp} \widehat{\phi} \subseteq [-\tfrac{\Omega}{2}, \tfrac{\Omega}{2}], \qquad \Omega > 0.$$

(i) $\Omega < 1$. *No such function exists.*
(ii) $\Omega = 1$. *The* sinc-*function is the unique example.*
(iii) $1 < \Omega \leq 3$. *There exist real-valued examples with rapid decay. If ϕ is integrable it cannot be symmetric.*
(iv) $\Omega > 3$. *There exist symmetric real-valued examples with rapid decay.*

We note that Theorem 1 can be viewed as the special case $a = 1$ of Theorem 2, which is formulated in Section 3 and proved in Section 4. The details of the general result are rather involved while those for the special case are quite transparent as indicated below.

Proof. Items (i) and (ii) are direct consequences of characterizations (1) and (2). Note that we use the Fourier transform normalized in such a way that $\text{sinc}^\wedge = 1_{[-1/2,1/2]}$.
(iii) The existence is verified by Example 1. The implication in the second statement is shown as follows. Suppose there exists such a function ϕ which is symmetric. Since ϕ is integrable, $\widehat{\phi}$ is continuous and we conclude that $\widehat{\phi}(\pm\frac{3}{2}) = 0$. Thus, by (1), (2) the complex numbers $\lambda_1 = \widehat{\phi}(-\frac{1}{2})$ and $\lambda_2 = \widehat{\phi}(\frac{1}{2})$ satisfy $\lambda_1 + \lambda_2 = 1$ and $|\lambda_1|^2 + |\lambda_2|^2 = 1$. Next, ϕ is symmetric, hence so is $\widehat{\phi}$ and, therefore, $\lambda := \lambda_2 = \lambda_1$. Hence, $2\lambda = 2\lambda^2 = 1$, which is impossible. Contradiction.
(iv) The existence is verified by Example 2. □

Bandlimited shift-orthonormal sampling functions with bandwidth constraint $\operatorname{supp} \widehat{\phi} \subseteq [-1, 1]$ are completely characterized in terms of their Fourier transform by the following lemma. The result is a reformulation of [6, Lemma 10.5].

Lemma 1. *Let $0 \le \varepsilon \le 1/2$ and suppose that $\phi \in L^2(\mathbb{R})$ and $\operatorname{supp}\widehat{\phi} \subseteq [-\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon]$.*
(i) *Then ϕ is a sampling function if and only if*

$$\widehat{\phi}(\xi) = \begin{cases} 1, & |\xi| \le \frac{1}{2}-\varepsilon, \\ \frac{1}{2} \pm \frac{1}{2}u(\xi \pm \frac{1}{2}), & |\xi \pm \frac{1}{2}| < \varepsilon, \\ 0, & |\xi| \ge \frac{1}{2}+\varepsilon, \end{cases}$$

for some square-integrable function $u\colon [-\varepsilon,\varepsilon] \to \mathbb{C}$.
(ii) *A sampling function ϕ so characterized is in addition shift-orthonormal if and only if u satisfies $|u(\xi)| = 1$, for all $\xi \in [-\varepsilon,\varepsilon]$.*

Proof. Both (i) and (ii) follow from the fact that $\widehat{\phi}$ enjoys relations (1) and (2). For example, if ϕ is a sampling function then (2) implies $\widehat{\phi}(r+\frac{1}{2}) + \widehat{\phi}(r-\frac{1}{2}) = 1$, for a. e. $0 < r < \varepsilon$. We conclude that $\widehat{\phi}(r \pm \frac{1}{2}) = \frac{1}{2} \pm \frac{1}{2}u$, for some $u = u(r) \in \mathbb{C}$. If in addition ϕ is shift-orthonormal, then by (1) there also holds $|\widehat{\phi}(r+\frac{1}{2})|^2 + |\widehat{\phi}(r-\frac{1}{2})|^2 = 1$. Thus $|\frac{1}{2}+\frac{1}{2}u|^2 + |\frac{1}{2}-\frac{1}{2}u|^2 = 1$ which is equivalent to $|u| = 1$. □

We state two cases where Lemma 1 yields an explicit form of ϕ. First, for $u(\xi) = \xi/\varepsilon \pm i\sqrt{1-(\xi/\varepsilon)^2}$, we obtain the integrable real-valued shift-orthonormal sampling functions

$$\phi(x) = \Big(\frac{\sin 2\pi\varepsilon x}{2\pi\varepsilon x} \pm \frac{\pi}{2} J_1(2\pi\varepsilon x)\Big)\frac{\sin \pi x}{\pi x}, \qquad x \in \mathbb{R}, \tag{3}$$

where J_1 denotes the Bessel function of the first kind of order one. The decay is $\phi(x) = O(|x|^{-3/2})$, as $|x| \to \infty$.

