Norms in weighted L^2-spaces and Hardy operators

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INTRODUCTION

Let w denote a *weight* function on $(0, \infty)$, i. e., a nonnegative measurable function on $(0, \infty)$. For $1 \le p \le \infty$ the weighted space $L^p(w)$ is the space of real functions generated by the norm

$$\|f\|_{L^p(w)} = \left(\int_0^\infty |f(x)|^p w(x)\,dx\right)^{1/p},$$

with the usual modification for $p = \infty$. The weighted Hardy operator H_w is defined by

$$H_w f(x) = \frac{1}{W(x)} \int_0^x f(t)w(t)\,dt,$$

when $0 < W(x) := \int_0^x w(t)\,dt < \infty$ for all $x > 0$ (cf. [8]). Note that for $w = 1$ the operator H_w is the usual Hardy operator $Hf(x) = (1/x)\int_0^x f(t)\,dt$.

It was recently proved in [7] that

(a) If $1 \le p \le \infty$, $\alpha > -1$ and $f \in L^p(x^{-\alpha p-1})$, then

$$\|f - Hf\|_{L^p(x^{-\alpha p-1})} \le \left(1 + \frac{1}{\alpha+1}\right)\|f\|_{L^p(x^{-\alpha p-1})}. \tag{0.1}$$

(b) If $1 \le p \le \infty$, $\alpha > -1$, $\alpha \ne 0$, and $f \in L^p(x^{-\alpha p-1})$, then

$$\|f\|_{L^p(x^{-\alpha p-1})} \le \left(1 + \frac{1}{|\alpha|}\right)\|f - Hf\|_{L^p(x^{-\alpha p-1})}. \tag{0.2}$$

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Moreover, it was pointed out in [7] that (0.2) implies Grisvard's inequality, the so-called fractional order Hardy inequality. Moreover, (0.1) and (0.2) applied for decreasing functions give a well-known characterization of Lorentz $L^{p,q}$-norms from [1] (cf. also [2]).

One special case of (0.1) and (0.2), i. e., when $p = 2$ and $\alpha = -1/2$, gives

$$\frac{1}{3}\|f - Hf\|_{L^2} \le \|f\|_{L^2} \le 3\|f - Hf\|_{L^2}.$$

In this paper we first prove a result which in fact shows that we even have the identity

$$\|f\|_{L^2} = \|f - Hf\|_{L^2}$$

for every function f in L^2. In fact, we prove that this formula holds even in the weighted spaces $L^2(w)$, where w is a weight function with $\int_0^\infty w(t)\,dt = \infty$, if the usual Hardy operator H is replaced by the weighted Hardy operator H_w; see Section 1. More generally, in Section 1 we present and discuss some representations of the norm in a weighted L^2-space of the form $\|f - Af\|_{L^2(w)}$, where A are averaging operators of Hardy type. The proofs are given directly via integrals. In Section 2 another proof is given by using well-known results about isometries in Hilbert spaces. Moreover, the inequalities (0.1) and (0.2) are sharpened for some cases and these new bounds are sharp (see Section 3). Finally, in Section 3 we also present some new applications to fractional order Hardy inequalities and equivalent representations of norms in Lorentz spaces.

1 THE MAIN RESULT

Our main result in this section reads:

THEOREM 1.1 Let w be a weight function on $(0, \infty)$.

(i) Suppose $0 < W(x) = \int_0^x w(t)\,dt < \infty$ for any $x > 0$ and $\int_0^\infty w(t)\,dt = \infty$. If $f \in L^2(w)$, then

$$\|f\|_{L^2(w)} = \|f - H_w f\|_{L^2(w)}, \tag{1.1}$$

where $H_w f(x) = \big(1/W(x)\big)\int_0^x f(t)w(t)\,dt$.

(ii) Suppose $0 < \widetilde{W}(x) = \int_x^\infty w(t)\,dt < \infty$ for any $x > 0$ and $\int_0^\infty w(t)\,dt = \infty$. If $f \in L^2(w)$, then

$$\|f\|_{L^2(w)} = \|f - \widetilde{H}_w f\|_{L^2(w)}, \tag{1.2}$$

where $\widetilde{H}_w f(x) = \big(1/\widetilde{W}(x)\big)\int_x^\infty f(t)w(t)\,dt$.

