On Dimension Invariance of Discrete Gabor Expansions

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Abstract

We reveal a dimension invariance property of the (discrete) Gabor expansion (DGE). This is an important and very useful result in applications where signal dimension is large, or signals are on-going with infinite length. As a consequence of this result, a DGE scheme developed in a given (relatively small) dimension may be applied in spaces of arbitrary large dimension by simply adjusting the number of translations accordingly.

Key Words: Discrete Gabor Expansions (DGE), Synthesis and analysis waveforms, Frames, Pseudo-duals, Dimension Invariance of the Gabor expansion.

1 Introduction

The Gabor expansion provides a very usual tool in time-frequency signal analysis and processing, e.g., [5], [6], [7]. Intriguing physiological studies also suggest that the majority of receptive field profiles of the mammalian visual system match quite well with Gabor type of functions [14].

The difficulties encountered in the past for finding Gabor coefficients have been, more or less, resolved by recent results on the subject, e.g., [1], [2], [9], [11], [12], [15], [16], [18], [19], [20]. Since most algorithms are derived in finite dimensional spaces, the computation of analysis windows becomes impractical and even impossible when signal dimension gets large.

There is a work on DGE for arbitrary length finite energy signals [16], where a Periodization and extension technique is used to generalize the finite DGE to infinite case based on the DGE scheme in [18]. [16] is a particular approach with certain restrictions. For instance, it is required that the length of analysis window is the same as that of synthesis window, which may affect the solution result and the existence of the solution of analysis windows.

We consider the characterization and computation of analysis waveforms in spaces of all finite energy signals (e.g., $l^2(\mathbb{Z})$) in a more general and natural way using the theory of general frame decompositions. We show that the (discrete) Gabor expansion has an intrinsic dimension invariance property. With this property, a pair of synthesis and analysis waveforms found in a given dimension remains valid for the Gabor transform as signal dimension changes, provided
a simple condition is satisfied. The only adjustment to be made is the number of translations. Therefore, in applications where large-size signal is encountered, one has a tool to develop a DGE scheme in a much smaller dimension and be sure that it can be safely used in signal spaces of any large size.

2 Background of The Discrete Gabor Expansion

A discrete Gabor expansion in a space of finite energy (e.g., $l^2(Z)$) is generally considered as a decomposition with two Gabor sequences $\{h_{mn}\}$ (synthesis) and $\{\gamma_{mn}\}$ (analysis) in the following way,

$$\forall x \in l^2(Z), \quad x = \sum_{m=0}^{T-1} \sum_{n=-\infty}^{\infty} \langle x, \gamma_{mn} \rangle h_{mn},$$

where $h_{mn}$ and $\gamma_{mn}$ are duals or pseudo-duals [9], [11] (sometimes biorthogonal [2]) to each other. They are translated and complex modulated window sequences, e.g.,

$$h_{mn}(\cdot) = h(\cdot - nT)e^{\frac{2\pi i}{T_1} m \cdot},$$

where $T, T_1$ are integers and $h$ is a discrete-time waveform (or window). $T$ is the time shifting step, and $\frac{T_1}{T}$ specifies the frequency shifting step, or $T_1$ is the number of frequency channels.

To find a Gabor expansion of a signal for the given family of Gabor waveforms $\{h_{mn}\}$, we need to find an associated analysis waveforms $\{\gamma_{mn}\}$. It is well known that a complete Gabor sequence $\{h_{mn}\}$ can be a frame, e.g., [3], [4]. A frame in a Hilbert space is certain complete sequence of functions $\{x_n\}$ that provides a decomposition of the space. Frames are generally overcomplete. Therefore, the frame decomposition is not generally unique [8]. This is precisely a property of Gabor expansions.

In fact, a discrete Gabor expansion is really a general frame decomposition, a point of view we elaborated in detail in, e.g., [9], [10], [11], [12], [13].

3 The Dimension Invariance of The Gabor Expansion

We now denote $l^2(Z_N)$ the N-dimensional space of all periodic finite energy signals. We shall always assume $N$ to be a multiple of $T$ and $T_1$ as it always is. As $N$ increases, $l^2(Z_N)$ eventually becomes $l^2(Z_\infty)$ or $l^2(Z)$. We shall take the argument of $h$ and $\gamma$ to be modulo $N$. When $N$ changes, $h$ and $\gamma$ will be padded with zeros accordingly.

In $l^2(Z_N)$, the DGE becomes

$$\forall x \in l^2(Z_N), \quad x = \sum_{m=0}^{T_1-1} \sum_{n=0}^{K-1} \langle x, \gamma_{mn} \rangle h_{mn},$$

where $K$ is determined by $KT = N$. 


It is extremely important to note that, with $T$ and $T_1$ fixed, the change of signal dimension only implies the change of translation parameter $K$ if $\{h_{mn}\}$ and $\{\gamma_{mn}\}$ remains valid.

We shall use a term \textit{essential support} of a window waveform $h$, an interval, denoted by $\text{ess supp } h$. For a negligible small number $\epsilon$, we define
\[
\text{ess supp } h \equiv \{\text{smallest interval } \Omega \subseteq \mathbb{Z}_N : \forall n \notin \Omega, |h(n)| < \epsilon\}.
\]

**Theorem 3.1** Let the synthesis sequence $\{h_{jk}\}$ be a Gabor frame for $l^2(\mathbb{Z}_N)$. Let $\{\gamma_{jk}\}$ be a pseudo-dual analysis sequence to $\{h_{jk}\}$ computed in $l^2(\mathbb{Z}_N)$. Assume that the length of the essential support of $h$ is $N_0$, and suppose

\[
|\text{ess supp } \gamma| \leq N - N_0.
\]

Now, let $N$ changes. Then, $\{h_{jk}\}$ and $\{\gamma_{jk}\}$ remain to be a pair of pseudo-duals of $l^2(\mathbb{Z}_N)$ for arbitrary $N$ satisfying (2) and being a multiple of $T$ and $T_1$ (with negligible reconstruction error). Here $|I|$ denotes for the measure (length) of the set $I$.