Using $u(\xi) = \sin(\pi\xi/\varepsilon) \pm i\cos(\pi\xi/\varepsilon)$ yields the integrable real-valued shift-orthonormal sampling function

$$\phi(x) = \frac{\cos 2\pi\varepsilon x}{1 \mp 4\varepsilon x}\,\frac{\sin \pi x}{\pi x} = \frac{\pi}{2}\operatorname{sinc}(2\varepsilon x \mp \tfrac{1}{2})\operatorname{sinc}(x), \qquad x \in \mathbb{R}. \tag{4}$$

Its decay is better than the previous example, $\phi(x) = O(|x|^{-2})$, as $|x| \to \infty$. This example was first obtained in the wavelet context [4, Example 2.3], also discussed in [6, p. 52 and p. 238], [7, 10]. In fact, for $\varepsilon \le \frac{1}{6}$, (3) and (4) are scaling functions, i.e., they satisfy the dyadic refinement equation of wavelet theory:

$$\phi(x/2) = \sum_{k\in\mathbb{Z}} c_k\,\phi(x-k), \qquad x \in \mathbb{R},$$

with $c_k = \phi(k/2)$. Indeed we note that Lemma 1 implies, if a shift-orthonormal sampling function ϕ satisfies $\operatorname{supp}\widehat{\phi} \subseteq [-\frac{2}{3}, \frac{2}{3}]$, then ϕ is refinable.

With suitable u we can improve the smoothness of $\widehat{\phi}$ and thus the decay properties of ϕ. To this end, observe that u can always be written

$$u(\xi) = e^{\pi i \nu_\varepsilon(\xi)/2} = \sin(\tfrac{\pi}{2}\nu_\varepsilon(\xi)) + i\cos(\tfrac{\pi}{2}\nu_\varepsilon(\xi)), \qquad \xi \in [-\varepsilon,\varepsilon],$$

where $\nu_\varepsilon(\xi) = \nu(\xi/\varepsilon)$, for some measurable function $\nu\colon [-1,1] \to \mathbb{R}$. By a clever choice of ν we can have that $\widehat{\phi}$ is C^∞, so that ϕ is of rapid decay. The next lemma describes the function ν that we will use in Example 1.

Lemma 2. *Define* $\nu\colon [-1,1] \to [-1,1]$ *by*

$$\nu(\xi) = \begin{cases} \tanh(\tan(\frac{\pi}{2}\xi)), & -1 < \xi < 1, \\ \pm 1, & \xi = \pm 1. \end{cases}$$

Then ν *is* C^∞ *on* $[-1,1]$ *and* $\nu^{(k)}(\pm 1) = 0$, *for* $k = 1, 2, \ldots$.

Proof. Note that all derivatives of $\tanh(\xi)$ vanish at infinity as can be seen by writing $\tanh(\xi) = (1 - \exp(-2\xi))/(1 + \exp(-2\xi))$ and using L'Hospital's rule. Thus the derivatives of ν at $\xi = \pm 1$ are obtained by elementary calculus, in the same way as one shows that all the derivatives of $\exp(-1/|\xi|)$ are zero at zero. $\square$

By making use of Lemma 2 we construct a bandlimited shift-orthonormal sampling function ϕ that has rapid decay.