Proof: (i) For arbitrary fixed $0 < b < \infty$ the function fw is Lebesgue integrable on $(0, b)$. In fact, by the Schwarz inequallity and the assumptions $f \in L^2(w)$, $W(b) < \infty$ it follows that

$$\left(\int_0^b f(t)w(t)\,dt\right)^2 \le \int_0^b f(t)^2 w(t)\,dt \int_0^b w(t)\,dt \le \|f\|^2_{L^2(w)} W(b) < \infty.$$

Then, for almost all $x \in (0, b)$,

$$\begin{aligned}
&\frac{d}{dx}\left[\frac{1}{W(x)}\left(\int_0^x f(t)w(t)\,dt\right)^2\right] \\
&\qquad = \frac{2f(x)w(x)}{W(x)}\int_0^x f(t)w(t)\,dt - \frac{w(x)}{W(x)^2}\left(\int_0^x f(t)w(t)\,dt\right)^2 \\
&\qquad = f(x)^2w(x) - \left[f(x) - \frac{1}{W(x)}\int_0^x f(t)w(t)\,dt\right]^2 w(x)
\end{aligned}$$

(cf. [3], p. 74). By integrating from 0 to b we obtain

$$\begin{aligned}
&\int_0^b f(t)^2w(t)\,dt - \int_0^b\left[f(x) - \frac{1}{W(x)}\int_0^x f(t)w(t)\,dt\right]^2 w(x)\,dx \\
&\qquad = \frac{1}{W(b)}\left(\int_0^b f(t)w(t)\,dt\right)^2 - \lim_{x\to 0+}\left[\frac{1}{W(x)}\left(\int_0^x f(t)w(t)\,dt\right)^2\right].
\end{aligned}$$

The last limit is zero. Really, in view of the Schwartz inequality and the assumption $f \in L^2(w)$, we have

$$0 \le \frac{1}{W(x)}\left(\int_0^x f(t)w(t)\,dt\right)^2 \le \int_0^x f(t)^2w(t)\,dt \to 0 \qquad \text{as } x \to 0^+.$$

Thus

$$\int_0^b f(t)^2w(t)\,dt - \int_0^b [f(x) - H_w f(x)]^2 w(x)\,dx = \frac{1}{W(b)}\left(\int_0^b f(t)w(t)\,dt\right)^2. \quad (1.3)$$

Now choose $\varepsilon > 0$ arbitrary. Since $f \in L^2(w)$ it follows that there exists $c > 0$ such that

$$\int_c^\infty f(t)^2w(t)\,dt < \frac{\varepsilon}{4}.$$

By using the Schwarz inequality, we obtain, for any $b > c$,

$$\left(\int_c^b f(t)w(t)\,dt\right)^2 \le \int_c^b f(t)^2w(t)\,dt \int_c^b w(t)\,dt \le \int_c^b w(t)\,dt\, W(b) < \frac{\varepsilon}{4}W(b).$$

Moreover, the assumption $\int_0^\infty w(t)\,dt = \infty$ gives that for large b we have

$$\frac{1}{W(b)}\left(\int_0^c f(t)w(t)\,dt\right)^2 < \varepsilon/4.$$

Therefore, for large b, we find that

$$\begin{aligned}
\frac{1}{W(b)}\left(\int_0^b f(t)w(t)\,dt\right)^2 &= \frac{1}{W(b)}\left(\int_0^c f(t)w(t)\,dt + \int_c^b f(t)w(t)\,dt\right)^2 \\
&\le 2\frac{1}{W(b)}\left[\left(\int_0^c f(t)w(t)\,dt\right)^2 + \left(\int_c^b f(t)w(t)\,dt\right)^2\right] \\
&\le 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \varepsilon
\end{aligned}$$

and conclude from (1.3) that

$$\lim_{b\to\infty}\left\{\int_0^b f(t)^2 w(t)\,dt-\int_0^b [f(x)-H_w f(x)]^2 w(x)\,dx\right\}=0,$$

or, equivalently,

$$\|f\|_{L^2(w)}=\|f-H_w f\|_{L^2(w)}.$$

(ii) For fixed $0<a<\infty$ the function fw is Lebesgue integrable on (a,∞) and for almost all $x\in(a,\infty)$ we have

$$\begin{aligned}\frac{d}{dx}&\left[-\frac{1}{\widetilde{W}(x)}\left(\int_x^\infty f(t)w(t)\,dt\right)^2\right]\\ &=\frac{2f(x)w(x)}{\widetilde{W}(x)}\int_x^\infty f(t)w(t)\,dt-\frac{w(x)}{\widetilde{W}(x)^2}\left(\int_x^\infty f(t)w(t)\,dt\right)^2\\ &=f(x)^2w(x)-\left[f(x)-\frac{1}{\widetilde{W}(x)}\int_x^\infty f(t)w(t)\,dt\right]^2 w(x).\end{aligned}$$

