

# **Gabor Analysis meets Non-commutative Geometry**

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**Mag. Franz LUEF**

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*Für Mimi*

## DANKSAGUNG

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# 1 Deutsche Zusammenfassung

Die Mathematik ist ein weites und grenzenloses Land, welches aus vielen unterschiedlichen Gebieten besteht. Wie im realen Leben unterscheiden sich diese Gebiete oftmals grundlegend. Die Hauptaufmerksamkeit der Forschung geht sinnvollerweise in die Erschliessung neuer Regionen, oder in die bessere Ausgestaltung von neu entdeckten Lebensräumen, viel seltener in die Herstellung von Querverbindungen zwischen bestehenden Gebieten, u.a. weil derartige Aktivitäten oft mühevoll sind und nicht immer die entsprechende Anerkennung mit sich bringen. Es kann aber natürlich immer wieder vorkommen, dass ein wichtiger Berggipfel an der Grenze mehrerer Gebiete liegt, was dann Mathematiker dieser unterschiedlichen Gebiete motiviert ihn von verschiedenen Seiten zu erklimmen.

Die vorgelegte Arbeit beschreibt eine solche Situation an Hand der Gabor Analysis und Nichtkommutativen Geometrie. Beide Gebiete entstanden vor einen guten Vierteljahrhundert und haben ihre Wurzeln in der Quantenmechanik von Heisenberg und Schrödinger. Jedoch ein Zusammenhang zwischen diesen Gebieten wurde bislang nicht hergestellt. Der Hauptgrund liegt in dem Umstand begründet, dass die Mathematiker nichts von einer gemeinsamen Grenze wußten. Die Beschreibung dieser Grenze wird an Hand von Rieffel's Arbeiten zur Konstruktion von projektiven Modulen über nichtkommutativen Tori beschrieben, welche in obigem Bild dem gemeinsamen Gipfel entspricht. Erste Hinweise auf einen Zusammenhang von Gabor Analysis und Nichtkommutativer Geometrie finden sich in einer Arbeit von I. Daubechies, wo sie ein grundlegendes Resultat der Gabor Analysis, das sogenannte Dichte-Resultat in einem Spezialfall beweist, und darauf hinweist, daß R. Howe und R. Steger meinen der allgemeine Fall folge aus einem Resultat von M.A. Rieffel über die Kopplungskonstante gewisser von Neumann Algebren. Dieser Hinweis von Howe und Steger fand jedoch keinen Widerklang unter den Wissenschaftlern der Gabor Analysis.

Einige Jahre später fanden verschiedene Gruppen von Mathematikern (Janssen; Ron-Shen) einen neuen Zugang zu der Struktur des Gabor Frameoperators, aber auch Daubechies in Zusammenarbeit mit H.J. Landau und Z. Landau. Diese neuen Einsichten erlaubte jeder dieser Gruppen einen Beweis des Dichte-Resultates zu geben, die alle von sehr unterschiedlicher Natur sind. Jede Gruppe sah in diesem Erfolg die Bestätigung, daß man Gabor Analysis vollkommen ohne Rieffel's Arbeiten über von Neumann Algebren betreiben kann. Dies mag einer der Gründe sein, wieso sich niemand die Mühe machten Rieffel's Arbeiten eingehend zu studieren. Aus irgendeinem unbekanntem Grund erweckten Rieffel's Arbeiten mein Interesse und das Verlangen aus seinen Resultaten das Dichte-Resultat abzuleiten. In meinem Ringen einen solchen Beweis zu geben, war ein Gipfel zu bezwingen. Nämlich zu

verstehen, was Rieffel meint, wenn er sagt, daß zwei  $C^*$ -Algebren *Morita äquivalent* sind. Zu meiner Verwunderung erkannte ich, daß Teilaspekte von Rieffel's Begriff der Morita Äquivalenz für  $C^*$ -Algebren von Zeit-Frequenz Verschiebungen erzeugten Gitter der Zeit-Frequenz Ebene die Ursache für die von Janssen und Daubechies-Landau-Landau beschriebenen Strukturen des Gabor Frame Operators sind, die sogenannte *Janssen Darstellung* des Gabor Frameoperators. Weiters entdeckte ich, daß Rieffel die grundlegenden Einsichten von Janssen und Daubechies-Landau-Landau in seiner Konstruktion von projektiven Modulen über mehrdimensionale Nichtkommutative Tori fast 10 Jahre zuvor kannte. Nach diesem prosaischen Einstieg möchte ich nun den Aufbau meiner Dissertation im Detail beschreiben.

Als Einstieg geben wir in Kapitel 3 einen kurzen historischen Abriss von der mathematischen Begründung der Quantenmechanik bis zu den Arbeiten von Janssen über Gabor's Theorie der Kommunikation, welche als Beginn der Gabor Analysis angesehen wird. Wir beschreiben unter anderem das Theorem von Stone-von Neumann und die Arbeiten von Weyl zur Quantenmechanik. Weiters beleuchten wir die Bedeutung von Dennis Gabor für die Zeit-Frequenz Analyse, der als erster eine Beziehung zwischen Quantenmechanik und der Übertragung von Signalen hergestellt hat.

Die Beziehung von Gabor Analysis und Nichtkommutativer Geometrie beruht auf der Diskussion verschränkter  $C^*$ -Gruppenalgebren lokalkompakter Gruppenn. Aus diesem Grund werden sie im Kapitel 4 detailliert beschrieben und zum besseren Verständnis haben wir die Konstruktion der  $C^*$ -Gruppenalgebren auch ausgeführt. Weiters werden die Zusammenhänge zwischen unitären Gruppendarstellungen und projektiven Gruppendarstellungen behandelt, wir in der Folge unsere Resultate für projektive Gruppendarstellungen formulieren. Die Entscheidung zu Gunsten projektiver Gruppendarstellungen findet ihre Motivation in dem Umstand, daß die Zeit-Frequenz Verschiebungen auf solche Darstellungen führen.

In Kapitel 5 stellen wir eine Beziehung zwischen der verschränkten  $C^*$ -Gruppenalgebra für die Zeit-Frequenz Ebene  $\mathbb{R}^{2d}$  und der Quantisierung von Operatoren her. Im wesentlichen folge ich in der Diskussion den Arbeiten von Feichtinger-Kozek und Gröchenig, die auf der sogenannten "integrierten Darstellung" der Heisenberggruppen beruhen. Aber ich verwende deren Einsichten, um zu zeigen, daß die verschränkte  $C^*$ -Gruppenalgebra von  $\mathbb{R}^{2d}$  (=  $C^*$ -Gruppenalgebra der Heisenberggruppe) die  $C^*$ -Algebra der kompakten Operatoren auf  $L^2(\mathbb{R}^d)$  ist. In weiterer Folge weisen wir daraufhin, daß die Irreduzibilität und die Quadratintegrierbarkeit der projektiven Darstellung durch Zeit-Frequenz Verschiebungen  $\{\pi(x, \omega) : (x, \omega) \in \mathbb{R}^{2d}\}$  in natürlicher Weise zur Inversionsformel für die Kurzzeitfouriertransformierte führt und daß eine wohlbekannt Methode diese Inversionsformel zu interpretieren, bloß ein Spezialfall der allgemgültigen Korrespondenz zwischen Darstellungen der Gruppe und involutiven Darstellungen der verschränkten Banachalgebra  $L^1(\mathbb{R}^{2d})$  ist. Zu guter letzt beschreibe ich die Projektionen in der verschränkten  $C^*$ -Gruppenalgebra von  $\mathbb{R}^{2d}$ , da diese die wesentlichen Bestandteile im Beweis des Satzes von Stone-von Neumann sind. In einem späteren Kapitel werden wir den Gedankengang als den Beweis der Morita Äquivalenz der komplexen Zahlen und der  $C^*$ -Algebra von kompakten Operatoren auf  $L^2(\mathbb{R}^d)$  interpretieren. Dieses Resultat von Rieffel nahm er

zum Anlass, um den Begriff der Morita Äquivalenz von Ringen auf  $C^*$ -Algebren zu verallgemeinern.

Im nächsten Kapitel behandeln wir die sogenannten Modualtionsräume von Feichtinger. Neben den wohlbekannten Eigenschaften geben wir eine operatoralgebraische Motivation für deren Bedeutung in der Behandlung von Operatoren mittels Zeit-Frequenz Analyse. Die Idee basiert darauf gewisse Derivationen der verschränkten  $C^*$ -Gruppenalgebra von  $\mathbb{R}^{2d}$  zu studieren. Derivationen von  $C^*$ -Algebren sind das nichtkommutative Analogon des Differentialoperators für Funktionen auf  $\mathbb{R}^d$ . Wir betrachten die nichtkommutativen Analoga der Differentiation entlang der Koordinatenachsen der Zeitfrequenzebene  $\mathbb{R}^{2d}$ . Derivationen sind im allgemeinen unbeschränkte Operatoren auf  $L^2(\mathbb{R}^{2d})$ , daher sind sie nur auf einem dichten Teilraum definiert. Die Bestimmung des Definitionsbereiches führt auf natürliche Weise zu den Modualtionsräumen  $M_{v_s}^1(\mathbb{R}^d)$  und die Bestimmung der glatten Elemente dieser Derivationen auf den wohlbekannten Sachverhalt, daß der Raum der schnellabklingenden glatten Funktionen  $\mathcal{S}(\mathbb{R}^d)$  gleich dem Durchschnitt  $\bigcap_{s \geq 0} M_{v_s}^1(\mathbb{R}^d)$  ist. Am Schluß weisen wir daraufhin, daß dies ein Spezialfall eines allgemeinen Resultates von O. Bratteli-G.A. Elliott und P.E. Jorgensen ist.

Im Kapitel 7 beginnen wir mit der Definition kontinuierlicher Frames für  $L^2(\mathbb{R}^d)$ , da die Quadratintegrierbarkeit von  $\{\pi(x, \omega) : (x, \omega) \in \mathbb{R}^{2d}\}$  impliziert, daß jede von Null verschiedene Funktion des  $L^2(\mathbb{R}^d)$  einen kontinuierlichen *tight* Gaborframes erzeugt. Danach führen wir den Begriff des *kontinuierlichen Banachframes* ein, welcher das kontinuierliche Analogon von Gröchenig's Banachframes ist. Auf diese Weise können wir die schwache Interpretation der Inversionsformel der Kurzzeitfouriertransformierten als die Existenz kontinuierlicher *tight* Banach-Gaborframes interpretieren. Weiters geben wir die grundlegenden Definitionen eines diskreten Gaborframes und beenden das Kapitel mit einer Diskussion der Fundamentalidentität der Gaboranalysis (FIGA), da sie der Schlüssel im Beweis der Morita Äquivalenz gewisser verschränkter  $C^*$ -Gruppenalgebren ist. In der Gaboranalysis wurde sie von Janssen verwendet, um die nach ihm benannte Darstellung eines Gaborframeoperators abzuleiten. Dieser Sachverhalt war Rieffel bereits im Jahre 1988 bekannt und wird die Grundlage für unsere Diskussion projektiver Module über verschränkten  $C^*$ -Gruppenalgebren sein.

In Vorbereitung auf unsere Hauptresultate über Morita-äquivalente verschränkter Gruppenalgebren behandeln wir in Kapitel 8 Hilbert  $C^*$ -module und beschreiben ihre wichtigsten Eigenschaften. Danach führen wir den Begriff der Morita Äquivalenz von  $C^*$ -Algebren ein und beschreiben Methoden zur Konstruktion von Äquivalenzbimodulen zwischen zwei  $C^*$ -Algebren. In unserer Darstellung behalten wir stets die Beziehung zu den traditionellen Begriffen der Hilbertraumtheorie im Auge. Als nichttriviale Anwendung der Theorie der Hilbert  $C^*$ -Module geben wir die wohlbekannte Definition der sogenannten Multiplialgebra einer nicht unitalen  $C^*$ -Algebra.

Im Kapitel 9 beweisen wir eines unserer Hauptresultate, daß die Konstruktion eines Gabor systems  $\mathcal{G}(g, \Lambda)$  für ein Gitter  $\Lambda \in \mathbb{R}^{2d}$  mit Gaboratom  $g$  in  $M_{v_s}^1(\mathbb{R}^d)$  gleichbedeutend zu der Konstruktion eines Hilbert  $C^*$ -Modules für die verschränkte  $C^*$ -Gruppenalgebra ist. In Vorbereitung auf dieses Resultate erinnern wir an das

wohlbekannte Resultate über die Beschränktheit des zugeordneten Gaborframeoperators. Weiters diskutieren wir die verschränkte  $C^*$ -Gruppenalgebra von  $\ell_{v_s}^1(\Lambda)$ . Die Relevanz der algebraischen Struktur für das Studium von Gaborframes ist in die wahre Einsicht von Janssen gewesen. Diese algebraische Struktur der verschränkten Gruppenalgebra  $\ell_{v_s}^1(\Lambda)$  war die Grundlage für viele Resultate von Feichtinger, Gröchenig und deren Koautoren. Im besonderen bei der Lösung einer Vermutung von Janssen über die Qualität von dualen Gaboratomen durch Gröchenig und Leinert im Jahr 2004 in [GL04]. Wir nutzen die Gelegenheit und zeigen, daß ein Großteil der Diskussion in [GL04] der Konstruktion der GNS-Darstellung der zugeordneten verschränkten  $C^*$ -Gruppenalgebra  $C^*(\Lambda)$  gewidmet ist. Wir machen dies explizit, in dem wir die natürliche Spur auf  $C^*(\Lambda)$  einführen und danach die Standardkonstruktion der GNS-Darstellung in diesem konkreten Fall angeben. Im letzten Teil des Kapitels zeigen wir, daß  $M_{v_s}^1(\mathbb{R}^d)$  ein Hilbert  $C^*(\Lambda)$ -Modul ist und daß die Rang eins Operatoren dieses Hilbert  $C^*(\Lambda)$ -Moduls gerade die Gaborframeoperatoren sind. Folglich können wir alle bekannten Resultate über Hilbert  $C^*$ -module in Resultate auf die von den Gaborframeoperatoren  $S_{g,\gamma,\Lambda}$  für  $g, \gamma \in \Lambda$  erzeugten Hilbert  $C^*(\Lambda)$ -Moduleabbildungen anwenden.

Nach der Konstruktion eines Hilbert  $C^*(\Lambda)$ -Moduls gehen wir einen Schritt weiter und geben  $M_{v_s}^1(\mathbb{R}^d)$  eine Äquivalenzbimodulstruktur, welche von der verschränkten  $C^*$ -Gruppenalgebra von  $\ell_{v_s}^1(\Lambda^0)$  induziert wird. Das Gitter  $\Lambda^0$  ist das zu  $\Lambda$  adjungierte Gitter. Der Beweis folgt dem von Rieffel für nichtkommutative Tori. Im Verlauf dieses Kapitels werden wir sehen, daß Rieffel's Assoziativbedingung die wohlbekannte Janssendarstellung des Gaborframeoperators ist. Das Hauptresultat dieses Kapitels ist die Morita Äquivalenz von  $C^*(\Lambda)$  und  $C^*(\Lambda^0)$ .

Um die Konsequenzen der Morita Äquivalenz von  $C^*(\Lambda)$  und  $C^*(\Lambda^0)$  besser zu verstehen, geben wir in Kapitel 10 die Definition eines Wienerpaares von Banachalgebren und diskutieren die entsprechenden Folgerung für den holomorphen Funktionalkalkül für Wienerpaare von Banachalgebren. Danach geben wir einen kurzen Ausblick auf Wienerpaare für Fréchetalgebren und den dazugehörigen holomorphen Funktionalkalkül. Damit kommen wir bei der Arbeit von A. Connes zu sogenannten *smooth algebras* von  $C^*$ -Algebren an. Am Ende des Kapitels geben wir eine kurze Einführung in den Begriff des topologischen Rangs einer Banachalgebra, der von Rieffel stammt.

Nach den Vorbereitungen im Kapitel 9 diskutieren wir den Äquivalenzbimodul  $M_{v_s}^1(\mathbb{R}^d)$   $C^*(\Lambda)$  und  $C^*(\Lambda^0)$  und die Konsequenzen im Kapitel 11 im Detail. Wir zeigen, daß dieser Äquivalenzbimodul ein endlich erzeugter projektiver Module ist und daß er auf natürliche Weise einen Multi - window Gaborframe erzeugt. Dazu erinnern wir an die Verallgemeinerung von Frames für Hilberträumen zu Modulframes für Hilbert  $C^*$ -Modul von M. Frank und D. Larson. Der Beweis dieser Tatsache beruht auf einer Konstruktion von Connes und dem Hauptresultat von Gröchenig-Leinert, dass die verschränkte Gruppenalgebra  $\ell_{v_s}^1(\Lambda)$  und  $C^*(\Lambda)$  ein Wienerpaar bilden. Als Folgerung erhalten wir die Morita-Äquivalenz von  $\ell_{v_s}^1(\Lambda)$  und  $\ell_{v_s}^1(\Lambda^0)$ . Laut Rieffel ist dies das erste nicht-triviale Resultat Morita-äquivalenter involutiver Banachalgebren. Weiters zeigen wir das Janssen's Resultat über die Güte

von Gaborframeoperatoren mit Schwartzatom bereits 1980 von Connes in einer Arbeit über nichtkommutative Tori bewiesen wurde. Am Ende geben wir ein allgemeine Methode für die Konstruktion von Projektionen in Morita-äquivalenten  $C^*$ -Algebren an. Als Spezialfall werden wir die Projektionen aus dem Beweis des Satzes von Stone-von Neumann erhalten. Für den Fall des Paares  $C^*(\Lambda)$  und  $C^*(\Lambda^0)$  zeigen wir das die Konstruktion einer Projektion äquivalent zur Konstruktion eines *tight Gaborframes* ist. Diese Beobachtung ist implizit im letzten Abschnitt von Rieffel's berühmter Arbeit [Rief81a] über irrationale Rotationsalgebren als Kuriosität angeführt. In der Sprache der Gaboranalysis hat Rieffel einen tight Gaborframe mit kompakten Fenster konstruiert und gezeigt, daß die Spur nicht grösser als eins sein kann.

Im folgenden Kapitel kehren wir zum Ausgangspunkt unserer Untersuchung zurück und geben Rieffel's Beweis der Kopplungskonstante der von den Zeit-Frequenz Verschiebungen von  $\Lambda$  und  $\Lambda^0$  von Neumann Algebren. Dies widerlegt Behauptungen von Bekka, Daubechies und anderen, daß man mit Hilfe der Methoden von Rieffel nicht den Fall eines allgemeinen Gitters in der Zeit-Frequenz Ebene behandeln kann. Als direkte Folgerung erhalten wir das Resultat von Bekka [Bek04] über die Existenz von tight Gaborframes für ein allgemeines Gitter in  $\mathbb{R}^{2d}$ .



## 2 Introduction

The main purpose of this investigations is to establish a connection between two new fields of mathematics: Gabor analysis and non-commutative geometry. Both fields have their origins in the struggle of mathematicians with the mathematical foundations of quantum mechanics. Especially, John von Neumann obtained deep results which have become crucial in the creation of Gabor analysis and non-commutative geometry. Our results have their origin in the observation of Dennis Gabor on the relation between quantum mechanics and signal analysis.

The central object in physics is the phase space which in signal analysis is called the *time-frequency plane*  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ . From the mathematical point of view the time-frequency plane is a manifold. Every manifold  $M$  can be considered from different points of view (1) as a measure space or (2) as a topological space. Each of these structures has a description in terms of certain algebras of functions. On the one hand, the measure space can be reconstructed from the commutative von Neumann algebra  $L^\infty(X)$  of essentially bounded measurable functions on  $X$ . On the other hand a locally compact space is determined by the  $C^*$ -algebra  $C_0(X)$ , the algebra of continuous functions vanishing at infinity.

These observations have led to generalizations of *measure space* and *topology* in the theory of operator algebras since 1940. We had to wait for Alain Connes that these non-commutative generalizations are of great use in many fields of mathematics and physics. Furthermore, he developed a non-commutative analog of differential geometry based on the fact that a smooth manifold  $V$  is given by the algebra of smooth functions with compact support  $C_c^\infty(V)$ .

We want to summarize the treated relations between commutative and non-commutative notions.

<b>Commutative</b>	<b>Non-commutative</b>
measure space	von Neumann algebras
topology	$C^*$ algebras

Since his ground-breaking work in the late 1970's Connes has created a vast field of research: **non-commutative geometry**. Connes gave as first example of a "non-commutative manifold" the **non-commutative torus** which is one of the most studied objects in mathematics and physics. The non-commutative torus and its higher-dimensional generalizations were introduced by G. A. Elliott to whom we owe much of our knowledge about the structure of non-commutative torus. Another expert on non-commutative tori is **M.A. Rieffel**. His work turns out to be the bridge between non-commutative geometry and Gabor analysis.

Gelfand's theory of commutative Banach algebras shows a one-to one correspondence between commutative Banach algebras and the algebra of continuous functions on a locally compact space. If one considers a general Banach algebra what kind of space will be the counterpart of a locally compact space? The most important example of non-commutative spaces arise from problems in harmonic analysis.

One of the central construction in harmonic analysis is the Pontrjagin dual of a locally compact abelian group. In the case of a discrete group  $\Gamma$  the Pontrjagin dual  $\widehat{\Gamma}$  is a compact abelian group and the duality is given by the Fourier transform. This shows that the algebra of functions on  $\widehat{\Gamma}$  can be identified with the reduced  $C^*$ -algebra of the group  $\Gamma$ ,

$$C_r^*(\Gamma) \cong C(\widehat{\Gamma}).$$

If  $\Gamma$  is non-abelian Pontrjagin duality *no longer* works in the traditional sense, but the left hand side still makes perfect sense and "behaves" like the algebra of functions on the dual group. In other words, the Pontrjagin dual  $\widehat{\Gamma}$  exists as a non-commutative space whose algebra of functions is the **twisted group  $CS$ -algebra**  $C_r^*(\Gamma)$ . Therefore noncommutative geometry provides a natural generalization of Pontrjagin duality, in the sense that duals of discrete groups are non-commutative spaces.

Recall that unital and non-unital commutative Banach algebras correspond to compact and locally compact spaces, respectively. In this sense unital group  $C^*$ -algebras provide us with a class of compact non-commutative spaces, e.g. the group  $C^*$ -algebra of a discrete non-abelian group. Consequently one thinks of a non-unital group  $C^*$ -algebra as a locally compact non-commutative space. The group  $C^*$ -algebra is the most prominent example of this class of objects. In non-commutative spaces are thought as the same object if they are Morita equivalent which is a weakening of the notion of isomorphism for Banach algebras.

In the present work we show that the notions of non-commutative geometry are implicitly provide a natural discussion of known-structures in Gabor analysis.

### 3 From Stone-von Neumann's theorem to Gabor analysis

In this section we present a short historical tour from Stone-von Neumann's theorem to Gabor analysis. Our motivation is two-fold (1) it allows a better understanding between Gabor analysis and harmonic analysis on the Heisenberg group (2) Rieffel's theory of Morita equivalence for  $C^*$ -algebras emerged from his attempt to understand Mackey's imprimitivity theorem (which is the ultimate generalization of Stone-von Neumann's theorem to locally compact groups). The main aim of this work is to present the connection between Connes's and Rieffel's work on projective modules over non-commutative tori and the work of Jannsen and Gröchenig on the quality of Gabor frames. All these results deal with a different aspect of the Heisenberg group which arose from Stone and von Neumann's work on the foundations of quantum mechanics.

In 1925 Heisenberg made a fundamental observation on the measurement in atomic systems which led to a revolution in physics and many considerable contributions in mathematics. Heisenberg gave a heuristic argument that the position and the momentum of an electron cannot be measured simultaneously. In a subsequent paper Heisenberg in collaboration with Born and Jordan the celebrated *Heisenberg commutation relations* were introduced via infinite-dimensional matrix equations. Therefore their work was the starting point of *matrix mechanics* in physics. Independently Schrödinger developed an alternative approach to the study of atomic systems which was based on de Broglie's wave-particle duality, the so-called "wave mechanics". At the first sight both theories look very different since on the one side Heisenberg used infinite dimensional matrices acting on  $\ell^2(\mathbb{Z})$  and on the other side Schrödinger modeled the dynamics of an atomic system by differential operators on  $L^2(\mathbb{R})$ . Shortly after these breakthrough Pauli and Schrödinger gave a heuristic argument for the equivalence of both theories which were based on the use of Fourier analysis and the theorem of Riesz-Fischer. These investigations in physics attracted the interest of some of the greatest mathematicians of the 20th century: *von Neumann, Stone* and *Weyl*. Their main motivation was to build a solid mathematical framework for matrix mechanics and wave mechanics and to settle the question if both theories are equivalent in a rigorous sense. These investigations led to the uniqueness of the solution of Heisenberg's commutation relations. As a byproduct von Neumann extended Hilbert's spectral theory to unbounded operators on a Hilbert space. In addition he gave the first definition of a Hilbert space in 1928, *five years* after Banach's seminal work on normed spaces.

In modern language Heisenberg's commutation relations, say for a free particle

with one degree of freedom:

$$PQ - QP = -2\pi iI, \quad (3.1)$$

where  $P$  and  $Q$  are supposed to be self-adjoint operators on a Hilbert space  $\mathcal{H}$ , representing momentum and position, respectively. The first observation to make is that by a simple trace argument (3.1) has no solutions if  $\mathcal{H}$  is finite-dimensional. In his work on quantum mechanics Weyl implicitly noticed that (3.1) also has no solutions with either  $Q$  or  $P$  bounded. Wielandt and Winter gave proofs of this result more than twenty years later. In some sense these results show the need for spectral theory of unbounded operators. In [Win47] Winter proved the following:

**Proposition 3.0.1.** *If  $P$  or  $Q$  are bounded operators on a Hilbert space  $\mathcal{H}$ , then (3.1) has no solutions.*

In [Wie49] Wielandt generalized Winter's result in the following sense.

**Proposition 3.0.2.** *Let  $V$  be an unital normed algebra. Then (3.1) has no solutions.*

*Proof.* Suppose that there are  $A, B \in V$  such that  $AB - BA = -2\pi iI$ , but  $AB^{n+1} - B^{n+1}A = (n+1)B^n$  for  $n \in \mathbb{N}$  implies

$$(n+1)\|B^n\| \leq 2\|A\|\|B^{n+1}\| \leq 2\|A\|\|B\|^{n+1}.$$

Therefore,  $B^n = 0$  for  $n$  sufficiently large, say  $N$ , then  $0 = B^N = B^{N-1} = \dots = B^0 = I$ , which is the desired contradiction.  $\square$

These results of Wielandt and Winter initiated a series of publications, see [Put67].

The most famous realization of (3.1) is the one by Schrödinger on the Hilbert space  $L^2(\mathbb{R}^d)$  of square-integrable functions where the position operator corresponds to the multiplication operator  $X$  such that

$$Xf(t) := tf(t), \quad f \in L^2(\mathbb{R}^d)$$

and the momentum operator  $P$  is given by

$$Pf(t) = \frac{1}{2\pi i}f'(t),$$

the differentiation operator. The operators  $Q$  and  $P$  are unbounded and therefore, the Heisenberg commutation relation only holds on a dense subspace of  $L^2(\mathbb{R}^d)$ , e.g. the Schwartz space  $\mathcal{S}(\mathbb{R}^d)$ .

Weyl found a brilliant way to circumvent the problem that the solutions of (3.1) are unbounded. He formally *integrated* Heisenberg's commutation relation. More concretely, he assumed that  $P$  and  $Q$  can be exponentiated to one-parameter unitary groups  $T_x = e^{-2\pi i x P}$  and  $M_\omega = e^{2\pi i \omega X}$ , respectively. For later reference we call the pair  $(T_x, M_\omega)$  the *Schrödinger representation*. This is possible by "Stone's Theorem":

**Theorem 3.0.3.** *Let  $\{U_t\}$  be a strongly continuous one-parameter unitary group on  $\mathcal{H}$ . Then there is an unbounded self-adjoint operator  $A$  on  $\mathcal{H}$  such that  $U_t = e^{2\pi itA}$ .*

In the following we use  $[P, Q]$  for the commutator  $PQ - QP$ . We now obtain formally from (3.1)

$$T_x Q T_{-x} = Q + 2\pi i x [[P, Q] + \frac{(2\pi i)^2}{2!} [P, [P, Q]] + \dots] = Q + 2\pi i x,$$

that  $Q$  is unitarily equivalent to  $P + q\pi i x$ . As a consequence the spectrum of  $Q$  must consist of  $\mathbb{R}$  and is therefore unbounded. In a similar way (3.1) implies that  $T_x f(Q) T_{-x} = f(Q + 2\pi i x)$ , for any real analytic function  $f$  on the spectrum of  $Q$ . In particular, for  $f(\omega) = e^{2\pi i \omega t}$  we have

$$\begin{aligned} T_x M_\omega T_{-x} &= T_x e^{2\pi i \omega Q} T_{-x} \\ &= e^{2\pi i \omega (Q+x)} = e^{2\pi i x \omega} M_\omega. \end{aligned}$$

Or equivalently,

$$T_x M_\omega = e^{2\pi i x \omega} M_\omega T_x, \tag{3.2}$$

which is *Weyl's commutation relation*, which had great impact on mathematics in the last 80 years. In [Wey28], which was the first textbook on quantum mechanics, Weyl conjectured that under the assumption of irreducibility the Heisenberg commutation relations necessarily follow from (3.2). Weyl's claim was the starting point for the work of Stone and von Neumann.

Since the  $Q$  and  $P$  in (3.1) are not defined on all of  $L^2(\mathbb{R}^d)$  while the right hand side is a multiple of the identity operator one gets into substantial problems. Dixmier, Friedrichs and others have considered the question under which conditions solutions of the Heisenberg commutation relation exist and satisfy a Stone-von Neumann theorem. We refer the reader to [Put67] for examples of solutions of (3.1) which do not integrate to unitary groups and in these cases the Stone-von Neumann theorem does not hold. These problems can be bypassed by going to the Weyl integrated form.

**Theorem 3.0.4** (Stone-von Neumann). *Let  $U$  and  $V$  be unitary representations of  $\mathbb{R}^d$  on a Hilbert space  $\mathcal{H}$ , satisfying the commutation relation*

$$U_x V_\omega = e^{2\pi i \theta(x, \omega)} V_\omega U_x, \quad x, \omega \in \mathbb{R}^d, \tag{3.3}$$

where  $\theta : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a bilinear and non-degenerate form. Then such a pair  $(U, V)$  is equivalent to multiples of the Schrödinger representation on  $L^2(\mathbb{R}^d)$ .

In the 1970's Rieffel showed that the Stone-von Neumann Theorem is actually an *equivalence of categories of representations* [Rief72]. This new interpretation of Stone-von Neumann motivated Rieffel's work on *Morita equivalence of  $C^*$ -algebras* which has turned out to be an extremely fruitful notion, e.g. Connes's theory of non-commutative geometry relies heavily on the Morita equivalence of  $C^*$ -algebras.

Following von Neumann's proof of the uniqueness of the solutions of Weyl's commutation relation he showed that the category of solutions is equivalent to the category of representations of  $\mathbb{C}$ . Furthermore, he concluded that on any Hilbert space representation of Weyl's commutation relation, there is a natural action of the compact operators. We will later present the details of Rieffel's construction in the language of time-frequency analysis.

After the invention of matrix mechanics and wave mechanics many physicists asked for the physical meaning of their objects and results. Many physicists preferred Schrödinger's wave mechanics since there is a physical interpretation of functions and differential operators. In contrast matrix mechanics dealt with infinite dimensional matrices which looked quite strange for a traditional physicist. But the physicists had much more trouble with the interpretation of the results emerging from the matrix and wave mechanics, respectively. The greatest problem came from Born's proposal to interpret the results only in a probabilistic way which had been never accepted by de Broglie, Einstein and Schrödinger. In the early thirties Wigner proposed a new description of a quantum mechanical system in terms of joint representation of momentum and position operators. More precisely, he associated with a function  $f$  a function  $Wf(x, \omega)$  which is aimed at yielding a description which is both accurate in position and momentum. But Heisenberg's commutation relation (3.1) yield an obstruction since it postulates the impossibility of a joint measurement of position and momentum. In other words Wigner wanted to invent a joint probability distribution for the position and momentum operator of a one-dimensional particle. But as a consequence from Heisenberg's commutation relation  $Wf(x, \omega)$  can attain negative values which cannot be interpreted as a probability for the occurrence of position and momentum. In modern language Wigner invented the *quantum plane* in his construction of the **Wigner distribution** of a function  $f \in L^2(\mathbb{R}^d)$  defined to be

$$Wf(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi i \omega t} f(x + t/2) \overline{f(x - t/2)} dt. \quad (3.4)$$

By polarization we define the **cross-Wigner distribution** of a function (signal)  $f \in L^2(\mathbb{R}^d)$  with respect to another function (window)  $g \in L^2(\mathbb{R}^d)$  to be

$$W(f, g)(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi i \omega t} f(x + t/2) \overline{g(x - t/2)} dt.$$

Later we will meet many other joint representations such as the short-time Fourier transform or the Rihaczek distribution. The Rihaczek distribution of a function  $f$  was actually introduced by the physicist Kirkwood who was inspired by Wigner's contribution. Later many researchers in a variety of fields reinvented this transform, e.g. by the engineer Rihaczek. Therefore we define the *Kirkwood-Rihaczek-distribution*  $Rf$  of a function  $f \in L^2(\mathbb{R}^d)$  to be

$$Rf(x, \omega) = f(x) \widehat{f(\omega)} e^{-2\pi i x \omega}. \quad (3.5)$$

Another physicist *Dennis Gabor* was fully aware of these investigations and proposed to apply these objects to describe speech signals in [Gab46]. In this work he gave an interpretation of the unitary groups generated by the position operator  $Q$

and momentum operator  $P$  in the Schrödinger representation. He considered a function  $f$  in  $L^2(\mathbb{R}^d)$  to be a speech signal which is concentrated in the **time-frequency plane**. Gabor took as coordinates *time* and *frequency* where a **translation** or *time-shift* of  $f \in L^2(\mathbb{R}^d)$  is defined by

$$T_x f(t) = e^{-2\pi i x P} f(t) = f(t - x) \quad (3.6)$$

and a **modulation** or *frequency shift* of  $f \in L^2(\mathbb{R}^d)$  is given by

$$M_\omega f(t) = e^{2\pi i \omega Q} f(t) = e^{2\pi i t \omega} f(t). \quad (3.7)$$

In this terminology the Weyl operators for the Schrödinger representation  $M_\omega T_x$  are called **time-frequency shifts** for a point  $(x, \omega)$  in the plane  $\mathbb{R}^{2d}$ . In the sequel we will often denote time-frequency shifts at  $(x, \omega)$  by  $\pi(x, \omega)$ .

In [Gab46] Gabor proposed the following method of transmission of a speech signal  $f$ . In modern language he discretized  $f$  into a sequence of bits, i.e. strings of the form "0100111". A natural way to transmit such strings is to send a pulse  $\varphi$  of length 1 at consequent time intervals and of size according to the amplitude of  $f$ , i.e.  $f = \sum_{k=1}^m a_k \varphi(t - k)$ . Now, a speech signal is a *band-limited* function, i.e.  $\text{supp}(\hat{f}) \subseteq [0, A]$  for some finite real  $A$ , we observe

$$\hat{f} = \sum_{k=1}^m a_k e^{-2\pi i k \omega} \hat{\varphi}(\omega) = \left( \sum_{k=1}^m a_k e^{-2\pi i k \omega} \right) \hat{\varphi}.$$

Therefore, the support of  $f$  and the support of  $\varphi$  are equal which suggests to be careful in the choice of the pulse  $\varphi$ . In practice we have to transmit more than one signal  $f$ , e.g. a conversation between a group of people. Gabor's brilliant idea was to shift each signal on a different frequency band. More precisely, if  $f_1, \dots, f_n$  are the band-limited signals we want to transmit, then he suggested to send  $f_l$  on the  $l$ -th frequency band

$$f_l(t) = \sum_{k=1}^m a_m^{(l)} e^{-2\pi i l \theta t} \varphi(t - k).$$

Therefore, the transmission of all signals  $f_1, \dots, f_n$  corresponds to

$$\sum_{l=1}^n f_l(t) = \sum_{l=1}^n \sum_{k=1}^m a_k^{(l)} e^{2\pi i l A t} \varphi(t - k) = \sum_{k,l} a_k^{(l)} M_{Al} T_k \varphi(t).$$

Heisenberg's commutation relation also yield that  $A$  has to be at least greater than 1 since each signals occupies an area of size greater than or equal to 1.

The preceding observations and the fact that the Gaussian  $\varphi(t) = e^{-\pi t^2}$  is well-localized in the time-frequency plane which is another consequence of Heisenberg's commutation relations (3.1). Furthermore, time-frequency shifts  $\pi(x, \omega)\varphi(t)$  are again well-localized since they act isometrically on  $L^2(\mathbb{R}^d)$ . Gabor proposed to decompose a signal  $f \in L^2(\mathbb{R}^d)$  into a series of time-frequency shifts of a Gaussian over  $\mathbb{Z}^{2d}$ :

$$f = \sum_{k,l \in \mathbb{Z}^d} a_{kl} M_\omega T_x e^{-\pi t^2} = \sum_{k,l \in \mathbb{Z}^d} a_{kl} \pi(k, l) e^{-\pi t^2}. \quad (3.8)$$

## Chapter 3

The last equation is the first example of an *atomic decomposition* which in the last thirty years has led to a new field of mathematical research with a vast literature. In honour of Gabor's lasting contribution we call decompositions of type (3.8) *Gabor series*. Gabor only gave heuristic arguments on the convergence of (3.8) for  $f \in L^2(\mathbb{R}^d)$  but a rigorous analysis of (3.8) had to wait for Janssen's contribution [Jan81] of the year 1981. The main result of Janssen says that the convergence of (3.8) holds only in a weak sense of distributions. Out of Janssen's paper [Jan81] emerged the new mathematical field of *Gabor analysis*. He realized that the coefficients in (3.8) are just samples of the Wigner transform of  $f$  with respect to the window  $\varphi$ . The structure and properties of Gabor series and its generalizations are the main motivation of our investigations.

The presentation in this section was motivated by the insightful article [Ros04] of Rosenberg on the history of the Stone-von Neumann Theorem and the presentation of Gabor analysis by Gröchenig in [Gr01].

# 4 Twisted Group $C^*$ -Algebras

In this chapter we set the stage for our investigations on the connection between Gabor analysis and non-commutative geometry. We recall some basic facts about locally compact groups and we give the definition of a *group  $C^*$ -algebra* for a locally compact group  $G$ . Then we restrict our attention to the group  $C^*$ -algebra of the Heisenberg group. Finally we formulate the Heisenberg group in terms of projective representations which leads naturally to the introduction of twisted group  $C^*$ -algebras.