Example 1. Given $0 < \varepsilon \leq \frac{1}{2}$, define $\phi \in L^2(\mathbb{R})$ by

$$\widehat{\phi}(\xi) = \begin{cases} 1, & |\xi| \leq \frac{1}{2} - \varepsilon, \\ \frac{1}{2} \pm \frac{1}{2}\sin(\frac{\pi}{2}\nu_\varepsilon(\xi \pm \frac{1}{2})) \pm i\,\frac{1}{2}\cos(\frac{\pi}{2}\nu_\varepsilon(\xi \pm \frac{1}{2})), & |\xi \pm \frac{1}{2}| < \varepsilon, \\ 0, & |\xi| \geq \frac{1}{2} + \varepsilon, \end{cases}$$

where $\nu(\xi) = \tanh(\tan(\frac{\pi}{2}\xi))$ and $\nu_\varepsilon(\xi) = \nu(\xi/\varepsilon)$. Then ϕ is a real-valued shift-orthonormal sampling function with rapid decay such that $\operatorname{supp}\widehat{\phi} = [-\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$.

While the construction of this example does not give rise to an elementary formula in time domain, the function ϕ can be readily implemented by the given formula in the Fourier domain. The specific case of ϕ in Example 1 with $\varepsilon = 0.1$ is illustrated in Figure 2 below. Note the lack of symmetry in this function.

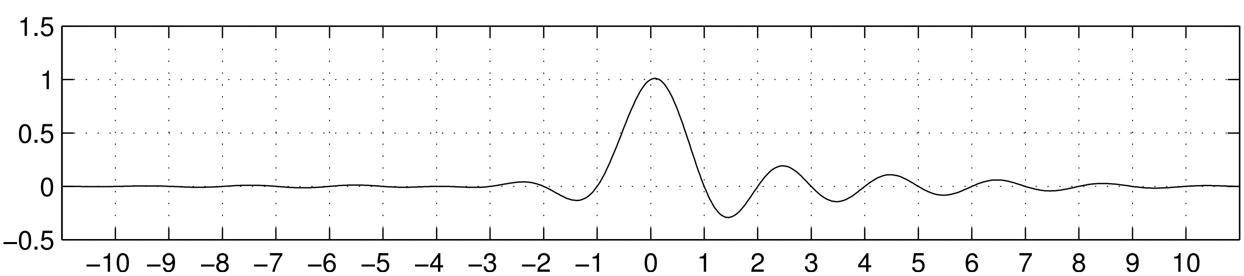

FIGURE 1. A bandlimited shift-orthonormal sampling function with rapid decay (Example 1 with $\varepsilon = 0.1$).

It can be deduced from Lemma 1 that a shift-orthonormal sampling function ϕ with $\operatorname{supp}\widehat{\phi} \subseteq [-1,1]$ cannot be symmetric when $\widehat{\phi}$ is continuous, indeed, item (iii) of Theorem 1 implies that this is true for bandwidths as large as 3. However, in (iv) we state that symmetric examples with rapid decay do exist if the bandwidth is is allowed to increase further. The next example verifies this observation.

Example 2. Given $0 < \varepsilon \le \frac{1}{2}$, define $\phi \in L^2(\mathbb{R})$ by

$$\widehat{\phi}(\xi) = \begin{cases} 1, & |\xi| \le \frac{1}{2} - \varepsilon, \\ \frac{1}{4} + \frac{\sqrt{3}}{4}\cos(\alpha\nu_\varepsilon(\xi \pm \frac{1}{2})) \pm \frac{1}{2}\sqrt{\frac{3}{2}}\sin(\alpha\nu_\varepsilon(\xi \pm \frac{1}{2})), & |\xi \pm \frac{1}{2}| < \varepsilon, \\ 0, & |\xi \pm 1| \le \frac{1}{2} - \varepsilon, \\ \frac{1}{4} - \frac{\sqrt{3}}{4}\cos(\alpha\nu_\varepsilon(\xi \pm \frac{3}{2})), & |\xi \pm \frac{3}{2}| < \varepsilon, \\ 0, & |\xi| \ge \frac{3}{2} + \varepsilon, \end{cases}$$

where $\alpha = \arccos\frac{1}{\sqrt{3}}$, $\nu(\xi) = \tanh(\tan(\frac{\pi}{2}\xi))$ and $\nu_\varepsilon(\xi) = \nu(\xi/\varepsilon)$. Then ϕ is a symmetric real-valued shift-orthonormal sampling function with rapid decay such that $\operatorname{supp}\widehat{\phi} \subseteq [-\frac{3}{2} - \varepsilon, \frac{3}{2} + \varepsilon]$.

This example is one of the main results of our analysis. By Theorem 1(iii) it cannot be improved in terms of smaller bandwidth. The specific case with $\varepsilon = 0.1$ is illustrated in Figure 2 below.