**Corollary 3.2** If $\{h_{jk}\}$ and $\{\gamma_{jk}\}$ are biorthogonal, the biorthogonality holds in spaces of arbitrary dimension $N$ whenever (2) is met and $N$ is a multiple of $T$ and $T_1$.

**Remark** If (2) is not satisfied in either pseudo-dual or biorthogonal relationship, we can slightly increase the dimension $N$, and re-compute the $\gamma$ until (2) is satisfied.

Figures 1 to 4 are for the demonstration of the dimension invariance property. Figure 1 is a pair of pseudo-dual windows computed in $N = 128$, where $h = \text{kaiser}(16,10)$ (dashed line), $T = 4$, $T_1 = 8$. In Figure 3, they are applied to $N = 256$ for the Gabor expansion and reconstruction. The original signal and the reconstruction are drawn in the same picture, but completely overlapped. The mean square error is in the order of $O(10^{-24})$. Figure 2 is a pair of biorthogonal windows computed in $N = 128$, where $h = \text{kaiser}(12,10)$ (dashed line) and $T = 8 = T_1$. They are applied to $N = 256$ in Figure 4 for the Gabor expansion and reconstruction. Again, the original and reconstructed signals are drawn in the same picture. The mean square reconstruction error is in the order of $O(10^{-16})$. 

![Figure 1. Pseudo-dual windows in $N = 128$.](image1.png)  
![Figure 2. Biorthogonal windows in $N = 128$.](image2.png)
Proof of Theorem 3.1: It can be verified easily that \( \{h_{jk}\} \) and \( \{\gamma_{jk}\} \) form a pair of pseudo-duals for the Gabor transform if and only if

\[
h_{mn} = \sum_{j=0}^{T_1-1} \sum_{k=0}^{K-1} \langle h_{mn}, \gamma_{jk} \rangle h_{jk}, \quad \forall 0 \leq m \leq T_1 - 1, \ 0 \leq n \leq K - 1,
\]

where \( K \) is such that \( KT = N \) in \( \mathcal{H}_N \). Assume (3) holds in \( \mathcal{H}_N \). As \( N \rightarrow N' \) and \( K \rightarrow K' \) with \( K'T = N' \), we wish to establish that a similar equation as (3) holds in \( \mathcal{H}_{N'} \) for the same set of Gabor waveforms \( \{h_{jk}\} \) and \( \{\gamma_{jk}\} \), i.e.,

\[
h_{mn} = \sum_{j=0}^{T_1-1} \sum_{k=0}^{K'-1} \langle h_{mn}, \gamma_{jk} \rangle h_{jk}, \quad \forall 0 \leq m \leq T_1 - 1, \ 0 \leq n \leq K' - 1.
\]

Now, for fixed \( m, n \), and a given index \( j \), the coefficients \( \{\langle h_{mn}, \gamma_{jk} \rangle\} \) are functions of the translation index \( k \). Since

\[|\text{ess supp } h| + |\text{ess supp } \gamma| \leq N,\]

the number of non-zero coefficients \( \langle h_{mn}, \gamma_{jk} \rangle \) remain the same, their values stay identical, and they happen at same locations where the \( h_{mn} \) is sited.

On the other hand, for fixed \( m, n \), and a given index \( k \), the role of index \( j \) in \( \mathcal{H}_N \) and in \( \mathcal{H}_{N'} \) are identical, coefficients \( \langle h_{mn}, \gamma_{jk} \rangle \) will not be changed for all \( j \).

Consequently, for fixed \( m, n \), the non-zero coefficients \( \{\langle h_{mn}, \gamma_{jk} \rangle\} \) in \( \mathcal{H}_{N'} \) contribute to \( h_{mn} \) identically in the same way, at the same positions and with the same amount as those do in \( \mathcal{H}_N \). Equation (4) clearly holds.

This establishes the result.

3.1 Algorithm

An algorithm can be summarized as follows. For details as how to compute \( \gamma \), we refer to, e.g., [9], [10], [17], [18].
Algorithm 3.3 Starting from a $N$-dimensional space, where $N$ depends upon the support of the window $h$.

- Step 1. Select $N$. Normally, $N = 2 \sim 4|\text{ess supp } h|$ are good numbers to start with.
- Step 2. Compute $\gamma$ in $H_N$.
- Step 3. Check if conditions in Theorem 3.1 hold. Stop if they do. Otherwise, go to step 4.
- Step 4. Slightly increase $N$. Make sure that $N$ is still the multiple of $T$ and $T_1$, and go to step 2.

4 Conclusion

We show that a (discrete) Gabor expansion has the dimension invariance property. This property allows users to compute an analysis waveform in a rather small dimension, and then use it in signal spaces of arbitrary large dimension by simply adjusting the number of translations. It is an important property of the Gabor expansion because signals have usually very large size in many practical applications. Knowing how to use this property to construct a discrete Gabor expansion for large-size signal spaces will make DGE a practically feasible tool for signal analysis and processing.

References