By integrating from a to ∞ we obtain

$$\begin{aligned}\int_a^\infty f(t)^2w(t)\,dt&-\int_a^\infty\left[f(x)-\frac{1}{\widetilde{W}(x)}\int_x^\infty f(t)w(t)\,dt\right]^2 w(x)\,dx\\ &=\frac{1}{\widetilde{W}(a)}\left(\int_a^\infty f(t)w(t)\,dt\right)^2-\lim_{x\to\infty}\left[\frac{1}{\widetilde{W}(x)}\left(\int_x^\infty f(t)w(t)\,dt\right)^2\right].\end{aligned}$$

The last limit is zero since, by the Schwartz inequality and the assumption $f\in L^2(w)$,

$$0\le\frac{1}{\widetilde{W}(x)}\left(\int_x^\infty f(t)w(t)\,dt\right)^2\le\int_x^\infty f(t)^2w(t)\,dt\to 0\qquad\text{as } x\to\infty.$$

Thus

$$\int_a^\infty f(t)^2w(t)\,dt-\int_a^\infty[f(x)-\widetilde{H}_w f(x)]^2w(x)\,dx=\frac{1}{\widetilde{W}(a)}\left(\int_a^\infty f(t)w(t)\,dt\right)^2. \tag{1.4}$$

Now choose $\varepsilon>0$ arbitrary. Since $f\in L^2(w)$ it follows that there exists $c>0$ such that

$$\int_c^\infty f(t)^2w(t)\,dt<\frac{\varepsilon}{4}.$$

Moreover, according to the Schwarz inequality, we obtain, for any $a<c$,

$$\left(\int_c^\infty f(t)w(t)\,dt\right)^2\le\int_c^\infty f(t)^2w(t)\,dt\int_c^\infty w(t)\,dt\le\int_c^\infty w(t)\,dt\cdot\widetilde{W}(a)<\frac{\varepsilon}{4}\widetilde{W}(a).$$

We also note that the assumption $\int_0^\infty w(t)\,dt=\infty$ gives that for small $a>0$ we have

$$\frac{1}{\widetilde{W}(a)}\left(\int_0^c f(t)w(t)\,dt\right)^2<\frac{\varepsilon}{4}.$$

Therefore, for small a, we find that

$$\begin{aligned}\frac{1}{\widetilde{W}(a)}\left(\int_a^\infty f(t)w(t)\,dt\right)^2 &= \frac{1}{\widetilde{W}(a)}\left(\int_a^c f(t)w(t)\,dt+\int_c^\infty f(t)w(t)\,dt\right)^2\\ &\le \frac{2}{\widetilde{W}(a)}\left[\left(\int_a^c f(t)w(t)\,dt\right)^2+\left(\int_c^\infty f(t)w(t)\,dt\right)^2\right]\\ &\le 2\left(\frac{\varepsilon}{4}+\frac{\varepsilon}{4}\right)=\varepsilon\end{aligned}$$

and conclude from (1.4) that

$$\lim_{a\to 0^+}\left\{\int_a^\infty f(t)^2w(t)\,dt-\int_a^\infty [f(x)-\widetilde{H}_w f(x)]^2 w(x)\,dx\right\}=0,$$

or

$$\|f\|_{L^2(w)}=\|f-\widetilde{H}_w f\|_{L^2(w)}.$$

□

REMARK 1.2 If $f\in L^2$, then

$$\|f\|_{L^2}=\|f-Hf\|_{L^2}\quad\text{or}\quad\int_0^\infty [Hf(x)]^2\,dx=2\int_0^\infty f(x)Hf(x)\,dx,$$

where $Hf(x)=(1/x)\int_0^x f(t)\,dt$.

REMARK 1.3 The assumption $\int_0^\infty w(t)\,dt=\infty$ on the weight w in Theorem 1.1 cannot be removed. In fact, if $0<\int_0^\infty w(t)\,dt<\infty$, then for the constant function $f_0(x)=c$ we have $H_w f_0=f_0$, $\widetilde{H}_w f_0=f_0$ and so $\|f_0-H_w f_0\|_{L^2(w)}=0$, $\|f_0-\widetilde{H}_w f_0\|_{L^2(w)}=0$ but $\|f_0\|_{L^2(w)}=c\left(\int_0^\infty w(t)\,dt\right)^{1/2}>0$.