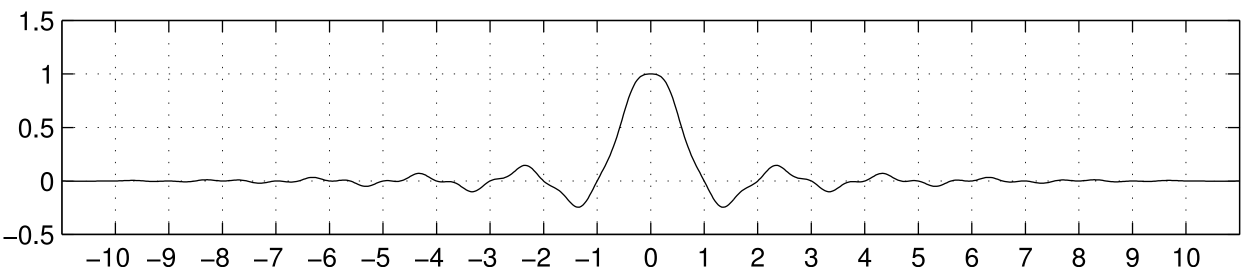

FIGURE 2. A symmetric bandlimited shift-orthonormal sampling function with rapid decay (Example 2 with $\varepsilon = 0.1$).

Remark 1. A shift-orthonormal sampling function ϕ which enjoys the property that $\widehat{\phi}(\xi) = 1$ in a neighborhood of the origin also satisfies $\widehat{\phi}(\xi) = 0$ in a neighborhood of the nonzero integers. If ϕ decays rapidly these properties imply that ϕ satisfies several features that are important in approximation theory. For example, ϕ reproduces algebraic polynomials of any order, that is, for all $N = 0, 1, 2, \dots$,

$$x^N = \sum_{k\in\mathbb{Z}} k^N \phi(x-k), \qquad x \in \mathbb{R}. \tag{5}$$

In particular, both Examples 1 and 2 enjoy this property.

3. General shift parameters

Let $a > 0$ and $\phi \in L^2(\mathbb{R})$.

(i) ϕ is $a\mathbb{Z}$-shift-orthonormal if

$$\int_{-\infty}^{\infty} \phi(x)\overline{\phi(x-ak)}dx = \delta_{k,0}, \qquad k \in \mathbb{Z}.$$

(ii) ϕ is an $a\mathbb{Z}$-sampling function if it is continuous and

$$\phi(ak) = \delta_{k,0}, \qquad k \in \mathbb{Z}.$$

As is well known, each of these notions can be related to the corresponding special case $a = 1$ by a dilation:

$$\phi(x) \text{ is shift-orthonormal } \Leftrightarrow\ a^{-1/2}\,\phi(x/a) \text{ is } a\mathbb{Z}\text{-shift-orthonormal,}$$
$$\phi(x) \text{ is a sampling function } \Leftrightarrow\ \phi(x/a) \text{ is an } a\mathbb{Z}\text{-sampling function.}$$

Note that that the above dilations do not lead directly to functions ϕ which are both $a\mathbb{Z}$-shift-orthonormal and $a\mathbb{Z}$-sampling functions when $a \neq 1$. While the factor of $a^{-1/2}$ may seem like a minor inconvenience it does raise the interesting question of whether this factor is indeed necessary, that is, whether there exist functions ϕ which are both shift-orthonormal and sampling functions when $a \neq 1$. Surprisingly the answer to this question is yes, see Theorem 2 below, but the development is not quite as slick as in the special case $a = 1$, see Section 4. Our interest in this question was prompted by the article [9].

We state the characterizations in the Fourier domain for the case of general $a > 0$. A function $\phi \in L^2(\mathbb{R})$ is $a\mathbb{Z}$-shift-orthonormal if and only if

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi - k/a)|^2 = a, \qquad \text{for a.e. } \xi \in \mathbb{R}. \tag{6}$$

A function $\phi \in L^2(\mathbb{R})$ with integrable Fourier transform is an $a\mathbb{Z}$-sampling function if and only if

$$\sum_{k\in\mathbb{Z}} \widehat{\phi}(\xi - k/a) = a, \qquad \text{for a.e. } \xi \in \mathbb{R}. \tag{7}$$

We formulate our main result for this general setting of arbitrary shift-parameters $a > 0$, a classification of bandlimited $a\mathbb{Z}$-shift-orthonormal $a\mathbb{Z}$-sampling functions in terms of the bandwidth. By $\lceil a \rceil$ we denote the least integer greater than or equal to a.