Following Connes twisted group  $C^*$ -algebra of non-discrete locally compact abelian groups provide us with a class of **non-compact non-commutative spaces** and the twisted group  $C^*$ -algebras of a discrete group are well-studied examples of **compact non-commutative spaces**. In the subsequent chapters we show the relevance of the twisted group  $C^*$ -algebra of  $\mathbb{R}^{2d}$  for the quantization of operators, the construction of tight continuous Gabor frames and Rieffel's interpretation of the Stone-von Neumann Theorem. In other words, *non-compact non-commutative spaces are the natural framework for time-frequency analysis*. Our main results on the construction of projective modules over twisted convolution  $C^*$ -algebras for a lattice  $\Lambda$  in  $\mathbb{R}^{2d}$  show that *compact non-commutative spaces are intrinsically connected to Gabor analysis*.

## 4.1 Basis Facts

Let  $G$  be a locally compact  $\sigma$ -compact group, i.e. there is a countable cover of  $G$  by compact sets. A locally compact group  $G$  supports a regular Borel measure  $\mu_G$  which is invariant under left translations; meaning that  $\mu_G(tB) = \mu_G(B)$  for every  $t \in G$  and Borel subset  $B$  of  $G$ . It is unique up to a scalar multiple, and is known as the **left Haar measure** on  $G$ . The measure is finite when  $G$  is compact. In this case we normalize so that  $\mu_G(G) = 1$ . If  $G$  is infinite and discrete, then we normalize so that  $\mu_G(\{e\}) = 1$ , where  $e$  is the identity element of  $G$ . We will write  $dt$  instead of  $d\mu_G(t)$  when there is no ambiguity.

In general, left Haar measure need not be right translation invariant. There is a continuous homomorphism  $\Delta$  of  $G$  into  $\mathbb{R}_+$  known as the **modular function** such that  $\mu_G(Bt) = \Delta(t)\mu_G(B)$  for any Borel subset of  $G$ . A group  $G$  is called **uni-modular** if  $\Delta$  is trivial. In particular we have the formula

$$d\mu_G(t^{-1}) = \Delta(t)^{-1}d\mu_G(t).$$

An understanding of the structure of a group relies often on the representation of the group on a function space which allows to build a concrete picture of the abstractly given group. The space of **square-integrable** functions on a locally compact group  $G$  plays a distinguished role in the theory of group representations. The Hilbert space

$$L^2(G) = \{f \text{ measurable} : \|f\|_2 = \left(\int_G |f(t)|^2 dt\right)^{1/2} < \infty\}$$

is *separable* since we have assumed our locally compact  $G$  to be  $\sigma$ -compact, see [HR63] Thm. 24.15. In the following we present the main definitions and results on group representations.

A **unitary representation** of a locally compact group  $G$  on a Hilbert space  $\mathcal{H}$  is a family of unitary operators  $(\pi(x))_{x \in G}$  on  $\mathcal{H}$  which satisfy

1.  $\pi(xy) = \pi(x)\pi(y)$  for all  $x, y \in G$ .
2. For all  $f$  in  $\mathcal{H}$  the mapping  $x \mapsto \pi(x)f$  is continuous from  $G$  into  $\mathcal{H}$ .

A subspace  $V$  of  $\mathcal{H}$  is  **$\pi$ -invariant**, if  $\pi(x)V \subseteq V$  for all  $x \in G$ . The representation  $\pi$  is **irreducible** if the only  $\pi$ -invariant closed subspaces of  $\mathcal{H}$  are  $\{0\}$  and  $\mathcal{H}$ . A **matrix coefficient** of  $\pi$  is a function of the form  $g \mapsto \langle f, \pi(x)g \rangle$  for any two  $f, g \in \mathcal{H}$ .

A representation  $\pi$  is irreducible if and only if  $\overline{\text{span}}(\pi(x)f)_{x \in G} = \mathcal{H}$  for any non-zero  $f \in \mathcal{H}$ . Therefore, if  $\pi$  is irreducible and  $g$  an arbitrary non-zero  $g \in \mathcal{H}$ , then any  $f \in \mathcal{H}$  can be approximated arbitray well by finite linear combinations of  $\pi(x)g$ .

Every locally compact group  $G$  has a distinguished representation called the **left regular representation** on  $L^2(G)$ . This is defined by

$$L_x g(t) = g(x^{-1}t), \quad g \in L^2(G).$$

In the sequel we need a few more definitions. A unitary irreducible representation  $\pi$  is **integrable** if there exists  $f \in \mathcal{H}$  such that

$$\int_G |\langle g, \pi(x)g \rangle| dx < \infty$$

and  $\pi$  is **square-integrable** if there exists  $f \in \mathcal{H}$  such that

$$\int_G |\langle g, \pi(x)g \rangle|^2 dx < \infty. \tag{4.1}$$

A non-zero element  $g \in \mathcal{H}$  satisfying (4.1) is called **admissible**. We denote the set of all admissible elements of  $\pi$  by  $\mathcal{A}$ . We associate to an admissible element  $g \in \mathcal{A}$  the operator  $T_{\pi, g}$  from  $\mathcal{H}$  to  $L^2(G)$  which is defined by

$$T_{\pi, g} f(x) = \langle f, \pi(x)g \rangle.$$

The domain  $\mathcal{D}(T_{\pi, g})$  is clearly  $\{f \in \mathcal{H} : T_{\pi, g} f \in L^2(G)\}$ .

The following lemma is an elementary consequence from the definitions.

**Lemma 4.1.1.** *Let  $\pi$  be a square-integrable, irreducible and unitary representation of a locally compact  $G$ . Then the following hold:*

1. *If  $g \in \mathcal{H}$  is admissible, then  $T_{\pi,g}f \in L^2(G)$  for all  $f \in \mathcal{H}$  and the map  $f \mapsto T_{\pi,g}f$  is an isometry.*
2. *The set of admissible vectors  $\mathcal{A}$  is a  $\pi$ -invariant, dense subspace of  $\mathcal{H}$ .*

*Proof.* 1. For  $f \in \mathcal{D}(T_{\pi,g})$  and  $x \in G$ , we have

$$\begin{aligned} \|T_{\pi,g}(\pi(x)f)\|_2^2 &= \int_G |\langle \pi(x)f, \pi(t)g \rangle| dt \\ &= \int_G |\langle f, \pi(x^{-1}t)g \rangle| dt \\ &= \int_G |\langle f, \pi(t)g \rangle| dt = \|T_{\pi,g}f\|_2^2, \end{aligned}$$

so  $\mathcal{D}(T_{\pi,g})$  is invariant under  $\pi$ . But  $\mathcal{D}(T_{\pi,g}) \neq \{0\}$ , so  $\mathcal{D}(T_{\pi,g})$  is dense in  $\mathcal{H}$ , and

$$T_{\pi,g}\pi(x) = L_x T_{\pi,g} \text{ on } \mathcal{D}(T_{\pi,g}).$$

In order to show that  $T_{\pi,g}$  is closed, consider a sequence  $(f_n)_{n \in \mathbb{N}}$  in  $\mathcal{D}(T_{\pi,g})$  converging to some  $f \in \mathcal{H}$ , with the property that  $T_{\pi,g}f_n$  converges to some  $F \in L^2(G)$ . By continuity of the scalar product  $\langle f_n, \pi(x)g \rangle$  converges to  $\langle f, \pi(x)g \rangle$  pointwise on  $G$ . On the other hand, convergence of  $T_{\pi,g}f_n$  to  $F$  implies the existence of a subsequence that converges pointwise almost everywhere to  $F$ , so  $F(x) = \langle f, \pi(x)g \rangle$  for almost all  $x \in G$ . In particular  $T_{\pi,g}f$  is in  $L^2(G)$  and thus  $f \in \mathcal{D}(T_{\pi,g})$  with  $T_{\pi,g}f = F$ , so  $T_{\pi,g}$  is closed. Consequently,  $\mathcal{D}(T_{\pi,g})$  is dense in  $\mathcal{H}$  and  $T_{\pi,g}$  is a multiple of an isometry.

2. Let  $g$  be admissible and  $x \in G$ . Then we have

$$\begin{aligned} \int_G |\langle \pi(x)g, \pi(tx)g \rangle|^2 dt &= \int_G |\langle g, \pi(x^{-1}tx)g \rangle|^2 dt \\ &= \int_G |\langle \pi(x)g, \pi(s)g \rangle|^2 \Delta(x^{-1}) ds < \infty, \end{aligned}$$

therefore  $\pi(x)g$  is admissible, too. □

The most important fact about square-integrable representations are the **orthogonality relations** for the matrix coefficients. This result goes back to Duflo and Moore, [DM76].

**Theorem 4.1.2.** *Let  $\pi$  be a square-integrable, irreducible and unitary representation of  $G$ . Then there exists a unique self-adjoint positive operator  $C$  such that the following hold:*

1. *The domain of  $C$  is the set of all admissible vectors.*

2. For any  $f_1, f_2 \in \mathcal{H}$  and any admissible vectors  $g_1, g_2 \in \mathcal{H}$ , we have

$$\int_G \langle f_1, \pi(x)g_1 \rangle \langle f_2, \pi(x)g_2 \rangle dx = \langle f_1, f_2 \rangle \overline{\langle Cg_1, Cg_2 \rangle}.$$

3. If  $G$  is unimodular, then  $C$  is a multiple of the identity, and thus all vectors are admissible.

As a corollary we get a **reproducing property** for representations of  $G$ .

**Corollary 4.1.3.** *Let  $\pi$  be a square-integrable, irreducible unitary representation of a locally compact  $G$ . Then the following hold:*

1. If  $g \in \mathcal{H}$  is admissible, then the adjoint map of  $T_{\pi, g}$

$$T_{\pi, g}^* : F \mapsto \int_G F(x) \pi(x) g dx$$

is a bounded linear operator from  $L^2(G)$  onto  $\mathcal{H}$ , where the integral converges weakly in  $\mathcal{H}$ .

2. **(Reproducing Property)**

If  $g_1, g_2 \in \mathcal{H}$  are admissible, then  $T_{\pi, g_2}^* T_{\pi, g_1} = \langle Cg_2, Cg_1 \rangle I_{\mathcal{H}}$ , i.e.

$$\int_G \langle f, \pi(x)g_1 \rangle \pi(x)g_2 dx = \langle Cg_2, Cg_1 \rangle f \quad \text{for all } f \in \mathcal{H},$$

where the integral converges weakly in  $\mathcal{H}$ , and  $C$  is the operator from Theorem 4.1.2.

## 4.2 Group $C^*$ -algebras

In the sequel we restrict our discussion to *unimodular* locally compact  $\sigma$ -compact group  $G$ . The space  $L^1(G)$  of absolutely integrable functions with respect to Haar measure, i.e.

$$L^1(G) = \{f : \|f\|_1 = \int_G |f(t)| dt < \infty\}.$$

Since  $G$  is  $\sigma$ -compact  $L^1(G)$  is *separable*, see [HR63] Theorem 24.15. The space  $L^1(G)$  becomes a  $\star$ -algebra with the operations of convolution and involution:

$$\begin{aligned} f * g(x) &= \int_G f(t)g(t^{-1}x)dx \\ f^*(x) &= \overline{f(x^{-1})}. \end{aligned}$$

The algebra  $L^1(G)$  is unital if and only if  $G$  is discrete, and then we write  $\ell^1(G)$ . In this case,  $\delta_e$ , the characteristic function of the identity element, is the unit. Moreover, the group algebra  $\mathbb{C}G$  consisting of all finite sums  $\sum_{x \in G} a_x \delta_x$  forms a dense subalgebra of  $\ell^1(G)$ .

In general  $L^1(G)$  has a norm one approximate identity as a substitute of a unit element. Despite the great importance of  $L^1(G)$  in functional analysis and harmonic analysis  $L^1(G)$  has one important drawback from an operator algebraic point of view: **The  $L^1(G)$ -norm  $\|f\|_1 = \int_G |f(t)|dt$  is not a  $C^*$ -algebra norm!**

In the following we present the construction of a naturally associated  $C^*$ -algebra which circumvents this deficiency: the *group  $C^*$ -algebra*. We start with the observation that a unitary and strongly continuous representation  $\pi$  of  $G$  on a Hilbert space  $\mathcal{H}$  induces a representation of  $L^1(G)$  by integration:

$$\tilde{\pi}(f) = \int f(t)\pi(t)dt.$$

We have that the operator norm of  $\tilde{\pi}(f)$  is dominated by the  $L^1$ -norm:

$$\|\tilde{\pi}(f)\| \leq \int |f(t)|dt = \|f\|_1.$$

**Proposition 4.2.1.** *The mapping  $f \mapsto \tilde{\pi}(f)$  is an involutive homomorphism.*

*Proof.* By Fubini's theorem we get

$$\begin{aligned} \tilde{\pi}(f \star g) &= \iint f(x)g(xt^{-1})dx\pi(t)dt \\ &= \int f(x)\pi(x)dx \int g(x^{-1}t)\pi(x^{-1}t)dt dx \\ &= \int f(x)\pi(x)dx \int g(u)\pi(u)du = \tilde{\pi}(f)\tilde{\pi}(g). \end{aligned}$$

To show that  $\tilde{\pi}(f)^* = \tilde{\pi}(f^*)$  we start with the definition of an adjoint operator on  $L^2(G)$ .

$$\begin{aligned} \langle \tilde{\pi}(f)^*x, y \rangle &= \langle x, \tilde{\pi}(f)y \rangle \\ &= \int \overline{f(t)} \langle x, \pi(t)y \rangle dt \\ &= \int \overline{f(t)} \langle \pi(t^{-1})x, y \rangle dt \\ &= \int \overline{f(u^{-1})} \langle \pi(u)x, y \rangle du \\ &= \left\langle \int f^*(u)x du, y \right\rangle = \langle \tilde{\pi}(f^*)x, y \rangle. \end{aligned}$$

□

Now, we want to study under which conditions a representation  $\tilde{\pi}$  of  $L^1(G)$  determines a unique unitary representation of  $G$ . It turns out that  $\tilde{\pi}$  has to be **non-degenerate** on  $\mathcal{H}$ , i.e.  $\tilde{\pi}(L^1(G))\mathcal{H} = \mathcal{H}$ .

**Proposition 4.2.2.** *Let  $\tilde{\pi}$  be a non-degenerate representation of  $L^1(G)$ . Then there exists a unitary representation of  $G$  which is uniquely determined by  $\tilde{\pi}$ .*

*Proof.* The argument relies heavily on the existence of norm one approximate identity  $(e_j)_{j \in J}$  of  $L^1(G)$  where the index set  $J$  is countable by the  $\sigma$ -compactness of  $G$ . For each open neighborhood  $U$  of  $e$ , we choose a positive function  $e_j$  in  $L^1(G)$  supported in  $U$  such that  $e_j^* = e_j$  and  $\|e_j\| = 1$  and the corresponding sequence  $(e_j)_{j \in J}$  is ordered by containment of sets. Then

$$\lim_{j \in J} \tilde{\pi}(e_j) \tilde{\pi}(g)x = \tilde{\pi}(g)x$$

for every  $g \in L^1(G)$  and  $x \in \mathcal{H}$ . Therefore, the contractions  $\tilde{\pi}(f)$  converge strongly to the identity operator.

We denote by  $T_s g(t) = g(s^{-1}t)$  the translation operator and define the following candidate for a unitary representation of  $G$ :

$$\pi(s) \tilde{\pi}(g)x := \tilde{\pi}(T_s g)x.$$

Then  $\pi(s) = \text{SOT} - \lim_{j \in J} \tilde{\pi}(T_s f_j)$ . Moreover, this implies that  $\pi$  is a contractive homomorphism of  $G$  into  $\mathcal{B}(\mathcal{H})$ . Since  $\|\pi(s)\|$  is dominated by the norm of the approximate unit we get  $\|\pi(s)\| \leq 1$  and  $\|\pi(s^{-1})\| \leq 1$ . As a consequence  $\pi(s)$  is unitary. These formulae are clearly necessary if  $\tilde{\pi}$  is to be induced by  $\pi$ , i.e. we obtained the desired uniqueness.  $\square$

The preceding propositions give the following important result.

**Theorem 4.2.3.** *Let  $G$  be a unimodular locally compact group  $G$ . Then there exists a bijective correspondence between unitary representations of  $G$  and involutive representations of  $L^1(G)$ .*

The **left regular representation** of  $G$  on  $L^2(G)$

$$\pi_L(x)g(t) := g(x^{-1}t)$$

is unitary and strongly-continuous on  $L^2(G)$ . Now, we define the reduced group  $C^*$  algebra of  $G$ .

**Definition 4.2.4.** *The reduced group  $C^*$ -algebra of  $G$  is defined to be  $C_r^*(G) := \overline{\tilde{\pi}_L(L^1(G))}$ .*

Now, we call the direct sum of all irreducible representations of  $G$  the **universal representation**  $\pi_u$  of  $G$ .

**Definition 4.2.5.** *Let  $G$  be a unimodular locally compact group. Then the group  $C^*$ -algebra  $C^*(G)$  of  $G$  is the norm closure of  $\pi_u(L^1(G))$ .*

Equivalently,  $C^*(G)$  is defined as the closure of  $L^1(G)$  by the  $C^*$ -norm

$$\|f\| = \sup\{\|\tilde{\pi}(f)\| : \tilde{\pi} \text{ is an involutive representation of } L^1(G)\}.$$

The supremum in the preceding equation is well-defined since  $\|f\| \leq \|f\|_1$ . Furthermore, the collection of involutive representations is non-empty because any irreducible representation of  $C_r^*(G)$  provides an irreducible representation of  $G$ . Now such representations exist by the GNS-construction.

By construction, there is a one-to-one correspondence between the irreducible representations of  $C^*(G)$  and  $L^1(G)$ , and hence with the irreducible representations of  $G$ . In particular, by the extension of the left regular representation  $\tilde{\pi}_L$  of  $C_r^*(G)$  we get an irreducible representation of  $C^*(G)$ .

All of you are quite familiar with the group  $C^*$ -algebra of an abelian group.

**Proposition 4.2.6.** *If  $G$  is an abelian group, then*

$$C^*(G) = C_r^*(G) = C_0(\hat{G}).$$

The proof relies on Gelfand's theory of commutative Banach algebras and Pontrajin duality. Since for abelian groups,  $L^1(G)$  is a commutative Banach algebra.

*Proof.* Since  $L^1(G)$  is a commutative Banach algebra the irreducible representations of  $L^1(G)$  are the one-dimensional representations corresponding to multiplicative linear functionals. By the argument above these functionals correspond to the one-dimensional representations of  $G$ , i.e. the characters  $\chi$  of  $G$ . Thus the Gelfand map sends  $f$  in  $L^1(G)$  to its Fourier transform  $\hat{f}$  in  $C_0(\hat{G})$ . Now, the range of the Gelfand map is self-adjoint and separates points; and thus is dense in  $C_0(\hat{G})$  by the Stone-Weierstraß Theorem.

The Fourier transform  $\mathcal{F}$  extends to a unitary operator  $\tilde{\mathcal{F}}$  from  $L^2(G)$  onto  $L^2(\hat{G})$ . We have

$$\tilde{F}\pi_L(f)\tilde{\mathcal{F}}^*\hat{g} = \tilde{\mathcal{F}}\pi_L(f)g = \mathcal{F}(f * g) = \hat{f}\hat{g}$$

for all  $f$  in  $L^1(G)$  and  $g \in L^1(G) \cap L^2(G)$ . Thus each  $\hat{f}$  in  $C_0(\hat{G})$  is sent to the multiplication operator by  $\hat{f}$ . This map is isometric; whence  $\pi_L$  is an isometric isomorphism. Therefore  $C^*(G) = C_r^*(G) = C_0(\hat{G})$ .  $\square$

What properties of  $G$  imply that  $C_r^*(G)$  is isomorphic to  $C^*(G)$ ? If the group  $G$  is **amenable**, then the reduced group  $C^*$ -algebra and the group  $C^*$ -algebra are isomorphic. Recall, that a group is **amenable** if there is a left translation invariant mean on  $L^\infty(G)$ . Compact and abelian groups are natural examples of amenable groups. Furthermore, every locally compact abelian group  $G$  is amenable. The following deep result is due to Hulanicki and states that  $C_r^*(G) \cong C^*(G)$  implies that  $G$  is amenable.

**Theorem 4.2.7.** *If  $G$  is a locally compact group, then the left regular representation  $\pi_L$  of  $C_r^*(G)$  onto  $C^*(G)$  is an isomorphism if and only if  $G$  is amenable.*

For more information on group  $C^*$ -algebras the interested reader will find many results in Dixmier's survey on  $C^*$ -algebras, [Dix77]. Our presentation followed closely Davidson's exposition in [Dav96].

### 4.3 Twisted group $C^*$ -algebras

Twisted group  $C^*$ -algebras were introduced by Auslander and Moore in [AM66]. We refer the reader to [PR03] for a detailed analysis of the structure of twisted group  $C^*$ -algebras.

In quantum mechanics and time-frequency analysis we have to deal with a particular locally compact group  $G$ : the **Heisenberg group**. The non-commutativity of the Heisenberg group is given in a very concrete form. Namely, it relies on the following notion. Let  $G$  be a locally compact abelian group. Then a **2-cocycle** on  $G$  is a continuous function  $c : G \times G \rightarrow \mathbb{C}$  satisfying

1.  $|c(x, y)| = 1$  for all  $x, y \in G$ ;
2.  $c(x, e) = c(e, x) = 1$ ;
3.  $c(x, yz)c(y, z) = c(yz, x)c(x, y)$  for all  $x, y, z \in G$ .

We denote the unimodular complex numbers by  $\mathbb{T}$ . We call the direct product  $G \times \mathbb{T}$  with the composition

$$(x, \omega)(y, \eta) = (xy, \omega\eta\overline{c(x, y)})$$

the Heisenberg group  $G_c$ . It is a locally compact group with respect to the product topology. Consequently, the Haar measure is the product measure from  $G$  and  $\mathbb{T}$ .

We want to emphasize the dependence of  $G_c$  on the 2-cocycle more explicitly. Following Weyl we say that a family of unitary operators  $(\rho(x))_{x \in G}$  is a **projective representation** of a locally compact abelian  $\sigma$ -compact group  $G$  on a Hilbert space  $\mathcal{H}$  if

1.  $\rho(e) = I_{\mathcal{H}}$  for  $e$  the unit of  $G$ .
2. There exists a continuous function  $G \times G \rightarrow \mathbb{C}$  such that

$$\rho(xy) = c(x, y)\rho(x)\rho(y) \quad \text{for all } x, y \in G.$$

3. The mapping  $x \mapsto \langle f, \rho(x)g \rangle$  is a Borel function on  $G$  for all  $f, g \in \mathcal{H}$ .

A subspace  $V$  of  $\mathcal{H}$  is  **$\rho$ -invariant**, if  $\rho(x)V \subseteq V$  for all  $x \in G$ . The representation  $\rho$  is **irreducible** if the only  $\rho$ -invariant closed subspaces of  $\mathcal{H}$  are  $\{0\}$  and  $\mathcal{H}$ . A **matrix coefficient** of  $\rho$  is a function of the form  $g \mapsto \langle f, \rho(x)g \rangle$  for any two  $f, g \in \mathcal{H}$ . A projective representation  $\rho$  on  $\mathcal{H}$  is **cyclic** if there exists a  $g \in \mathcal{H}$  such that

$$\overline{\text{span}}(\rho(x)g : x \in G) = \mathcal{H}.$$

A projective representation  $\rho$  for a 2-cocycle gives in a natural way a unitary representation of the Heisenberg group  $G_c$  by

$$\pi(x, \omega) = \omega\rho(x).$$

The following lemma allows us to prove that projective representations of  $G$  and unitary representations of the Heisenberg group  $G_c$  contain the same information.

**Lemma 4.3.1.** *Let  $G$  be a locally compact abelian group and  $c$  a cocycle of  $G$ . Then the following hold:*

1. *If the mapping  $x \mapsto \rho(x)g$  is continuous at  $e$ , then it is continuous at any group element  $y$ .*
2. *If  $x \mapsto \langle g, \rho(x)g \rangle$  is continuous for all  $g \in \mathcal{H}$ , then  $\rho$  is strongly continuous.*
3.  *$\rho$  is irreducible if and only if each non-zero  $g \in \mathcal{H}$  is cyclic.*

*Proof.* 1. Let  $x, y$  in  $G$  and  $g \in \mathcal{H}$ . Then

$$\begin{aligned}
 \|\rho(x)g - \rho(y)g\| &= \|\rho(yy^{-1}x)g - \rho(y)g\| \\
 &= \|\rho(y)(c(y, y^{-1}x)\rho(y^{-1}x))g - g\| \\
 &= \|c(y, y^{-1}x)\rho(y^{-1}x)g - g\| \\
 &\leq \|c(y, y^{-1}x)\rho(y^{-1}x)g - g\| + \|c(y, y^{-1}x)g - g\| \\
 &= \|\rho(y^{-1}x)g - g\| + |c(y, y^{-1}x) - 1|\|g\| \rightarrow 0
 \end{aligned}$$

for  $x \rightarrow y$ .

2. Let  $x, e \in G$  and  $g \in \mathcal{H}$ . Then

$$\begin{aligned}
 \|\rho(x)g - \rho(e)g\|^2 &= \|\rho(x)g - g\|^2 \\
 &= \|\rho(x)g\|^2 + \|g\|^2 - \langle g, \rho(x)g \rangle - \langle \rho(x)g, g \rangle \rightarrow 0
 \end{aligned}$$

for  $x \rightarrow e$ .

The irreducibility of  $\rho$  follows the same line of reasoning as for unitary representations.  $\square$

**Lemma 4.3.2.** *Let  $G$  be a locally compact abelian group and  $c$  a cocycle of  $G$ . Then the following hold:*

1. *The projective representation  $\rho$  of  $G$  is irreducible if and only if the unitary representation  $\pi$  of  $G_c$  is irreducible.*
2. *The projective representation  $\rho$  of  $G$  is strongly continuous if and only if the unitary representation  $\pi$  of  $G_c$  is strongly continuous.*

*Proof.* Let  $g \in \mathcal{H}$ .

1. The first assertion follows from the preceding lemma and

$$\begin{aligned}
 \text{span}(\rho(x)g : x \in G) &= \text{span}(\omega\rho(x)g : \omega \in \mathbb{T}, x \in G) \\
 &= \text{span}(\pi(x, \omega)g : (x, \omega) \in G_c).
 \end{aligned}$$

2. Since  $G_c$  is equipped with the product topology, the continuity of  $\rho$  follows from that of  $\pi$ . Now suppose  $\rho$  to be continuous. Then

$$\begin{aligned} \|\pi(x, \omega)g - g\| &= \|\omega\rho(x)g - g\| \\ &\leq \|\omega\rho(x)g - \rho(x)g\| + \|\rho(x)g - g\| \\ &= |1 - \omega| \cdot \|f\| + \|\rho(x)g - g\|, \end{aligned}$$

from which the desired result follows. □

An irreducible continuous projective representation  $\rho$  of a locally compact abelian group  $G$  is **integrable** if there exists a non-zero  $g \in \mathcal{H}$  such that

$$\int_{\mathcal{G}} |\langle g, \rho(x)g \rangle| dx < \infty.$$

Furthermore,  $\rho$  is called **square-integrable** if there exists a non-zero  $g \in \mathcal{H}$  such that

$$\int_{\mathcal{G}} |\langle g, \rho(x)g \rangle|^2 dx < \infty.$$

The trivial observation

$$|\langle g, \rho(x)g \rangle| = |\langle g, \pi(x, \omega)g \rangle|$$

yields

**Lemma 4.3.3.** *An irreducible continuous projective representation  $\rho$  of a locally compact abelian group  $G$  is (square) integrable if and only if the unitary representation  $\pi$  of  $G_c$  is (square) integrable.*

The preceding results between projective representations of  $G$  and unitary representations of  $G_c$  were in a different context independently obtained by Christensen, see [Chr96].

Now we have all ingredients for the discussion of twisted group  $C^*$ -algebras. The properties of a 2-cocycle allow the definition of the **twisted Banach convolution algebra**  $L^1(G, c)$  which consists of all absolutely integrable functions on  $L^1(G)$  with the following multiplication and involution:

1.  $(f \natural g)(t) = \int_G f(x)g(tx^{-1})c(x, tx^{-1})dx$ , the **twisted convolution** of  $f$  and  $g$ ,
2.  $f^*(t) = \overline{c(t, t^{-1})f(t^{-1})}$ , the **twisted involution** of  $f$ .

Like strongly continuous unitary representations of  $G$  induce a representation of  $L^1(G)$  a continuous projective representation  $\rho$  of  $G$  on a Hilbert space  $\mathcal{H}$  induces a representation of  $L^1(G, c)$  by integration:

$$\tilde{\rho}(f) = \int f(x)\rho(x)dx.$$

We have that the operator norm of  $\tilde{\rho}(f)$  is dominated by the  $L^1$ -norm:

$$\|\tilde{\rho}(f)\|_{\text{op}} \leq \int |f(x)| dx = \|f\|_1.$$

The mapping  $f \mapsto \tilde{\rho}(f)$  is an involutive homomorphism, i.e.

1.  $\tilde{\rho}(f \natural g) = \tilde{\rho}(f)\tilde{\rho}(g)$ ,
2.  $\tilde{\rho}(f)^* = \tilde{\rho}(f^*)$ .

The important fact is that under certain circumstances a representation  $\tilde{\rho}$  of  $L^1(G, c)$  determines a unique projective representation on  $G$ . It turns out the only condition on  $\tilde{\rho}$  is that  $\tilde{\rho}$  has to be **non-degenerate** on  $\mathcal{H}$ , i.e.  $\tilde{\rho}(L^1(G))\mathcal{H} = \mathcal{H}$ . On every locally compact abelian group  $G$  we have the **left regular  $c$ -representation** of  $G$  on  $L^2(G)$

$$\rho_L(x)g(t) := c(t, x^{-1}t)g(x^{-1}t),$$

which is a continuous projective representation on  $L^2(G)$ . Since  $L^1(G, c)$  possesses an approximate identity  $(e_j)_{j \in J}$  one gets that

$$\rho(x)\tilde{\rho}(f)g := \tilde{\rho}(\rho_L(x)f)g, \quad g \in L^2(G) \quad (4.2)$$

is the uniquely determined strongly continuous unitary projective representation of  $G$ . As in the case of group  $C^*$ -algebras we get the following important result.

**Theorem 4.3.4.** *Let  $G$  be a locally compact abelian group  $G$ . Then there exists a bijective correspondence between continuous projective representations of  $G$  and involutive representations of  $L^1(G, c)$ .*

Now, we define the reduced twisted group  $C^*$  algebra of  $G$ .

**Definition 4.3.5.** *The reduced twisted group  $C^*$ -algebra of  $G$  is defined to be  $C_r^*(G, c) := \overline{\tilde{\rho}_L(L^1(G, c))}$ .*

Now, we call the direct sum of all irreducible representations of  $G$  the **universal representation**  $\rho_u$  of  $G$ .

**Definition 4.3.6.** *Let  $G$  be a locally compact abelian group. Then the twisted group  $C^*$ -algebra  $C^*(G, c)$  of  $G$  is the norm closure of  $\rho_u(L^1(G))$ .*

Equivalently,  $C^*(G, c)$  is defined as the closure of  $L^1(G, c)$  by the  $C^*$ -norm

$$\|f\| = \sup\{\|\tilde{\rho}(f)\| : \tilde{\rho} \text{ is an involutive representation of } L^1(G, c)\}.$$

The supremum in the preceding equation is well-defined since  $\|f\| \leq \|f\|_1$ . Furthermore, the collection of involutive representations is non-empty because any irreducible representation of  $C_r^*(G, c)$  provides an irreducible projective representation of  $G$ . Now such representations exist by the GNS-construction.

By construction, there is a one-to-one correspondence between the irreducible representations of  $C^*(G, c)$  and  $L^1(G, c)$ , and hence with the irreducible representations of  $G$ . In particular, the extension of the twisted left regular representation  $\rho_L$  of  $C_r^*(G, c)$  we get an irreducible representation of  $C^*(G, c)$ .

Therefore the left-regular  $c$ -representation  $\rho_L$  of  $G$  on  $L^2(G)$  induces the involutive representation

$$\tilde{\rho}_L(f) = \int_G f(x)\rho_L(x)dx.$$

If  $g \in L^2(G)$ , then

$$\tilde{\rho}_L(f)g = \int_G f(x)\rho(x)gdx$$

which is just the twisted convolution of  $f$  and  $g$ .

Finally, we define the **right regular  $c$ -representation**  $\rho_R$  of  $G$  on  $L^2(G)$  by

$$\rho_R(x)g(t) = c(t, x)g(tx).$$

A straightforward application of the cocycle identity gives

**Lemma 4.3.7.** *Let  $\rho_R$  and  $\rho_L$  be the right/left  $c$ -representations. Then the following hold:*

1.  $\rho_R(x)\rho_R(y) = c(x, y)\rho_R(xy)$  and  $\rho_L(x)\rho_L(y) = c(x, y)\rho_L(xy)$  for all  $x, y \in G$ .
2. Let  $\bar{c}$  be the conjugate 2-cocycle of  $c$ . Then the  $c$ -regular representation commutes with the  $\bar{c}$ -right regular representation of  $G$ .

The construction of the twisted group  $CS$ -algebra  $C^*(G, c)$  obviously depends on the choice of the cocycle. A natural question arises when two group  $C^*$ -algebras  $C^*(G, c)$  of  $G$  are isomorphic. It is not a big surprise that this happens only if the involved cocycles are cohomologous. Recall, two cocycles  $c_1$  and  $c_2$  are called **cohomologous** if there exists some  $\tau : G \rightarrow \mathbb{C}$  such that

$$c_1(x, y) = c_2(x, y)\tau(x)\tau(y)\tau(xy)^{-1}.$$

**Proposition 4.3.8.** *Let  $c_1$  and  $c_2$  be two cocycles on a locally compact abelian group  $G$ . Then  $C^*(G, c_1)$  is isomorphic to  $C^*(G, c_2)$  if and only if  $c_1$  and  $c_2$  are cohomologous.*

*Proof.* If we replace  $c_1$  by a cohomologous cocycle  $c_2$ , then we get an involutive isomorphism between  $L^1(G, c_1)$  and  $L^1(G, c_2)$  by the following computations.

$$\begin{aligned} f\sharp_1 g(x) &= \int_G f(t)g(t^{-1}x)c_2(t, t^{-1}x)\tau(t)\tau(t^{-1}x)\tau(x)^{-1}\tau(t)dt \\ &= \tau(x)^{-1} \int_G \tau(t)f(t)g(t^{-1}x)\tau(t^{-1}x)c_2(t, t^{-1}x)dt \\ &= \tau(x)^{-1}(\tau f)\sharp_2(\tau g)(x) \end{aligned}$$

$$\begin{aligned}
 f^{*1}(x) &= \overline{c_2(x, x^{-1})\tau(x)\tau(x^{-1})\tau(e)^{-1}f(x^{-1})} \\
 &= \tau(x)^{-1}c_2(x, x^{-1})\tau(x^{-1})f(x^{-1}) \\
 &= \tau(x)^{-1}(\tau f)^{*2}(x).
 \end{aligned}$$

This involutive isomorphism extends to an involutive isomorphism of the twisted group  $C^*$ -algebras.  $\square$

Finally, we want to study in which sense

$$\tilde{\rho}(f) = \int_G f(x)\rho(x)dx$$

holds. Let  $g, h \in \mathcal{H}$ . Then

$$\langle h, \tilde{\rho}(f)g \rangle = \int_G f(x)\langle h, \rho(x)g \rangle dx$$

gives rise to a sesqui-linear functional

$$\ell(h) = \int_G f(x)\langle h, \rho(x)g \rangle dx.$$

Therefore the boundedness of  $\ell(h)$  depends on the "quality" of  $f$  and the matrix coefficients  $T_\rho(f, g)$ . Suppose  $F \in L^2(G)$ , then an application of Cauchy-Schwarz implies

$$|\ell(h)| \leq \|f\|_2 \|T_\rho(f, g)\|_2.$$

Or more generally, an application of Hölder's inequality gives

$$|\ell(h)| \leq \|f\|_p \|T_\rho(f, g)\|_p$$

for  $1/p + 1/q = 1$ . Consequently,  $\ell(h)$  is bounded if  $f \in L^p(G)$  and  $T_\rho(f, g) \in L^q(G)$ . Therefore  $T_\rho(f, g) \in L^\infty(G)$  and  $f \in L^1(G)$  are the weakest conditions to guarantee the boundedness of  $\ell(h)$ . Suppose  $f, g \in L^2(G)$ , then

$$|\langle h, \tilde{\rho}(f)g \rangle| \leq \|f\|_1 \sup_{x \in \mathcal{H}} |\langle h, \rho(x)g \rangle| \leq \|f\|_1 \|g\|_2 \|h\|_2.$$

is a bounded sesquilinear form on  $L^2(G)$ . Then Lax-Milgram's Theorem implies the boundedness of  $\rho(f)$  and

$$\|\tilde{\rho}(f)\|_{\text{op}} \leq \|f\|_1.$$

## Chapter 4

# 5 Twisted Group $C^*$ -Algebras and Time-Frequency Analysis

In our historical tour on the origins of Gabor analysis we have pointed out the intimate linking of quantum mechanics and time-frequency analysis. In his discussion of quantum mechanics Weyl proposed a method to associate an operator to a function on the time-frequency plane which generalizes linear differential operators. In quantum mechanics Weyl's correspondence is a so-called quantization rule. Many years later Kohn and Nirenberg described another quantization rule which appears naturally in electrical engineering for the description of time-varying systems.

## 5.1 Motivation

One possible way of introducing pseudodifferential operators on  $\mathbb{R}^d$  is as generalization of linear differential operators. As starting point we deal with operators  $A = \sum_{|\alpha| \leq N} \sigma_\alpha(x) D^\alpha$ , where  $D = (2\pi i)^{-1} \frac{\partial}{\partial x}$  denotes the differentiation operator and the non-constant coefficients  $(\sigma_\alpha(x))_{|\alpha| \leq N}$  belong to Schwartz class  $\mathcal{S}(\mathbb{R}^{2d})$  of rapidly decreasing  $C^\infty$ -functions on  $\mathbb{R}^d$ . Then for  $f \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\begin{aligned} (Af)(x) &= \sum \sigma_\alpha(x) D^\alpha f(x) = \int_{\mathbb{R}^d} \left( \sum \sigma_\alpha(x) \omega^\alpha \right) \hat{f}(\omega) e^{2\pi i x \omega} d\omega \\ &= \int_{\mathbb{R}^d} \sigma(x, \omega) \hat{f}(\omega) e^{2\pi i x \omega} d\omega \\ &= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \sigma(x, \omega) e^{2\pi i(x-y)\omega} d\omega \right) f(y) dy. \end{aligned}$$

The function  $\sigma(x, \omega) = \sum \sigma_\alpha(x) \omega^\alpha$  is called the *standard symbol* of  $A$ . Recall that an application of the operator  $\sigma_\alpha D^\alpha$  means first acting  $D^\alpha = ((2\pi i)^{-1} \frac{\partial}{\partial x})^\alpha$  on  $f$  and then multiplication by  $\sigma_\alpha$ . If we reverse the order we are led to substituting for  $A$  the differential operator  $\tilde{A}$  with

$$(\tilde{A}f)(x) = \int_{\mathbb{R}^{2d}} \sigma(y, \omega) e^{2\pi i(x-y)\omega} f(y) dy d\omega.$$

Then  $a(y, \omega)$  is called the *antistandard symbol* of  $\tilde{A}$ . Now a symbolic calculus is just a set of (simple) rules for computing the symbol of the composition of two given differential operators or that of the formal adjoint  $A^*$  on  $L^2(\mathbb{R}^d)$  of  $A$ . The formal adjoint  $A^*$  is given by  $A^*f = \sum D^\alpha(\overline{\sigma_\alpha})f$ . Therefore, the antistandard symbol of  $A^*$  is the function  $\bar{a}(y, \omega)$ . This approach makes it reasonable to represent operators  $A$  by their standard symbols, and their formal adjoints by their anti-standard symbols.