REMARK 1.4 According to Remark 1.3 the condition $\int_0^\infty w(t)\,dt=\infty$ is necessary but by analyzing the proof of Theorem 1.1 we see that if $f\in L^2(w)$ and $0<\int_0^\infty w(t)\,dt<\infty$, then

$$\begin{aligned}\|f\|^2_{L^2(w)}&=\|f-H_w f\|^2_{L^2(w)}+\left(\int_0^\infty f(t)w(t)\,dt\right)^2\Big/\int_0^\infty w(t)\,dt\\ &=\|f-\widetilde{H}_w f\|^2_{L^2(w)}+\left(\int_0^\infty f(t)w(t)\,dt\right)^2\Big/\int_0^\infty w(t)\,dt.\end{aligned}$$

COROLLARY 1.5

(i) Let f and w satisfy the assumptions in Theorem 1.1(i). Then

$$\|f\|_{L^2(w)}=\|f-S_w f\|_{L^2(w)},$$

where $S_w f(x)=\int_x^\infty f(t)w(t)/W(t)\,dt$.

(ii) Let f and w satisfy the assumptions in Theorem 1.1(ii). Then

$$\|f\|_{L^2(w)}=\|f-\widetilde{S}_w f\|_{L^2(w)},$$

where $\widetilde{S}_w f(x)=\int_0^x f(t)w(t)/\widetilde{W}(t)\,dt$.

Proof: (i) Consider the weight u defined by $u(t) = w(t)/W(t)^2$ and note that for $b > x$

$$\int_x^b u(t)\,dt = \int_x^b d\Big[-\frac{1}{W(t)}\Big] = \frac{1}{W(x)} - \frac{1}{W(b)} \to \frac{1}{W(x)} \quad \text{as } b \to \infty,$$

so that

$$U(x) := \int_x^\infty u(t)\,dt = \frac{1}{W(x)}.$$

This means that the weight u satisfies the assumptions in Theorem 1.1(ii). Moreover, we note that if $g(x) := f(x)W(x)$, then $f \in L^2(w)$ if and only if $g \in L^2(u)$. Thus, according to Theorem 1.1(ii),

$$\begin{aligned}\|f\|_{L^2(w)} &= \|g\|_{L^2(u)} = \|g - \widetilde{H}_u g\|_{L^2(u)} \\ &= \left(\int_0^\infty \left| f(x)W(x) - \frac{1}{U(x)}\int_x^\infty f(t)\frac{w(t)}{W(t)}\,dt\right|^2 \frac{w(x)}{W(x)^2}\,dx\right)^{1/2} \\ &= \left(\int_0^\infty |f(x) - S_w f(x)|^2 w(x)\,dx\right)^{1/2} = \|f - S_w f\|_{L^2(w)}.\end{aligned}$$

(ii) Let now the auxiliary weight v be defined by $v(t) = w(t)/\widetilde{W}(t)^2$ and note that as in the proof of (i) we find that

$$V(x) := \int_0^x v(t)\,dt = \frac{1}{\widetilde{W}(x)}.$$

We see that the weight v satisfies the assumptions in Theorem 1.1(i). Let $h(x) := f(x)\widetilde{W}(x)$. Then $f \in L^2(w)$ if and only if $h \in L^2(v)$. Therefore, by Theorem 1.1(i),

$$\begin{aligned}\|f\|_{L^2(w)} &= \|h\|_{L^2(v)} = \|h - H_v h\|_{L^2(v)} \\ &= \left(\int_0^\infty \left| f(x)\widetilde{W}(x) - \frac{1}{V(x)}\int_0^x f(t)\frac{w(t)}{\widetilde{W}(t)}\,dt\right|^2 \frac{w(x)}{\widetilde{W}(x)^2}\,dx\right)^{1/2} \\ &= \left(\int_0^\infty |f(x) - \widetilde{S}_w f(x)|^2 w(x)\,dx\right)^{1/2} = \|f - \widetilde{S}_w f\|_{L^2(w)}.\end{aligned}$$

The proof is complete. □

By applying Theorem 1.1 and Corollary 1.5 with the weights $w(t) = 1$ and $w(t) = t^{-2}$ we obtain the following (somewhat surprising) complements of the identity pointed out in Remark 1.2.

EXAMPLE 1.6 If $f \in L^2$, then

$$\|f - Hf\|_{L^2} = \|f\|_{L^2} = \|f - Sf\|_{L^2},$$

where $Hf(x) = (1/x)\int_0^x f(t)\,dt$ and $Sf(x) = \int_x^\infty \big(f(t)/t\big)\,dt$.