Theorem 2. *Given $a > 0$, let*

$$N = \begin{cases} 2, & \text{for } a < 1, \\ \lceil a \rceil, & \text{for } a \geq 1, \end{cases} \quad \text{and} \quad M = \begin{cases} 3, & \text{for } a < 2, \\ \lceil a \rceil, & \text{for } a \geq 2. \end{cases}$$

Suppose $\phi \in L^2(\mathbb{R})$ is an $a\mathbb{Z}$-shift-orthonormal $a\mathbb{Z}$-sampling function such that

$$\operatorname{supp} \widehat{\phi} \subseteq [-\tfrac{\Omega}{2}, \tfrac{\Omega}{2}], \qquad \Omega > 0.$$

(i) $\Omega < N/a$. *No such function exists.*
(ii) $\Omega = N/a$. *There exist symmetric real-valued examples. The function is unique if and only if a is an integer, where $\phi = \operatorname{sinc}$. There do not exist integrable examples.*
(iii) $N/a < \Omega \le M/a$. *There exist real-valued examples with rapid decay. If ϕ is integrable it cannot be symmetric.*
(iv) $\Omega > M/a$. *There exist symmetric real-valued examples with rapid decay.*

Remark 2. The case (iii) is void for $a \ge 2$, since in this case M and N coincide.

We illustrate (ii) by constructing an example, for general $a > 0$.

Example 3. Given $a > 0$, let $n = \max(2, \lceil a \rceil)$, and define $\phi_a \in L^2(\mathbb{R})$ by

$$\widehat{\phi}_a(\xi) = \begin{cases} \frac{a}{n} + \dfrac{1}{\sqrt{n-1}}\sqrt{\frac{a}{n}(1-\frac{a}{n})}, & |\xi| \le \frac{n-1}{2a}, \\ \frac{a}{n} - \sqrt{n-1}\sqrt{\frac{a}{n}(1-\frac{a}{n})}, & \frac{n-1}{2a} < |\xi| \le \frac{n}{2a}, \\ 0, & |\xi| > \frac{n}{2a}. \end{cases}$$

Then ϕ_a is a symmetric real-valued $a\mathbb{Z}$-shift-orthonormal $a\mathbb{Z}$-sampling function such that $\operatorname{supp} \widehat{\phi}_a \subseteq [-\frac{N}{2a}, \frac{N}{2a}]$ with N as in Theorem 2(ii). Indeed, for $a \ne 1$ we have $\operatorname{supp} \widehat{\phi}_a = [-\frac{n}{2a}, \frac{n}{2a}]$, while $a = 1$ yields $\operatorname{supp} \widehat{\phi}_1 = [-\frac{1}{2}, \frac{1}{2}]$. We note, when a is an integer, then $\phi_a = \operatorname{sinc}$. We also mention that $\phi_a \to \operatorname{sinc}$ in $L^2(\mathbb{R})$, as $a \to \infty$.

4. Preliminary lemmas and proof of Theorem 2

We provide the lemmas used below for proving Theorem 2. The first is a result that allows us to analyze bandlimited $a\mathbb{Z}$-shift-orthonormal $a\mathbb{Z}$-sampling functions by investigating classes of vectors in $\mathbb{C}^n$. Given $a > 0$ and $n \in \mathbb{N}$, we define $S_n^a \subset \mathbb{C}^n$ by

$$S_n^a = \{\lambda \in \mathbb{C}^n \colon \textstyle\sum_{k=1}^n |\lambda_k|^2 = \sum_{k=1}^n \lambda_k = a\}.$$