Weyl had the great idea neither choosing the standard or antistandard symbol but letting them meet halfway. Thus the operator  $L_\sigma$  with **Weyl symbol**  $\sigma$  is defined through

$$L_\sigma f(x) = \int_{\mathbb{R}^{2d}} \sigma\left(\frac{x+y}{2}, \omega\right) e^{2\pi i(x-y)\omega} dy d\omega. \quad (5.1)$$

Then, the Weyl symbol of  $L_\sigma^*$  is just  $\bar{\sigma}$ . Consequently,  $L_\sigma$  is a self-adjoint operator if and only if  $\sigma$  is real-valued.

The mapping  $\sigma \mapsto L_\sigma$  is the so-called **Weyl correspondence**, [Wey28]. What kind of assumptions one should make about the Weyl symbol  $\sigma$  to make (5.1) meaningful. The operator  $L_\sigma$  makes sense as a weakly continuous operator from  $\mathcal{S}(\mathbb{R}^d)$  to its dual space  $\mathcal{S}'(\mathbb{R}^d)$ , whenever  $\sigma$  is a tempered distribution on  $\mathbb{R}^{2d}$ . Furthermore, if  $\sigma \in \mathcal{S}'(\mathbb{R}^d)$  then  $L_\sigma$  extends as an operator on  $\mathcal{S}'(\mathbb{R}^d)$ .

A **pseudodifferential operator** are operators  $L_\sigma$  where  $\sigma$  is a tempered distribution on  $\mathbb{R}^{2d}$ .

By the same reasoning we can generalize the mapping arising from the standard symbol above. More precisely, let  $\tau$  a tempered distribution on  $\mathbb{R}^{2d}$ . Then the operator

$$K_\tau f(x) = \int_{\mathbb{R}^{2d}} \tau(x, \omega) e^{2\pi i(x-y)\omega} f(y) d\omega dy \quad (5.2)$$

gives another kind of pseudodifferential operator. The mapping  $\tau \mapsto K_\tau$  is the so-called **Kohn-Nirenberg correspondence** and  $\tau$  is the **Kohn-Nirenberg symbol**, since this kind of pseudodifferential operators was introduced by Kohn and Nirenberg in [KN65].

There is a concept dual to Weyl's correspondence  $\sigma \mapsto L_\sigma$  introduced by Wigner in 1928. We have already met the cross Wigner distribution  $W(f, g)$  of two functions  $f$  and  $g$ , as the first example of a time-frequency representation which was Wigner's original motivation. But its relation to the Weyl correspondence was immediately realized. It relies on the following identity

$$\langle L_\sigma f, g \rangle = \int_{\mathbb{R}^{2d}} \sigma(x, \omega) W(g, f)(x, \omega) dx d\omega \quad (5.3)$$

which is valid for every tempered distribution  $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ . Recall the definition of the *cross Wigner distribution*

$$W(f, g)(x, \omega) = 2^d \int_{\mathbb{R}^d} f(x+t) \overline{g(x-t)} e^{-4\pi i \omega t} dt.$$

Clearly  $W(f, g) \in \mathcal{S}'(\mathbb{R}^{2d})$  and thus for any  $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$  (5.3) is well-defined. Consequently,  $L_\sigma$  maps  $\mathcal{S}'(\mathbb{R}^d)$  into  $\mathcal{S}'(\mathbb{R}^d)$ .

Following Wigner's original motivation Kirkwood introduced another time-frequency representation which was later rediscovered by several researchers, e.g. by the engineer Rihaczek. If  $f, g \in \mathcal{S}'(\mathbb{R}^d)$ , then the so-called **Kirkwood-Rihaczek distribution** is defined by

$$R(f, g)(x, \omega) = e^{-2\pi i x \omega} g(x) \overline{\hat{f}(\omega)}$$

and  $R(f, g) \in \mathcal{S}'(\mathbb{R}^{2d})$ . The Kirkwood-Rihaczek distribution provides us with a dual rule for the Kohn-Nirenberg correspondence  $\tau \mapsto K_\tau$ , i.e.

$$\langle K_\tau f, g \rangle = \langle \tau, R(f, g) \rangle.$$

Consequently the Kohn-Nirenberg correspondence is defined for symbols  $\tau \in \mathcal{S}'(\mathbb{R}^{2d})$ . The preceding discussion should convince you that there is a close relation between time-frequency representations and pseudodifferential operators. In the following we want to gain a deeper understanding of this fact by the use of twisted group  $C^*$ -algebra of  $\mathbb{R}^{2d}$ .

## 5.2 The Twisted Group $C^*$ -Algebra of $\mathbb{R}^{2d}$

For further insight into the Kohn-Nirenberg and Weyl correspondences we rewrite them as integral operators. We begin with the Kohn-Nirenberg correspondence. By definition

$$\begin{aligned} K_\tau f(x) &= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \tau(x, \omega) e^{2\pi i(x-y)\omega} d\omega \right) f(y) dy \\ &= \int_{\mathbb{R}^d} k(x, y) f(y) dy. \end{aligned}$$

Let  $\mathcal{T}_a$  be the coordinate transformation  $\mathcal{T}_a F(x, y) = F(x, y - x)$  and let  $\mathcal{F}_2$  be the partial Fourier transform in the second variable,

$$\mathcal{F}_2 F(x, \omega) = \int_{\mathbb{R}^d} F(x, t) e^{-2\pi i t \omega} dt.$$

Then the integral kernel  $k$  of  $K_\tau$  can be expressed as

$$k(x, y) = \mathcal{F}_2 \tau(x, y - x) = \mathcal{T}_a \mathcal{F}_2 \tau(x, y).$$

Now,  $\mathcal{T}_a$  and  $\mathcal{F}_2$  are isomorphisms on  $L^2(\mathbb{R}^d)$ ,  $\mathcal{S}(\mathbb{R}^d)$  and by duality on  $\mathcal{S}'(\mathbb{R}^d)$  the transition between the pseudodifferential operator  $K_\tau$  and the integral operator is well-defined for symbols in  $\mathcal{S}'(\mathbb{R}^{2d})$ .

For the transition from the Weyl transform  $L_\sigma$  of our symbol  $\tau$  to its integral kernel we recall the relation between the Weyl transform and the cross Wigner distribution

$$W(f, g)(x, \omega) = \int_{\mathbb{R}^d} f(x + t/2) \overline{g(x - t/2)} e^{-2\pi i \omega t} dt.$$

Let  $\mathcal{T}_s$  be the symmetric coordinate transform  $\mathcal{T}_s F(x, y) = F(x + y/2, x - y/2)$  with inverse  $\mathcal{T}_s^{-1} F(x, y) = F((x + y)/2, x - y)$ . We denote the tensor product of  $f$  and  $g$  by  $f \otimes g(x, y) = f(x)g(y)$ . Then the cross Wigner distribution has the following close form:

$$W(f, g)(x, \omega) = \mathcal{F}_2 \mathcal{T}_s (f \otimes \bar{g})(x, \omega).$$

Now we use

$$\begin{aligned} \langle L_\sigma f, g \rangle &= \langle \sigma, W(g, f) \rangle = \langle \sigma, \mathcal{F}_2 \mathcal{T}_s (g \otimes \bar{f}) \rangle \\ &= \langle \mathcal{T}_s^{-1} \mathcal{F}_2^{-1} \sigma, g \otimes \bar{f} \rangle = \langle k, g \otimes \bar{f} \rangle. \end{aligned}$$

As a final result we obtain that the Weyl transform  $L_\sigma$  can be written as integral kernel with the kernel

$$k = \mathcal{T}_s^{-1} \mathcal{F}_2^{-1} \sigma.$$

These relations between the Kohn-Nirenberg and Weyl correspondence will be of importance later.

In our discussion of twisted group  $C^*$ -algebras we announced a relation to quantization rules which we investigate in the case of the Kohn-Nirenberg and Weyl correspondences. Time-frequency shifts provide the link between quantization and twisted group  $C^*$ -algebras.

We make a short digress on time-frequency shifts. More concretely, for  $f \in L^2(\mathbb{R}^d)$  we define the following operators on  $L^2(\mathbb{R}^d)$ :

1. the **translation** operator by  $T_x f(t) = f(t - x)$  for  $x \in \mathbb{R}^d$ ,
2. the **modulation** operator by  $M_\omega f(t) = e^{2\pi i t \cdot \omega} f(t)$  for  $\omega \in \mathbb{R}^d$ ,
3. the **time-frequency shifts** by  $\pi(x, \omega) f = M_\omega T_x f = e^{2\pi i \omega t} f(t - x)$  for  $(x, \omega) \in \mathbb{R}^{2d}$ .

The rich and beautiful structure of time-frequency analysis has its origin in the commutation relation for time-shifts and frequency shifts (3.2). Since it yields the following rule of composition for *time-frequency shifts*

$$M_\omega T_x M_{\omega'} T_{x'} = e^{2\pi i (x' \omega - x \omega')} M_{\omega'} T_{x'} M_\omega T_x \quad (5.4)$$

for  $(x, \omega), (x', \omega') \in \mathbb{R}^{2d}$ . The composition rule (5.4) led Weyl to the consideration of **projective representations** of  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$  which are ordinary representations of  $\mathbb{R}^{2d}$  up to a phase-factor. Therefore, the time-frequency shifts  $\{\pi(x, \omega) : (x, \omega) \in \mathbb{R}^{2d}\}$  are a *projective representation* of  $\mathbb{R}^{2d}$ .

The operators  $\pi(x, \omega)$  for  $(x, \omega) \in \mathbb{R}^{2d}$  are easily seen to be unitary and the mapping  $(x, \omega) \mapsto \pi(x, \omega)$  is strongly continuous. The condition that  $\pi(x, \omega)$  is irreducible requires some discussion. We need the *representation coefficient* of our projective representation  $\pi(x, \omega)$ . More precisely, if  $f, g \in L^2(\mathbb{R}^d)$  then the function  $(x, \omega) \mapsto \langle f, \pi(x, \omega) g \rangle$  is the associated representation coefficient. In time-frequency analysis we denote this representation coefficient by  $V_g f(x, \omega)$  and call it the **short-time Fourier transform** (STFT) of the signal  $f$  with respect to the window  $g$ . The representation coefficient has the following descriptions  $V_g f(x, \omega) = \langle f, M_\omega T_x \rangle = \widehat{(f \cdot T_x \bar{g})}(\omega) = \int_{\mathbb{R}^d} f(t) \overline{g(t - x)} e^{-2\pi i t \omega} dt$  which justify the term STFT. Since, let  $g$  be a compactly supported non-negative smooth function centered at the origin, then the product  $f \cdot T_x \bar{g}$  can be thought as a smooth cut-off of  $f$  to a neighborhood of  $x \in \mathbb{R}^d$ . Then  $V_g f(x, \omega)$  is the Fourier transform of the local piece of  $f$  near  $x$  and measures the amplitude of  $f$  in the frequency band near  $\omega \in \mathbb{R}^d$  at time  $x$ .

Moyal following Wigner explored the connection between Wigner transforms and the description of the dynamics of a quantum system in detail. One of his main tools was the following identity which is nowadays called **Moyal's identity**.

**Theorem 5.2.1** (Moyal's Identity). *Let  $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$ , then*

$$\iint_{\mathbb{R}^{2d}} V_{g_1} f_1(x, \omega) \overline{V_{g_2} f_2(x, \omega)} dx d\omega = \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle.$$

*In particular,  $V_g f \in L^2(\mathbb{R}^{2d})$  for  $f, g \in L^2(\mathbb{R}^d)$ .*

Moyal's identity expresses orthogonality relations for STFT and in a sense is equivalent to Parseval's theorem for the Fourier transform.

**Corollary 5.2.2.** *If  $f, g \in L^2(\mathbb{R}^d)$  and  $\|g\| = 1$ , then the STFT is an isometry from  $L^2(\mathbb{R}^d)$  into  $L^2(\mathbb{R}^{2d})$ , and*

$$\|V_g f\|_{L^2(\mathbb{R}^{2d})} = \|f\|_2, \quad \text{for all } f \in L^2(\mathbb{R}^d).$$

The last corollary says that  $f$  is completely determined by  $V_g f$ . Furthermore,  $\langle f, \pi(x, \omega)g \rangle = 0$  for all  $(x, \omega) \in \mathbb{R}^{2d}$  implies  $f = 0$  is equivalent to saying that for each fixed  $g \in L^2(\mathbb{R}^d)$  the set  $\{\pi(x, \omega)g : (x, \omega) \in \mathbb{R}^{2d}\}$  spans a dense subspace of  $L^2(\mathbb{R}^d)$ . But this is by Schur's lemma equivalent to the irreducibility of  $\{\pi(x, \omega) : (x, \omega) \in \mathbb{R}^{2d}\}$  on  $L^2(\mathbb{R}^d)$ .

**Proposition 5.2.3.** *The family of time-frequency shifts  $\{\pi(x, \omega) : (x, \omega) \in \mathbb{R}^{2d}\}$  is a continuous square-integrable projective representation of  $\mathbb{R}^{2d}$  on  $L^2(\mathbb{R}^d)$ .*

Let us return to the discussion of Kohn-Nirenberg and Weyl correspondences which we express as a superposition of time-frequency shifts.

Let  $\tau \in \mathcal{S}(\mathbb{R}^{2d})$ . Then

$$\begin{aligned} K_\tau f(x) &= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \tau(x, \omega) e^{-2\pi i x \omega} d\omega \right) f(y) dy \\ &= \int_{\mathbb{R}^d} \mathcal{F}_2^{-1} \tau(x, x - y) f(y) dy. \end{aligned}$$

If we express the kernel  $k$  of  $K_\tau$  as

$$k(x, y) = \mathcal{F}_2 \tau(x, y - x) = \mathcal{F}_1^{-1} \mathcal{F}_1 \mathcal{F}_2 \tau(x, y - x) = \mathcal{F}_1^{-1} \hat{\tau}(x, y - x),$$

then

$$\begin{aligned} K_\tau f(x) &= \iint_{\mathbb{R}^{2d}} \hat{\tau}(\eta, y - x) e^{2\pi i x \eta} f(y) d\eta dy \\ &= \iint_{\mathbb{R}^{2d}} \hat{\tau}(\eta, u) e^{2\pi i x \eta} f(x + u) du d\eta \\ &= \iint_{\mathbb{R}^{2d}} \hat{\tau}(\eta, u) M_\eta T_{-u} f(x) du d\eta. \end{aligned}$$

Consequently, the pseudodifferential operator is a superposition of time-frequency shifts:

$$K_\tau f(x) = \iint_{\mathbb{R}^{2d}} \hat{\tau}(\eta, u) M_\eta T_{-u} f(x) du d\eta. \quad (5.5)$$

The Weyl correspondence can be rewritten as a superposition of time-frequency shifts, too. Let  $\sigma$  be in  $\mathcal{S}(\mathbb{R}^d)$ . Then

$$k = \mathcal{T}_s^{-1} \mathcal{F}_2^{-1} \sigma = \mathcal{T}_s^{-1} \mathcal{F}_1 \hat{\sigma}$$

and

$$\begin{aligned} L_\sigma f(x) &= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \hat{\sigma}(\omega, y-x) e^{\pi i \omega(x+y)} d\omega \right) f(y) dy \\ &= \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\omega, u) e^{2\pi i \omega x} e^{\pi i \omega u} f(u+x) du d\omega \\ &= \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\omega, u) e^{\pi i \omega u} M_\omega T_{-u} f(x) du d\omega. \end{aligned}$$

In other words  $K_\tau$  and  $L_\sigma$  are elements of  $C^*(\mathbb{R}^{2d}, c)$ !

In Chapter 3 we showed that elements of  $C^*(\mathbb{R}^{2d}, c)$  have to be understood in the weak sense. In disguise we used this fact in the definition of Kohn-Nirenberg and Weyl correspondences where we met two important time-frequency representations. In our description of  $K_\tau$  and  $L_\sigma$  as superpositions of time-frequency shifts. For the Kohn-Nirenberg correspondence

$$\begin{aligned} \langle f, K_\tau g \rangle &= \iint_{\mathbb{R}^{2d}} \hat{\tau}(\eta, u) \langle f, \pi(-u, \eta) g \rangle du d\eta \\ &= \iint_{\mathbb{R}^{2d}} \hat{\tau}(\eta, u) V_g f(-u, \eta) du d\eta \end{aligned}$$

we arrive naturally at the STFT  $V_g f$ . And in the case of the Weyl correspondence

$$\begin{aligned} \langle f, L_\sigma g \rangle &= \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\eta, u) \langle f, \pi(-u, \eta) g \rangle du d\eta \\ &= \iint_{\mathbb{R}^{2d}} \hat{\sigma}(u, \eta) e^{-\pi i u \eta} V_g f(-u, \eta) du d\eta. \end{aligned}$$

The expression  $e^{-\pi i u \eta} V_g f(-u, \eta)$  is the so-called **cross-ambiguity function**  $A(f, g)$  of  $f$  and  $g$ . A simple change of variables gives the following expression for  $A(f, g)$

$$A(f, g)(x, \omega) = e^{\pi i x \omega} V_g f(x, \omega). \quad (5.6)$$

The preceding discussion might be a motivation for the study of the twisted group  $C^*$ -algebra  $C^*(\mathbb{R}^{2d}, c)$ .

**Theorem 5.2.4.** *The twisted group  $C^*$ -algebra  $C^*(\mathbb{R}^{2d}, c)$  is isomorphic to the  $C^*$ -algebra  $\mathcal{K}(L^2(\mathbb{R}^d))$  of compact operators on  $L^2(\mathbb{R}^d)$ .*

The proof relies on the Schwartz kernel theorem for  $\mathcal{S}(\mathbb{R}^d)$ .

**Theorem 5.2.5.** *If  $A$  is a continuous linear operator mapping  $\mathcal{S}(\mathbb{R}^d)$  into  $\mathcal{S}'(\mathbb{R}^d)$  (with the weak\* topology), then there exists a tempered distribution  $k \in \mathcal{S}'(\mathbb{R}^{2d})$  such that*

$$\langle Af, g \rangle = \langle k, g \otimes \bar{f} \rangle \quad \text{for } f, g \in \mathcal{S}(\mathbb{R}^d).$$

If  $k$  is a measurable function of polynomial growth, then

$$\langle k, g \otimes \bar{f} \rangle = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} k(x, y) f(y) dy \right) \overline{g(x)} dx$$

for  $f, g \in \mathcal{S}(\mathbb{R}^d)$ . Thus  $A$  is an integral operator in the traditional sense. As a consequence we get the following result.

**Theorem 5.2.6.** *Let  $A$  be a continuous linear operator mapping  $\mathcal{S}(\mathbb{R}^d)$  into  $\mathcal{S}'(\mathbb{R}^d)$ . Then there exists a tempered distribution  $F \in \mathcal{S}'(\mathbb{R}^{2d})$  such that  $A$  has the following representation:*

$$A = \iint_{\mathbb{R}^{2d}} F(x, \omega) M_\omega T_x dx d\omega.$$

As a corollary we get a result about Kohn-Nirenberg and Weyl correspondences.

**Corollary 5.2.7.** *Let  $A$  be a continuous linear operator mapping  $\mathcal{S}(\mathbb{R}^d)$  into  $\mathcal{S}'(\mathbb{R}^d)$ . Then there exist tempered distributions  $\sigma, \tau \in \mathcal{S}'(\mathbb{R}^{2d})$  such that  $A$  has the following representations:*

1. as a pseudodifferential operator  $A = K_\tau$  with Kohn-Nirenberg symbol  $\tau$ ; and
2. as a pseudodifferential operator  $A = L_\sigma$  with Weyl symbol  $\sigma$ .

As last ingredient we need that  $\tau, \sigma$  in  $\mathcal{S}'(\mathbb{R}^{2d})$  implies that  $K_\tau, L_\sigma$  are compact operators on  $L^2(\mathbb{R}^d)$  and therefore the subspace of  $\mathcal{K}(L^2(\mathbb{R}^d))$  generated by the pseudodifferential operators  $K_\tau, L_\sigma$  gives an involutive isomorphism between dense subspaces of  $C^*(\mathbb{R}^{2d}, c)$  and  $\mathcal{K}(L^2(\mathbb{R}^d))$ . This isomorphism extends by continuity and gives the desired result. The isomorphism between the  $C^*$ -algebra generated by  $K_\tau$  for  $\tau \in \mathcal{S}'(\mathbb{R}^{2d})$  and the  $C^*$ -algebra generated by  $L_\sigma$  for  $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$  also follows from our result that a twisted group  $C^*$ -algebra  $C^*(\mathbb{R}^{2d}, c)$  remains the same if we replace  $c$  by a cohomologous cocycle  $c'$ .

The previous result provides us with an abstract reason why time-frequency methods yield Schatten-class results for pseudo-differential operators, since Weyl and Kohn-Nirenberg correspondences have a natural description as elements of  $C^*(\mathbb{R}^{2d}, c)$ . Furthermore, we remark that the assertion of Theorem 5.2.4 holds true if we replace  $F \in \mathcal{S}'(\mathbb{R}^{2d})$  by  $F$  in the modulation space  $M^1(\mathbb{R}^{2d})$ , see Chapter 6 for the definition of modulation spaces. Since the proof of Theorem 5.2.4 relies on a Fourier invariant function space with a kernel theorem which are two of the most important properties of  $M^1(\mathbb{R}^{2d})$ !

Another remark concerns the decomposition of two pseudo-differential operators. We know from our general discussion of twisted group  $C^*$ -algebras that the product of two elements corresponds to another element of our twisted group  $C^*$ -algebra which is built from the twisted convolution of the two elements. More concretely, if  $A_F = \iint_{\mathbb{R}^{2d}} F(x, \omega) M_\omega T_x dx d\omega$  and  $A_G = \iint_{\mathbb{R}^{2d}} G(x, \omega) M_\omega T_x dx d\omega$  for  $F, G \in \mathcal{S}'(\mathbb{R}^{2d})$ , then

$$A_F A_G = \iint_{\mathbb{R}^{2d}} (F \natural G)(x, \omega) \pi(x, \omega) dx d\omega.$$

Recall that the *twisted convolution* of  $F$  and  $G$  is given by

$$(F \natural G)(x, \omega) = \iint_{\mathbb{R}^{2d}} F(y, \eta) G(x - y, \omega - \eta) e^{-2\pi i(\omega - \eta)y} dy d\eta.$$

### 5.3 The Inversion Formula for STFT

The square-integrability and irreducibility of our projective representation  $(\pi(x, \omega) : (x, \omega) \in \mathbb{R}^{2d})$  of  $\mathbb{R}^{2d}$  induces an additional structure on  $C^*(\mathbb{R}^{2d}, c)$ .

First we explore the consequences due to the irreducibility of  $\{\pi(x, \omega) : (x, \omega) \in \mathbb{R}^{2d}\}$ . We already know, Chapter 4, that every  $f \in L^2(\mathbb{R}^d)$  can be approximated well by finite linear combinations of  $\pi(x, \omega)g$  for any non-zero  $g \in \mathcal{H}$ . The equivalence of the two assertions is one form of **Schur's lemma**. Here we state another variant of Schur's lemma.

**Proposition 5.3.1.** *Let  $\{\rho(x, \omega) : (x, \omega) \in \mathbb{R}^{2d}\}$  be a continuous projective irreducible representation of  $\mathbb{R}^{2d}$ . Then the following are equivalent:*

- (1)  $\rho$  is irreducible.
- (2) If a bounded operator  $S : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  satisfies  $\rho(x, \omega)S = S\rho(x, \omega)$  for all  $(x, \omega) \in \mathbb{R}^{2d}$ , then  $S = \lambda I_{L^2(\mathbb{R}^d)}$  for some  $\lambda \in \mathbb{C}$ .

*Proof.* (1)  $\Rightarrow$  (2) : Suppose that  $\rho$  is irreducible and that  $S\rho(x, \omega) = \rho(x, \omega)S$  for all  $(x, \omega) \in L^2(\mathbb{R}^d)$ . Then  $S^*S$  also commutes with all  $\rho(x, \omega)$  and we may assume without loss of generality that the operator  $S$  is self-adjoint. If  $S \neq \lambda I_{L^2(\mathbb{R}^d)}$ , then the spectral theorem implies the existence of an orthogonal projection  $P \neq I_{L^2(\mathbb{R}^d)}$ , that commutes with  $S$  and with all  $\rho(x, \omega)$ . Then  $PL^2(\mathbb{R}^d)$  is invariant under  $\rho$  and  $PL^2(\mathbb{R}^d) \neq \{0\}$  and  $PL^2(\mathbb{R}^d) \neq L^2(\mathbb{R}^d)$ , contradicting the irreducibility of  $\rho$ .

(2)  $\Rightarrow$  (1). Suppose that  $V$  is a closed invariant subspace for  $\rho$  and let  $P$  be the orthogonal projection onto  $L^2(\mathbb{R}^d)$ . Since  $\langle f, \rho(x, \omega)g \rangle = \langle \rho(-x, -\omega)f, g \rangle = 0$  for all  $f \in V, g \in V^\perp$  and  $g \in L^2(\mathbb{R}^d)$ , we conclude that  $V^\perp$  is also invariant under  $\rho$ . Therefore  $P\rho(x, \omega)(I - P)f = 0$  for all  $f \in L^2(\mathbb{R}^d)$ , and consequently

$$P\rho(x, \omega)f = P\rho(x, \omega)Pf = \rho(x, \omega)Pf$$

Therefore, by assumption  $P = \lambda I_L^2(\mathbb{R}^d)$ . But since  $P = P^2$ , we must have either  $\lambda = 0$  and  $V = \{0\}$  or  $\lambda = 1$  and  $V = \mathcal{H}$ . Thus  $\rho$  is irreducible.  $\square$

Let  $A_g$  be the linear operator defined by

$$A_g F = \iint_{\mathbb{R}^{2d}} F(x, \omega) \pi(x, \omega) g dx d\omega, \quad F \in L^2(\mathbb{R}^{2d}).$$

Then  $A_g$  is a bounded operator from  $L^2(\mathbb{R}^{2d})$  onto  $L^2(\mathbb{R}^d)$ . Since, the conjugate-linear functional  $\ell(h) = \iint_{\mathbb{R}^{2d}} F(x, \omega) \overline{\langle h, \pi(x, \omega)g \rangle} dx d\omega$  is a bounded functional on  $L^2(\mathbb{R}^{2d})$  which follows from an application of Cauchy-Schwarz:

$$|\ell(h)| \leq \|F\|_2 \|V_g h\|_2 = \|F\|_2 \|g\|_2 \|h\|_2.$$

At the end of Chapter 4 we already discussed the boundedness of operators like  $A_g F$  in greater generality. We only repeated our argument because it is an elementary and very useful argument. Moreover,  $A_g$  is the adjoint operator of the STFT  $V_g$ .

**Proposition 5.3.2.** *Let  $g \in L^2(\mathbb{R})$ ,  $F \in L^2(\mathbb{R}^{2d})$  and  $V_g$  the mapping from  $L^2(\mathbb{R}^d)$  to  $L^2(\mathbb{R}^{2d})$  which assigns  $f \mapsto V_g f$ . Then*

$$A_g F = \iint_{\mathbb{R}^{2d}} F(x, \omega) \pi(x, \omega) g dx d\omega$$

is the adjoint operator of the STFT  $V_g$ .

The proof is an elementary computation:

$$\begin{aligned} \langle A_g F, h \rangle &= \iint_{\mathbb{R}^{2d}} F(x, \omega) \overline{\langle \pi(x, \omega) g, h \rangle} dx d\omega \\ &= \langle F, V_g h \rangle = \langle V_g^* F, h \rangle. \end{aligned}$$

As another corollary we state an inversion formula which is a straightforward consequence of Moyal's identity.

**Corollary 5.3.3.** *Let  $g, \gamma \in L^2(\mathbb{R}^d)$  and  $\langle g, \gamma \rangle \neq 0$ . Then for all  $f \in L^2(\mathbb{R}^d)$*

$$\frac{1}{\langle g, \gamma \rangle} V_\gamma^* V_g = I_{L^2(\mathbb{R}^d)} \quad (5.7)$$

or more explicitly,

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) M_\omega T_x \gamma dx d\omega. \quad (5.8)$$

In quantum mechanics the inversion formula is often called *resolution of the identity*.

**Proposition 5.3.4.** *Fix  $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$  and assume that  $F(x, \omega)$  has rapid decay on  $\mathbb{R}^{2d}$ . Then the integral*

$$f(t) = \iint_{\mathbb{R}^{2d}} F(x, \omega) \pi(x, \omega) dx d\omega$$

defines a function  $f$  in  $\mathcal{S}(\mathbb{R}^d)$ .

Recall that a function  $F$  on  $\mathbb{R}^{2d}$  is of rapid decay if for all  $n \geq 0$ , there is a constant  $C_n > 0$  such that  $|F(x, \omega)| \leq C_n (1 + |x| + |\omega|)^{-n}$ . The following characterization of Schwartz class  $\mathcal{S}(\mathbb{R}^d)$  is a corollary of the preceding proposition and the inversion formula for the STFT (5.8).

**Theorem 5.3.5.** *Fix a non-zero  $g \in \mathcal{S}(\mathbb{R}^d)$ . The following are equivalent:*

1.  $f \in \mathcal{S}(\mathbb{R}^d)$ .

2.  $V_g f \in \mathcal{S}'(\mathbb{R}^{2d})$ .

3. There is  $C_n > 0$  such that  $|V_g f(x, \omega)| \leq C_n(1 + |x| + |\omega|)^{-n}$  for all  $n \geq 0$ .

**Corollary 5.3.6.** *If  $f \in \mathcal{S}'(\mathbb{R}^d)$ , then there is  $C_n > 0$  such that  $V_g f(x, \omega) \leq (1 + |x| + |\omega|)^N$  for some  $N \geq 0$ .*

In the following we denote by  $v_s(x, \omega)$  the weight  $v_s(x, \omega) = (1 + |x| + |\omega|)^s$  on the time-frequency plane  $\mathbb{R}^{2d}$ . Therefore, if  $g \in \mathcal{S}'(\mathbb{R}^d)$ , then the collection of seminorms

$$\|V_g f\|_{L_{v_s}^\infty} = \sup_{(x, \omega) \in \mathbb{R}^{2d}} (1 + |x| + |\omega|)^s |V_g f(x, \omega)|, \quad s \geq 0.$$

The preceding observations yield an extension of inversion formula for the STFT (5.8).

**Theorem 5.3.7.** *Assume that  $g, \gamma$  are non-zero functions in  $\mathcal{S}'(\mathbb{R}^d)$ .*

1. *If  $|F(x, \omega)| \leq C(1 + |x| + |\omega|)^N$  for some constant  $C > 0$  and  $N \geq 0$ , then the integral  $\iint_{\mathbb{R}^{2d}} F(x, \omega) \pi(x, \omega) dx d\omega$  defines a tempered distribution  $f \in \mathcal{S}'(\mathbb{R}^d)$  in the sense that for all  $\varphi \in \mathcal{S}'(\mathbb{R}^d)$*

$$\langle f, \varphi \rangle = \iint_{\mathbb{R}^{2d}} F(x, \omega) \langle \pi(x, \omega) g, \varphi \rangle dx d\omega.$$

2. *In particular, if  $F = V_g f$  for some  $f \in \mathcal{S}'(\mathbb{R}^d)$ , then*

$$\langle f, \varphi \rangle = \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) \langle \pi(x, \omega) g, \varphi \rangle dx d\omega.$$

*Consequently, the short-time Fourier transform is one-to-one on  $\mathcal{S}'(\mathbb{R}^d)$ .*

In our discussion of twisted group  $C^*$ -algebras we showed that continuous projective representations of  $G$  are in one-to-one correspondence with involutive representations of  $L^1(G)$ . The proof relied on an application of a approximate unit of norm 1. If one takes as approximate unit of  $L^1(\mathbb{R}^d)$  the characteristic functions of the cubes centered at the origin, then one recovers the traditional method of interpretation of the inversion formula for the STFT.

## 5.4 The Stone-von Neumann Theorem

In this section we give the proof of the Stone-von Neumann Theorem stated in Chapter 3.

In terms of projective representations the Stone-von Neumann theorem asserts that

**Theorem 5.4.1** (Stone-von Neumann). *Every irreducible continuous projective representation of  $\mathbb{R}^{2d}$  is equivalent to*

$$\rho_\lambda(x, \omega) = e^{2\pi i x \omega} M_\omega T_{\lambda x}$$

for some  $\lambda \in \mathbb{R}^d \setminus \{0\}$ .

We close the section with an indication of von Neumann's proof of the Stone-von Neumann Theorem 5.4.1. In modern language von Neumann constructed a specific projection in  $C^*(\mathbb{R}^{2d}, c)$  and he showed that the mapping  $F \mapsto \tilde{\pi}(F)$  is injective on  $L^1(\mathbb{R}^{2d})$ . We want to use the developed machinery of twisted group  $C^*$ -algebras to present a general method for the construction of self-adjoint projections in  $C^*(\mathbb{R}^{2d}, c)$ .

We start with an element of  $C^*(\mathbb{R}^{2d}, c)$

$$P_F = \iint_{\mathbb{R}^{2d}} F(x, \omega) \pi(x, \omega) dx d\omega$$

with  $F \in \mathcal{S}(\mathbb{R}^{2d})$ . Then  $P_F$  is a projection in  $C^*(\mathbb{R}^{2d}, c)$  if  $P_F = P_F^* = P_F^2$  are satisfied. In more explicit terms this means:

1.  $F^*(x, \omega) = F(x, \omega)$ ,
2.  $F \natural F(x, \omega) = F(x, \omega)$ .

We recall the definition of the STFT  $V_g f(x, \omega) = \langle f, \pi(x, \omega)g \rangle$ . Then

$$\begin{aligned} V_g f(x, \omega) &= e^{-2\pi i x \omega} \langle \pi(-x, -\omega) f, g \rangle \\ &= e^{-2\pi i x \omega} \overline{V_f g(-x, -\omega)} = V_f^* g(x, \omega). \end{aligned}$$

Furthermore, we have the following formula for the twisted convolution of two STFT's

$$V_{g_1} f_1 \natural V_{g_2} f_2 = \langle g_1, f_2 \rangle V_{g_2} f_1, \quad \text{for } f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d). \quad (5.9)$$

The proof is a straightforward consequence of the definition of the twisted convolution and of the STFT. These two observations provides us with a large number of self-adjoint projections in  $C^*(\mathbb{R}^{2d}, c)$ .

**Proposition 5.4.2.** *Let  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ . Then*

$$P_\varphi = \iint_{\mathbb{R}^{2d}} V_\varphi \varphi \pi(x, \omega) dx d\omega \quad (5.10)$$

yields a self-adjoint projection in  $C^*(\mathbb{R}^{2d}, c)$ .

The specific projection constructed by von Neumann corresponds to the choice  $\varphi_0(t) = e^{-\pi t^2}$ . Then

$$V_{\varphi_0} \varphi_0(x, \omega) = e^{-\pi i x \omega} e^{-\frac{\pi}{2}(x^2 + \omega^2)}.$$

Furthermore, one finds with this choice that

$$P_{\varphi_0}\pi(y, \eta)P_{\varphi_0} = e^{-\pi(y^2+\eta^2)}P_{\varphi_0}.$$

From the irreducibility of the projective representation  $\pi(x, \omega)$  of  $\mathbb{R}^{2d}$  on  $L^2(\mathbb{R}^d)$ , it follows that  $P_{\varphi_0}$  must be a rank-one projection. If the rank of  $P_{\varphi_0}$  would be greater than 1, then we can write  $P_{\varphi_0}$  as the sum of two proper subprojections, which would generate proper invariant subspaces of  $L^2(\mathbb{R}^d)$ , contradicting irreducibility. The Stone-von Neumann Theorem follows, since given two irreducible projective representations  $\rho$  and  $\rho'$  of  $\mathbb{R}^{2d}$  on  $L^2(\mathbb{R}^d)$ , the map sending a unit vector in the range of  $P_{\varphi_0}$  to a unit vector in the range of the corresponding operator  $P'_{\varphi_0}$  will extend uniquely to a unitary interwining operator.

# 6 Function Spaces for Time-Frequency Analysis

Over the last twenty years Feichtinger and Gröchenig have shown that time-frequency analysis and Gabor analysis are actually *harmonic analysis over the Heisenberg group*. Their work placed the results of Janssen in a proper mathematical framework and provided us with the correct class of function spaces for a rigorous treatment of Gabor analysis: the so-called *modulation spaces*.

## 6.1 Motivation

The Heisenberg commutation relation (3.1) has another striking consequence: *A signal cannot be concentrated on small sets in phase space*. Actually, a signal is concentrated on sets with area greater or equal than 1. Therefore, the time-frequency plane carries a partition into a disjoint union of rectangles of area greater or equal than 1.

This localization of a signal  $f \in L^2(\mathbb{R}^d)$  in the time-frequency plane is often expressed in terms of *uncertainty inequalities*. The most famous uncertainty inequality arises from the commutation relation between the position and momentum operator

**Theorem 6.1.1.** *Let  $f \in L^2(\mathbb{R}^d)$ . Then*

$$\|Qf\|_2^2 + \|Pf\|_2^2 \geq \frac{1}{2\pi} \|f\|_2^2 \quad (6.1)$$

*with equality if and only if  $f(t) = ce^{-\pi t^2}$  for some  $c \in \mathbb{R}$ .*

As a consequence the Gaussians  $e^{-\pi t^2}$  are optimally concentrated signals in the time-frequency plane. A simple computation shows that time-frequency shifts of a Gaussian  $\pi(x, \omega)e^{-\pi t^2}$  minimize (6.1), too.

Another great input had a class of function spaces introduced by Feichtinger in the early 1980's which he called **modulation spaces**. They have turned out to be the correct class of function spaces for Gabor analysis because they have an intrinsic characterization by Gabor series. The connection between Gabor analysis and modulation spaces was not known at the very beginning of the study of modulation spaces.

Based on the observation that the uncertainty principle of Theorem 6.1.1 is actually an embedding theorem for certain function spaces Gröchenig showed that

embeddings of a function space in a modulation space gives a variety of new uncertainty principles. Another motivation for the relevance of modulation spaces comes from Lieb's formulation of uncertainty principles in terms of time-frequency representations, especially for the Wigner distribution. Lieb obtained his results in the year 1991 which is almost 60 years after the invention of the Wigner distribution. More precisely, Lieb proved the following theorem:

**Theorem 6.1.2.** *Let  $f, g \in L^2(\mathbb{R}^d)$ . Then the following inequalities hold*

1. *If  $p \in [1, 2]$ , then*

$$\iint_{\mathbb{R}^{2d}} |W(f, g)(x, \omega)|^p dx d\omega \geq \frac{1}{p} (\|f\|_2 \|g\|_2)^p.$$

2. *If  $p \in [2, \infty)$ , then*

$$\iint_{\mathbb{R}^{2d}} |W(f, g)(x, \omega)|^p dx d\omega \leq \frac{1}{p} (\|f\|_2 \|g\|_2)^p.$$

*Furthermore, equality holds only for Gaussians.*

Lieb's proof of Theorem [?] is based on his work of sharp constants and minimizers in the Hausdorff-Young inequality.