If $f \in L^2(x^{-2})$, then

$$\|f - \widetilde{H}f\|_{L^2(x^{-2})} = \|f\|_{L^2(x^{-2})} = \|f - \widetilde{S}f\|_{L^2(x^{-2})},$$

where $\widetilde{H}f(x) = x\int_x^\infty \big(f(t)/t^2\big)\,dt$ and $\widetilde{S}f(x) = \int_0^x \big(f(t)/t\big)\,dt$.

REMARK 1.7 The first statement in Example 1.6 means that for each function in L^2 it yields that

$$\|f\|_{L^2} = \|f - Af\|_{L^2}$$

for $A = H$ and $A = S$ so it should be interesting to find *all* (averaging) operators having this remarkable property.

2 AN ALTERNATIVE PROOF VIA ISOMETRIES IN HILBERT SPACES

We will give here another proof of Theorem 1.1 by using the following result about isometries in an arbitrary real Hilbert space $\mathcal{H}$ (concerning *complex* Hilbert spaces see e. g. [5], Chapters 3.9-10; [9], Chapter 4; [10], Lemma 2.6 and Theorem 2.26):

LEMMA 2.1 Let $\mathcal{H}$ be a real Hilbert space with the norm $\|x\| = \|x\|_{\mathcal{H}} = \langle x, x\rangle^{1/2}$. Consider $T: \mathcal{H} \to \mathcal{H}$ with dual operator T^*.

- (i) The operator T is an isometry, i. e., $\|Tx\| = \|x\|$ for any $x \in \mathcal{H}$ if and only if $T^*T = I$.
- (ii) The operator T is a surjective isometry (or unitary operator) if and only if $T^*T = TT^* = I$.

The proof from the complex case need to be modified so we give the details.

Proof: (i) If $T^*T = I$, then, for every $x, y \in \mathcal{H}$,

$$\langle Tx, Ty\rangle = \langle T^*Tx, y\rangle = \langle x, y\rangle$$

and, in particular, for $y = x \in \mathcal{H}$ we obtain

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, x\rangle = \|x\|^2$$

or $\|Tx\| = \|x\|$.

Conversely, if T is an isometry in $\mathcal{H}$, then

$$\begin{aligned}\langle Tx, Ty\rangle &= \frac{1}{2}[\langle T(x+y), T(x+y)\rangle - \langle Tx, Tx\rangle - \langle Ty, Ty\rangle] \\ &= \frac{1}{2}[\|T(x+y)\|^2 - \|Tx\|^2 - \|Ty\|^2] = \frac{1}{2}[\|x+y\|^2 - \|x\|^2 - \|y\|^2] = \langle x, y\rangle\end{aligned}$$

for all $x, y \in \mathcal{H}$. This gives that

$$\langle T^*Tx, y\rangle = \langle x, y\rangle \text{ or } \langle (T^*T - I)x, y\rangle = 0$$

for all $x, y \in \mathcal{H}$. Thus, choosing $y = (T^*T - I)x$ we get

$$\|(T^*T - I)x\|^2 = \langle (T^*T - I)x, (T^*T - I)x\rangle = 0$$

or $(T^*T - I)x = 0$ for every $x \in \mathcal{H}$ and so $T^*T = I$.

(ii) If T is a surjective isometry, then $T^*T = I$ by (i). Also the inverse T^{-1} exists and maps $T(\mathcal{H}) = \mathcal{H}$ into $\mathcal{H}$ since the equalities $\|Tx - Ty\| = \|T(x-y)\| = \|x - y\|$ give that T is one-to-one. Therefore,

$$T^* = (T^*T)T^{-1} = IT^{-1} = T^{-1}$$

and so

$$TT^* = TT^{-1} = I.$$

Thus $T^*T = TT^* = I$.

Conversely, assume that $T^*T = I = TT^*$. The first equality gives, by (i), that T is an isometry. The second equality implies that

$$T^* = (T^{-1}T)T^* = T^{-1}(TT^*) = T^{-1}I = T^{-1},$$

and therefore T is onto so that T is a surjective isometry. □

Second proof of Theorem 1.1. We look now at the space $L^2(w)$ as a Hilbert space with the inner product given by