Lemma 3. *Let $a > 0$. Given a bandlimited function $\phi \in L^2(\mathbb{R})$, with*

$$\operatorname{supp} \widehat{\phi} \subseteq [-\tfrac{n}{2a} - \varepsilon, \tfrac{n}{2a} + \varepsilon], \qquad n \in \mathbb{N}, \quad 0 \le \varepsilon < \tfrac{1}{2a},$$

we define $v_\xi \in \mathbb{C}^n$ and $w_\xi \in \mathbb{C}^{n+1}$, for $\xi \in \mathbb{R}$, by

$$v_\xi = \big(\widehat{\phi}(-\tfrac{n-1}{2a} + \xi), \widehat{\phi}(-\tfrac{n-3}{2a} + \xi), \dots, \widehat{\phi}(\tfrac{n-1}{2a} + \xi)\big), \quad \textit{and}$$
$$w_\xi = \big(\widehat{\phi}(-\tfrac{n}{2a} + \xi), \widehat{\phi}(-\tfrac{n-2}{2a} + \xi), \dots, \widehat{\phi}(\tfrac{n}{2a} + \xi)\big), \quad \textit{respectively.}$$

Then ϕ is an $a\mathbb{Z}$-shift-orthonormal $a\mathbb{Z}$-sampling function if and only if

$$v_\xi \in S_n^a, \qquad \textit{for a.e. } \xi \in [-\tfrac{1}{2a} + \varepsilon, \tfrac{1}{2a} - \varepsilon], \quad \textit{and}$$
$$w_\xi \in S_{n+1}^a, \qquad \textit{for a.e. } \xi \in (-\varepsilon, \varepsilon).$$

The equivalence follows from (6) and (7). We omit the details.
In the sequel we analyze properties of the set S_n^a.

Lemma 4. *Given $a > 0$ and $n = 1, 2, \dots,$ the following hold.*
(i) *Let $a < n$. If $n = 1$, then $S_n^a = \varnothing$. If $n = 2, 3, \dots,$ then S_n^a is the $(n-2)$-sphere in $\mathbb{C}^n$ with center $p = (\frac{a}{n}, \dots, \frac{a}{n}) \in \mathbb{C}^n$ and radius $r = \sqrt{a(1 - \frac{a}{n})}$, oriented along the hyperplane through p that is orthogonal to the position vector of p.*
(ii) *For $a = n$, we have $S_n^a = \{(1, \dots, 1) \in \mathbb{C}^n\}$.*
(iii) *For $a > n$, we have $S_n^a = \varnothing$.*

Proof. (i),(ii),(iii) Case I. $n \geq 2$. The equation $\sum_{k=1}^n \lambda_k = a$ defines the $(n-1)$-hyperplane in $\mathbb{C}^n$ through p and orthogonal to the position vector of p. The equation $\sum_{k=1}^n |\lambda_k|^2 = a$ defines the $(n-1)$-sphere of radius $R = \sqrt{a}$ around the origin in $\mathbb{C}^n$. The set S_n^a is thus the intersection of the hyperplane with this sphere, it yields the $(n-2)$-sphere described in the lemma. The radius r is obtained by $r^2 + \|p\|^2 = R^2$. Finally, we identify the cases $a = n$, where the sphere degenerates to a single point, and $a > n$, where the intersection is empty.
Case II. $n = 1$. While $S_1^1 = \{1\}$, we have for any other $a \neq 1$ that $S_1^a = \varnothing$. □

For easy referencing, we summarize when S_n^a is non-empty.

Lemma 5. (i) *Let $a < 1$. Then $S_n^a \neq \varnothing$ if and only if $n \geq 2$.*
(ii) *Let $a \geq 1$. Then $S_n^a \neq \varnothing$ if and only if $n \geq \lceil a \rceil$.*

We will need to know when S_n^a contains symmetric vectors. A vector $v \in \mathbb{C}^n$ is called symmetric if $(v_1, v_2, \dots, v_n) = (v_n, v_{n-1}, \dots, v_1)$.

Lemma 6. (i) *Let $a < 2$, $a \neq 1$. Then S_n^a contains a symmetric vector if and only if $n \geq 3$.*
(ii) *Let $a = 1$. Then S_n^a contains a symmetric vector if and only if $n = 1$ or $n \geq 3$.*
(iii) *Let $a \geq 2$. Then S_n^a contains a symmetric vector if and only if $n \geq \lceil a \rceil$.*