Since Lieb's work many researchers have discussed uncertainty principles in terms of time-frequency representation, e.g. [Gr96].

As a motivation for the introduction of modulation spaces we state a characterization of Schwartz class  $\mathcal{S}(\mathbb{R}^d)$  in terms of the short-time Fourier transform which is based on the study of derivations on  $C^*(\mathbb{R}^{2d}, c)$ . Recall that an operator  $\delta$  on a  $C^*$ -algebra  $\mathcal{A}$  is called a **derivation** if

$$\delta(AB) = \delta(A)B + A\delta(B), \quad \text{for all } A, B \in \mathcal{A}.$$

Recall that the unitary groups  $T_x = e^{-2\pi xiP}$  and  $M_\omega = e^{2\pi\omega iQ}$  generated by the momentum operator  $P$  and position operator  $Q$  on  $L^2(\mathbb{R}^d)$  play the role of coordinate translations on the time-frequency plane. We can use these groups to take derivatives of operators on  $L^2(\mathbb{R})$  in the following way. For any bounded operator  $A$  on  $L^2(\mathbb{R}^d)$ , the operators  $e^{-2\pi xiP} A e^{2\pi xiP}$  and  $e^{2\pi\omega iQ} A e^{-2\pi\omega iQ}$  are thought of as its shift by  $x$  and  $\omega$  units in the horizontal and vertical directions, respectively. Therefore, the "derivative of  $A$  in the horizontal direction" should be the operator

$${}_{nc}\partial_x(A) := \lim_{x \rightarrow 0} \frac{e^{-2\pi xiP} A e^{2\pi xiP} - A}{2\pi x}$$

and its "derivative in the vertical direction" should be

$${}_{nc}\partial_y(A) := \lim_{\omega \rightarrow 0} \frac{e^{2\pi\omega iQ} A e^{-2\pi\omega iQ} - A}{2\pi\omega}.$$

Heuristically, we expect that

$$\begin{aligned} {}_{nc}\partial_x(A) &= \lim_{x \rightarrow 0} \frac{e^{-2\pi xiP} A e^{2\pi xiP} - A}{2\pi x} = \lim_{x \rightarrow 0} \frac{(I - 2\pi xiP)A(I + 2\pi xiP) - A}{2\pi x} \\ &= -i(PA - AP) = -i[P, A] \end{aligned}$$

and similarly

$${}_{nc}\partial_y(A) = \lim_{\omega \rightarrow 0} \frac{e^{2\pi\omega iQ} A e^{-2\pi\omega iQ} - A}{2\pi\omega} = -i[Q, A].$$

Observe that  ${}_{nc}\partial_x$  and  ${}_{nc}\partial_y$  satisfy a Leibniz-rule

$${}_{nc}\partial_j(AB) = {}_{nc}\partial_j(A)B + A{}_{nc}\partial_j B, \quad A, B \in \mathcal{B}(\mathcal{H}), j = x, y.$$

In other words,  ${}_{nc}\partial_x, {}_{nc}\partial_y$  are derivations on  $\mathcal{B}(L^2(\mathbb{R}^d))$ . These allows us to define a derivation on  $C^*(\mathbb{R}^{2d}, c)$  by  $\delta := I + {}_{nc}\partial_x + {}_{nc}\partial_y$ . In a heuristic way the action of  $\delta$  on a time-frequency shift  $\pi(x, \omega) = M_\omega T_x$  is given by

$$\delta(\pi(x, \omega)) = (1 - i(x + \omega))\pi(x, \omega).$$

There are several possibilities to make these heuristic considerations rigorous. For example, we can take integral operators with a smooth kernel as domain for  $\delta$ . Our discussion of the twisted group  $C^*$ -algebra  $C^*(\mathbb{R}^{2d}, c)$  suggests another approach.

Let  $\tilde{\pi}(F) = \int_{\mathbb{R}^{2d}} F(x, \omega)\pi(x, \omega)dx d\omega$  in  $C^*(\mathbb{R}^{2d}, c)$ . Then we define

$$\delta(\tilde{\pi}(F)) = \int_{\mathbb{R}^{2d}} (1 - i(x + \omega))F(x, \omega)\pi(x, \omega)dx d\omega$$

in a weak sense by

$$\langle \delta(\tilde{\pi}(F))f, g \rangle = \int_{\mathbb{R}^{2d}} (1 - i(x + \omega))F(x, \omega)\langle \pi(x, \omega)f, g \rangle dx d\omega.$$

for  $f, g \in \mathcal{S}(\mathbb{R}^d)$ . The operator norm of  $\delta$  satisfies the following estimate:

$$\|\delta\|_{\text{op}} \leq \|V_g f\|_\infty \int_{\mathbb{R}^{2d}} (1 + |x| + |\omega|)|F(x, \omega)|dx d\omega.$$

By Theorem 5.3.7 the set

$$\{F \in \mathcal{S}'(\mathbb{R}^{2d}) : \int_{\mathbb{R}^{2d}} (1 + |x| + |\omega|)|F(x, \omega)|dx d\omega < \infty\}$$

gives a space of distributions. Therefore, boundedness conditions on  $\delta$  are intrinsically tied with spaces of distributions. Recall our extension of the inversion formula for the STFT to tempered distributions, i.e. the choice  $F(x, \omega) = V_g f(x, \omega)$  for  $g \in \mathcal{S}(\mathbb{R}^d)$  and  $f \in \mathcal{S}'(\mathbb{R}^d)$  should be a good candidate for a description of  $f$  in terms of time-frequency analysis. This space is a weighted version of Feichtinger's algebra and belongs to the class of modulation spaces which we discuss in the following section.

**Definition 6.1.3.** Fix, a non-zero window  $g \in \mathcal{S}(\mathbb{R}^d)$  and  $1 \leq p \leq \infty$ . Then the modulation space  $M_{v_s}^1(\mathbb{R}^d)$  consists of all tempered distributions  $f \in \mathcal{S}'(\mathbb{R}^d)$  such that  $V_g f$  is in the weighted space  $L_{v_s}^1(\mathbb{R}^d)$ . The norm on  $M_{v_s}^1(\mathbb{R}^d)$  is given by

$$\|f\|_{M_{v_s}^1} = \iint_{\mathbb{R}^{2d}} (1 + |x| + |\omega|)^s |V_g f(x, \omega)| dx d\omega$$

The space  $\cap_{s \geq 0} \text{dom}(\delta^s)$  is the Frechet algebra  $\mathcal{S}'(\mathbb{R}^d)$  by well-known properties of  $M_{v_s}^1(\mathbb{R}^d)$ . But it also follows from general results on domains of derivations on a  $C^*$ -algebra by Bratteli, Elliott and Jorgensen, [BEJ84].

## 6.2 Modulation spaces

In 1983 Feichtinger introduced a class of Banach spaces (see [Fei03]), which allow a measurement of the time-frequency concentration of a function or distribution  $f$  on  $\mathbb{R}^d$ , the so called *modulation spaces*. We choose the STFT  $V_g f$  of  $f$  with respect to a window  $g$  with a good time-frequency concentration and as a measure we take the norm of a function space which is (isometrically) invariant under translations in the time-frequency plane  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . For our investigations we restrict our study to weighted mixed-norm spaces  $L_m^{p,q}$  on  $\mathbb{R}^{2d}$ , [Fei03]. But for the translation invariance of  $L_m^{p,q}$  Feichtinger showed that the weight  $m$  has to be a moderate weight on  $\mathbb{R}^{2d}$  with respect to a positive and rotational symmetric submultiplicative weight  $v$  on  $\mathbb{R}^{2d}$ , i.e  $m(z_1 + z_2) \leq C m(z_1) v(z_2)$  for  $z_1, z_2 \in \mathbb{R}^{2d}$ . Now for  $1 \leq p, q \leq \infty$  we define a function or tempered distribution  $f$  to be an element of the *modulation space*  $M_m^{p,q}(\mathbb{R}^d)$  if for a fixed  $g$  in Schwartz space  $\mathcal{S}(\mathbb{R}^d)$  the norm

$$\|f\|_{M_m^{p,q}} := \|V_g f\|_{L_m^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \omega)|^p m(x, \omega)^p dx \right)^{q/p} d\omega \right)^{1/q}$$

is finite. Then  $M_m^{p,q}(\mathbb{R}^d)$  is a Banach space whose definition is independent of the choice of the window  $g$ . We always measure the  $M_m^{p,q}$ -norm with a fixed non-zero window  $g \in \mathcal{S}(\mathbb{R}^d)$  and that for any non-zero  $g_1 \in M_v^1(\mathbb{R}^d)$  the norm equivalence of  $\|f\|_{M_m^{p,q}}$  with  $\|V_{g_1} f\|_{L_m^{p,q}}$  holds.

One reason for the usefulness of modulation spaces is that many well-known function spaces can be identified with modulation spaces for certain weights:

1.  $M^{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ .
2.  $M^1(\mathbb{R}^d)$  is *Feichtinger's algebra*, which is sometimes denoted by  $S_0(\mathbb{R}^d)$ .
3. If  $m(x, \omega) = (1+x^2)^{s/2}$  then  $M_m^{2,2}(\mathbb{R}^d) = L_s^2(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : (\int_{\mathbb{R}^d} |f(x)|^2 (1+x^2)^s dx)^{1/2} < \infty\}$  is a weighted  $L^2$ -space.
4. If  $m(x, \omega) = (1+\omega^2)^{s/2}$  then  $M_m^{2,2}(\mathbb{R}^d) = H_s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : (\int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 (1+\omega^2)^s d\omega)^{1/2} < \infty\}$  is a *Sobolev space*.

5. If  $m(x, \omega) = (1 + x^2 + \omega^2)^{s/2}$  then  $M_m^{2,2}(\mathbb{R}^d) = Q_s(\mathbb{R}^d) = L_s^2(\mathbb{R}^d) \cap H^s(\mathbb{R}^d)$ , where  $Q_s$  is the *Shubin class*, see [Shu01].

Modulation spaces inherit many properties from the mixed norm spaces, e.g. duality. In the following theorem we state some of their properties, that are of interest in the later discussion.

**Theorem 6.2.1.** *Let  $1 \leq p, q < \infty$  and  $m$  a  $v$ -moderate weight on  $\mathbb{R}^{2d}$ .*

1. *The dual space of  $M_m^{p,q}(\mathbb{R}^d)$  is  $M_{1/m}^{p',q'}(\mathbb{R}^d)$  with  $1/p + 1/p' = 1$  and  $1/q + 1/q' = 1$  and the duality is given by*

$$\langle f, h \rangle = \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) \overline{V_g h(x, \omega)} dx d\omega,$$

for  $f \in M_m^{p,q}(\mathbb{R}^d)$  and  $h \in M_{1/m}^{p',q'}(\mathbb{R}^d)$ .

2.  *$M_m^{p,q}(\mathbb{R}^d)$  is invariant under time-frequency shifts:*

$$\|\pi(u, \eta)f\|_{M_m^{p,q}} \leq Cv(u, \eta)\|f\|_{M_m^{p,q}} \quad \text{for } (u, \eta) \in \mathbb{R}^{2d}.$$

3. *If  $p = q$  and  $m(\omega, -x) \leq Cm(x, \omega)$  then  $M_m^{p,p}(\mathbb{R}^d)$  is invariant under Fourier transform.*

*Proof.* All these statements are well-known and the interested reader may find a proof of statement (1) in Chapter 11 of [Gr01]. We only give the arguments for statements (2) and (3), because they provide the reader with some insight about our choice of weights.

Another reason for the usefulness of modulation spaces is that most of their properties are consequences from manipulations of the STFT. In the following lemma we collect the most important one:

The following lemma expresses elementary properties of the STFT.

**Lemma 6.2.2.** *Let  $f, g \in L^2(\mathbb{R}^d)$  and  $(u, \eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . Then*

1. *Covariance Property of the STFT*

$$V_g(\pi(u, \eta)f)(x, \omega) = e^{2\pi i u \cdot (\omega - \eta)} V_g f(x - u, \omega - \eta).$$

2. *Basic Identity of Time-Frequency Analysis*

$$V_g f(x, \omega) = e^{-2\pi i x \cdot \omega} V_g \hat{f}(\omega, -x).$$

- (2) The time-frequency invariance of  $M_m^{p,q}(\mathbb{R}^d)$  is a direct consequence of the definition of moderate weights and the Covariance Property of the STFT, see

Lemma 6.2.2. Let  $z = (u, \eta)$  be a point of the time-frequency plane  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . Then the following holds:

$$\begin{aligned}
 \|\pi(u, \eta)f\|_{M_m^{p,q}} &= \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x - u, \omega - \eta)|^p m(x, \omega)^p dx \right)^{q/p} d\omega \right)^{1/q} \\
 &= \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \omega)|^p m(x + u, \omega + \eta)^p dx \right)^{q/p} d\omega \right)^{1/q} \\
 &\leq C \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \omega)|^p v(u, \eta)^p m(x, \omega)^p dx \right)^{q/p} d\omega \right)^{1/q} \\
 &= Cv(z) \|f\|_{M_m^{p,q}}.
 \end{aligned}$$

- (3) The key of the argument is an application of the basic identity of Gabor analysis, see Lemma 6.2.2, to a Fourier invariant window  $g$  and the independence of the definition of  $M_m^{p,q}$  for  $g \in \mathcal{S}(\mathbb{R}^d)$ . For simplicity we choose  $g$  to be the standard Gaussian  $g_0(x) = 2^{-d/4} e^{-\pi x^2}$ .

$$\begin{aligned}
 \|\hat{f}\|_{M_m^{p,p}} &= \left( \iint_{\mathbb{R}^{2d}} |V_{g_0} f(x, \omega)|^p m(x, \omega)^p dx d\omega \right)^{1/p} \\
 &= \left( \iint_{\mathbb{R}^{2d}} |V_{\widehat{g_0}} \hat{f}(x, \omega)|^p m(x, \omega)^p dx d\omega \right)^{1/p} \\
 &= \left( \iint_{\mathbb{R}^{2d}} |V_{g_0} f(-\omega, x)|^p m(x, \omega)^p dx d\omega \right)^{1/p} \\
 &= \left( \iint_{\mathbb{R}^{2d}} |V_{g_0} f(x, \omega)|^p m(\omega, -x)^p dx d\omega \right)^{1/p} \\
 &= C \|f\|_{M_m^{p,p}}.
 \end{aligned}$$

□

In the following Corollary, we state some of the properties of the modulation space  $M^{1,1}(\mathbb{R}^d)$ . In harmonic analysis  $M^{1,1}(\mathbb{R}^d)$  is the so-called *Feichtinger algebra* and some authors use the notation  $S_0(\mathbb{R}^d)$  to indicate that Feichtinger's algebra is a Segal algebra, too. There is another reason for this notation, because  $S_0(\mathbb{R}^d)$  shares many properties with the Schwartz space  $\mathcal{S}(\mathbb{R}^d)$  of test functions, e.g., Feichtinger's algebra is invariant under Fourier transform. In the rest of our paper we will denote Feichtinger's algebra by  $M^1(\mathbb{R}^d)$ .

**Corollary 6.2.3.** *Feichtinger's algebra  $M^1(\mathbb{R}^d)$  has the following properties:*

1.  $M^1(\mathbb{R}^d)$  is a Banach algebra under convolution.
2.  $M^1(\mathbb{R}^d)$  is a Banach algebra under pointwise multiplication.
3.  $M^1(\mathbb{R}^d)$  is invariant under time-frequency shifts.
4.  $M^1(\mathbb{R}^d)$  is invariant under Fourier transform.

Before we present the proof we recall that the STFT can be written as a convolution. Namely, let  $g^*(x) = \overline{g(-x)}$  be the involution of  $g \in L^2(\mathbb{R}^d)$ . Then, STFT of  $f \in L^2(\mathbb{R}^d)$  has the following form

$$V_g f(x, \omega) = e^{-2\pi i x \cdot \omega} (f * M_\omega g^*)(x). \quad (6.2)$$

For other formulations of the STFT and its relation to other time-frequency representations, such as the Wigner distribution or the ambiguity function we refer the reader to Gröchening's book [Gr01].

*Proof.* 1. By (6.2) the  $M^1$ -norm of  $f$  is given by

$$\|f\|_{M^1} = \int_{\mathbb{R}^d} \|f * M_\omega g^*\|_{L^1} d\omega.$$

Therefore, we get the following estimate for  $h * f$ , where  $f \in M^1(\mathbb{R}^d)$  and  $h \in L^1(\mathbb{R}^d)$ :

$$\begin{aligned} \|h * f\|_{M^1} &= \int_{\mathbb{R}^d} \|h * f * M_\omega g^*\|_{L^1} d\omega \\ &\leq \int_{\mathbb{R}^d} \|h\|_{L^1} \|f * M_\omega g^*\|_{L^1} d\omega \\ &= \|h\|_{L^1} \|f\|_{M^1}. \end{aligned}$$

2. The statement follows from (1) by applying Fourier transforms.
3. The statement is a special case of our general result for modulation spaces, Theorem 6.2.1.
4. The statement is again a special case of our general result for modulation spaces, Theorem 6.2.1.

□

Despite the above stated properties, Feichtinger observed that  $M^1(\mathbb{R}^d)$  is the *minimal* time-frequency homogenous Banach space, [Fei81]. Another pleasant property of Feichtinger's algebra  $M^1(\mathbb{R}^d)$  is that it is the largest Banach space which allows an application of Poisson's summation formula, [Fei81]. Our main results about FIGA rely heavily on this fact. We will discuss this topic further after the introduction of the symplectic Fourier transform.

A deeper analysis of functions in a modulation space is linked with a better understanding of the inversion formula for the STFT.

Recall the definition of the adjoint operator of the STFT. Given a non-zero window  $\gamma$  and a function  $F$  on  $\mathbb{R}^{2d}$ , then

$$V_\gamma^* F = \iint_{\mathbb{R}^{2d}} F(x, \omega) M_\omega T_x \gamma dx d\omega = \iint_{\mathbb{R}^{2d}} F(x, \omega) \pi(x, \omega) dx d\omega.$$

The integral is understood in the weak sense and the following proposition states that for a function  $F$  in  $L_m^{p,q}(\mathbb{R}^{2d})$   $V_\gamma^*$  is well-defined.

**Proposition 6.2.4.** *Assume that  $m$  is  $v$ -moderate and that  $\gamma \in \mathcal{S}(\mathbb{R}^d)$ . Then  $V_\gamma^*$  maps  $L_m^{p,q}(\mathbb{R}^{2d})$  into  $M_m^1(\mathbb{R}^d)$  and satisfies*

$$\|V_\gamma^* F\|_{M_m^{p,q}} \leq C \|V_{g_0} \gamma\|_{L_v^1} \|F\|_{L_m^{p,q}}.$$

*Proof.* The first step is to show that  $V_\gamma^* F \in \mathcal{S}(\mathbb{R}^d)$  for  $F \in L_m^{p,q}(\mathbb{R}^d)$ . Let  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  and we use the weight  $v_n(z) = (1 + |z|)^n$ . Then for sufficiently large  $n$  the following expression is finite.

$$\begin{aligned} |\langle V_\gamma^* F, \varphi \rangle| &= |\langle F, V_\gamma \varphi \rangle| \\ &= \|F\|_{L_m^{p,q}} \|V_\gamma \varphi\|_{L_{1/m}^{p',q'}} \\ &\leq \|F\|_{L_m^{p,q}(\mathbb{R}^d)} \|v_n V_\gamma \varphi\|_\infty \|v_n^{-1}\|_{L_{1/m}^{p',q'}}. \end{aligned}$$

Consequently  $V_\gamma^* F = \iint_{\mathbb{R}^{2d}} F(x, \omega) \pi(x, \omega) dx d\omega$  is well defined as a tempered distribution and  $V_\gamma^* F$  has a continuous STFT:

$$\begin{aligned} V_g V_\gamma^* F(u, \eta) &= \langle V_\gamma^* F, \pi(u, \eta) g \rangle \\ &= \iint_{\mathbb{R}^{2d}} F(x, \omega) \overline{V_\gamma(M_\eta T_u g)(x, \omega)} dx d\omega \\ &= \iint_{\mathbb{R}^{2d}} F(x, \omega) V_g \gamma(u - x, \eta - \omega) e^{-2\pi i x(\eta - \omega)} dx d\omega. \end{aligned}$$

Therefore we obtain the very important pointwise estimate

$$|V_g V_\gamma^* F(u, \eta)| \leq (|F| \star |V_g \gamma|)(u, \eta).$$

By the convolution theorem for weighted mixed norm spaces yields

$$\|V_g V_\gamma^* F\|_{L_m^{p,q}} \leq C \|F\|_{L_m^{p,q}} \|V_g \gamma\|_{L_v^1}. \quad (6.3)$$

If  $g, \gamma \in \mathcal{S}(\mathbb{R}^d)$  then  $V_g \gamma \in \mathcal{S}(\mathbb{R}^{2d})$  and consequently  $V_g \gamma \in L_v^1(\mathbb{R}^d)$ . Let  $g_0$  be the window to measure the modulation space norm, i.e.

$$\|V_\gamma^* F\|_{M_m^{p,q}} = \|V_{g_0} V_\gamma^* F\|_{L_m^{p,q}} \leq C \|F\|_{L_m^{p,q}} \|V_{g_0} \gamma\|_{L_v^1}.$$

In other words  $V_\gamma^* F \in M_m^1(\mathbb{R}^d)$ . □

**Corollary 6.2.5.** *The definition of  $M_m^1(\mathbb{R}^d)$  is independent of the window  $g \in \mathcal{S}(\mathbb{R}^d)$  which yield equivalent norms.*

*Proof.* By equation (6.3) with  $g = \gamma$  and  $\|g\|_2 = 1$ , we obtain

$$\|f\|_{M_m^{p,q}} = \|V_{g_0} f\|_{L_m^{p,q}} = \|V_{g_0} (V_g^* V_g f)\|_{L_m^{p,q}} \leq C \|V_{g_0} g\|_{L_v^1} \|V_g f\|_{L_m^{p,q}}.$$

We now interchange the roles of  $g$  and  $g_0$ ,

$$\|V_g f\|_{L_m^{p,q}} \leq C \|V_g g_0\|_{L_v^1} \|V_{g_0} f\|_{L_m^{p,q}} = C_1 \|f\|_{M_m^{p,q}}.$$

Thus  $f \in M_m^1(\mathbb{R}^d)$  if and only if  $V_g f \in L_m^{p,q}(\mathbb{R}^d)$  for some and hence for all  $g \in \mathcal{S}(\mathbb{R}^d)$ , and the norms  $\|V_{g_0} f\|_{L_m^{p,q}}$  and  $\|V_g f\|_{L_m^{p,q}}$  are equivalent on  $M_m^1(\mathbb{R}^d)$ . □

Another consequence is an extension of the inversion formula for STFT.

**Corollary 6.2.6.** *The inversion formula*

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) \pi(x, \omega) \gamma dx d\omega$$

holds in  $M_m^1(\mathbb{R}^d)$ .

*Proof.* If  $V_g f \in L_m^{p,q}(\mathbb{R}^d)$ , then  $\tilde{f} = \frac{1}{\langle g, \gamma \rangle} V_\gamma^* V_g f \in M_m^1(\mathbb{R}^d)$ . The equality  $\tilde{f} = f$  follows from the inversion formula for  $\mathcal{S}'(\mathbb{R}^{2d})$ .  $\square$

## 6.3 Wiener Amalgam Spaces

Around 1980 Feichtinger introduced a class of Banach spaces, which allow measurement of local and of global properties of functions, see [Fei81a]. Feichtinger's work was motivated by some spaces Wiener had used in his study of the Fourier transform, see [Wie33]. Nowadays those spaces are called *Wiener amalgam spaces* and they are a generalization of Fournier and Stewart's amalgam spaces, [FS85]. Wiener amalgam spaces have turned out to be very useful in harmonic analysis and time-frequency analysis, e.g. [Fei90, FZ98, Gr01].

More concretely, let  $g$  be an element of the space of test functions  $\mathcal{D}(\mathbb{R}^{2d})$ , whose integer translates generate a partition of unity over  $\mathbb{R}^{2d}$ , i.e.,  $\sum_{m \in \mathbb{Z}^{2d}} T_m g \equiv 1$ . Let  $V(\mathbb{R}^{2d})$  be a translation invariant Banach space of functions (or distributions) over  $\mathbb{R}^{2d}$  with the property that  $\mathcal{D} \cdot V \subset V$ . Then the *Wiener amalgam space*  $W(X, L_m^{p,q})$  with local component  $X$  and global component  $L_m^{p,q}$  is defined as the space of all functions resp. distributions for which the norm

$$\|f\|_{W(X, L_m^{p,q})} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \|f \cdot T_{(z_1, z_2)} \bar{g}\|_X^p m(z_1, z_2)^p dz_1 \right)^{q/p} dz_2 \right)^{1/q}$$

is finite. We note that different choices of  $g \in \mathcal{D}$  give the same space and yield equivalent norms [Fei81a].

In [Fei81a] also an extension of Hölder's inequality to Wiener amalgam spaces is given: If  $X$  be a Banach algebra with respect to pointwise multiplication, then for  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\frac{1}{q} + \frac{1}{q'} = 1$  one has

$$\|f \cdot g\|_{W(X, L^1)} \leq \|f\|_{W(X, L_m^{p,q})} \|g\|_{W(X, L_{1/m}^{p',q'})} \quad (6.4)$$

There are many characterizations of Feichtinger's algebra  $M^1(\mathbb{R}^d)$ . Later we shall need a result of Feichtinger that  $M^1(\mathbb{R}^d) = W(\mathcal{FL}^1, L^1)$  with equivalent norms [Fei81]. One of these norms has a formulation by means of the STFT:

$$\begin{aligned} \|f\|_{W(\mathcal{FL}^1, L^1)} &= \iint_{\mathbb{R}^{2d}} \|f \cdot T_{(z_1, z_2)} g\|_{\mathcal{FL}^1} dz_1 dz_2 \\ &= \iint_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^d} |(f \cdot T_{(z_1, z_2)} \bar{g})^\wedge(\omega)| d\omega \right) dz_1 dz_2 \\ &= \iint_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^d} |V_g f(z_1, z_2, \omega)| d\omega \right) dz_1 dz_2. \end{aligned}$$

Analogous expressions for the norm of Wiener amalgam spaces  $W(X, Y)$  with local component  $\mathcal{FL}^1$  and global component  $L_m^{p,q}$  can be derived in terms of the STFT.

We follow the presentation of Cordero and Gröchenig to derive local properties of the STFT. To this end we have to compute the STFT of a STFT. Since the STFT of a function on  $\mathbb{R}^{2d}$  is a function on  $\mathbb{R}^{4d}$ , we distinguish between the STFT  $V_g f(x, \omega)$  for  $(x, \omega) \in \mathbb{R}^{2d}$  of  $f \in \mathcal{S}'(\mathbb{R}^d)$  and the STFT  $\mathcal{V}_\Phi F(z, \zeta)$  for  $(z, \zeta) \in \mathbb{R}^{4d}$  of  $F \in \mathcal{S}'(\mathbb{R}^{2d})$  where  $z = (z_1, z_2) \in \mathbb{R}^{2d}$  and  $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$ .

**Proposition 6.3.1.** *Fix a non-zero  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  and let  $f, g \in \mathcal{S}(\mathbb{R}^d)$ . Set  $\Phi = V_\varphi \varphi$ . Then the STFT of  $V_g f$  with respect to the window  $\Phi$  is given by*

$$\mathcal{V}_\Phi(V_g f)(z, \zeta) = e^{-2\pi i z \zeta} \overline{V_\varphi g(-z_1 - \zeta_2, \zeta_1)} V_\varphi f(-\zeta_2, z_2 + \zeta_1).$$

*Proof.* Recall the definition of STFT  $\mathcal{V}_\Phi(V_g f)(z, \zeta) = \langle V_g f, M_\zeta T_z \Phi \rangle = \mathcal{F}(V_g f T_z \overline{\Phi})(\zeta)$ . We start with the evaluation of  $T_z \Phi$  using the covariance property of STFT and find that

$$T_z \Phi(x, \omega) = V_\varphi \varphi(x - z_1, \omega - z_2) = e^{2\pi i(\omega - z_2)z_1} V_\varphi(M_{z_2} T_{z_1} \varphi)(x, \omega).$$

Now we substitute this expression into the formula for  $\mathcal{V}_\Phi(V_g f)$ :

$$\begin{aligned} \mathcal{V}_\Phi(V_g f)(z, \zeta) &= \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) \overline{V_\varphi(M_{z_2} T_{z_1} \varphi)(x, \omega)} e^{-2\pi i[(x\zeta_1 + \omega\zeta_2) + (\omega - z_2)z_1]} dx d\omega \\ &= e^{2\pi i z_1 z_2} \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) \overline{V_\varphi(M_{z_2} T_{z_1} \varphi)(x, \omega)} e^{-2\pi i[x\zeta_1 + (z_1 + \zeta_2)\omega]} dx d\omega \\ &= e^{2\pi i z_1 z_2} \mathcal{F}(V_g f \overline{V_\varphi(M_{z_2} T_{z_1} \varphi)})(\zeta_1, z_1 + \zeta_2) \\ &= e^{2\pi i z_1 z_2} \mathcal{F}(V_{(M_{z_2} T_{z_1} f)} \varphi \overline{V_\varphi g})(-\zeta_1 - \zeta_2, \zeta_1) \\ &= e^{-2\pi i z_2 \zeta_2} V_\varphi(f)(-\zeta_2, z_2 + \zeta_1) \overline{V_\varphi g(-z_1 - \zeta_2, \zeta_1)}. \end{aligned}$$

□

The preceding result is the key ingredient of a result by Cordero and Gröchenig on the local properties of STFT.

By assumption our weight  $m$  is  $v$ -moderate for  $v$  a submultiplicative weight on  $\mathbb{R}^{2d}$ . The space  $M_v^1(\mathbb{R}^d)$  is now the correct weighted version of Feichtinger's algebra  $M^1(\mathbb{R}^d)$ .

**Proposition 6.3.2** (Cordero/Gröchenig). *Let  $1 \leq p, q \leq \infty$ . If  $f \in M_m^{p,q}(\mathbb{R}^d)$  and  $g \in M_v^1(\mathbb{R}^d)$ , then  $V_g f \in W(\mathcal{FL}^1, L_m^{p,q})$  with*

$$\|V_g f\|_{W(\mathcal{FL}^1, L_m^{p,q})} \leq C \|f\|_{M_m^{p,q}} \|g\|_{M_v^1}. \quad (6.5)$$

In other words, the norms of  $L_m^{p,q}$  and of  $W(\mathcal{FL}^1, L_m^{p,q})$  are equivalent on the range of  $V_g$ . The interested reader is referred to [CG03] for a proof.

For an analysis of the local properties of STFT we only need a special class of Wiener amalgam spaces. Namely, the class of those continuous functions where the

sequence of local suprema belongs to a weighted mixed norm sequence space over  $\mathbb{Z}^{2d}$ . More precisely, let  $m$  be a  $v$ -moderate weight  $m$  on  $\mathbb{Z}^{2d}$ , then  $\ell_m^{p,q}(\mathbb{Z}^{2d})$  is the space of all sequences  $\mathbf{a} = (a(k, l))_{k, l \in \mathbb{Z}^d}$  such that

$$\|\mathbf{a}\|_{\ell_m^{p,q}} = \left( \sum_{l \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |a(k, l)|^p m(k, l)^p \right)^{q/p} \right)^{1/q}$$

is finite. A continuous function  $F$  on  $\mathbb{R}^{2d}$  belongs to the **Wiener amalgam space**  $W(C, L_m^{p,q}(\mathbb{R}^d))$ , if the sequence of local suprema

$$a(k, l) = \|FT_{(k,n)}\chi_Q\|_{\sup} = \sup_{x, \omega \in \mathbb{R}^{2d}} |F(x + k, \omega + l)|$$

belongs to  $\ell_m^{p,q}(\mathbb{Z}^{2d})$ , where  $\chi_Q$  denotes the characteristic function of the unit cube  $Q = [0, 1]^{2d}$ . The norm on  $W(C, L_m^{p,q}(\mathbb{R}^d))$  is defined by

$$\|F\|_{W(L_m^{p,q}(\mathbb{R}^d))} = \|\mathbf{a}\|_{\ell_m^{p,q}}.$$

We use Wiener amalgam spaces to control the samples of a continuous function  $F$  on  $\mathbb{R}^{2d}$ .

**Proposition 6.3.3.** *Let  $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$  be a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$  for  $\alpha, \beta > 0$  and let  $F \in W(C, \ell_m^{p,q})$  for  $p \in [1, \infty)$  where  $\widehat{m}(k, l) = m(\alpha k, \beta l)$ . Then there exists a constant  $C$  such that*

$$\|F|_{\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d}\|_{\ell_m^{p,q}} \leq C \|f\|_{W(C, \ell^p)}.$$

*Proof.* The characteristic function  $\chi_Q$  generates a bounded partition of unity (BUPU) in  $\mathcal{FL}^1(\mathbb{R}^d)$ , i.e. we have  $\text{supp}\chi_Q \subseteq B_R(0)$  for some  $R > 0$ . We only need that we have a uniform control of the number of lattice points contained in any ball of radius centered at some point  $n \in \mathbb{Z}^d$ , i.e. there exists a constant  $C > 0$  such that  $\#(\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d \cap B_R(n)) \leq C_{\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d}$  for all  $n \in \mathbb{Z}^d$ . Furthermore  $m(\alpha k, \beta l) \leq C m(r, s)$  for all  $r, s \in \mathbb{Z}^d$ . As a consequence we conclude that

$$|F(\alpha k, \beta l)| m(\alpha k, \beta l) \leq C \|FT_{(r,s)}\chi_Q\|_{\infty} m(r, s).$$

Since there are at most  $C_{\alpha} < \infty$  points  $\alpha k \in r + [0, 1]^d$ , the  $L^p$ -norm over  $k$  is bounded by

$$\left( \sum_{k \in \mathbb{Z}^d} |F(\alpha k, \beta l)|^p m(\alpha k, \beta l)^p \right)^{1/p} \leq \left( C C_{\alpha} \sum_{r \in \mathbb{Z}^d} \|F \cdot T_{(r,s)}\chi_Q\|_{\infty}^p m(r, s)^p \right)^{1/p}.$$

This estimate is valid for all  $\beta l \in s + [0, 1]^d$ , of which there are at most  $C_{\beta} < \infty$  many points. Now taking the  $L^q$ -norm over  $l$  yields the desired estimate

$$\begin{aligned} \|F|_{\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d}\|_{\ell_m^{p,q}} &\leq \left( \sum_{s \in \mathbb{Z}^d} C_{\beta} \left( C C_{\alpha} \sum_{r \in \mathbb{Z}^d} \|F \cdot T_{(r,s)}\chi_Q\|_{\infty}^p m(r, s)^p \right)^{q/p} \right)^{1/q} \\ &= C_{\beta}^{1/q} (C C_{\alpha})^{1/p} \|F\|_{W(L_m^{p,q}(\mathbb{R}^d))}. \end{aligned}$$

which is the desired result. □

## Chapter 6

An innocently looking assertion about the local behaviour of the STFT for  $f, g \in M^1(\mathbb{R}^d)$  will be the key to give  $M^1(\mathbb{R}^d)$  the structure of an equivalence bimodule for twisted convolution algebras.

**Lemma 6.3.4.** *Let  $f, g \in M^1(\mathbb{R}^d)$ . Then  $V_g f \in W(C, \ell^1)$ , and we have the following estimate*

$$\|V_g f\|_{W(C, \ell^1)} \leq C \|f\|_{M^1} \|g\|_{M^1}.$$

For a detailed account on this topics we refer the reader to [FZ98, Gr01].

# 7 Gabor Analysis

## 7.1 Motivation

Instead of the original definition of a frame for a Hilbert space  $\mathcal{H}$  due to Duffin and Schaeffer, [DS52], we present a generalization proposed by Kaiser [Ka94] and independently by Ali, Antoine and Gazeau [AAG93] since it allows a nice interpretation of the inversion formula of the STFT, see Chapter 5.

**Definition 7.1.1.** Let  $\mathcal{H}$  be a complex Hilbert space and  $M$  a measure space with a positive measure  $\mu$ . A **continuous frame** is a family of vectors  $(f_k)_{k \in M}$  such that

1. for all  $f \in \mathcal{H}$ ,  $k \mapsto \langle f, f_k \rangle$  is a measurable function on  $M$ ;
2. there exist constants  $A, B > 0$  such that

$$A\|f\|^2 \leq \int_M |\langle f, f_k \rangle|^2 d\mu(k) \leq B\|f\|^2, \quad \text{for all } f \in \mathcal{H}.$$

By the inequality of Cauchy-Schwarz the integral  $\int_M \langle f, f_k \rangle \langle f_k, g \rangle d\mu(k)$  is well-defined for all  $f, g \in \mathcal{H}$ . Furthermore, for a fixed  $f \in \mathcal{H}$  the sesqui-linear form

$$\ell(g) = \int_M \langle f, f_k \rangle \langle f_k, g \rangle d\mu(k)$$

is bounded. By Riesz's representation theorem there exists a unique element  $h$  in  $\mathcal{H}$  such that

$$\langle h, g \rangle = \int_M \langle f, f_k \rangle \langle f_k, g \rangle d\mu(k)$$

for all  $g \in \mathcal{H}$ . Therefore we have defined the **frame operator**  $S$  by

$$Sf = \int_M \langle f, f_k \rangle f_k d\mu(k). \quad (7.1)$$

Now,  $S$  is a linear and bounded mapping on  $\mathcal{H}$ . By definition of a continuous frame  $S$  is positive

$$A\|f\|^2 \leq \langle Sf, f \rangle \leq B\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \quad (7.2)$$

Furthermore the frame operator  $S$  is invertible and every  $f \in \mathcal{H}$  has the following representations

1.  $f = S^{-1}Sf = \int_M \langle f, f_k \rangle S^{-1}f_k d\mu(k)$ ,
2.  $f = SS^{-1}f = \int_M \langle f, S^{-1}f_k \rangle f_k d\mu(k)$ ,

hold at least in a weak sense.

We observe that the inversion formula for the STFT (5.8) is equivalent to the following statement.

**Theorem 7.1.2.** *Let  $g$  be a non-zero function in  $L^2(\mathbb{R}^d)$ . Then  $\{\pi(x, \omega)g : (x, \omega) \in \mathbb{R}^{2d}\}$  is a continuous frame for  $L^2(\mathbb{R}^d)$  with respect to the measure space  $L^2(\mathbb{R}^d)$  with the Lebesgue measure.*

But the inversion formula for the STFT holds not only for functions in  $L^2(\mathbb{R}^d)$  it actually gives a reconstruction formula for the whole class of modulation spaces, see Chapter 6. This fact suggests the following elementary generalization of a continuous frame for a Hilbert space.