$$\langle f, g\rangle = \int_0^\infty f(x)g(x)w(x)\,dx.$$

Then, by using the Fubini theorem, we find that

$$\begin{aligned}\langle H_w f, g\rangle &= \int_0^\infty H_w f(x)g(x)w(x)\,dx = \int_0^\infty \left(\frac{1}{W(x)}\int_0^x f(t)w(t)\,dt\right)g(x)w(x)\,dx\\ &= \int_0^\infty \left(\int_t^\infty g(x)\frac{w(x)}{W(x)}\,dx\right)f(t)w(t)\,dt = \langle f, H_w^* g\rangle,\end{aligned}$$

where

$$H_w^* f(x) = \int_x^\infty f(t)\frac{w(t)}{W(t)}\,dt.$$

Now, if $0 < W(x) < \infty$ for any $x > 0$ and $W(\infty) = \infty$, then

$$H_w^* \circ H_w = H_w \circ H_w^* = H_w + H_w^*. \tag{2.1}$$

In fact, by again using the Fubini theorem, we obtain that

$$\begin{aligned}H_w^* \circ H_w f(x) &= \int_x^\infty H_w f(t)\frac{w(t)}{W(t)}\,dt = \int_x^\infty \left(\int_0^t f(s)w(s)\,ds\right)\frac{w(t)}{W(t)^2}\,dt\\ &= \int_0^x \left(\int_x^\infty \frac{w(t)}{W(t)^2}\,dt\right)f(s)w(s)\,ds + \int_x^\infty \left(\int_s^\infty \frac{w(t)}{W(t)^2}\,dt\right)f(s)w(s)\,ds\\ &= \int_0^x \left(\int_x^\infty d\left[-\frac{1}{W(t)}\right]\right)f(s)w(s)\,ds + \int_x^\infty \left(\int_s^\infty d\left[-\frac{1}{W(t)}\right]\right)f(s)w(s)\,ds\\ &= \frac{1}{W(x)}\int_0^x f(s)w(s)\,ds + \int_x^\infty \frac{1}{W(s)}f(s)w(s)\,ds = H_w f(x) + H_w^* f(x),\end{aligned}$$

and

$$\begin{aligned}
H_w \circ H_w^* f(x) &= \frac{1}{W(x)} \int_0^x H_w^* f(t) w(t)\, dt \\
&= \frac{1}{W(x)} \int_0^x \left(\int_t^\infty f(s) \frac{w(s)}{W(s)}\, ds \right) w(t)\, dt \\
&= \frac{1}{W(x)} \int_0^x \left(\int_0^s w(t)\, dt \right) f(s) \frac{w(s)}{W(s)}\, ds \\
&\quad + \frac{1}{W(x)} \int_x^\infty \left(\int_0^x w(t)\, dt \right) f(s) \frac{w(s)}{W(s)}\, ds \\
&= \frac{1}{W(x)} \int_0^x f(s) w(s)\, ds + \int_x^\infty f(s) \frac{w(s)}{W(s)}\, ds = H_w f(x) + H_w^* f(x).
\end{aligned}$$

For the operator $T_w = I - H_w$ we obtain from (2.1) that

$$T_w^* \circ T_w = (I - H_w)^* \circ (I - H_w) = (I - H_w^*) \circ (I - H_w) = I - H_w^* - H_w + H_w^* \circ H_w = I$$

and

$$T_w \circ T_w^* = (I - H_w) \circ (I - H_w)^* = (I - H_w) \circ (I - H_w^*) = I - H_w - H_w^* + H_w \circ H_w^* = I.$$

By using Lemma 2.1 we obtain that T_w is a surjective isometry in $L^2(w)$, i. e., equality (1.1) holds. Similarly we can prove equality (1.2). □

REMARK 2.2 Corollary 1.5 follows also from the above proof since $H_w^* = S_w$ and the operator $T_w = I - H_w$ is a surjective isometry in $L^2(w)$ if and only if the dual operator $T_w^* = I - H_w^*$ is a surjective isometry in $L^2(w)$.

3 FURTHER RESULTS AND REMARKS

First we state and prove the following sharp inequalities of the type (0.1) and (0.2) for the case $p = 2$:

COROLLARY 3.1 If $f \in L^2(x^\beta)$, $\beta < 1$ and $\beta \neq -1$, then

$$\min\left(1, \frac{|1+\beta|}{1-\beta}\right) \|f\|_{L^2(x^\beta)} \le \|f - Hf\|_{L^2(x^\beta)} \le \max\left(1, \frac{|1+\beta|}{1-\beta}\right) \|f\|_{L^2(x^\beta)}. \quad (3.1)$$

Both inequalities are sharp.