Proof. (i) Case I. $n = 1$. Then $S_n^a = \varnothing$ by Lemma 5.
Case II. $n = 2$. Suppose there exists a symmetric vector $v = (\lambda, \lambda) \in S_2^a$. Then $\lambda \in S_1^{a/2}$. However, since $a \neq 2$, we have by Lemma 5 that $S_1^{a/2} = \varnothing$. Contradiction.
Case III. $n \geq 3$. Then S_n^a contains, for example, the symmetric vector $v = (\lambda, \mu, \dots, \mu, \lambda)$ defined by

$$\lambda = \tfrac{a}{n} \pm \sqrt{\tfrac{n-2}{2}} \sqrt{\tfrac{a}{n}(1 - \tfrac{a}{n})} \quad \text{and} \quad \mu = \tfrac{a}{n} \mp \sqrt{\tfrac{2}{n-2}} \sqrt{\tfrac{a}{n}(1 - \tfrac{a}{n})}.$$

(ii) For $n = 1$, we have $S_1^1 = \{1\}$.
For $n = 2$, see (i), Case II.
For $n \geq 3$, see (i), Case III.
(iii) For $n < \lceil a \rceil$, $S_n^a = \varnothing$ by Lemma 5.
For $n = \lceil a \rceil$, we have $S_n^a = \{(1, \dots, 1)\}$ by Lemma 4.
For $n > \lceil a \rceil$, see (i), Case III. □

Lemma 7. *Let $a > 0$ and $n = 1, 2, \dots$ be such that $S_n^a \neq \varnothing$. Then S_n^a contains a Hermitean vector, i.e., a vector v of the form $(v_1, v_2, \dots, v_n) = (\overline{v}_n, \overline{v}_{n-1}, \dots, \overline{v}_1)$.*

Proof. We only need to consider the case $n = 2$ and $a < 2$, since for all other cases we find by inspecting the proof of Lemma 6 that S_n^a indeed contains real-valued symmetric vectors, such are Hermitean in particular. For $n = 2$ and $a < 2$, let $s = \sqrt{\frac{a}{2}(1-\frac{a}{2})}$. Then not just the real-valued vector $(\frac{a}{2}+s, \frac{a}{2}-s)$ belongs to S_2^a but also the complex vector $v = (\frac{a}{2}+is, \frac{a}{2}-is)$, which is Hermitean. □

Proof of Theorem 2. First, note that the numbers N and M are defined in such a way that

$$N = \min\{n \in \mathbb{N} \colon S_n^a \neq \varnothing\} \quad \text{and} \tag{8}$$

$$M = \min\big\{n \in \mathbb{N} \colon \text{ both } S_n^a \text{ and } S_{n+1}^a \text{ contain symmetric vectors}\big\}; \tag{9}$$

as can be verified by Lemma 5 and Lemma 6, respectively.
(i) Suppose such ϕ exists. By Lemma 3 there exist vectors $v = v_\xi$, for a. e. $\xi \in [-\frac{1}{2a}+\varepsilon, \frac{1}{2a}-\varepsilon]$, that belong to S_N^a. Since $\widehat{\phi}$ is supported on a subset of $[-N\frac{1}{2a}, N\frac{1}{2a}]$ of smaller measure, we have that at least one (in fact, many) v_ξ has a zero component. This implies that the vector of length $N-1$ obtained by removing the zero entry of v_ξ belongs to S_{N-1}^a. However, S_{N-1}^a is empty by (8). Contradiction.
(ii) A.-Existence: By (8) S_N^a is non-empty. Therefore, Lemma 3 with $\varepsilon = 0$ provides a valid construction of such a function ϕ.
B.-Uniqueness: In view of item A above, uniqueness holds if and only if S_N^a consists of a single vector. By Lemma 4 this is the case if and only if $a = N$ is an integer.
C.-Non-integrability: Case I: If $a = 1$, then $N = 1$ and the statement follows since the only sampling function $\phi \in L^2(\mathbb{R})$ with $\operatorname{supp}\widehat{\phi} \subseteq [-\frac{1}{2}, \frac{1}{2}]$ is the cardinal sine (Theorem 1(ii)).
Case II: If $a \neq 1$, then we have $N \geq 2$. Suppose there exists such a function ϕ with continuous $\widehat{\phi}$. By Lemma 3 the vector

$$v := \big(\widehat{\phi}(-\tfrac{N}{2a}), \widehat{\phi}(-\tfrac{N-2}{2a}), \dots, \widehat{\phi}(\tfrac{N}{2a})\big) \in \mathbb{C}^{N-1}$$