**Definition 7.1.3.** *Let  $B$  be a dense Banach subspace of  $\mathcal{H}$  and  $M$  a measure space with a positive measure  $\mu$ . A **continuous Banach frame** is a family of vectors  $(f_k)_{k \in M}$  in  $B$ , if the following holds:*

1. for all  $f \in B'$ ,  $k \mapsto \langle f, f_k \rangle$  is a weakly continuous function on  $M$ ;
2. there exist constants  $A, B > 0$  such that

$$A\|f\|^2 \leq \int_M |\langle f, f_k \rangle|^2 d\mu(k) \leq B\|f\|^2, \quad \text{for all } f \in B',$$

where  $\langle \cdot, \cdot \rangle$  is interpreted in a weak sense.

Our discussion in Chapter 6 on the validity of the inversion formula for the STFT translates into the following statement.

**Proposition 7.1.4.** *Let  $g$  be a non-zero element of  $M_m^1(\mathbb{R}^d)$ , then  $\{\pi(x, \omega)g : (x, \omega) \in \mathbb{R}^{2d}\}$  is a Banach frame.*

The preceding statement will serve as a motivation for Gröchenig's definition of a discrete Banach frame. The transition from the continuous to the discrete setting causes serious subtleties. Since their introduction [Gr91] they have been studied by many researchers. The validity of the inversion formula for the STFT on the whole class of modulation spaces may be formulated as follows: If  $\{\pi(x, \omega)g : (x, \omega) \in \mathbb{R}^{2d}\}$  generates a continuous frame for  $L^2(\mathbb{R}^d)$ , then it automatically is a frame for the whole scale of modulation spaces  $M_m^1(\mathbb{R}^d)$ . The answer for discrete Gabor frames is an intriguing difficult problem which led to the notion of "localization" of a frame in [Gr04]. This new field of frame theory relies heavily on the study of non-commutative matrix algebras.

## 7.2 Basic Definitions

Since  $L^2(\mathbb{R}^d)$  is a separable Hilbert space one is tempted to look for discrete analogues of the inversion formula (5.8)

$$f = \sum_{j \in J} \langle f, \pi(x_j, \omega_j)g \rangle \pi(x_j, \omega_j)\gamma$$

such that the system  $\{\pi(x_j, \omega_j)g : (x_j, \omega_j) \in \mathbb{R}^{2d}\}_J$  is a frame for  $L^2(\mathbb{R}^d)$ . From the work of Feichtinger and Gröchenig on irregular sampling we know that under certain conditions on  $\{(x_j, \omega_j) : j \in J\}$  the system  $\{\pi(x_j, \omega_j)g : (x_j, \omega_j) \in \mathbb{R}^{2d}\}_J$  is a frame for  $L^2(\mathbb{R}^d)$ , [FG89a]. We restrict our presentation to discrete sets with the structure of a lattice in  $\mathbb{R}^{2d}$  since this yields to a beautiful structure of the associated frames.

Let  $\Lambda$  be a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  and let  $g \in L^2(\mathbb{R}^d)$  be a Gabor atom then  $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$  is a *Gabor system*. Since the Balian-Low principle tells us that it is not possible to construct (this is in contrast to the situation with wavelets) an orthonormal basis for  $L^2(\mathbb{R}^d)$  of this form, starting e.g. from a Schwartz function  $g$ , interest in Gabor frames arose. A milestone was the paper by Daubechies, Grossmann and Meyer, [DGM86], where the “painless use” of (tight) Gabor frames was suggested.

In our case the Gabor frame operator has the following form:

$$S_{g,\Lambda}f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g, \quad f \in L^2(\mathbb{R}^d).$$

A Gabor system  $\mathcal{G}(g, \Lambda)$  is called a *Gabor frame* if the Gabor frame operator  $S_{g,\Lambda}$  is invertible, i.e., if there exist some finite, positive real numbers  $A, B$  such that

$$A \cdot \mathbf{I} \leq S_{g,\Lambda} \leq B \cdot \mathbf{I}$$

or equivalently,

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B\|f\|^2,$$

for all  $f$  in  $L^2(\mathbb{R}^d)$ . Gabor frames  $\mathcal{G}(g, \Lambda)$  allow the following reconstruction formulas

$$f = (S_{g,\Lambda})^{-1}S_{g,\Lambda}f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)(S_{g,\Lambda})^{-1}g \quad (7.3)$$

$$f = S_{g,\Lambda}(S_{g,\Lambda})^{-1}f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)(S_{g,\Lambda})^{-1}g \rangle \pi(\lambda)g. \quad (7.4)$$

Due to its appearance in the reconstruction formulas  $\gamma_0 := (S_{g,\Lambda})^{-1}g$  is called the (canonical) *dual Gabor atom*. Note that the non-orthogonality of the time-frequency shifts yields that the coefficients in the reconstruction formula (7.3) are not unique and therefore there are other *dual atoms*  $\gamma \in L^2(\mathbb{R}^d)$  with  $S_{g,\gamma,\Lambda} := \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g = \mathbf{I}$ . Some authors call  $(g, \gamma)$  a *dual pair* of Gabor atoms if  $S_{g,\gamma,\Lambda} = \mathbf{I}$ .

The main reason for the beautiful structure of Gabor frames is the invariance of the Gabor frame operator under time-frequency shifts  $\pi(\lambda)$  from the lattice  $\Lambda$ . The argument is a fairly elementary but its consequences are far reaching, see Chapter 9.

**Proposition 7.2.1.** *Let  $f, g, \gamma \in L^2(\mathbb{R}^d)$ . If  $\Lambda$  is a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ , then the Gabor frame type operator*

$$S_{g,\gamma,\Lambda}f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)\gamma$$

*commutes with  $\pi(\lambda)$  for  $\lambda \in \Lambda$ .*

*Proof.* Given  $f \in L^2(\mathbb{R}^d)$  and  $\mu \in \Lambda$ ,

$$\begin{aligned} \pi(\mu)^* S_{g,\gamma,\Lambda} \pi(\mu) f &= \sum_{\lambda \in \Lambda} \langle \pi(\mu) f, \pi(\lambda) g \rangle \pi(\mu)^{-1} \pi(\lambda) \gamma \\ &= \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda - \mu) g \rangle \pi(\mu)^{-1} \pi(\lambda - \mu) \gamma = S_{g,\gamma,\Lambda} f. \end{aligned}$$

□

Furthermore,

**Theorem 7.2.2.** *If  $\mathcal{G}(g, \Lambda)$  is a Gabor frame for  $L^2(\mathbb{R}^d)$ , then  $S_{g,g,\Lambda}$  is a positive operator.*

As a consequence the continuous functional calculus allows us to take the square-root of  $S_{g,g,\Lambda}$ , see 13. We call  $h_0 = S_{g,g,\Lambda}^{-1/2}$  the **canonical tight Gabor atom**. Then we get

$$f = S_{g,g,\Lambda}^{-1/2} S_{g,g,\Lambda} S_{g,g,\Lambda}^{-1/2} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) h_0 \rangle \pi(\lambda) h_0,$$

i.e.  $h_0$  generates a tight Gabor frame. In our discussion of projections in non-commutative tori tight Gabor frames appear quite naturally, see 12. For an extensive study of Gabor frames we refer the reader to the excellent monograph [Gr01] by Gröchenig.

If one considers Gabor systems  $\mathcal{G}(\Lambda, g)$  beyond  $L^2$ -setting, then the operator identity  $S_{g,\gamma,\Lambda} = I$  has to be interpreted in weak sense. This approach was suggested by Feichtinger and Zimmermann in [FZ98]. They called two elements  $g, \gamma \in L^2(\mathbb{R}^d)$  a *weakly dual pair* with respect to  $\Lambda$ , if

$$\langle f, h \rangle = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) \gamma \rangle \langle \pi(\lambda) g, h \rangle \quad (7.5)$$

holds. One can show that absolute convergence of the series on the right, for all  $f, h \in M^1(\mathbb{R}^d)$ . Although the absolute convergence of the series seems very restrictive at first sight also for the case that  $g \in M^1(\mathbb{R}^d)$  and  $\gamma \in M^\infty(\mathbb{R}^d)$ . By the symmetry of the definition in  $g$  and  $\gamma$  we get the same result for  $\gamma \in M^1(\mathbb{R}^d)$  and  $g \in M^\infty(\mathbb{R}^d)$ . Therefore, in the sequel we will only state our results for one setting. We also mention without proof that under the above assumption on the pair  $(g, \gamma)$  the operator mapping the pair  $(f, h)$  to  $\sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) \gamma \rangle \langle \pi(\lambda) g, h \rangle$  is continuous on  $M^1(\mathbb{R}^d) \times M^1(\mathbb{R}^d)$  and that the corresponding Gabor frame operator  $S_{g,\gamma,\Lambda}$  maps  $M^1(\mathbb{R}^d)$  into  $M^\infty(\mathbb{R}^d)$ , [FZ98]. Another direct consequence of the definition is that a pair  $(g, \gamma)$  is weakly dual if and only if  $S_{g,\gamma,\Lambda} = I$ . For further properties we refer the interested reader to [FZ98].

The preceding discussion leads in a natural way to the study of the product of two STFT's  $V_\gamma f \cdot \bar{V}_g h$  restricted to a lattice  $\Lambda$ . In Gabor analysis Tolimieri and Orr realized (in the one-dimensional case and for a product lattice  $\alpha\mathbb{Z} \times \beta\mathbb{Z}$ ,  $\alpha, \beta > 0$ ) that such sums should be evaluated with the help of Poisson's summation formula. But in his research on Morita equivalence of noncommutative tori [Rief88], Rieffel

had used this identity for functions  $f, g, h, \gamma$  in Schwartz-Bruhat space  $\mathcal{S}(G)$  for an elementary locally compact abelian group  $G$  and restrictions to a closed subgroup  $D$  of  $G \times \widehat{G}$  in 1988! Only recently the author has realized the connection between Rieffel's results and Gabor analysis [Lu05c]. Therefore the research in Gabor analysis has been undertaken independently of Rieffel's work, despite its great relevance for Gabor analysis. We will explore this further in a subsequent paper.

### 7.3 The Fundamental Identity of Gabor Analysis

In the following we define the symplectic Fourier transform and some of its basic properties, which was implicitly used by Rieffel in his derivation of the Fundamental Identity of Gabor Analysis. Due to the work of Feichtinger and Kozek [FK98] we also have gained the insight that the symplectic Fourier transform might be of some relevance in this context.

Recall the composition rule for time-frequency shifts

$$\begin{aligned} \pi(x, \omega)\pi(y, \eta) &= e^{-2\pi i x \eta} \pi(x + y, \omega + \eta) \\ &= e^{2\pi i (y\omega - x\eta)} \pi(y, \eta)\pi(x, \omega) \end{aligned} \tag{7.6}$$

for  $(x, \omega), (y, \eta) \in \mathbb{R}^{2d}$ . We denote the by  $\rho((x, \omega), (y, \eta)) = e^{2\pi i (y\omega - x\eta)}$  the skew-symmetric phase-factor which is induced by the standard symplectic form on  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . We start to investigations of the character  $\rho$  of the commutation relation (7.6). The antisymmetry of  $\Omega$  implies that  $\rho$  is a skew-bicharacter of  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . Nevertheless,  $\rho$  gives a Fourier transform  $\widehat{F}^s$  on the time-frequency plane  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$

$$\begin{aligned} \widehat{F}^s(z) &= \iint_{\mathbb{R}^{2d}} \rho(z, z') F(z') dz' \\ &= \iint_{\mathbb{R}^{2d}} e^{2\pi i \Omega(z, z')} F(z') dz' \\ &= \iint_{\mathbb{R}^{2d}} e^{2\pi i (x \cdot \eta - y \cdot \omega)} F(y, \eta) dy d\eta, \end{aligned}$$

for  $z = (x, \omega)$  and  $z' = (y, \eta)$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . We call  $\widehat{F}^s$  the *symplectic Fourier transform* of a function  $F$  in  $L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ , because it is induced by the symplectic form  $\Omega$  of  $\mathbb{R}^{2d}$ .

The Poisson summation formula is one of the most powerful tools in harmonic analysis. In our derivation of FIGA we need a Poisson summation formula for the symplectic Fourier transform, which relates values of a function  $F$  on a lattice  $\Lambda$  in the time-frequency plane with the samples of its symplectic Fourier transform on the adjoint lattice  $\Lambda^0$ .

In the following theorem we give some properties of the symplectic Fourier transform.

**Theorem 7.3.1.** *Let  $M^1(\mathbb{R}^{2d})$  be Feichtinger's algebra over the time-frequency plane  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ .*

1. The symplectic Fourier transform is selfinverse on  $L^2(\mathbb{R}^{2d})$ .
2.  $M^1(\mathbb{R}^{2d})$  is invariant under the symplectic Fourier transform.

*Proof.*

1. The statement is a consequence of the anti-symmetry of the symplectic form  $\Omega$ .
2. We make the observation that  $\rho$  arises from  $e^{2\pi i(x\cdot\omega+y\cdot\eta)}$  by a rotation of  $\pi/2$ , i.e. the symplectic Fourier transform is a rotated version of the Fourier transform on  $\mathbb{R}^{2d}$ . Therefore the result follows from the main properties of Feichtinger's algebra, see Theorem 6.2.1.

□

Traditionally a harmless use of the Poisson summation formula is only known for Schwartz functions. That for Feichtinger's algebra Poisson summation holds pointwise and with absolute convergence is quite unexpected. We only remind you on the work of Katznelson and of Kahane et al., where they give striking examples for the failure of Poisson's summation formula, [KL94, Kat67]. In [Gr96] Gröchenig pointed out the relevance of Poisson's summation formula for  $M^1(\mathbb{R}^d)$  in his study of uncertainty principles and embeddings of various function spaces into  $M^1(\mathbb{R}^d)$ .

In the following theorem we state Poisson's summation formula for the symplectic Fourier transform.

**Theorem 7.3.2.** *[Poisson Summation] Let  $F \in M^1(\mathbb{R}^{2d})$  and  $\Lambda$  a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  then*

$$\sum_{\lambda \in \Lambda} F(\lambda) = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} \widehat{F}^s(\lambda^0) \quad (7.7)$$

*holds pointwise and with absolute convergence of both sums.*

In the last theorem we have denoted the volume of a fundamental domain of  $\Lambda$  by  $|\Lambda|$ . For a proof we refer the reader to [FK98]. In [BP04] Benedetto/Pfander constructed a function  $g \in L^2(\mathbb{R})$  such that  $|V_g g|^2 \notin M_1(\mathbb{R}^2)$  and therefore the symplectic Fourier transform is not valid for  $|V_g g|^2$ .

Before we present our results on the validity of the FIGA, we compute the symplectic Fourier transform of  $V_g f$  for  $f, g \in M^1(\mathbb{R}^d)$ .

**Lemma 7.3.3.** *Let  $f, g \in M^1(\mathbb{R}^d)$  then the following holds:*

1.  $V_g f \in M^1(\mathbb{R}^{2d})$ .
2.  $\widehat{V_g f}^s(z) = f(-x)\overline{\widehat{g}(-\omega)}e^{-2\pi i x \cdot \omega}$  for  $z = (x, \omega) \in \mathbb{R}^{2d}$ .

*Proof.* 1. Follows from the functorial properties and the minimality of  $M^1(\mathbb{R}^d)$ , see [Fei81]. Feichtinger and Kozek give a different proof in [FK98].

2. Straightforward computation. □

We recall that *Rihaczek's distribution* for  $f, g \in L^2(\mathbb{R}^d)$  is defined as  $R(f, g)(x, \omega) = f(x)\overline{\hat{g}(\omega)}e^{-2\pi i x \cdot \omega}$  for  $z = (x, \omega) \in \mathbb{R}^{2d}$ . It is a very popular time-frequency representation in engineering, [HM01, Gr04]. An application of Theorem 7.3.2 yields a generalization of a formula of Kaiblinger [Ka05] which also established a connection to the Rihaczek distribution.

**Proposition 7.3.4.** *Let  $f, g$  be in  $M^1(\mathbb{R}^d)$  and  $\Lambda$  a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . Then the following relation holds:*

$$\sum_{\lambda \in \Lambda} V_g f(\lambda) = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} R(f, g)(-\lambda^0),$$

An application of Theorem (7.3.2) to a product of two STFT's combined with Lemma (4.3)(2) yields the FIGA.

**Theorem 7.3.5** (Basic FIGA). *Assume that  $f_1, f_2, g_1, g_2 \in M^1(\mathbb{R}^d)$ . Then*

$$\sum_{\lambda \in \Lambda} V_{g_1} f_1(\lambda) \cdot \overline{V_{g_2} f_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_{g_1} g_2(\lambda^0) \cdot \overline{V_{f_1} f_2(\lambda^0)} \quad (7.8)$$

*Proof.* Our argument is just the computation of the symplectic Fourier transform of  $F = V_{g_1} f_1(\lambda) \cdot \overline{V_{g_2} f_2(\lambda)}$  and a use of Theorem (7.3.2).

$$\begin{aligned} \widehat{F}^s(Y) &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_{g_1} f_1(X) \overline{V_{g_2} f_2(X)} \rho(Y, X) dX \\ &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle \pi(Y) f_1, \pi(Y) \pi(X) g_1 \rangle \overline{\langle f_2, \pi(X) g_2 \rangle} \rho(X, Y) dX \\ &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle \pi(Y) f_1, \pi(X) \pi(Y) g_1 \rangle \overline{\langle f_2, \pi(X) g_2 \rangle} dX \\ &= \overline{\langle f_2, \pi(Y) f_1 \rangle} \langle g_2, \pi(Y) g_1 \rangle, \end{aligned}$$

where in the last step we used Moyal's formula ,see Theorem 5.2.1. □

The Fourier invariance of  $M^1(\mathbb{R}^d)$  and the basic identity of Gabor analysis (2) yield the following reformulation of (9.3):

$$\sum_{\lambda \in \mathcal{J}\Lambda} V_{\hat{g}_1} \hat{f}_1(\lambda) \cdot \overline{V_{\hat{g}_2} \hat{f}_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0 \in \mathcal{J}\Lambda^\perp} V_{g_1} g_2(\lambda^0) \cdot \overline{V_{f_1} f_2(\lambda^0)}$$

where  $\mathcal{J} = \begin{pmatrix} 0 & \mathbf{I}_d \\ -\mathbf{I}_d & 0 \end{pmatrix}$  denotes a rotation by  $\pi/2$  of the time-frequency plane  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . We have stated this reformulation of (9.3) for two reasons: (1) It is another

manifestation of the fact that  $V_g f$  encodes information about  $f$  and  $\hat{f}$ , (2) It shows that an application of the Fourier transform on the level of signals and windows corresponds to a rotation of the lattice  $\Lambda$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  and that the application of the symplectic Fourier transform to a time-frequency representation of  $f$  yields a rotation of the dual lattice  $\Lambda^\perp$  in  $\widehat{\mathbb{R}}^d \times \mathbb{R}^d$ .

What properties of  $f, g \in M_1(\mathbb{R}^d)$  imply that  $V_g f \in M_1(\mathbb{R}^{2d})$ ? Recall that  $M_1(\mathbb{R}^d) = W(\mathcal{FL}^1, L^1)$ . This fact suggests that for  $f \in M_m^{p,q}(\mathbb{R}^d)$  and  $g \in M^1(\mathbb{R}^d)$  the STFT  $V_g f \in W(\mathcal{FL}^1, L_m^{p,q})$ . In [CG03] Cordero and Gröchenig have proved this result in their investigations of localization operators. Before we state their result on local properties of the STFT, we introduce a weighted version of Feichtinger's algebra  $M^1(\mathbb{R}^d)$ .

By assumption our weight  $m$  is  $v$ -moderate for  $v$  a submultiplicative weight on  $\mathbb{R}^{2d}$ . The space  $M_v^1(\mathbb{R}^d)$  is now the correct weighted version of Feichtinger's algebra  $M^1(\mathbb{R}^d)$ .

**Proposition 7.3.6** (Cordero/Gröchenig). *Let  $1 \leq p, q \leq \infty$ . If  $f \in M_m^{p,q}(\mathbb{R}^d)$  and  $g \in M_v^1(\mathbb{R}^d)$ , then  $V_g f \in W(\mathcal{FL}^1, L_m^{p,q})$  with*

$$\|V_g f\|_{W(\mathcal{FL}^1, L_m^{p,q})} \leq C \|f\|_{M_m^{p,q}} \|g\|_{M_v^1}. \quad (7.9)$$

In other words, the norms of  $L_m^{p,q}$  and of  $W(\mathcal{FL}^1, L_m^{p,q})$  are equivalent on the range of  $V_g$ .

We presented a proof in the previous section.

The remaining part of this section we look for conditions on the quadruple  $f_1, f_2, g_1, g_2$ , which imply that  $V_{g_1} f_1 \cdot \overline{V_{g_2} f_2}$ , belongs to Feichtinger's algebra  $M^1(\mathbb{R}^d)$ , because then a careless application of Poisson formula summation is allowed.

The following theorem is our main result, which is an extension of the validity of range of FIGA.

**Theorem 7.3.7.** *[Main Result] There exists  $C > 0$  independent of  $p, q, v, m$  such that for  $f_1 \in M_m^{p,q}$ ,  $f_2 \in M_{1/m}^{p',q'}$  and  $g_1, g_2 \in M_v^1$ , then the following holds:*

$$\sum_{\lambda \in \Lambda} V_{g_1} f_1(\lambda) \cdot \overline{V_{g_2} f_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_{g_1} g_2(\lambda^0) \cdot \overline{V_{f_1} f_2(\lambda^0)}.$$

*Proof.* By Proposition 7.3.6 we have that  $V_{g_1} f_1 \in W(\mathcal{FL}^1, L_m^{p,q})$  and  $V_{g_2} f_2 \in W(\mathcal{FL}^1, L_{1/m}^{p',q'})$ . Therefore an application of Hölder's inequality (6.4) for Wiener amalgam spaces yields that  $V_{g_1} f_1 \cdot \overline{V_{g_2} f_2} \in W(\mathcal{FL}^1, L^1)$ . The inequalities (6.4) and (7.9) imply the desired norm estimate:

$$\begin{aligned} \|V_{g_1} f_1 \cdot \overline{V_{g_2} f_2}\|_{M^1} &\leq \|V_{g_1} f_1 \cdot \overline{V_{g_2} f_2}\|_{W(\mathcal{FL}^1, L^1)} \\ &\leq C \|V_{g_1} f_1\|_{W(\mathcal{FL}^1, L_m^{p,q})} \|V_{g_2} f_2\|_{W(\mathcal{FL}^1, L_{1/m}^{p',q'})} \\ &\leq C \|g_1\|_{M_v^1} \|g_2\|_{M_v^1} \|f_1\|_{M_m^{p,q}} \|f_2\|_{M_{1/m}^{p',q'}}. \end{aligned}$$

Therefore our object of interest is in  $M^1(\mathbb{R}^d)$  and an application of Poisson summation yields the desired result.  $\square$

As an application of Theorem 7.3.7 we derive the known results about the validity of the FIGA. The first result was obtained by Feichtinger and Zimmermann in their discussion of weakly dual pairs [FZ98].

**Corollary 7.3.8** (Feichtinger-Zimmermann). *Let  $g_1, g_2$  be in  $M^1(\mathbb{R}^d)$ . If  $f_1 \in M^1(\mathbb{R}^d)$  and  $f_2 \in M^\infty(\mathbb{R}^d)$  or  $f_1, f_2 \in L^2(\mathbb{R}^d)$  then*

$$\sum_{\lambda \in \Lambda} V_{g_1} f_1(\lambda) \cdot \overline{V_{g_2} f_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_{g_1} g_2(\lambda^0) \cdot \overline{V_{f_1} f_2(\lambda^0)}.$$

holds.

*Proof.* The corollary covers the cases  $f_1 \in M^1(\mathbb{R}^d)$  and  $f_2 \in (M^1(\mathbb{R}^d))' = M^\infty(\mathbb{R}^d)$  and  $f_1, f_2 \in M^{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ . Therefore the proof is a direct consequence of Theorem 7.3.7  $\square$

The second result covers the case of Tolimieri/Orr of the validity of the FIGA for Schwartz functions [TO95]. The proof consists of the well-known fact [?] that the modulation spaces  $M_{v_s}^1(\mathbb{R}^d)$  for  $v_s(x, \omega) = (1 + x^2 + \omega^2)^{s/2}$  are the building blocks of the Schwartz class  $\mathcal{S}(\mathbb{R}^d)$ , namely

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{s \geq 0} M_{v_s}^1(\mathbb{R}^d).$$

By duality we get a description of tempered distributions

$$\mathcal{S}'(\mathbb{R}^d) = \bigcup_{s \geq 0} M_{1/v_s}^\infty(\mathbb{R}^d).$$

**Corollary 7.3.9** (Tolimieri-Orr). *Let  $f_1, g_1, g_2$  be in  $\mathcal{S}(\mathbb{R}^d)$  and  $f_2 \in \mathcal{S}'(\mathbb{R}^d)$  then we have the following identity:*

$$\sum_{\lambda \in \Lambda} V_{g_1} f_1(\lambda) \cdot \overline{V_{g_2} f_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_{g_1} g_2(\lambda^0) \cdot \overline{V_{f_1} f_2(\lambda^0)}.$$

*Proof.* The statement is true for every building block of  $\mathcal{S}(\mathbb{R}^d)$  and of  $\mathcal{S}'(\mathbb{R}^d)$ , respectively. Therefore our statement is a direct consequence of our main result Theorem 7.3.7.  $\square$

As a consequence we get the continuity of the Gabor frame type operator  $S_{g, \gamma, \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d}$  for  $g, \gamma \in M^1(\mathbb{R}^d)$  which is the composition of the analysis operator and the synthesis operator. Recall, that a Gabor frame type operator  $S_{g, \gamma, \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d}$  is the composition of the analysis mapping  $C_\gamma$ , which is sampling of  $V_\gamma f$  on the lattice  $\alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$ , and the synthesis operator  $D_g$ , which is given by  $\sum_{k, l \in \mathbb{Z}^d} a(k, l) \pi(\alpha k, \beta l) g$  for a sequence  $(a(k, l))_{k, l \in \mathbb{Z}^d}$ .

**Proposition 7.3.10.** *Let  $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$  be a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$  with  $\alpha, \beta > 0$ .*

1. *Then for  $\gamma \in M^1(\mathbb{R}^d)$  the analysis operator  $T_\gamma$  is a bounded mapping from  $M^1(\mathbb{R}^d)$  to  $\ell^1(\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d)$ .*
2. *Then for  $g \in M^1(\mathbb{R}^d)$  the synthesis operator  $D_g$  is a bounded mapping from  $\ell^1(\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d)$  to  $M^1(\mathbb{R}^d)$ .*

As a straightforward consequence of the preceding lemma we get the boundedness  $S_{g,\gamma,\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d}$  for  $g, \gamma \in M^1(\mathbb{R}^d)$ .

**Corollary 7.3.11.** *Let  $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$  be a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$  with  $\alpha, \beta > 0$  and  $g, \gamma \in M^1(\mathbb{R}^d)$ . Then the Gabor frame type operator  $S_{g,\gamma,\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d}$  is a bounded operator on  $M^1(\mathbb{R}^d)$  with absolute convergence of the series. Furthermore, we have*

$$\|S_{g,\gamma,\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d}\|_{M^1} \leq C \|g\|_{M^1} \|\gamma\|_{M^1}.$$

In other words the mapping  $(g, \gamma) \mapsto S_{g,\gamma,\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d}$  is continuous for  $g, \gamma \in M^1(\mathbb{R}^d)$ .

# 8 Hilbert $C^*$ -modules

## 8.1 Basic Facts

In his study of Mackey's imprimitivity theorem Rieffel was led to introduce Hilbert  $C^*$ -modules which are a generalization of Hilbert spaces in which the field of scalars is replaced by a  $C^*$ -algebra. As a consequence Rieffel found a new approach to the uniqueness of Weyl's commutation relations which is based on a generalization of Morita equivalence for rings to  $C^*$ -algebras. In a later chapter we use Hilbert  $C^*$ -modules to discuss Gabor frames. Just as with Hilbert spaces (or more generally with function spaces) we are interested in the operators acting on Hilbert  $C^*$ -modules. Gabor frame operators appear quite naturally as finite rank operators for certain Hilbert  $C^*$ -modules. All this is intended as motivation that Hilbert  $C^*$ -modules are a useful framework for the discussion of Gabor frames.

Our presentation follows the excellent monograph of Raeburn and Williams, [RW98], and the seminal work of Rieffel [Rief74a, Rief74b]. Before we begin, we want to mention that Paschke independently of Rieffel defined Hilbert  $C^*$ -modules in [Pas73].

Let  $\mathcal{A}$  be a  $C^*$ -algebra. Then by a right  $\mathcal{A}$ -module, we shall mean a vector space  $V$  together with a bilinear pairing  $(f, A) \mapsto f \cdot A$  from  $V \times \mathcal{A} \rightarrow V$  satisfying the usual consistency conditions and the extra condition  $(\lambda f) \cdot A = f \cdot (\lambda A)$ . Since we want our operators act on the left on our Hilbert  $\mathcal{A}$ -modules we have to work with *right* Hilbert modules.

**Definition 8.1.1.** *A right inner-product  $\mathcal{A}$ -module is a right  $\mathcal{A}$ -module  $V$  with a pairing  $\langle \cdot, \cdot \rangle_{\mathcal{A}} : V \times V \rightarrow \mathcal{A}$  satisfying the following conditions:*

1.  $\langle f, \lambda g + \mu h \rangle_{\mathcal{A}} = \lambda \langle f, g \rangle_{\mathcal{A}} + \mu \langle f, h \rangle_{\mathcal{A}}$ ;
2.  $\langle f, g \cdot A \rangle_{\mathcal{A}} = \langle f, g \rangle_{\mathcal{A}} A$ ;
3.  $\langle f, g \rangle_{\mathcal{A}}^* = \langle g, f \rangle_{\mathcal{A}}$ ;
4.  $\langle f, f \rangle_{\mathcal{A}} \geq 0$ ;
5.  $\langle f, f \rangle_{\mathcal{A}} = 0$  implies that  $f = 0$ .

**Remark 8.1.2.** *We often write  $V_{\mathcal{A}}$  to emphasize that  $V$  is a right  $\mathcal{A}$ -module. In condition (4) positivity of  $\langle f, f \rangle_{\mathcal{A}} \geq 0$  is understood as an element of  $\mathcal{A}$ . Furthermore, the above conditions imply that  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$  is conjugate linear in the first variable and that*

$$\langle f \cdot A, g \rangle_{\mathcal{A}} = A^* \langle f, g \rangle_{\mathcal{A}}.$$

From these observations it follows that

$$\text{span}\{\langle f, g \rangle_{\mathcal{A}} : f, g \in V_{\mathcal{A}}\} \quad (8.1)$$

is a two-sided ideal in  $\mathcal{A}$ .

The perhaps simplest example of an inner-product  $\mathcal{A}$ -module is the  $C^*$ -algebra  $\mathcal{A}$ .

**Example 8.1.3.** *If  $\mathcal{A}$  acts by right multiplication on  $\mathcal{A}$  and with  $\langle A, B \rangle_{\mathcal{A}} = A^*B$ , the  $C^*$ -algebra  $\mathcal{A}$  is an inner-product  $\mathcal{A}$ -module. The verification of the axioms is elementary, e.g. the last axiom follows from the  $C^*$ -identity:*

$$\langle A, A \rangle_{\mathcal{A}} = 0 \Leftrightarrow A * A = 0 \Leftrightarrow \|A^*A\| = 0 \Leftrightarrow \|A\|^2 = 0 \Leftrightarrow A = 0.$$

Before we define a Hilbert  $\mathcal{A}$ -module to be an inner-product  $\mathcal{A}$ -module which is complete in the norm given by

$$\|f\|_{\mathcal{A}} := \|\langle f, f \rangle_{\mathcal{A}}\|^{1/2}.$$

We need a **Cauchy-Schwarz inequality** for  $C^*$ -valued inner-products which implies that  $\|\cdot\|_{\mathcal{A}}$  really defines a norm on  $V_{\mathcal{A}}$ .

**Theorem 8.1.4** (Cauchy-Schwarz). *Let  $V$  be an inner-product  $\mathcal{A}$ -module. If  $f, g$  are in  $V$ , then*

$$\langle f, g \rangle_{\mathcal{A}}^* \langle g, f \rangle_{\mathcal{A}} \leq \|\langle f, f \rangle_{\mathcal{A}}\| \langle g, g \rangle_{\mathcal{A}} \quad (8.2)$$

as elements of the  $C^*$ -algebra  $\mathcal{A}$ .

**Remark 8.1.5.** 1. *The statement of (8.2) is actually valid when  $V$  is a right  $\mathcal{A}_0$ -module for a dense  $\star$ -subalgebra  $\mathcal{A}_0$  of  $\mathcal{A}$ , and  $V$  has a pairing satisfying (1) – (4) of Definition 8.1.1, provided we interpret the inequalities in (4) as holding in the completion  $\mathcal{A}$  of  $\mathcal{A}_0$ , i.e.  $A \geq 0$  means  $A = B^*B$  for some  $B \in \mathcal{A}$ .*

2. *Recall that an element  $A$  of a  $C^*$ -algebra  $\mathcal{A}$  is positive if  $\psi(A) \geq 0$  for every state  $\psi$  of  $\mathcal{A}$ .*

*Proof.* We show that

$$\psi(\langle f, g \rangle_{\mathcal{A}}^* \langle g, f \rangle_{\mathcal{A}}) \leq \|\langle f, f \rangle_{\mathcal{A}}\| \psi(\langle g, g \rangle_{\mathcal{A}}) \quad (8.3)$$

holds for every state  $\psi$  of  $\mathcal{A}$ . Let  $\psi$  be a fixed state of  $\mathcal{A}$ . Then  $(x, y) \mapsto \psi(\langle x, y \rangle_{\mathcal{A}})$  is a semi-definite positive sesquilinear form on  $V$ , and the ordinary Cauchy-Schwarz inequality implies that

$$|\psi(\langle x, y \rangle_{\mathcal{A}})| \leq \psi(\langle x, x \rangle_{\mathcal{A}})^{1/2} \psi(\langle y, y \rangle_{\mathcal{A}})^{1/2}. \quad (8.4)$$

An application of (8.4) to  $x = f\langle f, g \rangle_{\mathcal{A}}$  and  $y = g$  gives

$$\begin{aligned} \psi(\langle f, g \rangle_{\mathcal{A}}^* \langle g, f \rangle_{\mathcal{A}}) &= \psi(\langle f \langle f, g \rangle_{\mathcal{A}}, g \rangle_{\mathcal{A}}) \\ &\leq \psi(\langle f \langle f, g \rangle_{\mathcal{A}}, f \langle f, g \rangle_{\mathcal{A}} \rangle_{\mathcal{A}})^{1/2} \psi(\langle g, g \rangle_{\mathcal{A}})^{1/2} \\ &= \psi(\langle f, g \rangle_{\mathcal{A}}^* \langle f, f \rangle_{\mathcal{A}} \langle f, g \rangle_{\mathcal{A}})^{1/2} \psi(\langle g, g \rangle_{\mathcal{A}})^{1/2}. \end{aligned}$$

For any  $B \in \mathcal{A}$  and a positive  $C \in \mathcal{A}$  we have the inequality  $B^*CB \leq \|C\|B^*B$  which for  $B = \langle f, g \rangle_{\mathcal{A}}$  and  $C = \langle f, f \rangle_{\mathcal{A}}$  gives

$$\psi(\langle f, g \rangle_{\mathcal{A}}^* \langle f, g \rangle_{\mathcal{A}}) \leq \|\langle f, f \rangle_{\mathcal{A}}\|_{\mathcal{A}} \psi(\langle f, g \rangle_{\mathcal{A}}^* \langle f, g \rangle_{\mathcal{A}})^{1/2} \psi(\langle g, g \rangle_{\mathcal{A}})^{1/2}.$$

After squaring and cancelling a factor of  $\psi(\langle f, g \rangle_{\mathcal{A}}^* \langle f, g \rangle_{\mathcal{A}})$  gives the desired assertion.  $\square$

**Corollary 8.1.6.** *If  $V$  is an inner-product  $\mathcal{A}$ -module, then  $\|f\|_{\mathcal{A}} := \|\langle f, f \rangle_{\mathcal{A}}\|^{1/2}$  defines a norm on  $V$  such that  $\|f \cdot A\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}}\|A\|$ .*

*Proof.* It is easy to check that  $\|\lambda \cdot A\|_{\mathcal{A}} = |\lambda|\|A\|_{\mathcal{A}}$ , and  $\|f\|_{\mathcal{A}} \geq 0$  and  $\|f\|_{\mathcal{A}} = 0 \Rightarrow f = 0$ . The Cauchy-Schwarz inequality (8.2) implies that  $\|\langle f, g \rangle_{\mathcal{A}}\| \leq \|f\|_{\mathcal{A}}\|g\|_{\mathcal{A}}$ , and as in the Hilbert space case, we get the triangle-inequality

$$\begin{aligned} \|f + g\|_{\mathcal{A}}^2 &\leq \|f\|_{\mathcal{A}}^2 + \|g\|_{\mathcal{A}}^2 + \|\langle f, g \rangle_{\mathcal{A}}\| + \|\langle g, f \rangle_{\mathcal{A}}\| \\ &\leq \|f\|_{\mathcal{A}}^2 + \|g\|_{\mathcal{A}}^2 + 2\|f\|_{\mathcal{A}}\|g\|_{\mathcal{A}} \\ &= (\|f\|_{\mathcal{A}} + \|g\|_{\mathcal{A}})^2. \end{aligned}$$

Thus,  $\|\cdot\|_{\mathcal{A}}$  is a norm.

The following chain of inequalities

$$\|f \cdot A\|_{\mathcal{A}}^2 \leq \|A^* \langle f, f \rangle_{\mathcal{A}} A\| \leq \|A\| \|f\|_{\mathcal{A}}^2 \|A^*\| = \|A\|^2 \|f\|_{\mathcal{A}}^2$$

implies the second assertion.  $\square$

**Corollary 8.1.7.** *The normed module  $(V_{\mathcal{A}}, \|\cdot\|_{\mathcal{A}})$  is non-degenerate in the sense that the elements of the form  $f \cdot A$  span a dense subspace of  $V$ . Indeed,*

$$V \cdot \langle V, V \rangle_{\mathcal{A}} := \text{span}\{f \cdot \langle g, h \rangle_{\mathcal{A}} : f, g, h \in \mathcal{A}\}$$

is  $\|\cdot\|_{\mathcal{A}}$ -dense in  $V_{\mathcal{A}}$ .