Proof: First we prove that if $\beta < 1$ and $\beta \neq -1$, then

$$\|f\|_{L^2(x^\beta)} = \|f - (1-\beta) Hf\|_{L^2(x^\beta)}. \quad (3.2)$$

In fact, by using Theorem 1.1 with $w(t) = t^{-\beta}$ and $g(t) = f(x)x^\beta$ we obtain that

$$\begin{aligned}
\|f\|_{L^2(x^\beta)} &= \|g\|_{L^2(x^{-\beta})} = \|g - Hg\|_{L^2(x^{-\beta})} \\
&= \left(\int_0^\infty \left| f(x)x^\beta - x^{\beta-1}(1-\beta) \int_0^x f(t)\, dt \right|^2 x^{-\beta}\, dx \right)^{1/2} \\
&= \|f - (1-\beta) Hf\|_{L^2(x^\beta)}.
\end{aligned}$$

Next we note that, according to (3.2) and the Minkowski inequality or the reversed Minkowski inequality, we have

$$\begin{aligned}(1-\beta)\|f-Hf\|_{L^2(x^\beta)} &= \|-\beta f+f-(1-\beta)Hf\|_{L^2(x^\beta)}\\ &\le |\beta|\,\|f\|_{L^2(x^\beta)}+\|f-(1-\beta)Hf\|_{L^2(x^\beta)}\\ &= (|\beta|+1)\|f\|_{L^2(x^\beta)}\end{aligned}$$

or

$$\begin{aligned}(1-\beta)\|f-Hf\|_{L^2(x^\beta)} &= \|-\beta f+f-(1-\beta)Hf\|_{L^2(x^\beta)}\\ &\ge \Big|\,|\beta|\,\|f\|_{L^2(x^\beta)}-\|f-(1-\beta)Hf\|_{L^2(x^\beta)}\Big|\\ &= \big|\,|\beta|-1\big|\,\|f\|_{L^2(x^\beta)}.\end{aligned}$$

Thus

$$\frac{\big|\,|\beta|-1\big|}{1-\beta}\|f\|_{L^2(x^\beta)}\le\|f-H\|_{L^2(x^\beta)}\le\frac{|\beta|+1}{1-\beta}\|f\|_{L^2(x^\beta)}$$

and (3.1) follows.

For $r>(-\beta-1)/2$ the functions

$$f_r(x):=x^r\chi_{[0,1]}(x)$$

are in $L^2(x^\beta)$ and

$$Q_r:=\|f_r-Hf_r\|_{L^2(x^\beta)}/\|f_r\|_{L^2(x^\beta)}=\left(\left(\frac{r}{r+1}\right)^2+\frac{2r+\beta+1}{(r+1)^2(1-\beta)}\right)^{1/2}.$$

Since $Q_r\to|1+\beta|/(1-\beta)$ as $r\to[(-\beta-1)/2]^+$ and $Q_r\to 1$ as $r\to\infty$ we conclude that both inequalities in (3.1) are sharp. The proof is complete. ☐

REMARK 3.2 We note that the crucial formula (3.2) holds trivially also for the case $\beta=1$ but for $\beta>1$ it can never hold for any nontrivial function f.

Second we consider a result in Lorentz $L^{p,q}$-spaces. In [1] Bennet-DeVore-Sharpley used the functional $f^{**}-f^*$ in the definition of the "weak-L^∞" space and also proved an interpolation result in this connection (see also [2]). Moreover, they proved that if $f\in L^{p,q}$, $1<p<\infty$, $1\le q\le\infty$, and $f^{**}(\infty)=\lim_{t\to\infty}f^{**}(t)=0$, then

$$\|f\|_{L^{p,q}}:=\left(\int_0^\infty\left(t^{1/p}f^*(t)\right)^q\frac{dt}{t}\right)^{1/q}\approx\left(\int_0^\infty\left(t^{1/p}[f^{**}(t)-f^*(t)]\right)^q\frac{dt}{t}\right)^{1/q},\quad(3.3)$$

where f^* denotes the nonincreasing rearrangement of a measurable function f on a σ-finite measure space (Ω,μ) and $f^{**}(t)=Hf^*(t)=(1/t)\int_0^t f^*(s)\,ds$. By applying Corollary 3.1 with $\beta=2/p-1$ we obtain the following more precise statement for $q=2$:

EXAMPLE 3.3 If $f \in L^{p,2}$, $1 < p < \infty$ and $f^{**}(\infty) = \lim_{t \to \infty} f^{**}(t) = 0$, then

$$\min\left(1, \frac{1}{p-1}\right)\left(\int_0^\infty (t^{1/p} f^*(t))^2 \frac{dt}{t}\right)^{1/2} \le \left(\int_0^\infty (t^{1/p}[f^{**}(t) - f^*(t)])^2 \frac{dt}{t}\right)^{1/2}$$
$$\le \max\left(1, \frac{1}{p-1}\right)\left(\int_0^\infty (t^{1/p} f^*(t))^2 \frac{dt}{t}\right)^{1/2}.$$

The first inequality is sharp for $1 < p \le 2$ and the second one for $2 \le p < \infty$.