belongs to S_{N+1}^a. The continuity of $\widehat{\phi}$ and the constraint on its support imply $\widehat{\phi}(\pm\frac{N}{2a}) = 0$. That is, the first and last entries of v vanish. Thus we conclude that the vector of length $N-2$ that is left when deleting these two entries belongs to S_{N-1}^a. However, by (8) S_{N-1}^a is empty. Contradiction.
(iii) A.-Existence: Since $S_N^a \neq \varnothing$, there exists $v \in S_N^a$. We note that the augmented vectors $w_1 = (v, 0) \in \mathbb{C}^{N+1}$ and $w_2 = (0, v) \in \mathbb{C}^{N+1}$ belong to S_{N+1}^a. According to Lemma 4 S_{N+1}^a is always a connected set, hence we find a continuous mapping $w \colon [-\varepsilon, \varepsilon] \to S_{N+1}^a$ such that $w(-\varepsilon) = w_1$ and $w(\varepsilon) = w_2$. Now define ϕ with $\operatorname{supp}\widehat{\phi} \subseteq [-N\frac{1}{2a} - \varepsilon, N\frac{1}{2a} + \varepsilon]$ by

$$\begin{aligned} &\big(\widehat{\phi}(-\tfrac{N}{2a}+\xi), \widehat{\phi}(-\tfrac{N-2}{2a}+\xi), \dots, \widehat{\phi}(\tfrac{N}{2a}+\xi)\big) = w(\xi), &&\text{for } \xi \in (-\varepsilon, \varepsilon), \quad \text{and} \\ &\big(\widehat{\phi}(-\tfrac{N-1}{2a}+\xi), \widehat{\phi}(-\tfrac{N-3}{2a}+\xi), \dots, \widehat{\phi}(\tfrac{N-1}{2a}+\xi)\big) \equiv v, &&\text{for } \xi \in [-\tfrac{1}{2a}+\varepsilon, \tfrac{1}{2a}-\varepsilon]. \end{aligned}$$

Then by Lemma 3 we have that ϕ is $a\mathbb{Z}$-shift-orthonormal and $a\mathbb{Z}$-sampling. By Lemma 7 we can assume that v is a Hermitean vector—see Lemma 7 for this notion—and construct w in such a way that $w(-\xi) = \overline{w(\xi)}$. From this property we obtain that $\widehat{\phi}$ satisfies $\widehat{\phi}(-\xi) = \overline{\widehat{\phi}(\xi)}$,

$\xi \in \mathbb{R}$, and hence ϕ is real-valued. Without further details, we finally note that by chosing w appropriately we also obtain that $\widehat{\phi}$ is not just continuous but indeed a C^∞ function.
B.-Non-symmetry if integrable: Suppose there exists such a function ϕ which is symmetric.
Case I: If $a > 2$, then $M = \lceil a \rceil$ and we have that $\widehat{\phi}$ cannot be continuous. Contradiction.
Case II: If $a < 2$, then $M = 3$. Therefore, since $\widehat{\phi}$ is continuous we have $\widehat{\phi}(\pm\frac{3}{2a}) = 0$. Hence, since ϕ is symmetric we have that $\widehat{\phi}$ is symmetric and we obtain $\lambda := \widehat{\phi}(-\frac{1}{2a}) = \widehat{\phi}(\frac{1}{2a})$. Then from Lemma 3 we obtain that $(\lambda, \lambda) \in S_2^a$, i.e., S_2^a contains a symmetric vector, in contradiction to Lemma 6(ii).
Case III: If $a = 2$, then $M = 2$. Therefore, since $\widehat{\phi}$ is continuous we have $\widehat{\phi}(\pm\frac{1}{2}) = 0$. Hence, $\lambda := \widehat{\phi}(0)$ satisfies $\lambda \in S_1^2$ or $\lambda^2 = \lambda = 2$, which is impossible. Contradiction.
(iv) A.-Existence: Such a function is constructed as described above with the following additional properties. First, the vector v is chosen to be symmetric. Secondly, the function w is chosen to be a symmetric function. This is possible by letting $w(0) = w_0$, where w_0 is a symmetric vector in S_N^a. □

Note that the proof of (iii)-A provides a construction.

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