*Proof.* Let  $\mathcal{B} = \overline{\text{span}}\{\langle f, g \rangle_{\mathcal{A}} : f, g \in X\}$  and  $(x_{\lambda})$  be an approximate identity for the ideal  $\mathcal{B}$ . Since

$$\|f - f \cdot x_{\lambda}\|_{\mathcal{A}}^2 = \|\langle f, f \rangle_{\mathcal{A}} - \langle f, f \rangle_{\mathcal{A}} x_{\lambda} - x_{\lambda} \langle f, f \rangle_{\mathcal{A}} + x_{\lambda} \langle f, f \rangle_{\mathcal{A}} x_{\lambda}\|_{\mathcal{A}},$$

given any  $\varepsilon > 0$  there exists  $x_{\lambda}$  such that  $\|f - f \cdot x_{\lambda}\|_{\mathcal{A}} < \varepsilon/2$ . There are also sequences  $(f_i), (g_i) \in V$  such that  $\|\sum_i \langle f_i, g_i \rangle_{\mathcal{A}} - x_{\lambda}\|_{\mathcal{A}} < \varepsilon/2\|f\|_{\mathcal{A}}$ . Thus, we have

$$\|f - f(\sum_i \langle f_i, g_i \rangle_{\mathcal{A}})\|_{\mathcal{A}} < \varepsilon.$$

$\square$

Now, we are ready to introduce the notion of a Hilbert  $\mathcal{A}$ -module.

**Definition 8.1.8.** A Hilbert  $\mathcal{A}$ -module is an inner-product  $\mathcal{A}$ -module  $V$  which is complete in the norm  $\|\cdot\|_{\mathcal{A}}$ . It is a **full** Hilbert  $\mathcal{A}$ -module if the ideal

$$I = \text{span}\{\langle f, g \rangle_{\mathcal{A}} : f, g \in V\}$$

is dense in  $\mathcal{A}$ .

The next examples are given to relate the new notion to the traditional Hilbert space and to the  $C^*$ -algebra of compact operators on a Hilbert space  $\mathcal{H}$ .

**Example 8.1.9.** The Hilbert  $\mathbb{C}$ -modules are the Hilbert spaces over  $\mathbb{C}$  with the usual scalar multiplication and inner-product which is conjugate linear in the first variable, i.e.  $\langle f, g \rangle_{\mathbb{C}} = \langle f, g \rangle$  (which denotes the inner-product for Hilbert spaces).

**Example 8.1.10.** Let  $\mathcal{H}$  be a Hilbert space and  $\mathcal{K}(\mathcal{H})$  the  $C^*$ -algebra of compact operators on  $\mathcal{H}$ . Let  $f \otimes \bar{g}$  denote the rank-one operator  $h \mapsto f \cdot \langle g, h \rangle_{\mathbb{C}} = \langle h, g \rangle f$ . Then with the left action and inner-product

$$T \cdot f = T(f) \quad \text{and} \quad \kappa \langle f, g \rangle = f \otimes \bar{g}$$

$\mathcal{H}$  is a left Hilbert  $\mathcal{K}$ -module. The following identities allow a straightforward verification of the axioms:

$$(f \otimes \bar{g})^* = g \otimes \bar{f} \quad (f \otimes \bar{g})(k \otimes \bar{l}) = \langle k, g \rangle (f \otimes \bar{l}).$$

Furthermore, the operator norm of  $f \otimes \bar{f}$  is  $\|f\|^2$  and the  $\mathcal{K}(\mathcal{H})$ -norm  $\|\cdot\|_{\mathcal{K}}$  is just the usual norm on  $\mathcal{H}$ . Therefore  $\mathcal{H}$  is  $\|\cdot\|_{\mathcal{K}}$ -complete. It is full because  $\text{span}\{\kappa \langle f, g \rangle : f, g \in \mathcal{H}\}$  is the ideal of finite rank operators on  $\mathcal{H}$ .

In the following we give some elementary examples which will be of importance later.

**Example 8.1.11.** If  $\mathcal{A}$  is a  $C^*$ -algebra, then  $\mathcal{A}_{\mathcal{A}}$  is a full Hilbert module with  $A \cdot B = AB$  and  $\langle A, B \rangle_{\mathcal{A}} = A^*B$ . The norm on  $\mathcal{A}_{\mathcal{A}}$  is  $\|A\|_{\mathcal{A}}^2 = \|\langle A, A \rangle_{\mathcal{A}}\| = \|A^*A\| = \|A\|^2$ . Since every  $C^*$ -algebra possesses approximate identities, this gives the fullness of  $\mathcal{A}_{\mathcal{A}}$ .

**Example 8.1.12.** Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra and that  $p$  is a projection in  $\mathcal{A}$ . Then  $\mathcal{A}p = \{Ap : A \in \mathcal{A}\}$  is a Hilbert  $p\mathcal{A}p$ -module with inner-product  $\langle Ap, Bp \rangle_{p\mathcal{A}p} = pA^*Bp$ . The algebraic properties are easily checked, and the norm coincides with the usual norm on  $\mathcal{A}p$  as a subset of  $\mathcal{A}$ :

$$\|Ap\|_{p\mathcal{A}p}^2 = \|\langle Ap, Ap \rangle_{p\mathcal{A}p}\| = \|pA^*Ap\| = \|Ap\|^2.$$

Since  $\mathcal{A}p$  is a closed linear subspace of  $\mathcal{A}$ , it is complete. The Hilbert module is full because the products of the form  $A^*B$  are dense on  $\mathcal{A}$ .

**Example 8.1.13** (Direct Sum). *Suppose  $V$  and  $W$  are Hilbert  $\mathcal{A}$ -modules. Then the direct sum  $Z = V \otimes W := \{(f, g) : f \in V, g \in W\}$  is obviously a right Hilbert  $\mathcal{A}$ -module with respect to componentwise multiplication and the following  $\mathcal{A}$ -valued inner-product*

$$\langle (f_1, g_1), (f_2, g_2) \rangle_{\mathcal{A}} = \langle f_1, f_2 \rangle_{\mathcal{A}} + \langle g_1, g_2 \rangle_{\mathcal{A}}.$$

*The completeness of  $Z$  follows from the completeness of  $V$  and  $W$  due to the inequalities*

$$\max\{\|f\|_{\mathcal{A}}, \|g\|_{\mathcal{A}}\} \leq \|(f, g)\|_{\mathcal{A}} \leq \sqrt{\|f\|_{\mathcal{A}}^2 + \|g\|_{\mathcal{A}}^2}.$$

*Of particulare importance are the Hilbert modules  $\mathcal{A}^n$  of  $n$  copies and of infinite many copies of  $\mathcal{A}$ . Note that an application of the Cauchy-Schwarz inequality (8.2) gives*

$$\left\| \sum_{i=1}^n a_i^* b_i \right\|^2 \leq \left\| \sum_{i=1}^n a_i^* a_i \right\| \left\| \sum_{i=1}^n b_i^* b_i \right\|.$$

*Like in the Hilbert space case we define an analog of  $\ell^2(\mathbb{Z})$ .*

**Proposition 8.1.14.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra. Then*

$$\mathbf{H}_{\mathcal{A}} = \left\{ \mathbf{A} = (A_i) \in \prod_{i=1}^{\infty} \mathcal{A} : \sum_{i=1}^{\infty} A_i^* A_i \text{ converges in } \mathcal{A} \right\}$$

*is a right Hilbert  $\mathcal{A}$ -module with*

$$\mathbf{A} \cdot A = (A_i A) \quad \text{and} \quad \langle \mathbf{A}, \mathbf{B} \rangle_{\mathcal{A}} = \sum_{i=1}^{\infty} A_i^* B_i.$$

*The proof follows similar lines as for  $\ell^2(\mathbb{Z})$  but is much more involved due to the treatment of  $C^*$ -algebras.*

The next result is one of the main tools in our investigations of Gabor frames because it enables us to construct Hilbert  $C^*$ -modules from pre-inner function spaces.

**Proposition 8.1.15.** *Suppose that  $\mathcal{A}_0$  is a dense  $\star$ -subalgebra of a  $C^*$ -algebra  $\mathcal{A}$ , and that  $V_0$  is a right  $\mathcal{A}_0$ -module. We further suppose that  $V_0$  is a pre-inner product  $\mathcal{A}_0$ -module in the sense that there is a bilinear form  $\langle \cdot, \cdot \rangle_0 : V_0 \times V_0 \rightarrow \mathcal{A}_0$  which satisfies the conditions (1) – (4) in Definition (8.1.1) with the exception that we require that  $\langle f, f \rangle_{\mathcal{A}} \geq 0$  in the completion  $\mathcal{A}$ .*

*Then there is a Hilbert  $\mathcal{A}$ -module  $V$  and a linear map  $q : V_0 \rightarrow V$  such that*

1.  $q(V_0)$  is dense in  $V$ ;
2.  $q(f) \cdot A = q(f \cdot A)$  for all  $f \in V_0$  and  $A \in \mathcal{A}_0$ ;
3.  $\langle q(f), q(g) \rangle_{\mathcal{A}} = \langle f, g \rangle_0$ .

*We call  $V$  the **completion** of  $V_0$ .*

*Proof.* Let  $N = \{f \in V_0 : \langle f, f \rangle_0 = 0\}$ , and let  $q : V_0 \rightarrow V_0/N$  be the quotient map. By an application of Cauchy-Schwarz (8.2) in the present setting we obtain

$$\langle f, g \rangle_0 = 0 = \langle g, f \rangle_0 \text{ whenever } g \in V_0, f \in N.$$

Therefore,  $N$  is a vector subspace of  $V_0$ , which is actually an  $\mathcal{A}_0$ -submodule. Furthermore, the formulas

$$\langle q(f), q(g) \rangle_{\mathcal{A}} := \langle f, g \rangle_0 \text{ and } q(f) \cdot A := q(f \cdot A)$$

give a well-defined pairing and module structure on  $V_0/N$ , and these trivially satisfy conditions (1) – (5) of Definition (8.1.1). Furthermore,  $\|q(f)\| = \|\langle f, f \rangle_{\mathcal{A}}\|^{1/2}$  is a norm on  $V_0/N$ . Therefore, we can form the completion  $V$  and identify  $V_0/N$  with the corresponding subspace of  $V$ . The right multiplication by  $A \in \mathcal{A}_0$  is a bounded operator on  $V_0/N$  follows from

$$\|q(f) \cdot A\|^2 = \|\langle f \cdot A, f \cdot A \rangle_0\| = \|A^* \langle f, f \rangle_0 A\| \leq \|A\|^2 \|q(f)\|^2.$$

Now we can extend the right multiplication operator to an operator on  $V$ . If  $(q(f_n))$  converges to  $f$  and  $q(g_n)$  converges to  $g$ , then

$$\langle f, g \rangle_{\mathcal{A}} := \lim_n \langle q(f_n), q(g_n) \rangle_{\mathcal{A}}.$$

The implication that  $\langle f, f \rangle_{\mathcal{A}} = 0$  implies  $f = 0$  is a little bit tricky. One has to use that the positive cone  $\mathcal{A}^+$  is a closed set. (If  $A_n \rightarrow B$  and  $\lambda \notin \sigma(B)$ , then  $\lambda \in \sigma(A_n)$  for large  $n$ . Thus,  $\sigma(A_n) \subset [0, \infty)$  for all  $n$  implies  $\sigma(B) \subset [0, \infty)$ .) Finally, if  $\langle f, f \rangle_{\mathcal{A}} = 0$ , then there is a sequence  $q(f_n) \rightarrow f$  with  $\|q(f_n)\| \rightarrow 0$ , and  $f$  must be the zero element of  $V$ .  $\square$

## 8.2 Bounded Maps on Hilbert Modules

The greater generality of Hilbert modules causes some serious problems on the properties of operators on them. The most striking difference between Hilbert module maps and operators on a Hilbert space is that adjoints of bounded maps do not automatically exist.

**Definition 8.2.1.** *Suppose  $V$  and  $W$  are Hilbert  $\mathcal{A}$ -modules. A mapping  $T : V \rightarrow W$  is **adjointable** if there is a function  $T^* : W \rightarrow V$  such that*

$$\langle Tf, g \rangle_{\mathcal{A}} = \langle f, T^*g \rangle_{\mathcal{A}} \text{ for all } f, g \in V.$$

*We denote the space of all adjointable mappings from  $V$  to  $W$  by  $\mathcal{L}(V, W)$ . We write  $\mathcal{L}(V)$  for  $\mathcal{L}(V, V)$ .*

$\mathcal{L}(V)$  is a subalgebra of the Banach algebra  $\mathcal{B}(V)$  of bounded operators on  $V$ .

**Lemma 8.2.2.** *Let  $V$  be a Hilbert  $\mathcal{A}$ -module. Then  $f = g$  in  $V$  if and only if  $\langle f, h \rangle_{\mathcal{A}} = \langle g, h \rangle_{\mathcal{A}}$  for all  $h \in V$ .*

The above statement is a consequence of

$$\|h\|_{\mathcal{A}} = \sup\{\|\langle h, f \rangle_{\mathcal{A}}\| : f \in V, \|f\|_{\mathcal{A}} \leq 1\}.$$

**Lemma 8.2.3.** *Every adjointable map  $T : V \rightarrow W$  between Hilbert  $\mathcal{A}$ -modules is a bounded linear  $\mathcal{A}$ -module map from  $V$  to  $W$ .*

*Proof.*

$$\begin{aligned} \langle T(f \cdot A), g \rangle_{\mathcal{A}} &= \langle f \cdot A, T^*g \rangle_{\mathcal{A}} \\ &= A^* \langle f, T^*g \rangle_{\mathcal{A}} \\ &= A^* \langle Tf, g \rangle_{\mathcal{A}} \\ &= \langle T(f) \cdot A, g \rangle_{\mathcal{A}}, \end{aligned}$$

which shows that  $T(f \cdot A) = T(f) \cdot A$ . Therefore,  $T$  and  $T^*$  are  $\mathcal{A}$ -linear.

To see that any adjointable map  $T$  is bounded, we invoke the closed-graph theorem. Suppose  $f_n \rightarrow f$  in  $V$  and  $T(f_n) \rightarrow h$  in  $W$ . Then for every  $g \in W$ , we have  $\langle Tf_n, g \rangle_{\mathcal{A}} \rightarrow \langle h, g \rangle_{\mathcal{A}}$  and  $\langle f_n, T^*g \rangle_{\mathcal{A}} \rightarrow \langle f, T^*g \rangle_{\mathcal{A}} = \langle Tf, g \rangle_{\mathcal{A}} = \langle h, g \rangle_{\mathcal{A}}$  for all  $f \in V, g \in W$  and hence  $Tf = h$ . Thus,  $T$  has a closed graph, and is bounded.  $\square$

**Lemma 8.2.4.** *Let  $T \in \mathcal{L}(V)$ . Then  $T^*$  is unique,  $T^* \in \mathcal{L}(V)$  and  $T^{**} = T$ . Furthermore,  $T \rightarrow T^*$  is an involution on  $\mathcal{L}(V)$ .*

**Proposition 8.2.5.** *Let  $V$  be a Hilbert  $\mathcal{A}$ -module. Then  $\mathcal{L}(V)$  is a  $C^*$ -algebra with respect to the operator norm.*

*Proof.* Since  $\mathcal{B}(V)$  is a Banach algebra,  $\|TT^*\| \leq \|T\|\|T^*\|$ . On the other hand from the Cauchy-Schwarz inequality we have

$$\|T^*T\| \geq \sup_{\|f\|_{\mathcal{A}} \leq 1} \|\langle T^*Tf, f \rangle_{\mathcal{A}}\| = \sup_{\|f\|_{\mathcal{A}} \leq 1} \|\langle Tf, Tf \rangle_{\mathcal{A}}\| = \|T\|_{\mathcal{A}}^2.$$

Therefore, we have  $\|T^*\| \geq \|T\|$ . Furthermore, we have  $T = T^{**}$  which gives  $\|T\| = \|T^*\|$ . Consequently, we have

$$\|T\|^2 \leq \|TT^*\| \leq \|T\|\|T^*\|$$

which yields  $\|T^*T\| = \|T\|^2$ .

The continuity of the involution implies that  $\mathcal{L}(V)$  is closed in  $\mathcal{B}(V)$ , and hence is a  $C^*$ -algebra.  $\square$

The next result is of great importance in the construction of equivalence-bimodules which is the basic object in Morita-equivalence of  $C^*$ -algebras.

**Corollary 8.2.6.** *If  $V$  is a Hilbert  $\mathcal{A}$ -module and  $T \in \mathcal{L}(V)$ , then*

$$\langle Tf, Tf \rangle_{\mathcal{A}} \leq \|T\|^2 \langle f, f \rangle_{\mathcal{A}}$$

*as elements of the  $C^*$ -algebra  $\mathcal{A}$ .*

*Proof.* Since  $\|T\|^2 I - T^*T \geq 0$  in the  $C^*$ -algebra  $\mathcal{L}(V)$ , there exists  $S \in \mathcal{L}(V)$  such that  $\|T\|^2 I - T^*T = S^*S$ . Therefore, we have

$$\|T\|^2 \langle f, f \rangle_{\mathcal{A}} - \langle Tf, Tf \rangle_{\mathcal{A}} = \langle S^*Sf, f \rangle_{\mathcal{A}} = \langle Sf, Sf \rangle_{\mathcal{A}} \geq 0.$$

□

In the following lemma we give a characterization of positive adjointable operators on a Hilbert  $\mathcal{A}$ -module.

**Lemma 8.2.7.** *Let  $V$  be a Hilbert  $\mathcal{A}$ -module and  $T$  a linear operator from  $V$  to  $V$ . Then  $T$  is a positive operator of  $\mathcal{L}(V)$  if and only if  $\langle Tf, f \rangle_{\mathcal{A}} \geq 0$  for all  $f \in V$ .*

*Proof.* One direction is trivial. Let  $T \geq 0$  in  $\mathcal{L}(V)$ . Then  $T = S^*S$  for some  $S \in \mathcal{L}(V)$  and  $\langle Tf, f \rangle_{\mathcal{A}} = \langle Sf, Sf \rangle_{\mathcal{A}} \geq 0$ .

The proof of the reverse implication makes use of the polarization identity

$$\langle Tf, g \rangle_{\mathcal{A}} = \frac{1}{4} \sum_{i=1}^4 i^k \langle f + i^k g, T(f + i^k g) \rangle_{\mathcal{A}}.$$

Suppose  $\langle Tf, f \rangle_{\mathcal{A}} \geq 0$  for all  $f \in V$  then by the polarization identity and  $\langle Tf, g \rangle_{\mathcal{A}} = \langle f, Tg \rangle_{\mathcal{A}}$  for all  $f, g \in V$  implies that  $\langle Tf, g \rangle_{\mathcal{A}} = \langle f, Tg \rangle_{\mathcal{A}}$  for all  $f, g \in V$ . Therefore,  $T$  is adjointable with  $T^* = T$ . By standard results we write  $T = S - R$  with  $S, R \geq 0$  in  $\mathcal{L}(V)$  and  $RS = SR = 0$ . For all  $f \in V$  we have  $0 \leq \langle TRf, Rf \rangle_{\mathcal{A}} = -\langle R^3 f, f \rangle_{\mathcal{A}}$  but  $R^3 \geq 0$  implies that  $\langle R^3 f, f \rangle_{\mathcal{A}} = 0$  for all  $f \in V$ . Consequently,  $R^3 = 0$  by the polarization identity, and  $R = 0$ . This gives  $T = S \geq 0$ . □

In a Hilbert space  $\mathcal{H}$ , i.e. a Hilbert  $\mathbb{C}$ -module, every bounded operator  $T$  possesses an adjoint. In this case  $\mathcal{L}(V)$  is just the  $C^*$ -algebra of bounded operators  $\mathcal{B}(\mathcal{H})$ . But the  $C^*$ -algebra  $\mathcal{K}(\mathcal{H})$  of compact operators is of greater relevance in the theory of  $C^*$ -algebras. We therefore look for an analogue of  $\mathcal{K}(\mathcal{H})$  in  $\mathcal{L}(V)$ . Since  $\mathcal{K}(\mathcal{H})$  is the closed span of rank-one operators Rieffel proposed the following analogue of the **algebra of compact operators** on  $V$ . Let  $V$  and  $W$  be Hilbert  $\mathcal{A}$ -modules. Then for  $f \in V$  and  $g \in W$ , we define a **rank-one operator**  $K_{g,f} : V \rightarrow W$  by

$$K_{g,f}h := g \cdot \langle f, h \rangle_{\mathcal{A}}. \tag{8.5}$$

**Definition 8.2.8.** *Let  $V, W$  be Hilbert  $\mathcal{A}$ -modules. Then the **algebra of compact operators** from  $V$  to  $W$ ,  $\mathcal{K}(V, W)$ , is defined to be the closed linear subspace of  $\mathcal{L}(V, W)$  spanned by  $\{K_{g,f} : f \in V, g \in W\}$ .*

The definition makes sense since every rank-one operator  $K_{g,f}$  is an adjointable operator.

**Lemma 8.2.9.** *Let  $V$  and  $W$  be Hilbert  $\mathcal{A}$ -modules. Then if  $f \in V$  and  $g \in W$   $K_{g,f}$  is adjointable and  $K_{g,f}^* = K_{f,g}$ .*

*Proof.* The argument consists of an elementary calculation.

$$\begin{aligned} \langle K_{g,f}h, l \rangle_{\mathcal{A}} &= \langle g \cdot \langle f, h \rangle_{\mathcal{A}}, l \rangle_{\mathcal{A}} = \langle f, h \rangle_{\mathcal{A}}^* \langle g, l \rangle_{\mathcal{A}} \\ &= \langle h, f \rangle_{\mathcal{A}} \langle g, l \rangle_{\mathcal{A}} = \langle h, f \cdot \langle g, l \rangle_{\mathcal{A}} \rangle_{\mathcal{A}} \\ &= \langle h, K_{f,g}l \rangle_{\mathcal{A}}. \end{aligned}$$

□

The elements of  $\mathcal{K}(V) := \mathcal{K}(V, V)$  are usually not compact operators in the usual sense but they are a closed two-sided ideal in  $\mathcal{L}(V)$ .

**Proposition 8.2.10.** *For any Hilbert  $\mathcal{A}$ -module  $V$ , the algebra of compact operators  $\mathcal{K}(V)$  is a closed two-sided ideal in  $\mathcal{L}(V)$ .*

*Proof.* Let  $T$  be an element of  $\mathcal{L}(V)$  and  $K_{g,f} \in \mathcal{K}(V)$ . Then

$$TK_{g,f}h = T(g \cdot \langle f, h \rangle_{\mathcal{A}}) = Tg \cdot \langle f, h \rangle_{\mathcal{A}} = K_{Tg,f}h$$

gives that  $\mathcal{K}(V)$  is a left ideal in  $\mathcal{L}(V)$ . But  $K_{g,f}^* = K_{f,g}$  implies that  $\mathcal{K}(V)$  is closed under involution and therefore  $\mathcal{K}(V)$  is a two-sided ideal. □

The observation, that if  $V$  is a right  $\mathcal{A}$ -module, then  $V$  is also a full left Hilbert  $\mathcal{K}(V)$ -module, is of interest in the construction of equivalence bimodules.

**Theorem 8.2.11.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra, and suppose that  $V$  is a right Hilbert  $\mathcal{A}$ -module. Then  $V$  is a full Hilbert  $\mathcal{K}(V)$ -module with respect to the left action  $T \cdot f := Tf$  and the inner-product  $\langle f, g \rangle_{\mathcal{A}} := K_{g,f}$ . Moreover, the norms  $\|f\|_{\mathcal{A}} = \|\langle f, f \rangle_{\mathcal{A}}\|^{1/2}$  and  $\|f\|_{\mathcal{K}} = \|\kappa\langle f, f \rangle\|^{1/2}$  coincide.*

*Proof.* We verify that  $\kappa\langle \cdot, \cdot \rangle$  is a left inner-product.

1.  $\kappa\langle \lambda f + \mu g, h \rangle = \lambda \cdot \kappa\langle f, h \rangle + \mu \cdot \kappa\langle g, h \rangle$  is an obvious consequence of the properties of a rank-one operator.
2.  $\kappa\langle Tf, g \rangle = T\kappa\langle f, g \rangle$  follows from  $TK_{f,g} = K_{Tf,g}$ .
3. By definition  $\kappa\langle f, g \rangle = \kappa\langle g, f \rangle^*$  is equivalent to  $K_{f,g} = K_{g,f}^*$ .
4.  $\kappa\langle f, f \rangle \geq 0$  is equivalent to

$$\langle \kappa\langle f, f \rangle \cdot g, g \rangle_{\mathcal{A}} = \langle f, g \rangle_{\mathcal{A}}^* \langle f, g \rangle_{\mathcal{A}} \geq 0.$$

Therefore, the assertion is a consequence of the above stated characterization of positive module maps.

5. The assertion that  $\kappa\langle f, f \rangle = 0$  implies  $f = 0$  is equivalent to

$$\|\langle \kappa\langle f, f \rangle \cdot g, g \rangle_{\mathcal{A}}\| = \|\langle f, g \rangle_{\mathcal{A}}\| = 0$$

for all  $f, g \in V$ . But this implies that  $\langle f, g \rangle_{\mathcal{A}} = 0$  for all  $g \in V$  and therefore  $f = 0$ .

The equivalence of the norms relies on the Cauchy-Schwarz inequality.

$$\langle \kappa \langle f, f \rangle \cdot g, g \rangle_{\mathcal{A}} = \langle f, g \rangle_{\mathcal{A}}^* \langle f, g \rangle_{\mathcal{A}} \leq \| \langle f, f \rangle_{\mathcal{A}} \| \langle g, g \rangle_{\mathcal{A}},$$

and hence that  $\| \kappa \langle f, f \rangle \| \leq \| \langle f, f \rangle_{\mathcal{A}} \|$ . On the other hand, the choice  $g = f$  gives

$$\| \kappa \langle f, f \rangle f, f \rangle_{\mathcal{A}} \| = \| \langle f, f \rangle_{\mathcal{A}} \|^2.$$

This implies that  $\| \kappa \langle f, f \rangle \| \geq \| \langle f, f \rangle_{\mathcal{A}} \|$ .  $\square$

### 8.3 Rieffel-Morita equivalence

In the 1970's Rieffel showed in a series of seminal papers that Mackey's Imprimitivity Theorem is actually a statement about Morita equivalence of  $C^*$ -algebras. Recall that Mackey's Imprimitivity Theorem characterizes the induced unitary representations of locally compact groups.

**Definition 8.3.1.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $C^*$ -algebras. Then  $\mathcal{A}$ - $\mathcal{B}$ -equivalence bimodule is an  $\mathcal{A}$ - $\mathcal{B}$ -bimodule  $V$  such that:*

1.  $V$  is a full left Hilbert  $\mathcal{A}$ -module and a full right Hilbert  $\mathcal{B}$ -module;
2. for all  $f, g \in V$ ,  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ ,

$$\langle A \cdot f, g \rangle_{\mathcal{B}} = \langle f, A^* \cdot g \rangle_{\mathcal{B}} \quad \text{and} \quad {}_{\mathcal{A}} \langle f \cdot B, g \rangle = {}_{\mathcal{A}} \langle f, g \cdot B^* \rangle,$$

3. **Associativity Condition:**  ${}_{\mathcal{A}} \langle f, g \rangle \cdot h = f \cdot \langle g, h \rangle_{\mathcal{B}}$ .

**Remark 8.3.2.** *Condition (2) says that  $\mathcal{A}$  acts by adjointable operators on  $V_{\mathcal{B}}$  and that  $\mathcal{B}$  acts by adjointable operators on  ${}_{\mathcal{A}}V$ . As a consequence the module actions satisfy*

$$(A \cdot f) \cdot B = A \cdot (f \cdot B) \quad \text{and} \quad (\lambda A) \cdot B = A \cdot (f \cdot (\lambda B)).$$

*Thus, it is redundant to require that  $V$  is an  $\mathcal{A}$ - $\mathcal{B}$ -bimodule.*

**Example 8.3.3.** *A Hilbert space  $\mathcal{H}$  is a  $\mathcal{K}(\mathcal{H})$ - $\mathbb{C}$  equivalence bimodule with  $\kappa \langle f, g \rangle_{\mathbb{C}} := f \otimes \bar{g}$ . The axioms are easily checked. Observe that the Associativity Condition is just the definition of the rank-one operator.*

**Example 8.3.4.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra. Then  $\mathcal{A}$  is an  $\mathcal{A}$ - $\mathcal{A}$ -equivalence bimodule for the bimodule structure given by the multiplication in  $\mathcal{A}$ , with the inner-products  ${}_{\mathcal{A}} \langle A, B \rangle = AB^*$  and  $\langle A, B \rangle_{\mathcal{A}} = A^*B$ .*

We already know that a full right Hilbert  $\mathcal{B}$ -module  $V_{\mathcal{B}}$  is a full left Hilbert  ${}_{\kappa}V$ -module. In the next proposition we will complete the knowledge of  $\mathcal{A}$ - $\mathcal{B}$ -equivalence bimodules.

**Proposition 8.3.5.** *Every full Hilbert  $\mathcal{B}$ -module  $V_{\mathcal{B}}$  is a  $\mathcal{K}(V) - \mathcal{B}$ -equivalence bimodule with respect to  $\kappa\langle f, g \rangle := K_{f,g}$ . Conversely, if  $V$  is an  $\mathcal{A} - \mathcal{B}$ -equivalence bimodule, then there is an isomorphism  $\Phi$  of  $\mathcal{A}$  onto  $\mathcal{K}(V)$  such that  $\Phi({}_{\mathcal{A}}\langle f, g \rangle) = \kappa\langle f, g \rangle$  for all  $f, g \in V$ .*

*Proof.* Suppose  $V_{\mathcal{B}}$  is a full right Hilbert  $\mathcal{B}$ -module then  ${}_{\mathcal{K}(V)}V$  is a full left Hilbert  $\mathcal{K}(V)$ -module and  $\mathcal{K}(V)$  acts by adjointable operators on  $V$ . Furthermore, for  $B \in \mathcal{B}$  and  $f, g, h \in V$  we have

$$\kappa\langle f \cdot B, g \rangle h = (f \cdot B) \cdot \langle g, h \rangle_{\mathcal{B}} = f \langle g \cdot B^*, h \rangle_{\mathcal{B}} = \mathcal{K}\langle f, g \cdot B \rangle h.$$

The Associativity Condition is a consequence of the definition of the rank-one operator  $\kappa\langle f, g \rangle = K_{f,g}$ . As a consequence of the preceding observations we get the first assertion.

Now we assume that  $V$  is an  $\mathcal{A} - \mathcal{B}$ -equivalence-bimodule. The map  $\Phi : \mathcal{A} \mapsto \mathcal{L}(V)$  defined by  $\Phi(A)f := A \cdot f$  is a homomorphism between  $C^*$ -algebras, and hence has closed range. By Axiom (3) of Definition 8.3.1 we have that  $\Phi({}_{\mathcal{A}}\langle f, g \rangle) = \mathcal{K}\langle f, g \rangle$  for all  $f, g \in V$ , so  $\Phi(\mathcal{A})$  is all of  $\mathcal{K}(V)$ . For the injectivity of  $\Phi$  we suppose  $\Phi(A) = 0$  for some  $A \neq 0$ . Since  ${}_{\mathcal{A}}\langle V, V \rangle$  spans a dense ideal, we can approximate  $A$  by an element of the form  $A \sum_i {}_{\mathcal{A}}\langle f_i, g_i \rangle$ . But if  $\Phi(A) = 0$ , then  $\sum_i {}_{\mathcal{A}}\langle A \cdot f_i, g_i \rangle = 0$ , i.e.  $\Phi$  is injective.  $\square$

Most equivalence bimodules appear as completions of modules over pre- $C^*$ -algebras as our equivalence bimodule over twisted convolution algebras. Therefore, we have to discuss an alternate version of equivalence bimodules.

**Definition 8.3.6.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $C^*$ -algebras and  $\mathcal{A}_0 \subset \mathcal{A}$  and  $\mathcal{B}_0 \subset \mathcal{B}$  dense  $*$ -subalgebras. An  $\mathcal{A}_0 - \mathcal{B}_0$ -pre-equivalence bimodule is a vector space  $V_0$  which is an  $\mathcal{A}_0 - \mathcal{B}_0$ -bimodule such that*

1.  $V_0$  is a left pre-inner product  $\mathcal{A}_0$ -module and a right pre-inner product  $\mathcal{B}_0$ -module.
2.  ${}_{\mathcal{A}_0}\langle V_0, V_0 \rangle$  and  $\langle V_0, V_0 \rangle_{\mathcal{B}_0}$  span dense ideals of  $\mathcal{A}$  and  $\mathcal{B}$ , respectively;
3. for all  $A \in \mathcal{A}_0, B \in \mathcal{B}_0$ , and  $f \in V_0$ ,

$$\begin{aligned} \langle A \cdot f, A \cdot f \rangle_{\mathcal{B}_0} &\leq \|A\|^2 \langle f, f \rangle_{\mathcal{B}_0} \\ {}_{\mathcal{A}_0}\langle f \cdot B, f \cdot B \rangle &\leq \|B\|_{\mathcal{A}_0}^2 \langle f, f \rangle. \end{aligned}$$

*in the completions of  $\mathcal{B}$  and  $\mathcal{A}$ , respectively;*

4. for all  $f, g, h \in V_0$ ,

$${}_{\mathcal{A}_0}\langle f, g \rangle \cdot h = f \cdot \langle g, h \rangle_{\mathcal{B}_0}.$$

Then,

$$\|f\|_{\mathcal{A}}^2 = \|{}_{\mathcal{A}_0}\langle f, f \rangle\| = \|\langle f, f \rangle_{\mathcal{B}_0}\| = \|f\|_{\mathcal{B}}^2$$

for all  $f \in V$ .

The following results are elementary consequences of the preceding discussion of Hilbert  $C^*$ -module and the definition of equivalence bimodules.

**Proposition 8.3.7.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $C^*$ -algebras and suppose that  $V$  is an  $\mathcal{A} - \mathcal{B}$ -bimodule satisfying (1) and (3) from the above definition. Then  $V$  is an  $\mathcal{A} - \mathcal{B}$ -bimodule if and only if  $V$  satisfies for all  $A \in \mathcal{A}, B \in \mathcal{B}$ , and  $f \in V$ ,*

$$\begin{aligned}\langle A \cdot f, A \cdot f \rangle_{\mathcal{B}} &\leq \|A\|^2 \langle f, f \rangle_{\mathcal{B}} \\ {}_{\mathcal{A}}\langle f \cdot B, f \cdot B \rangle &\leq \|B\|_{\mathcal{A}}^2 \langle f, f \rangle.\end{aligned}$$

*Proof.* If  $V$  satisfies (1) of Definition 8.3.1 then  $\mathcal{A}$  acts as adjointable operators on the Hilbert module  $V_{\mathcal{B}}$ , and by an application of Lemma 8.2.7.

By (1) and (3) of Definition 8.3.1 we have

$$\begin{aligned}\langle {}_{\mathcal{A}}\langle f, g \rangle \cdot h, k \rangle_{\mathcal{B}} &= \langle f \cdot \langle g, h \rangle_{\mathcal{B}}, k \rangle_{\mathcal{B}} = \langle k, g \rangle_{\mathcal{B}} \langle f, k \rangle_{\mathcal{B}} \\ &= \langle h, g \cdot \langle f, k \rangle_{\mathcal{B}} \rangle_{\mathcal{B}} = \langle h, {}_{\mathcal{A}}\langle f, g \rangle^* \cdot k \rangle_{\mathcal{B}},\end{aligned}$$

in other words

$$\langle A \cdot h, k \rangle_{\mathcal{B}} = \langle h, A^* \cdot k \rangle_{\mathcal{B}} \quad \text{for all } h, k \in \mathcal{B} \quad (8.6)$$

and all  $A$  in the dense ideal  $\langle V, V \rangle_{\mathcal{A}}$ . By assumption  $f \mapsto A \cdot f$  has norm at most  $\|A\|$ , thus by Cauchy-Schwarz inequality that  $\|\langle A \cdot h, k \rangle_{\mathcal{B}}\| \leq \|A\| \|h\|_{\mathcal{B}} \|k\|_{\mathcal{B}}$ . Thus (8.3) extends by continuity to all  $A \in \mathcal{A}$ . An analogous computation for the  $\mathcal{B}$ -action completes the proof.  $\square$

The preceding Proposition 8.3.7 says that an  $\mathcal{A} - \mathcal{B}$ -equivalence bimodule is an  $\mathcal{A} - \mathcal{B}$ -pre-equivalence bimodule. Furthermore, we have two semi-norms on  $V_0$  defined by  $\|f\|_{\mathcal{A}} := \|{}_{\mathcal{A}}\langle f, f \rangle\|^{1/2}$  and  $\|f\|_{\mathcal{B}} := \|\langle f, f \rangle_{\mathcal{B}_0}\|^{1/2}$  which allow a completion of  $V_0$  in a canonical way. The next proposition establishes that these two semi-norms agree.

**Proposition 8.3.8.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $C^*$ -algebras and  $\mathcal{A}_0 \subset \mathcal{A}$  and  $\mathcal{B}_0 \subset \mathcal{B}$  dense  $\star$ -algebras. If  $V_0$  is an  $\mathcal{A}_0 - \mathcal{B}_0$ -equivalence bimodule, then*

$$\|f\|_{\mathcal{A}} = \|{}_{\mathcal{A}_0}\langle f, f \rangle\|^{1/2} = \|\langle f, f \rangle_{\mathcal{B}_0}\|^{1/2} = \|f\|_{\mathcal{B}} \quad \text{for all } f \in V.$$

*Proof.* By Cauchy-Schwarz inequality we have

$$\begin{aligned}\|\langle f, f \rangle_{\mathcal{B}_0}\|^2 &= \|\langle f, f \rangle_{\mathcal{B}_0} \langle f, f \rangle_{\mathcal{B}_0}\| = \|\langle f, f \cdot \langle f, f \rangle_{\mathcal{B}_0} \rangle_{\mathcal{B}_0}\| \\ &= \|\langle f, \langle f, f \rangle_{\mathcal{A}_0} \cdot f \rangle_{\mathcal{B}_0}\| \\ &\leq \|\langle f, f \rangle_{\mathcal{B}_0}\|^{1/2} \|\langle \langle f, f \rangle_{\mathcal{A}_0} \cdot f, \langle f, f \rangle_{\mathcal{A}_0} \cdot f \rangle_{\mathcal{B}_0}\|^{1/2}.\end{aligned}$$

Therefore,

$$\|\langle f, f \rangle_{\mathcal{B}_0}\|^2 \leq \|\langle f, f \rangle_{\mathcal{B}_0}\|_{\mathcal{A}_0}^{1/2} \|\langle f, f \rangle_{\mathcal{B}_0}\|^{1/2},$$

i.e.

$$\|\langle f, f \rangle_{\mathcal{B}_0}\| \leq \|{}_{\mathcal{A}_0}\langle f, f \rangle\|.$$

The same argument for  $\mathcal{A}_0$  and  $\mathcal{B}_0$  yields the opposite inequality. This gives the desired equivalence of the semi-norms.  $\square$

By a combination of the preceding statements we get the following result.