REMARK 3.4 By using the estimates (0.1) and (0.2) instead of Corollary 3.1 we obtain another and more precise (than in [1]) variant of (3.3) also for the general case $p > 1$. Note also that Example 3.3 with $p = 2$ in particular gives another remarkable complement of the equality in Remark 1.2 as follows: if $f \in L^2$, then

$$\|f\|_{L^2} = \|f - Hf\|_{L^2} = \|f^* - Hf^*\|_{L^2}.$$

Third we recall that it is well-known that if $f \in C^1(0, \infty)$ with $f(0) = f(\infty) = 0$, then the *fractional order Hardy inequality*

$$\left(\int_0^\infty \left|\frac{f(x)}{x^\theta}\right|^p dx\right)^{1/p} \le C(\theta, p)\left(\int_0^\infty \int_0^\infty \frac{|f(x) - f(y)|^p}{|x - y|^{\theta p + 1}} dx\, dy\right)^{1/p}, \quad (3.4)$$
$$0 < \theta < 1,$$
$$\theta \ne 1/p$$

holds. An elementary proof together with references and historical remarks can be found in [7] (see also [4]). In particular, the following (so far least) constant $C(\theta, p) = 2^{-1/p}\left(1 + 1/|\theta - 1/p|\right)$ was pointed out. By applying our Corollary 3.1 with $\beta = -2\theta$ the following improvement (for $\theta < 1/2$) of the above constant can be obtained for the case $p = 2$:

$$C(\theta, 2) = \frac{1}{\sqrt{2}} \frac{2\theta + 1}{|2\theta - 1|}, \qquad \theta \ne 1/2.$$

For the readers convenience we include the proof. By Jensen's inequality

$$\left|f(x) - \frac{1}{x}\int_0^x f(y)\, dy\right|^2 = \left|\frac{1}{x}\int_0^x [f(x) - f(y)]\, dy\right|^2 \le \frac{1}{x}\int_0^x |f(x) - f(y)|^2\, dy,$$

and so

$$\int_0^\infty \left|f(x) - \frac{1}{x}\int_0^x f(y)\, dy\right|^2 x^{-2\theta}\, dx \le \int_0^\infty \int_0^x |f(x) - f(y)|^2\, dy\, x^{-2\theta - 1}\, dx$$
$$\le \int_0^\infty \int_0^x \frac{|f(x) - f(y)|^2}{|x - y|^{2\theta + 1}}\, dy\, dx$$
$$= \frac{1}{2}\int_0^\infty \int_0^\infty \frac{|f(x) - f(y)|^2}{|x - y|^{2\theta + 1}}\, dx\, dy,$$

where in the last equality we have used the symmetry of the integral. The proof now follows by using this estimate together with the left hand side inequality (3.1) with $\beta = -2\theta$.

REMARK 3.5 The exception $\theta \ne 1/p$ in the formula (3.4) was analyzed and explained in [6]. The key is to study interpolation of closed subspaces, that is, the real method of interpolation $(\cdot)_{\theta,p}$ applied to closed subspaces of X_0 and X_1, which need not necessarily be a closed subspace of $(X_0, X_1)_{\theta,p}$. This gives also an explanation of the corresponding restriction $\beta \ne -1$ in Corollary 3.1.

Open problem Find the best possible constant $C(\theta, p)$ in inequality (3.4).

REMARK 3.6 The results given in this paper for functions on $(0, \infty)$ can also be formulated for functions on $I = (a, b)$, where $-\infty \leq a < b \leq \infty$. For example, Theorem 1.1(i) will then have the following form: *If* w *is a nonnegative function on* I *such that* $0 < W(x) := \int_a^x w(t)\,dt < \infty$ *for any* $x \in I$ *and* $\int_a^b w(t)\,dt = \infty$, *then* $\int_a^b [f(x) - H_w f(x)]^2 w(x)\,dx = \int_a^b f(x)^2 w(x)\,dx$, *where* $H_w f(x) = \big(1/W(x)\big) \int_a^x f(t)w(t)\,dt$.

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