**Proposition 8.3.9.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $C^*$ -algebras and  $\mathcal{A}_0 \subset \mathcal{A}$  and  $\mathcal{B}_0 \subset \mathcal{B}$  dense  $*$ -subalgebras. If  $V_0$  is an  $\mathcal{A}_0 - \mathcal{B}_0$ -equivalence bimodule, then there is an  $\mathcal{A} - \mathcal{B}$ -equivalence bimodule  $V$  and an  $\mathcal{A}_0 - \mathcal{B}_0$ -equivalence bimodule homomorphism  $q : V_0 \rightarrow V$  such that  $q(V_0)$  is dense in  $V$  and*

$$\langle q(f), q(g) \rangle_{\mathcal{B}} = \langle f, g \rangle_{\mathcal{B}_0} \quad \text{and} \quad {}_{\mathcal{A}}\langle f, g \rangle = {}_{\mathcal{A}_0}\langle f, g \rangle$$

for all  $f, g \in V_0, A \in \mathcal{A}_0$  and  $B \in \mathcal{B}_0$ .

We have reached the point to introduce the most important notion of this exposition: *Morita equivalence* of  $C^*$ -algebras. Since Rieffel's seminal work [Rief74a], who called it *strong Morita equivalence*, the concept of Morita equivalence has found many applications in mathematics and physics. In a subsequent chapter we show that the notion of Morita equivalence appears very naturally in the construction of Gabor frames.

**Definition 8.3.10.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $C^*$ -algebras. Then  $\mathcal{A}$  and  $\mathcal{B}$  are called **Morita equivalent** if there is an  $\mathcal{A} - \mathcal{B}$ -equivalence bimodule  $V$ .*

One says that  $V$  implements the Morita equivalence of  $\mathcal{A}$  and  $\mathcal{B}$ . Rieffel was able to establish that Morita equivalence is actually an equivalence relation on  $C^*$ -algebras. Since we will not need this result we omit the very difficult and technically involved proof.

**Theorem 8.3.11.** *Morita equivalence is an equivalence relation on  $C^*$ -algebras.*

Morita equivalence between  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  is weaker than the notion of isomorphism.

**Proposition 8.3.12.** *Let  $\Phi$  be an isomorphism of  $\mathcal{A}$  onto  $\mathcal{B}$ . Then  $\mathcal{A}$  and  $\mathcal{B}$  are Morita equivalent.*

*Proof.* The desired equivalence bimodule is  ${}_{\mathcal{A}}V_{\mathcal{B}}$  with underlying space  $\mathcal{B}$  by

$$f \cdot B := fB, \quad A \cdot f := \Phi(A)f$$

and

$$\langle f, g \rangle_{\mathcal{B}} := f^*g, \quad {}_{\mathcal{A}}\langle f, g \rangle := \Phi^{-1}(fg^*).$$

□

We proceed to give another characterization of Morita-equivalence of  $C^*$ -algebras. This is based on the notion of the *multiplier algebra* of a  $C^*$ -algebra  $\mathcal{A}$ .

**Definition 8.3.13.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra.*

1. An ideal  $I$  in  $\mathcal{A}$  is **essential** if  $I$  has non-zero intersection with every other non-zero ideal in  $\mathcal{A}$ .
2. A **unitization** of a  $\mathcal{A}$  is a  $C^*$ -algebra  $\mathcal{B}$  with identity and an injective homomorphism  $\iota : \mathcal{A} \rightarrow \mathcal{B}$  such that  $\iota(\mathcal{A})$  is an essential ideal of  $\mathcal{B}$ .

The multiplier algebra of compact operators  $\mathcal{K}(\mathcal{H})$  is of great relevance since it allows to discuss the automorphisms of  $\mathcal{K}(\mathcal{H})$ . All automorphisms of  $\mathcal{K}(\mathcal{H})$  are implemented by inner automorphisms of unitary operators  $U$  on  $\mathcal{H}$  which are not elements of  $\mathcal{K}(\mathcal{H})$ .

**Definition 8.3.14.** A unitization  $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$  is called **maximal** if for every embedding  $j$  of  $\mathcal{A}$  as an essential ideal of a  $C^*$ -algebra  $\mathcal{C}$ . There is a homomorphism  $\varphi : \mathcal{C} \rightarrow \mathcal{B}$  such that  $\iota = j \circ \varphi$ .

After one has shown that up to isomorphism there is only one maximal unitization, one calls  $M(\mathcal{A})$  the **multiplier algebra** of  $\mathcal{A}$ . We now present an alternative definition in terms of Hilbert  $C^*$ -modules.

Let  $L : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A}_{\mathcal{A}})$  be the embedding of  $\mathcal{A}$  by left multiplication  $L_A(B) = AB$ . Then  $(\mathcal{L}(\mathcal{A}_{\mathcal{A}}), L)$  is a unitization of  $\mathcal{A}$ . Since the range of  $L$  is the ideal  $\mathcal{K}(\mathcal{A}_{\mathcal{A}})$  of compact operators. Then  $\mathcal{K}(\mathcal{A})$  is an essential ideal of  $\mathcal{L}(\mathcal{A})$ . But if  $T \in \mathcal{L}(\mathcal{A})$  satisfies  $TK = 0$  for all  $K \in \mathcal{K}(\mathcal{A})$ , then  $0 = TK_{B,C} = K_{TB,C}$  for all  $B, C \in \mathcal{A}$ , which implies  $TB = 0$  for all  $B \in \mathcal{A}$ , and  $T = 0$ .

**Definition 8.3.15.** The  $C^*$ -algebra  $\mathcal{L}(\mathcal{A}_{\mathcal{A}})$  will be called the **multiplier algebra** of  $\mathcal{A}$  and we denote it by  $M(\mathcal{A})$ .

That  $M(\mathcal{A})$  is well-defined follows from the next theorem.

**Theorem 8.3.16.** For any  $C^*$ -algebra  $\mathcal{A}$  the unitization  $L : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A}_{\mathcal{A}})$  is maximal. Furthermore, it is unique: If  $j : \mathcal{A} \hookrightarrow \mathcal{B}$  is another maximal unitization, then there is an isomorphism  $\varphi$  of  $\mathcal{B}$  onto  $\mathcal{L}(\mathcal{A}_{\mathcal{A}})$  such that  $\varphi \circ j = L$ .

We need the notion of the corner of a  $C^*$ -algebra before we are able to state the other characterization of Morita equivalence of  $C^*$ -algebras.

A  $C^*$ -subalgebra  $\mathcal{B}$  of a  $C^*$ -algebra  $\mathcal{A}$  is called a **corner** if there is a projection  $p \in M(\mathcal{A})$  such that  $\mathcal{B} = p\mathcal{A}p$ . Two corners  $p\mathcal{A}p$  and  $q\mathcal{A}q$  are called **complementary** if  $p + q$  is the identity  $1_{M(\mathcal{A})}$  of  $M(\mathcal{A})$ . A corner is called **full** if it is not contained in any proper closed two-sided ideal, i.e. the span $\{p\mathcal{A}p\}$  is dense in  $\mathcal{A}$ .

**Proposition 8.3.17.** Any full corner of  $\mathcal{A}$  is Morita equivalent to  $\mathcal{A}$  via  $\mathcal{A}p$ .

*Proof.* Let  $p$  be a projection in  $M(\mathcal{A})$ . Then  $\mathcal{A}p = \{Ap : A \in \mathcal{A}\}$  is a Hilbert  $p\mathcal{A}p$ -module with inner-product  $\langle pA, pB \rangle_{p\mathcal{A}p} = pA^*Bp$ . The Hilbert module is full since products of the form  $A^*B$  are dense in  $\mathcal{A}$ .  $\square$

Therefore, it follows from the transitivity of the Morita equivalence relation that complementary full corners are Morita equivalent.

**Theorem 8.3.18.** *Two  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  are Morita equivalent if and only if there is a  $C^*$ -algebra  $\mathcal{C}$  with complementary full corners isomorphic to  $\mathcal{A}$  and  $\mathcal{B}$ , respectively.*

This result is of great importance in the theory of operator algebras. In our treatment of the Stone-von Neumann theorem we will use the above notions to discuss Rieffel's approach to this fundamental result.

## 8.4 Frames in Hilbert $C^*$ -modules

**Definition 8.4.1.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and  $J$  a at most countable index set. A sequence  $\{g_j : j \in J\}$  of elements in a Hilbert module  $V$  is said to be a **frame** if there are constants  $C, D > 0$  such that*

$$C\langle f, f \rangle_{\mathcal{A}} \leq \sum_{j \in J} \langle f, g_j \rangle_{\mathcal{A}} \langle g_j, f \rangle_{\mathcal{A}} \leq D\langle f, f \rangle_{\mathcal{A}}$$

for every  $f \in V$ . If the sum in the middle of the above inequality converges in norm, we call  $\{g_j : j \in J\}$  a **standard frame**.

As in the classical case the optimal constants are called **frame bounds**. Furthermore, the frame  $\{g_j : j \in J\}$  is said to be a **tight frame** if  $C$  equals  $D$  and it is called **normalized tight** if  $C = D = 1$ .

As in the Hilbert space case a set  $\{g_j : j \in J\}$  is a normalized tight frame if and only if

$$\langle f, f \rangle_{\mathcal{A}} = \sum_{j \in J} \langle f, g_j \rangle_{\mathcal{A}} \langle g_j, f \rangle_{\mathcal{A}}$$

holds for every  $f \in V$ .

**Theorem 8.4.2.** *Every algebraically finitely generated Hilbert  $\mathcal{A}$ -module  $V$  over a unital  $C^*$ -algebra is projective and any algebraically generating set  $\{g_j : j = 1, \dots, n\}$  of  $V$  is a frame.*

The positive finite rank  $\mathcal{A}$ -module operator  $S$  is strictly positive. Since the set of all  $\mathcal{A}$ -compact module operators on finitely generated Hilbert  $C^*$ -modules is a unital  $C^*$ -algebra  $S$  has to be invertible and therefore the upper and lower frame bounds as  $\|S\|^2$  and  $\|S^{-1}\|^{-2}$ .

One of Rieffel's main contribution to Morita equivalence between  $C^*$ -algebras is the existence of *module frames* for finitely generated Hilbert  $C^*$ -module  $V$  over a unital  $C^*$ -algebra  $\mathcal{A}$ .

Frank and Larson have shown that if  $\{g_j : j = 1, \dots, n\}$  is a standard module frame for  $V$ , then the reconstruction formula

$$v = \sum_{j=1}^n g_j \langle g_j, v \rangle_{\mathcal{A}} \quad (8.7)$$

holds for all  $v \in V$ , where the sum converges in norm.

**Theorem 8.4.3.** *Let  $V_0$  be a submodule of some Hilbert  $C^*$ -module  $V$  over some unital  $C^*$ -algebra  $\mathcal{A}$  which has a standard module frame  $\{g_j\}$  of finite cardinality. Then  $V_0$  is an  $\mathcal{A}$ -module direct summand of  $V$ . A projection operator,  $P$ , from  $V$  onto  $V_0$  is given by*

$$Pv = \sum_{j=1}^n g_j \langle g_j, v \rangle_{\mathcal{A}} \quad (8.8)$$

The previous result implies the following theorem.

**Theorem 8.4.4.** *Let  $V$  be a Hilbert  $C^*$ -module over some unital  $C^*$ -algebra  $\mathcal{A}$  which has a standard module frame  $\{g_j\}$  of finite cardinality. Then  $V$  can be embedded into some free module  $\mathcal{A}^n$ -module with preservation of the  $\mathcal{A}$ -valued inner product. Furthermore,  $V$  is a projective  $\mathcal{A}$ -module.*

More precisely:

**Definition 8.4.5.** *Let  $V$  be a finitely generated Hilbert  $C^*$ -module  $V$  over a unital  $C^*$ -algebra  $\mathcal{A}$ . By a standard module frame for  $V$  we mean a finite collection  $g_1, \dots, g_n$  of elements of  $V$  for which the reconstruction formula*

$$v = \sum_{j=1}^n g_j \langle g_j, v \rangle_{\mathcal{A}} \quad (8.9)$$

holds for all  $v \in V$ .

By definition a *projective module*  $V$  is isomorphic to a direct summand of a free module  $\mathcal{A}^n$  with standard basis  $\{e_j\}$ , i.e. there is a self-adjoint  $n \times n$ -matrix  $P$  with entries in  $\mathcal{A}$  which is a projection, such that  $V = P\mathcal{A}^n$ . Rieffel proved that if  $\mathcal{A}$  and  $\mathcal{B}$  are unital  $C^*$ -algebras and if  $V$  is a  $(\mathcal{B}, \mathcal{A})$ -equivalence bimodule, then  $V$  is a projective right  $\mathcal{B}$ -module, and a projective left  $\mathcal{A}$ -module. Furthermore,  $\mathcal{A}$  is equivalent to the  $C^*$ -algebra  $\mathcal{K}(V, \mathcal{B})$  of compact Hilbert  $\mathcal{B}$ -module operators.

We refer to the very technical proofs of the general result on frames in Hilbert  $C^*$ -modules to their seminal paper [FL02]. We only stated the results we need in our discussion of Hilbert  $C^*$ -module frames for  $C^*(\Lambda, c)$  in Chapter 11.

# 9 Hilbert $C^*$ -modules in Gabor Analysis

## 9.1 Motivation

It is well-known that the  $C^*$ -algebra of continuous functions on the  $2d$ -torus is the universal  $C^*$ -algebra generated by  $2d$  commuting unitaries, which can be considered as the coordinate functions. This suggests considering the universal  $C^*$ -algebra  $\mathcal{A}_\Theta$  generated by  $2d$  unitaries  $U_1, \dots, U_{2d}$  with the simple commutation relation

$$U_j U_k = e^{2\pi i \theta_{jk}} U_k U_j, \quad k, j = 1, \dots, 2d,$$

where the  $\theta_{jk}$  form a skew-symmetric matrix  $\Theta = (\theta_{jk})$  with real entries. This algebra was called the **non-commutative torus**.

We regard  $\Theta$  as a real skew-bilinear form on  $\mathbb{Z}^{2d}$ , with entries given by  $\Theta(e_j, e_k) = \theta_{jk}$ . Then a noncommutative  $2d$ -torus  $\mathcal{A}_\Theta$  is the twisted group  $C^*$ -algebra  $C^*(\mathbb{Z}^{2d}, c)$ , where  $c : \mathbb{Z}^{2d} \times \mathbb{Z}^{2d} \rightarrow \mathbb{T}$  is a 2-cocycle. We remark that non-commutative 2-tori  $A_\theta$  are often called *rotation algebras*. The commutation rules for the unitary operators  $U$  and  $V$  generating  $A_\theta$  read as

$$UV = e^{2\pi i \theta} VU,$$

for a real number  $\theta$ . Let  $\theta = \alpha\beta$  then the  $C^*$ -algebra generated by time-frequency shifts  $\pi(\alpha k, \beta l) = M_{\beta l} T_{\alpha k}$  for  $k, l \in \mathbb{Z}^d$  is a representation of  $\mathcal{A}_\theta$  on  $L^2(\mathbb{R}^d)$ .

In this paper we want to treat the general case of  $2d$ -noncommutative tori or equivalently  $C^*$ -algebras  $C^*(\Lambda, c)$  of time-frequency shifts  $\pi(\lambda)$  for a lattice  $\Lambda$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  with

$$\pi(\lambda)\pi(\lambda') = c(\lambda, \lambda')\pi(\lambda + \lambda'), \quad \lambda, \lambda' \in \Lambda.$$

Therefore, an element of  $C^*(\Lambda, c)$  is given by

$$\sum_{\lambda \in \Lambda} a(\lambda)\pi(\lambda)$$

for a bounded complex-valued sequence  $\mathbf{a} = (a(\lambda))_{\lambda \in \Lambda}$ .

Moreover, this representation is **faithful** on  $L^2(\mathbb{R}^d)$ , which is of *great importance in our proofs*. One consequence, is that it suffices to establish statements for a dense subspace of  $L^2(\mathbb{R}^d)$ . For an operator algebraic proof see [Rief88] and in [GL04] a proof is given using time-frequency methods.

The choice of a sequence space on  $\Lambda$  induces on the noncommutative torus an additional structure. The space  $\mathcal{S}(\Lambda)$  of sequences on  $\Lambda$  with decay faster than the inverse of any polynomial yields a *smooth* structure on the algebra of functions on  $C^*(\Lambda, c)$ , i.e.

$$\mathcal{A}^\infty(\Lambda) = \{A \in \mathcal{B}(L^2(\mathbb{R}^d)) : A = \sum_{\lambda} a(\lambda)\pi(\lambda), \mathbf{a} = (a(\lambda))_{\lambda \in \Lambda} \in \mathcal{S}(\Lambda)\}.$$

In the present paper we introduce another structure on  $C^*(\Lambda, c)$ . Namely,

$$\mathcal{A}^1(\Lambda) = \{A \in \mathcal{B}(L^2(\mathbb{R}^d)) : A = \sum_{\lambda} a(\lambda)\pi(\lambda), \mathbf{a} = (a(\lambda))_{\lambda \in \Lambda} \in \ell^1(\Lambda)\}. \quad (9.1)$$

We consider the involutive Banach algebra  $\mathcal{A}^1(\Lambda)$  as **non-commutative Wiener algebra** since for trivial cocycle  $c$  the set  $\mathcal{A}^1(\Lambda)$  is the well-known Wiener algebra of *absolutely summable Fourier series*.

We have already explored such objects in the construction of the twisted group  $C^*$ -algebra in Chapter 4. In the following section we give a detailed treatment of the twisted convolution algebra  $\ell^1(\Lambda)$ .

## 9.2 Twisted Convolution Algebras

The work of Janssen, Gröchenig and Leinert on twisted convolution algebras was inspired by certain problems in Gabor analysis. Our main source of motivation is to establish the Morita equivalence of certain twisted convolution algebras and their group  $C^*$ -algebras. In our work on this problem we realized that group  $C^*$ -algebras appear implicitly in the proofs of the main results by Janssen, Gröchenig and Leinert. In this section we want to discuss the basic properties of twisted group  $C^*$ -algebras for discrete groups.

Recall the definition of a twisted group  $C^*$ -algebra from Chapter 4. In the following we discuss the twisted group  $C^*$ -algebra of a lattice  $\Lambda$  in the time-frequency plane and a cocycle  $c$  on the discrete group  $\Lambda$ . Then  $\ell^1(\Lambda)$  with **twisted convolution** of  $\mathbf{a}, \mathbf{b}$

$$(\mathbf{a} \natural_c \mathbf{b})(\lambda) = \sum_{\mu \in \Lambda} a(\mu)b(\lambda - \mu)c(\mu, \lambda - \mu)$$

as multiplication and with **twisted involution**

$$a(\lambda)^* = \overline{c(\lambda, -\lambda)a(-\lambda)}$$

as involution a **twisted convolution algebra**. Janssen's observation that

$$\sum_{\lambda} a(\lambda)\pi(\lambda) \sum_{\mu} b(\mu)\pi(\mu) = \sum_{\mu} (\mathbf{a} \natural_c \mathbf{b})(\mu)\pi(\mu)$$

means that the projective representation  $\{\pi(\lambda) : \lambda \in \Lambda\}$  of  $\Lambda$  induces an involutive representation

$$\tilde{\pi}(\mathbf{a}) = \sum_{\lambda \in \Lambda} a(\lambda)\pi(\lambda)$$

on the twisted group algebra  $(\ell^1(\Lambda), \natural_c, \star)$ . On  $(\ell^1(\Lambda), \natural_c, \star)$  we have two kinds of translation (1)*right  $c$ -translation* and (2)*left  $c$ -translation*.

Let  $\delta_\lambda(\mu)$  be the Kronecker delta for  $\lambda, \mu \in \Lambda$ . Then the twisted convolution of two Kronecker deltas is given by

$$\delta_\lambda \natural_c \delta_\mu = c(\lambda, \mu) \delta_{\lambda+\mu},$$

which is a deformation of the traditional convolution rule  $\delta_\lambda * \delta_\mu = \delta_{\lambda+\mu}$ . Let  $\mathbf{a}$  be in  $\ell^2(\Lambda)$ . Then we introduce the **right  $c$ -translation**  $R_\lambda^{(c)}$  of  $\mathbf{a}$  by

$$R_\lambda^{(c)} \mathbf{a}(\mu) = \tilde{\pi}(\delta_\lambda)(\mathbf{a})(\mu) = c(\lambda, \mu) \mathbf{a}(\lambda + \mu)$$

which fullfills

$$\begin{aligned} R_\lambda^{(c)} R_\mu^{(c)} &= \tilde{\pi}(\delta_\lambda) \tilde{\pi}(\delta_\mu) = \tilde{\pi}(\delta_\lambda \natural_c \delta_\mu) \\ &= c(\lambda, \mu) R_{\lambda+\mu}^{(c)}. \end{aligned}$$

In a similar way we define **left  $c$ -translation**  $L^{(c)}$  for  $\mathbf{a} \in \ell^2(\Lambda)$  by

$$L_\lambda^{(c)} \mathbf{a}(\mu) = c(\lambda, \lambda - \mu) \mathbf{a}(\mu - \lambda).$$

In the commutative case  $c \equiv 1$  we recapture the well-known right and left translations on  $\Lambda$  which allow to express the convolution of  $\mathbf{a}$  and  $\mathbf{b}$  as

$$\mathbf{a} * \mathbf{b}(\mu) = \sum_{\lambda \in \Lambda} a(\lambda) L_\lambda b(\mu).$$

In the non-commutative setting the twisted convolution of  $\mathbf{a}$  and  $\mathbf{b}$  is a sum of left  $c$ -translations,

$$(\mathbf{a} \natural_c \mathbf{b})(\mu) = \sum_{\lambda \in \Lambda} a_\lambda L_\lambda^{(c)} \mathbf{a}(\mu).$$

We call the representations on  $\ell^2(\Lambda)$  generated by the *left/right  $c$ -translations* the **left/right  $c$ -regular** representations of  $\Lambda$ . By an application of the cocycle identity

$$c(\lambda, \mu) c(\lambda + \mu, \nu) = c(\lambda, \mu + \nu) c(\mu, \nu)$$

we get that the left  $c$ -regular representation commutes with the right  $\bar{c}$ -regular representation. Also the left  $\bar{c}$ -regular representation commutes with the right  $c$ -regular representation, where  $\bar{c}$  denotes the conjugate cocycle.

The group  $C^*$ -algebra  $\mathbf{C}^*(\Lambda, c)$  is generated by the twisted right translations. If  $\mathbf{a}$  is in  $\ell^1(\Lambda)$ , then  $\mathbf{a} = \sum_{\lambda \in \Lambda} a(\lambda) \delta_\lambda$  and  $\tilde{\pi}(\mathbf{a}) = \sum_{\lambda \in \Lambda} a(\lambda) R_\lambda^{(c)}$ . The system  $(\delta_\lambda)_{\lambda \in \Lambda}$  is an orthonormal basis of  $\ell^2(\Lambda)$  and we have

$$\begin{aligned} \|\tilde{\pi}(\mathbf{a})\|_{\mathcal{B}(\ell^2(\Lambda))} &\geq \|\tilde{\pi}(\mathbf{a}) \delta_0\|_{\ell^2(\Lambda)} \\ &= \left\| \sum_{\lambda \in \Lambda} a(\lambda) R_\lambda^{(c)} \delta_0 \right\|_{\ell^2(\Lambda)} \\ &= \left\| \sum_{\lambda \in \Lambda} a(\lambda) c(-\lambda, \lambda) \delta_{-\lambda} \right\|_{\ell^2(\Lambda)} = \|\mathbf{a}\|_{\ell^2(\Lambda)}. \end{aligned}$$

The preceding observations proof the next lemma.

**Lemma 9.2.1.** *Let  $C^*(\Lambda, c)$  be the twisted group  $C^*$ -algebra of  $\Lambda$ . Then we have the embeddings of Banach spaces*

$$\ell^1(\Lambda) \subset C^*(\Lambda, c) \subset \ell^2(\Lambda).$$

The twisted convolution algebras  $\ell^1(\Lambda, \natural_c, \star)$  possess a natural trace  $\tau_c$ .

**Lemma 9.2.2.** *Let  $\mathbf{a}$  be in  $\ell^1(\Lambda, \natural_c, \star)$ . Then  $\mathbf{a} \mapsto \tau_c(\mathbf{a}) = a(0)$  defines a trace on  $\ell^1(\Lambda, \natural_c, \star)$ , which has a continuous extension to  $C^*(\Lambda, c)$ .*

*Proof.* The linearity and additivity of  $\tau_c$  is trivial. The computation

$$\begin{aligned} (\mathbf{a} \natural_c \mathbf{b})(0) &= \sum_{\lambda \in \Lambda} a(\lambda) b(-\lambda) c(\lambda, -\lambda) \\ &= \sum_{\lambda \in \Lambda} b(\lambda) a(-\lambda) c(\lambda, -\lambda) \\ &= (\mathbf{b} \natural_c \mathbf{a})(0) \end{aligned}$$

gives the tracial property of  $\tau_c$ .

The continuity of  $\tau_c$  is a consequence of  $\tilde{\pi}(\mathbf{a})(\delta_0)(\lambda) = \overline{c(\lambda, -\lambda)a(-\lambda)}$  and the computation

$$\begin{aligned} a(0) &= \langle \delta_0, \mathbf{a} \rangle_{\ell^2(\Lambda)} \\ &= \langle \delta_0, \tilde{\pi}(\mathbf{a})(\delta(0)) \rangle_{\ell^2(\Lambda)} \\ &\leq \|\tilde{\pi}(\mathbf{a})\|_{\mathcal{B}(\ell^2(\Lambda))}. \end{aligned}$$

Furthermore  $\tau_c$  is faithful on  $\ell^1(\Lambda) \subset \ell^2(\Lambda)$  since

$$\begin{aligned} \tau_c(\tilde{\pi}(\mathbf{a})^* \tilde{\pi}(\mathbf{a})) &= \mathbf{a}^* \natural_c \mathbf{a}(0) \\ &= \sum_{\lambda \in \Lambda} \overline{c(\lambda, -\lambda)a(-\lambda)} a(-\lambda) c(\lambda, -\lambda) \\ &= \sum_{\lambda \in \Lambda} \overline{a(\lambda)} a(\lambda) = \|\mathbf{a}\|_{\ell^2(\Lambda)}. \end{aligned}$$

By Lemma 9.2.1 the trace  $\tau_c$  is faithful on  $C^*(\Lambda, c)$ . □

If  $c \equiv 1$ , then  $\tau(\mathbf{a}) = a(0) = \int_{\hat{\Lambda}} \hat{f}(\omega) d\omega$ .

The GNS-representation of  $\ell^1(\Lambda, \natural_c, \star)$  for the trace  $\tau_c$  is the content of the next lemma.

**Lemma 9.2.3.** *The GNS-representation of  $\ell^1(\Lambda, \natural_c, \star)$  for the trace  $\tau_c$  is the right  $\bar{c}$  representation on  $\ell^2(\Lambda)$ .*

*Proof.* For  $\mathbf{a} \in \ell^1(\Lambda)$  we have that  $\tau_c(\mathbf{a}^* \mathbf{a}) = \|\mathbf{a}\|_{\ell^2(\Lambda)}$ . By construction  $\ell^1(\Lambda) \subset C^*(\Lambda, c)$  and by Lemma 9.2.1  $C^*(\Lambda, c) \subset \ell^2(\Lambda)$ . Since  $\ell^1(\Lambda) \subset \ell^2(\Lambda)$  is dense, the Hilbert space for the GNS-construction is  $\ell^2(\Lambda)$ .

By definition the GNS-representation  $\pi_c$  is faithful since  $\tau_c$  is faithful. Let  $\mathbf{a} \in \ell^1(\Lambda)$ . Then

$$\begin{aligned} (\delta_\lambda \natural_c \mathbf{a})(\mu) &= c(\lambda, \mu - \lambda) a(\mu - \lambda) \\ &= c(\lambda, -\lambda) \bar{c}(-\lambda, \mu) a(\mu - \lambda) \\ &= c(\lambda, -\lambda) L_\lambda^{(\bar{c})} a(\mu) \end{aligned}$$

and in analogy

$$(\mathbf{a} \natural_c \delta_\lambda)(\mu) = c(-\lambda, \lambda) R_\lambda^{(\bar{c})} a(\mu).$$

Therefore,  $C^*(\Lambda, c)$  is isomorphic to the  $C^*$ -algebra generated by the twisted left translations for the conjugate cocycle  $\bar{c}$  in  $\mathcal{B}(\ell^2(\Lambda))$ .  $\square$

### 9.3 $M^1(\mathbb{R}^d)$ as Equivalence-Bimodule

For the construction of a Hilbert  $C^*(\Lambda, c)$ -module  $V$  we are looking for a time-frequency homogenous Banach space, with properties similar to  $\mathcal{S}(\mathbb{R}^d)$ . In his seminal paper [Fei81] Feichtinger has introduced such a space, nowadays called **Feichtinger's algebra** which we denote by  $M^1(\mathbb{R}^d)$ .

In Chapter 6 we have extensively studied the properties of  $M^1(\mathbb{R}^d)$  which we need for the application of Rieffel's construction of projective modules for non-commutative tori. Let  $f, g$  in  $M^1(\mathbb{R}^d)$ .

1. If  $(x, \omega) \in \mathbb{R}^{2d}$ , then  $\pi(x, \omega)f \in M^1(\mathbb{R}^d)$ .
2. Then  $V_g f \in M^1(\mathbb{R}^{2d})$ .
3. Let  $\Lambda$  be a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ . Then  $\sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle| < \infty$ .
4. If  $\mathbf{a} = (a(\lambda))_{\lambda \in \Lambda}$  is in  $\ell^1(\Lambda)$ , then the function  $\mathbf{a}g = \sum_{\lambda \in \Lambda} a(\lambda)\pi(\lambda)g$  is also an element of  $M^1(\mathbb{R}^d)$ .

In Section 6.3 we discussed local properties of the STFT. As a consequence we get the continuity of the Gabor frame type operator  $S_{g, \gamma, \Lambda}$  for  $g, \gamma \in M^1(\mathbb{R}^d)$  which is the composition of the analysis operator and the synthesis operator. Recall, that a Gabor frame type operator  $S_{g, \gamma, \Lambda}$  is the composition of the analysis mapping  $C_\gamma$ , which is sampling of  $V_\gamma f$  on the lattice  $\Lambda$ , and the synthesis operator  $D_g$ , which is given by  $\sum_{\lambda \in \Lambda} a(\lambda)\pi(\lambda)g$  for a sequence  $(a(\lambda))_{\lambda \in \Lambda}$ .

**Proposition 9.3.1.** *Let  $\Lambda$  be a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ .*

1. *Then for  $\gamma \in M^1(\mathbb{R}^d)$  the analysis operator  $T_\gamma$  is a bounded mapping from  $M^1(\mathbb{R}^d)$  to  $\ell^1(\Lambda)$ .*
2. *Then for  $g \in M^1(\mathbb{R}^d)$  the synthesis operator  $D_g$  is a bounded mapping from  $\ell^1(\Lambda)$  to  $M^1(\mathbb{R}^d)$ .*

As a straightforward consequence of the preceding lemma we get the boundedness  $S_{g,\gamma,\Lambda}$  for  $g, \gamma \in M^1(\mathbb{R}^d)$ .

**Corollary 9.3.2.** *Let  $\Lambda$  be a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  and  $g, \gamma \in M^1(\mathbb{R}^d)$ . Then the Gabor frame type operator  $S_{g,\gamma,\Lambda}$  is a bounded operator on  $M^1(\mathbb{R}^d)$  with absolute convergence of the series. Furthermore, we have*

$$\|S_{g,\gamma,\Lambda}\|_{op} \leq C \|g\|_{M^1} \|\gamma\|_{M^1}.$$

In other words the mapping  $(g, \gamma) \mapsto S_{g,\gamma,\Lambda}$  is continuous for  $g, \gamma \in M^1(\mathbb{R}^d)$ . The reader is directed to browse through [FZ98] for an extensive study of continuity results for Gabor frame type operators in the setting of modulation spaces.

Rieffel's original idea was to define a left action of  $C^*(\Lambda, c)$  on  $M^1(\mathbb{R}^d)$  by a *Gabor expansion* for the window  $g$  and the lattice  $\Lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ :

$$\mathbf{a}g = \sum_{\lambda \in \Lambda} a(\lambda) \pi(\lambda)g, \quad \mathbf{a} = (a(\lambda)) \in \ell^1(\Lambda).$$

In the last equation we used that for discrete groups  $M^1(\Lambda)$  is just  $\ell^1(\Lambda)$ . The invariance of  $M^1(\mathbb{R}^d)$  under time-frequency shifts implies that this action is well-defined on  $M^1(\mathbb{R}^d)$ .

**Proposition 9.3.3.** *Let  $\mathbf{a} \in \ell^1(\Lambda, c)$  and  $g \in M^1(\mathbb{R}^d)$ , then*

$$\left\| \sum_{\lambda \in \Lambda} a(\lambda) \pi(\lambda)g \right\|_{M^1} \leq \|\mathbf{a}\|_1 \|g\|_{MO}.$$

*Proof.*

$$\begin{aligned} \left\| \sum_{\lambda \in \Lambda} a(\lambda) \pi(\lambda)g \right\|_{M^1} &\leq \sum_{\lambda \in \Lambda} |a(\lambda)| \|\pi(\lambda)g\|_{M^1} \\ &= \sum_{\lambda \in \Lambda} |a(\lambda)| \|g\|_{MO} \\ &= \|\mathbf{a}\|_1 \|g\|_{M^1}. \end{aligned}$$

□

Therefore for  $f, g, h \in M^1(\mathbb{R}^d)$  the left action given by  $\mathbf{a}(\lambda) = (V_g f(\lambda))_{\lambda \in \Lambda}$  on  $h$  by

$$\langle f, g \rangle_{\Lambda} h := \sum_{\lambda \in \Lambda} V_g f(\lambda) \pi(\lambda)h$$

is well-defined on  $M^1(\mathbb{R}^d)$ . In Gabor analysis the operator defined by  $(f, g, h) \mapsto \langle f, g \rangle_{\Lambda} h$  is called a **Gabor frame type operator** for a lattice  $\Lambda$  in  $\mathbb{R}^{2d}$

$$S_{h,g,\Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)h.$$

The above discussion shows that the Gabor type frame operator  $S_{g,\Lambda}$  is continuous on  $M^1(\mathbb{R}^d)$  for  $f, g \in M^1(\mathbb{R}^d)$ .

Following Rieffel we define a  $C^*(\Lambda, c)$ -valued innerproduct for  $C^*(\Lambda, c)$  by the Janssen representation of a Gabor frame type operator

$$\langle f, g \rangle_{\mathcal{A}} := \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda) \quad (9.2)$$

for  $f, g$  in Feichtinger's algebra  $M^1(\mathbb{R}^d)$ . In the sequel we denote by  $\mathcal{A}$  the group  $C^*$ -algebra of  $\ell^1(\Lambda, c)$ .

First we show that (9.2) is compatible with the action of  $\ell^1(\Lambda, c)$  on  $M^1(\mathbb{R}^d)$ . More precisely, we prove the following proposition.

**Proposition 9.3.4.** *Let  $f, g \in M^1(\mathbb{R}^d)$  and let  $\mathbf{a} \in \ell^1(\Lambda, c)$ . Then*

$$\langle \mathbf{a}f, g \rangle_{\Lambda} = \mathbf{a} \natural_c \langle f, g \rangle_{\Lambda}.$$

*Proof.*

For  $\lambda \in \Lambda$  we have

$$\begin{aligned} \langle \mathbf{a}f, \pi(\lambda)g \rangle &= \sum_{\lambda' \in \Lambda} a(\lambda') \langle \pi(\lambda')f, \pi(\lambda)g \rangle \\ &= \sum_{\lambda' \in \Lambda} a(\lambda') \langle f, \pi^*(\lambda')\pi(\lambda)g \rangle \\ &= \sum_{\lambda' \in \Lambda} a(\lambda') \langle f, \pi(\lambda - \lambda')g \rangle c(\lambda', \lambda - \lambda') \\ &= \mathbf{a} \natural_{\Lambda} \langle f, \pi(\lambda)g \rangle, \end{aligned}$$

since  $\pi(\lambda')\pi(\lambda - \lambda') = c(\lambda', \lambda - \lambda')\pi(\lambda)$ . □

Now we reached the point to state one of our main theorems.

**Theorem 9.3.5.** *Feichtinger's algebra  $M^1(\mathbb{R}^d)$  is a left inner-product  $C^*(\Lambda, c)$ -module with respect to the inner product, given for  $f, g$  in  $M^1(\mathbb{R}^d)$  by*

$$\langle f, g \rangle_{\mathcal{A}} = \sum_{\lambda \in \Lambda} V_g f(\lambda) \pi(\lambda).$$

The proof of Theorem 9.3.5 relies on the **Fundamental Identity of Gabor Analysis**, which we discussed in Section 7.3. We postpone the proof after the discussion of the group  $C^*$ -algebra of the adjoint lattice  $\Lambda^\circ$ . Instead we apply the results of Chapter 8 to our setting.

The adjointable maps of the  $C^*(\Lambda, c)$ -module  $M^1(\mathbb{R}^d)$  are those bounded operators  $T : M^1(\mathbb{R}^d) \rightarrow M^1(\mathbb{R}^d)$  where  $T^*$  commutes with all time-frequency shifts  $\{\pi(\lambda) : \lambda \in \Lambda\}$ . By definition of the  $C^*(\Lambda, c)$ -innerproduct we have

$$\begin{aligned} \langle Tf, g \rangle_{\mathcal{A}} &= \sum_{\lambda \in \Lambda} \langle Tf, \pi(\lambda)g \rangle \pi(\lambda) \\ &= \sum_{\lambda \in \Lambda} \langle f, T^* \pi(\lambda)g \rangle \pi(\lambda) \\ &= \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)T^*g \rangle \pi(\lambda) \\ &= \langle f, T^*g \rangle_{\mathcal{A}}. \end{aligned}$$

In [FK98] Feichtinger and Kozek treated self-adjoint operators on  $M^1(\mathbb{R}^d)$  that commute with all time-frequency shifts of  $\{\pi(\lambda) : \lambda \in \Lambda\}$ . They called those operators  $\Lambda$ -invariant.

**Proposition 9.3.6.** *The set of all selfadjoint adjointable operators of the  $C^*(\Lambda, c)$ -module  $M^1(\mathbb{R}^d)$  are the bounded  $\Lambda$ -invariant operators of  $M^1(\mathbb{R}^d)$ .*

Now, we investigate the set of rank-one module operators for our Hilbert  $C^*(\Lambda, c)$ -module  $M^1(\mathbb{R}^d)$ . By definition a rank-one module operator is given by

$$\begin{aligned} K_{g,f\gamma} &= \langle f, \gamma \rangle_{\mathcal{A}} g \\ &= \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g = S_{g,\gamma,\Lambda}f, \end{aligned}$$

for  $f, g, \gamma \in M^1(\mathbb{R}^d)$ . The operator  $S_{g,\gamma,\Lambda}$  is called a Gabor frame operator with analysis window  $\gamma$  and synthesis window  $g$  for a lattice  $\Lambda$ .

Consequently, a finite-rank module operator for the  $C^*(\Lambda, c)$ -module  $M^1(\mathbb{R}^d)$  is a finite sum of Gabor frame operators, a so-called **multi-window** Gabor frame operator. Furthermore, a rank-one module operator  $S_{g,\gamma,\Lambda}$  is an adjointable operator, i.e. it is a  $\Lambda$ -invariant operator.

**Proposition 9.3.7.** *The algebra of compact operators on the Hilbert  $C^*(\Lambda, c)$ -module  $M^1(\mathbb{R}^d)$  is generated by multi-window Gabor frame operators  $\sum_{j=1}^n S_{g_j,\gamma_j}$  for  $g_j, \gamma_j \in M^1(\mathbb{R}^d)$ .*

In Section 7.3 we defined the **adjoint lattice** of  $\Lambda$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  as the set of all points  $X = (x, \omega)$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  such that  $c(\lambda, X)c(\lambda, X) = 1$ , which by the commutation relation of time-frequency shifts (7.6) is equivalent to

$$\Lambda^0 = \{\lambda^0 \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d : \pi(\lambda)\pi(\lambda^0) = \pi(\lambda^0)\pi(\lambda) \text{ for all } \lambda \in \Lambda\}.$$

Therefore, the set of all bounded operators on  $L^2(\mathbb{R}^d)$  commuting with elements from  $C^*(\Lambda, c)$  is the  $C^*$ -algebra generated by time-frequency shifts  $\pi(\lambda^0)$  for  $\lambda^0$  in  $\Lambda^0$ .

**Proposition 9.3.8.** *The commutant of  $C^*(\Lambda, c)$  is the group  $C^*$ -algebra of  $\ell^1(\Lambda, c)$ .*

Therefore  $C^*(\Lambda^0, c)$  is a good candidate for inducing on  $M^1(\mathbb{R}^d)$  another  $C^*$ -module structure. Consequently,  $C^*(\Lambda^0, c)$  has to act from the right on  $M^1(\mathbb{R}^d)$ , where the right action is induced by the opposite algebra of  $C^*(\Lambda^0, c)$ . Following Rieffel in [Rief88],  $C^*(\Lambda^0, c)$  can be generated by  $\pi^*(\lambda^0)$  acting on the left on  $M^1(\mathbb{R}^d)$ , which commutes with the right action of  $C^*(\Lambda, c)$  on  $M^1(\mathbb{R}^d)$ . Therefore, the opposite algebra of  $C^*(\Lambda, c)^{\text{opp}}$  is generated by  $\pi^*(\lambda^0)$  with  $\bar{c}(X, Y) = \overline{c(X, Y)}$  for  $X = (x, \omega)$  and  $Y = (y, \eta)$  as cocycle, i.e.  $C^*(\Lambda^0, \bar{c})$ . Note, that cohomologous cocycles yield isomorphic  $C^*$ -algebras, see Chapter 4.

As for the twisted convolution algebra  $\ell^1(\Lambda, c, \star)$  we equip  $\ell^1(\Lambda^0)$  with the structure of an involutive algebra given by twisted convolution and involution with respect to the cocycle  $\bar{c}$ :

1. Let  $\mathbf{b}_1, \mathbf{b}_2 \in \ell^1(\Lambda^\circ, \bar{c})$ . Then twisted convolution of  $\mathbf{b}_1, \mathbf{b}_2$  is defined by

$$\mathbf{b}_1 \sharp_{\bar{c}} \mathbf{b}_2(\lambda^\circ) = \sum_{\mu^\circ \in \Lambda^\circ} b_1(\mu^\circ) b_2(\lambda^\circ) \bar{c}(\mu^\circ, \lambda^\circ - \mu^\circ).$$

2. Let  $\mathbf{b} \in \ell^1(\Lambda^\circ, \bar{c})$ . Then twisted involution of  $\mathbf{b}$  is defined by

$$b^*(\lambda^\circ) = c(\lambda^\circ, -\lambda^\circ) b(\lambda^\circ).$$

By definition the opposite algebra of  $C^*(\Lambda, c)$  gives a right action on  $M^1(\mathbb{R}^d)$  by a Gabor expansion with respect to the lattice  $\Lambda^0$  by

$$g\mathbf{b} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} b(\lambda^0) \pi^*(\lambda^0) g, \quad g \in M^1(\mathbb{R}^d), \mathbf{b} \in S_0(\Lambda^0, \bar{c}).$$

By a reasoning similar to the one used for the left action of  $C^*(\Lambda, c)$  on  $M^1(\mathbb{R}^d)$  yields that the right action is well-defined on  $M^1(\mathbb{R}^d)$ .

In a manner analogous to discussion of Theorem 9.3.5 we get introduce a right action of defines a right inner-product  $C^*(\Lambda^0, \bar{c})$ -module structure on  $M^1(\mathbb{R}^d)$  with respect to the  $\mathcal{B} := C^*(\Lambda^0, \bar{c})$ -inner-product

$$\langle f, g \rangle_{\mathcal{B}} := |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} \langle \pi(\lambda^0) g, f \rangle \pi(\lambda^0), \quad f, g \in M^1(\mathbb{R}^d).$$

Consequently, we have

**Theorem 9.3.9.** *Feichtinger's algebra  $M^1(\mathbb{R}^d)$  is a right inner-product  $C^*(\Lambda^0, \bar{c})$ -module with respect to the inner-product, given by  $f, g$  in  $M^1(\mathbb{R}^d)$  by*

$$\langle f, g \rangle_{\mathcal{A}} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_g f(\lambda^0) \pi(\lambda^0).$$

The proof follows the same lines as for Theorem 9.3.5 but the proof of the positivity of the  $C^*(\Lambda, c)$  is still missing. We use the Fundamental Identity of Gabor Analysis to close this gap. In Section 7.3 we presented an extensive study of FIGA. In the proof of the positivity we use the following form of the FIGA.

**Theorem 9.3.10** (Basic FIGA). *Assume that  $f_1, f_2, g_1, g_2 \in M^1(\mathbb{R}^d)$ . Then*

$$\sum_{\lambda \in \Lambda} V_{g_1} f_1(\lambda) \cdot \overline{V_{g_2} f_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_{g_1} g_2(\lambda^0) \cdot \overline{V_{f_1} f_2(\lambda^0)} \quad (9.3)$$

Following Rieffel we give the following proof for the positivity of  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ .

**Proposition 9.3.11.** *Let  $f \in M^1(\mathbb{R}^d)$  then  $\langle f, f \rangle_{\mathcal{A}}$  is a positive element of  $C^*(\Lambda, c)$ .*

*Proof.* The representation of time-frequency shifts of  $C^*(\Lambda, c)$  is faithful, therefore, it suffices to establish positivity for a dense subspace of  $L^2(\mathbb{R}^d)$ . Of course we choose  $M^1(\mathbb{R}^d)$  as dense subspace. Let  $g \in M^1(\mathbb{R}^d)$

$$\begin{aligned} \langle \langle f, f \rangle_{\Lambda} g, g \rangle &= \left\langle \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) f \rangle \pi(\lambda) g, g \right\rangle \\ &= \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) f \rangle \overline{\langle g, \pi(\lambda) g \rangle} \\ &= \sum_{\lambda^0 \in \Lambda^0} \langle f, \pi(\lambda^0) g \rangle \overline{\langle f, \pi(\lambda^0) g \rangle} \geq 0. \end{aligned}$$

□

In an analogous manner  $\langle f, f \rangle_{\mathcal{B}}$  is a positive element of  $C^*(\Lambda^{\circ}, \bar{c})$ .

The left and right  $C^*$ -module structure are compatible in the sense of Rieffel, see Chapter 8.

In our setting Rieffel's associativity condition is just the Janssen representation of a Gabor frame type operator  $S_{g, \gamma}$  for  $g, \gamma \in M^1(\mathbb{R}^d)$ .

**Theorem 9.3.12.** *Let  $\mathcal{A} = C^*(\Lambda, c)$  and  $\mathcal{B} = C^*(\Lambda^{\circ}, \bar{c})$  with the above defined actions and inner-products  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$  and  $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ , respectively. Then*

$$S_{g, \gamma, \Lambda} f = |\Lambda|^{-1} S_{f, \gamma, \Lambda^0} g$$

for all  $f, g, \gamma \in M^1(\mathbb{R}^d)$ .

*Proof.* As in the proof of positivity of  $\langle f, f \rangle_{\mathcal{A}}$  for  $f \in M^1(\mathbb{R}^d)$  it suffices to show that for all  $\gamma, h \in M^1(\mathbb{R}^d)$

$$\langle S_{g, \gamma, \Lambda} f, h \rangle = |\Lambda|^{-1} \langle S_{f, \gamma, \Lambda^0} g, h \rangle.$$

$$\begin{aligned} \langle \langle f, g \rangle_{\mathcal{A}} \gamma, h \rangle &= \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \overline{\langle h, \pi(\lambda) \gamma \rangle} \\ &\stackrel{FIGA}{=} |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} \langle f, \pi(\lambda^0) h \rangle \overline{\langle g, \pi(\lambda^0) \gamma \rangle} \\ &= \langle f \langle g, \gamma \rangle_{\mathcal{B}}, h \rangle. \end{aligned}$$

□

Recall that a Hilbert  $C^*$ -module  $V$  over  $\mathcal{A}$  is called *full* when the collection  $\{\langle f, g \rangle_{\mathcal{A}} : f, g \in V\}$  is dense in  $\mathcal{A}$ . We state our main result on the Morita equivalence of  $C^*(\Lambda, c)$  and  $C^*(\Lambda^{\circ}, \bar{c})$ .

**Theorem 9.3.13.** *Let  $M^1(\mathbb{R}^d)$  be given a bimodule structure as defined above. Let  $\mathcal{A} = C^*(\Lambda, c)$  and  $\mathcal{B} = C^*(\Lambda^{\circ}, \bar{c})$  then*

1.  $\{\langle f, g \rangle_{\mathcal{A}} : f, g \in M^1(\mathbb{R}^d)\}$  is dense in  $\mathcal{A}$ , i.e.  $M^1(\mathbb{R}^d)$  is a full Hilbert  $\mathcal{A}$ -module.
2.  $\{\langle f, g \rangle_{\mathcal{B}} : f, g \in M^1(\mathbb{R}^d)\}$  is dense in  $\mathcal{B}$ , i.e.  $M^1(\mathbb{R}^d)$  is a full Hilbert  $\mathcal{B}$ -module.
3. For all  $f \in M^1(\mathbb{R}^d)$  and  $A \in \mathcal{A}$ , we have

$$\langle fA, fA \rangle_{\mathcal{A}} \leq \|A\|^2 \langle f, f \rangle_{\mathcal{A}},$$

i.e. boundedness of the right action.

4. For all  $f \in M^1(\mathbb{R}^d)$  and  $B \in \mathcal{B}$ , we have

$$\langle Bf, Bf \rangle_{\mathcal{B}} \leq \|B\|^2 \langle f, f \rangle_{\mathcal{B}},$$

i.e. boundedness of the left action.

implies that  $M^1(\mathbb{R}^d)$  is an equivalence bimodule  $(\mathcal{A}, \mathcal{B})$  with norm  $\|f\| := \langle f, f \rangle_{\mathcal{A}}^{1/2}$ .

*Proof.* Our proof follows Rieffel's approach, see [Rief88].

1. The linear span of the range of  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$  is an ideal in  $\mathcal{A}$ . Then the norm closure  $I$  of this linear span is an ideal in  $\mathcal{A}$ . Furthermore  $I$  is invariant under modulation and because  $\pi(\lambda)$  is a faithful representation of  $\mathcal{A}$ , we get the desired conclusion.
2. By similar arguments as in (1).
3. It suffices to verify the inequality for a dense subspace of  $L^2(\mathbb{R}^d)$ . Let  $h \in M^1(\mathbb{R}^d)$  and  $A \in C^*(\Lambda, c)$ , then

$$\begin{aligned} \langle h \langle Af, Af \rangle_{\mathcal{A}}, h \rangle &= \langle \langle h, Af \rangle_{\mathcal{B}} Af, h \rangle \\ &= \langle Af, \langle Af, h \rangle_{\mathcal{B}} h \rangle \\ &= \langle Af, Af \langle h, h \rangle_{\mathcal{A}} \rangle \\ &= \langle A(f \langle h, h \rangle_{\mathcal{A}})^{1/2}, A(f \langle h, h \rangle_{\mathcal{A}})^{1/2} \rangle \\ &\leq \|A\|^2 \langle f, f \rangle_{\mathcal{A}} \\ &= \|A\|^2 \langle h \langle f, f \rangle_{\mathcal{A}}, h \rangle \end{aligned}$$

holds for all  $f$  in  $M^1(\mathbb{R}^d)$ . A standard density argument yields the desired result.

4. By similar arguments as in (3).

□

**Corollary 9.3.14.**  $C^*(\Lambda, c)$  and  $C^*(\Lambda^0, \bar{c})$  are Morita equivalent.

The consequences of Theorem 9.3.13 offer a new approach to Gabor frames which we present in the subsequent chapters.

We close this section with an interpretation of the FIGA in terms of traces on  $C^*(\Lambda, c)$  and  $C^*(\Lambda^\circ, \bar{c})$ . In Section 9.2 we defined on  $C^*(\Lambda, c)$  a trace  $\tau_{\mathcal{A}}$  by

$$\tau_{\mathcal{A}}(\mathbf{a}) = a(0) \quad \text{for } \mathbf{a} \in \ell^1(\Lambda).$$

In an analogous manner  $C^*(\Lambda^\circ, \bar{c})$  may be equipped with a trace  $\tau_{\mathcal{B}}$ :

$$\tau_{\mathcal{B}}(\mathbf{b}) = |\Lambda|^{-1}b(0) \quad \text{for } \mathbf{b} \in \ell^1(\Lambda^\circ).$$

Therefore, for  $f, g \in M^1(\mathbb{R}^d)$ ,

$$\tau_{\mathcal{A}}(\langle f, g \rangle_{\mathcal{A}}) = \langle g, f \rangle_{\mathcal{A}}(0) = \langle g, f \rangle$$

and

$$\tau_{\mathcal{B}}(\langle f, g \rangle_{\mathcal{B}}) = |\Lambda|^{-1}\langle g, f \rangle_{\mathcal{A}}(0) = |\Lambda|^{-1}\langle g, f \rangle.$$

Therefore,

**Theorem 9.3.15.** *Let  $f, g \in M^1(\mathbb{R}^d)$ . Then*

$$\tau_{\mathcal{B}}(\langle f, g \rangle_{\mathcal{B}}) = |\Lambda|^{-1}\tau_{\mathcal{A}}(\langle f, g \rangle_{\mathcal{A}}). \quad (9.4)$$

**Corollary 9.3.16.** *Let  $f_1, f_2, g_1, g_2 \in M^1(\mathbb{R}^d)$ . Then*

$$\tau_{\mathcal{B}}(\langle f_1, f_2 \rangle_{\mathcal{B}} \langle g_1, g_2 \rangle_{\mathcal{B}}) = |\Lambda|^{-1}\tau_{\mathcal{A}}(\langle g_2, f_1 \rangle_{\mathcal{A}} \langle f_2, g_1 \rangle_{\mathcal{A}}). \quad (9.5)$$

Recalling the definition of  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$  and  $\langle \cdot, \cdot \rangle_{\mathcal{B}}$  yields the FIGA:

$$\sum_{\lambda \in \Lambda} V_{g_1} f_1(\lambda) \cdot \overline{V_{g_2} f_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_{g_1} g_2(\lambda^0) \cdot \overline{V_{f_1} f_2(\lambda^0)}.$$

The preceding innocent looking relations between the traces  $\tau_{\mathcal{A}}$  and  $\tau_{\mathcal{B}}$  on  $C^*(\Lambda, c)$  and  $C^*(\Lambda^\circ, \bar{c})$  is implicit in the works of [Jan95, DLL95, ?]. Therefore the Morita equivalence of  $C^*(\Lambda, c)$  and  $C^*(\Lambda^\circ, \bar{c})$  is the very reason for the Wexler-Raz duality, the relation between the frame bounds of the frame operator seen as element of  $\ell^1(\Lambda, c)$  or  $\ell^1(\Lambda^\circ, \bar{c})$ .

Theorem 9.3.15 is a particular case of a general fact, again due to Rieffel [Rief74b].

**Theorem 9.3.17.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two unital  $C^*$ -algebras with traces  $\tau_{\mathcal{A}}$  and  $\tau_{\mathcal{B}}$  and  $V$  an equivalence-bimodule. Then there exists a positive scalar  $C$  such that*

$$\tau_{\mathcal{B}}(\langle f, g \rangle_{\mathcal{B}}) = C\tau_{\mathcal{A}}(\langle f, g \rangle_{\mathcal{A}}), \quad \text{for all } f, g \in V.$$

The rich structure of  $C^*(\Lambda, c)$  and  $C^*(\Lambda^\circ, \bar{c})$  allowed Rieffel to determine  $C$  in this specific setting. We have already met another example of relations between traces of Morita equivalent  $C^*$ -algebras: **Moyal's Identity**. How does this fit into our framework. Observe that in (FIGA) we can replace the lattice by a closed subgroup

of  $\mathbb{R}^{2d}$ . Then take this subgroup to be  $\mathbb{R}^{2d}$ , which implies  $\mathbb{R}^{2d^\circ} = \{0\}$ . Consequently for this pairing the FIGA is just Moyal's identity.

Recall that the square-integrability and irreducibility of  $(\pi(x, \omega) : (x, \omega) \in \mathbb{R}^{2d})$  are the group theoretic reasons for the existence of Moyal's formula and the existence of the inversion formula for the STFT. Now  $\{\pi(\lambda) : \lambda \in \Lambda\}$  and  $\{\pi(\lambda) : \lambda \in \Lambda^\circ\}$  are merely **reducible** representations of  $\Lambda$  and  $\Lambda^\circ$ , respectively. This causes the failure of a reconstruction formula as in the continuous case. Therefore we have to suppose the existence the boundedness from above and below of

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B\|f\|_2^2 \quad (9.6)$$

to get inversion formulas ala STFT. Furthermore the failure of the irreducibility has to be fixed by the assumption that all terms other than 0-th on the right-hand side in FIGA vanish. But that is the so-called **Wexler-Raz condition**. As a summary, if one restricts an irreducible group representation of  $G$  to a subgroup  $D$  the situation changes dramatically.

## Chapter 9

# 10 Involutive Banach and Frechet Algebras

## 10.1 Wiener Pairs of Banach Algebras and Frechet Algebras

The real insight of the results in [GL04] is not the resolution of an outstanding open problem in Gabor analysis but their use of inverse-closed Banach algebras. In Gabor analysis this lead to a series of unexpected results.

In this section we want to discuss the analogous notion of inverse-closed Fréchet algebras which emerged from Connes's work on non-commutative geometry, especially on non-commutative tori. The main motivation was to attract the attention of workers in Gabor analysis to a variety of generalizations of inverse-closed Banach algebras.

In the early eighties of the last century he drew the attention of researchers in operator algebras to *smooth* subalgebras of  $C^*$ -algebras. The adjective "smooth" recalls the well-known fact that for a locally compact Hausdorff space  $M$  the space of smooth functions  $C^\infty(M)$  is a dense subalgebra of the  $C^*$ -algebra  $C(M)$  of continuous functions on  $M$ . In the last two decades "smooth subalgebras" of  $C^*$ -algebras have been studied by many researchers, e.g. Badea, Bost, Cuntz, Ji and Schweitzer, [Sch92, Sch93, Sch94].

In the following the notion of topological algebras will be our framework. Recall that a topological algebra is a complex topological vector space endowed with an algebra structure where the multiplication is jointly continuous. In the sequel we only treat unital topological algebras.

Let  $\mathcal{A}$  be a topological algebra. Then we denote the set of inverse elements of  $\mathcal{A}$  by  $\text{Inv}(\mathcal{A})$ .

**Definition 10.1.1.** *Let  $\mathcal{A}$  be a topological algebra.*

1. *Then  $\mathcal{A}$  is called **Q-algebra** if  $\text{Inv}(\mathcal{A})$  is open.*
2. *Then  $\mathcal{A}$  is called **good** if it is a Q-algebra and the mapping  $A \mapsto A^{-1}$  is continuous on  $\text{Inv}(\mathcal{A})$ .*
3. *If  $\mathcal{A}$  is a Banach space and  $\|AB\| \leq \|A\|\|B\|$  then  $\mathcal{A}$  is a **Banach algebra**.*

4. If  $\mathcal{A}$  is a Fréchet space then  $\mathcal{A}$  is called a **Fréchet algebra**.
5. If  $\mathcal{A}$  is a Fréchet space then  $\mathcal{A}$  is **m-convex** if there exists a family of submultiplicative seminorms on  $\mathcal{A}$  which give the topology on  $\mathcal{A}$ .

A well-known property of Banach algebra is that  $\mathcal{GL}(\mathcal{A})$  is open but in general a Fréchet algebra is not necessarily a Q-algebra.

We continue our discussion with the notion of inverse-closedness for involutive Banach algebras.

**Definition 10.1.2.** Let  $\mathcal{A}$  and  $\mathcal{B}$  involutive Banach algebras with a common unit  $I$ . Then  $\mathcal{A}$  is called **inverse-closed** if  $A \in \mathcal{A}$  and  $A^{-1} \in \mathcal{B}$  then  $A^{-1} \in \mathcal{A}$ .

**Remark 10.1.3.** Many authors use instead of inverse-closedness of  $\mathcal{A}$  in  $\mathcal{B}$  the name **spectral invariance** or that  $(\mathcal{A}, \mathcal{B})$  form a **Wiener pair**.

A well-known example of a Wiener pair is the case where  $\mathcal{A}$  is a unital  $C^*$ -subalgebra of a  $C^*$ -algebra  $\mathcal{B}$ . This observations is another manifestation of the differences between a general involutive Banach algebra and a  $C^*$ -algebra.

**Proposition 10.1.4.** If  $\mathcal{A}$  is a unital  $C^*$ -algebra of a  $C^*$ -algebra  $\mathcal{B}$  and  $A \in \mathcal{A}$ , then

$$\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A).$$

*Proof.* Since  $\sigma_{\mathcal{B}}(A) \subset \sigma_{\mathcal{A}}(A)$ , it suffices to show that  $A \in \mathcal{A}$  is invertible in  $\mathcal{B}$ , then the inverse lies in  $\mathcal{A}$ .

If  $A \in \mathcal{A}$  is selfadjoint, then  $\mathcal{C} = C^*(\{A, A^{-1}\})$  is a subalgebra of  $\mathcal{B}$  isomorphic to some  $C(X)$  for a compact Hausdorff space  $X$ . Let  $\hat{A}$  denote the image of  $A$  under this isomorphism. Then  $\hat{A} \neq 0$  on  $X$ . So  $0 \notin \sigma_{\mathcal{C}}(A)$  which is a subset of the real line. Choose polynomials  $p_n$  such that  $p_n(x)$  converges uniformly to  $x^{-1}$  on  $\sigma_{\mathcal{C}}(A)$ . Then since  $\widehat{A^{-1}} = \lim_{n \rightarrow \infty} p_n(\hat{A})$ , we see that  $A^{-1} = \lim_{n \rightarrow \infty} p_n(A)$  belongs to  $C^*(\mathcal{A})$ , which is contained in  $\mathcal{A}$ .

For general  $A \in \mathcal{A}$ , if  $A^{-1}$  belongs to  $\mathcal{B}$ , then  $(A^*A)^{-1} = A^{-1}A^{-1*}$  lies in  $\mathcal{B}$  and therefore also in  $\mathcal{A}$ . Thus  $(A^*A)^{-1}A^* = A^{-1}$  belongs to  $\mathcal{A}$ .  $\square$

As another example of the great importance of spectral invariance we give a characterisation of positivity in a  $C^*$ -algebra  $\mathcal{A}$ .

**Proposition 10.1.5.** Let  $\mathcal{A}$  be a  $C^*$ -algebra. Then  $A \in \mathcal{A}$  is positive if  $\psi(A) \geq 0$  for every state  $\psi$  of  $\mathcal{A}$ .

*Proof.* We assume that  $\hat{\psi}(A) \geq 0$  for all states  $\psi$  of  $\mathcal{A}$ , and we choose a faithful representation  $\rho$  of  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$ . Since  $A \mapsto \langle \psi(A)f, f \rangle$  is a state for all  $f \in \mathcal{H}$ . In other words  $\pi(A)$  is a positive operator on  $\mathcal{H}$ . Since the positive operators are precisely the positive elements of  $\mathcal{B}(\mathcal{H})$ , we have that  $\psi(A) \geq 0$  in  $\mathcal{B}(\mathcal{H})$ . But  $\psi(A) \geq 0$  in  $\mathcal{B}(\mathcal{H})$  exactly when  $\sigma_{\mathcal{B}(\mathcal{H})}(\psi(A)) \subset [0, \infty)$ . By spectral invariance, this is equivalent to  $\sigma_{\mathcal{A}}(A) \subset [0, \infty)$ , and hence to  $A \geq 0$  in  $\mathcal{A}$ .  $\square$

In [Bo90] Bost stated the following elementary condition which is very useful in the discussion of inverse-closedness of algebras.

**Lemma 10.1.6.** *Let  $\mathcal{A}$  be any subalgebra of an algebra  $\mathcal{B}$ . If  $\mathcal{A}\mathcal{B}\mathcal{A} \subseteq \mathcal{A}$ , then  $\mathcal{A}$  is inverse-closed in  $\mathcal{B}$ .*

*Proof.* If  $\mathcal{A}$  is unital, then  $\mathcal{A} = \mathcal{B}$ . Therefore we assume that  $\mathcal{A}$  is nonunital. Let  $A + \lambda I \in \overline{\mathcal{A}}$  for  $\lambda \neq 0$ , and we assume that  $(A + \lambda I)^{-1} \in \overline{\mathcal{B}}$ . Then there is a  $B \in \mathcal{B}$  such that  $B + \lambda^{-1} = (A + \lambda I)^{-1} \in \overline{\mathcal{B}}$ . Since  $(A + \lambda I)(B + \lambda^{-1}) = (B + \lambda^{-1})(A + \lambda I) = 1$ , we have  $B = -A\lambda^{-2} - AB\lambda^{-1} = -A\lambda^{-2} - BA\lambda^{-1}$ . This implies that  $B \in \mathcal{A}\mathcal{B}\mathcal{A} \subseteq \mathcal{A}$ . Thus  $A + \lambda I$  is an invertible element of  $\overline{\mathcal{A}}$  and  $\mathcal{A}$  is inverse-closed in  $\mathcal{B}$ .  $\square$

In non-commutative geometry one needs a notion of inverse-closedness for Fréchet subalgebras of a Banach algebra. L. Schweitzer proposed the following extension of spectral invariance to Fréchet algebras.

**Definition 10.1.7.** *Let  $\mathcal{A}$  be a dense Fréchet subalgebra of a Banach algebra  $\mathcal{B}$ .*

1. Let  $\{\|\cdot\|_m\}$  be a family of seminorms giving the topology of  $\mathcal{A}$ , and arrange that  $\|\cdot\|_0$  is a norm giving the topology of  $\mathcal{B}$ . We say that  $\mathcal{A}$  is **strongly spectral invariant** in  $\mathcal{B}$  if there exists constants  $C > 0$  and  $(D_i)_{i=1}^{\infty}$  such that for all  $A_1, \dots, A_n \in \mathcal{A}$

$$\|A_1 \cdots A_n\| \leq D_m C^n \sum_{k_1 + \cdots + k_n \leq p_m} \|A_1\|_{k_1} \cdots \|A_n\|_{k_n}, \quad (10.1)$$

where  $p_m$  are integers greater than  $m$  for all  $m \in \mathbb{N}$ .

2. We say that  $\mathcal{A}$  satisfies the **Blackadar-Cuntz condition** in  $\mathcal{B}$  if there exists a family of seminorms  $\{\|\cdot\|_m\}$  for  $\mathcal{A}$  such that there exists a constant  $C > 0$  such that

$$\|A_1 A_2\|_m \leq C \sum_{i+j=m} \|A_1\|_i \|A_2\|_j$$

holds for all  $A_1, A_2 \in \mathcal{A}$  and all  $m \in \mathbb{Z}$ .

**Proposition 10.1.8.** *Let  $\mathcal{A}$  be a dense Fréchet subalgebra of a Banach algebra  $\mathcal{B}$ . Then  $\mathcal{A}$  is strongly spectral invariant if and only if condition (10.1) holds for any/every family of seminorms  $\{\|\cdot\|_m\}$  on  $\mathcal{A}$ .*

**Theorem 10.1.9.** *If  $\mathcal{A}$  is strongly spectral invariant in  $\mathcal{B}$ , then  $\mathcal{A}$  is spectral invariant in  $\mathcal{B}$ .*

**Lemma 10.1.10.** *If  $\mathcal{A}$  is strongly spectral invariant in  $\mathcal{B}$ , then  $\text{Inv}(\mathcal{A})$  is spectral invariant in  $\text{Inv}(\mathcal{B})$ .*

**Proposition 10.1.11.** *Let  $\mathcal{A}$  be a dense Fréchet subalgebra of a Banach algebra  $\mathcal{B}$ . If  $\mathcal{A}$  satisfies the Blackadar-Cuntz condition in  $\mathcal{B}$ , then  $\mathcal{A}$  is strongly spectral invariant in  $\mathcal{B}$ .*

In [Mic52] Micheal has proved the following statement about Fréchet algebras: For a Fréchet algebra  $(\mathcal{A}, \|\cdot\|_s)$  we denote the completion of  $\mathcal{A}$  in the seminorm  $\|\cdot\|_s$  by  $\mathcal{A}_s$ .

**Theorem 10.1.12.** *Let  $\mathcal{A}$  be a Fréchet algebra with submultiplicative seminorms  $\|\cdot\|_s$ . Then  $\mathcal{A}$  is spectral invariant in a Banach algebra  $\mathcal{B}$  if  $\mathcal{A}_s \cap \mathcal{B}$  is spectral invariant in  $\mathcal{B}$ .*

## 10.2 Holomorphic functional calculus

At the moment we draw some general conclusions from the definition of inverse-closedness. We will focus on the spectral radius, the spectrum and holomorphic functional calculus for a general Wiener pair  $(\mathcal{A}, \mathcal{B})$ . Since  $A$  is an element of  $\mathcal{A}$  and  $\mathcal{B}$  we can talk about the spectrum of  $A$  with respect to the algebra  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. Recall that the spectrum of  $A$  in  $\mathcal{A}$  is defined as

$$\sigma_{\mathcal{A}}(A) = \{z \in \mathbb{C} : A - zI \text{ is not invertible in } \mathcal{A}\}. \quad (10.2)$$

An elementary argument yields the following lemma.

**Lemma 10.2.1.** *For  $\mathcal{A} \subset \mathcal{B}$  with common unit  $I$  then*

$$\sigma_{\mathcal{B}}(A) \subset \sigma_{\mathcal{A}}(A).$$

*Proof.* Let  $z \notin \sigma_{\mathcal{A}}(A)$  then  $(A - zI)^{-1}$  exists in  $\mathcal{A} \subseteq \mathcal{B}$ . Then  $z \notin \sigma_{\mathcal{B}}(A)$  □

In combination with the preceding observation we get the following innocent looking lemma which has great consequences.

**Lemma 10.2.2.** *Let  $\mathcal{A} \subset \mathcal{B}$  be Banach algebras with common unit  $I$ . Then the following statements are equivalent:*

1.  $(\mathcal{A}, \mathcal{B})$  is a Wiener pair.
2.  $\text{Inv}(\mathcal{A}) = \text{Inv}(\mathcal{A}) \cap \mathcal{B}$ .
3.  $\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A)$ .

In many situations we have to invert or take a square root of an element  $a$  in a Banach algebra  $\mathcal{A}$  with unit element  $I$ . In the early years F. Riesz, one of the pioneers of functional analysis, had the idea to build functions  $f(T)$  of a compact operator  $T \in \mathcal{B}(L^2(\mathbb{R}))$  in analogy to Cauchy's formula in complex analysis. Later this kind of reasoning had been persuaded by Wiener and substantially generalized by Dunford. Therefore the calculus goes by the names: *Riesz functional calculus*, *Dunford calculus* or *holomorphic calculus*.

More precisely, let  $\mathcal{A}$  be a unital Banach algebra and  $A \in \mathcal{A}$ . Then

$$\text{Hol}(A) = \{f : f \text{ is holomorphic on an open neighborhood } G \text{ of } \sigma_{\mathcal{A}}(A)\} \quad (10.3)$$

is the reservoir of functions which allow to form new elements  $\tilde{f}(A)$  in the Banach algebra  $\mathcal{A}$ . Therefore, we choose a neighborhood  $G$  of  $\sigma_{\mathcal{A}}(A)$  and a contour  $\Gamma$  of  $\sigma_{\mathcal{A}}(A)$  in  $G$ . Then  $f \in \text{Hol}(A)$  with domain  $G$  allows to define

$$\tilde{f}(A) := \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI - A)^{-1} dz \quad (10.4)$$

as a Banach algebra valued integral. The basic results on Cauchy's formula in complex analysis imply that the definition of  $\tilde{f}(A)$  is independent of  $\Gamma$  and that the integral exists as a Riemann integral and  $\tilde{f}(A)$  is an element of  $\mathcal{A}$ . Recall, that  $R(z, \mathcal{A}) = (zI - A)^{-1}$  is called the *resolvent function* of  $A$  which is defined on the *resolvent set*  $\rho_{\mathcal{A}}(A) = \mathbb{C}/\sigma_{\mathcal{A}}(A)$ . The resolvent function  $R(\cdot, \mathcal{A})$  is analytic on  $\rho_{\mathcal{A}}(A)$  and therefore  $z \mapsto f(z)(zI - A)^{-1}$  is analytic from  $G \cap \rho_{\mathcal{A}}(A)$  into  $\mathcal{A}$ .

The main theorem in this context says that for a fixed  $A \in \mathcal{A}$ . The mapping  $A \mapsto \tilde{f}(A)$  is an algebra homomorphism and that this mapping is continuous from  $\text{Hol}(A)$  under uniform convergence on compact sets to  $\mathcal{A}$  with the norm topology.

The following result is what we need in the construction of a projective module for twisted group  $C^*$ -algebras.

**Theorem 10.2.3.** *Let  $\mathcal{A} \subset \mathcal{B}$  be Banach algebras with common unit  $I$ . If  $(\mathcal{A}, \mathcal{B})$  is a Wiener pair, then the Riesz functional calculus for  $\mathcal{A}$  coincides with the one for  $\mathcal{B}$ .*

*Proof.* Since  $\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A)$ , the resolvent function  $R(z, \cdot)$  is defined in  $\mathcal{A}$  and  $\mathcal{B}$ . Consequently, we get that  $\text{Hol}_{\mathcal{A}}(A)$  and  $\text{Hol}_{\mathcal{B}}(A)$  coincide.  $\square$

The preceding discussion tells us that a Wiener pair  $(\mathcal{A}, \mathcal{B})$  has many nice properties. But how can we decide if two Banach algebras  $\mathcal{A} \subset \mathcal{B}$  with common unit  $I$  form a Wiener pair? That's actually the hard part in this topic and the only known tool is the so-called *lemma of Hulanicki* which relates the inverse-closedness of  $(\mathcal{A}, \mathcal{B})$  with the symmetry of  $\mathcal{B}$ .

The notion of symmetry of an involutive Banach algebra  $\mathcal{A}$  with unit  $I$  is the generalization of the fact that a positive bounded operator  $T$  on a Hilbert space  $\mathcal{H}$  has spectrum  $\sigma(T) \subset [0, \infty)$ .

**Definition 10.2.4.** *An involutive Banach algebra  $\mathcal{A}$  with unit  $I$  is called **symmetric**, if  $\sigma_{\mathcal{A}}(AA^*) \subset [0, \infty)$ .*

An element  $A \in \mathcal{A}$  is called **positive** if  $A = CC^*$  for some  $c \in \mathcal{A}$ . In this terminology we can say that positive elements  $A$  in an involutive Banach algebra  $\mathcal{A}$  with unit have "positive" spectrum. Therefore symmetry of an involutive Banach algebra measures how close  $\mathcal{A}$  is to a  $C^*$ -algebra.

**Remark 10.2.5.** *An involutive Banach algebra  $\mathcal{A}$  with unit is symmetric if and only if  $\sigma_{\mathcal{A}}(A) \subset \mathbb{R}$  for all  $A = A^*$  in  $\mathcal{A}$ , if  $A = A^*$  then  $A$  is **hermitian**. This fact is the **Ford-Kiraly lemma** and has been an open question for many years in the early days of normed algebras.*

The following theorem is the lemma of Hulanicki from the early 70's.

**Theorem 10.2.6** (Lemma of Hulanicki). *Assume that  $\mathcal{B}$  is a symmetric Banach algebra and  $\mathcal{A}$  a subalgebra of  $\mathcal{B}$  with common unit  $I$ . Then  $\mathcal{A}$  is inverse-closed in  $\mathcal{B}$  if and only if the spectral radii for all  $A = A^*$  with respect to  $\mathcal{A}$  and  $\mathcal{B}$  are equal, i.e.  $\text{spr}_{\mathcal{A}}(A) = \text{spr}_{\mathcal{B}}(A)$*

This fundamental result gives rise to some comments.

**Remark 10.2.7.**

*In the case of Hulanicki's lemma  $\mathcal{A}$  is symmetric, too.*

*Recall Gelfand's formula for the spectral radius  $\text{spr}_{\mathcal{A}}(a) = \lim_{n \rightarrow \infty} \|A^n\|_{\mathcal{A}}^{1/n}$  which a few years earlier had been proved for the Wiener algebra  $\mathcal{A}(\mathbb{T})$  by Beurling.*

A recent unpublished result by Leinert expresses the equivalence between inverse-closedness and symmetry. Recall the definition of a group  $C^*$ -algebra for a locally compact group  $G$  in Chapter 4. The construction is a special case of the notion of an **enveloping  $C^*$ -algebra** for an involutive Banach algebra  $\mathcal{A}$  with norm  $\|\cdot\|_{\mathcal{A}}$ . Then any involutive representation  $\rho$  of  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}_{\rho}$  satisfies  $\|\rho(A)\| \leq \|A\|_{\mathcal{A}}$ . Since  $\mathcal{B}(\mathcal{H}_{\rho})$  is a  $C^*$ -algebra  $\rho$  is norm decreasing. Therefore the supremum over all such involutive representations is bounded by  $\|A\|_{\mathcal{A}}$ . Therefore

$$\|A\| := \sup\{\rho(A) : \rho \text{ involutive}\} \leq \|A\|_{\mathcal{A}}$$

defines a seminorm on  $\mathcal{A}$ . Since  $\|\rho(A^*A)\| = \|\rho(A)\|^2$  for each  $\rho$ , it is a  $C^*$ -norm which we denote by  $\|\cdot\|_{C^*(\mathcal{A})}$ . The *completion* of  $\mathcal{A}$  in this norm is a  $C^*$ -algebra, the so-called **enveloping  $C^*$ -algebra** of  $\mathcal{A}$ .

**Theorem 10.2.8** (Leinert). *let  $C^*(\mathcal{A})$  be the enveloping  $C^*$ -algebra of an involutive Banach algebra  $\mathcal{A}$  with unit  $e$  then  $\mathcal{A}$  is symmetric if and only if  $\mathcal{A}$  is inverse-closed in  $C^*(\mathcal{A})$ .*

In [GL04] the main result is the symmetry of the twisted convolution algebra  $\ell^1(\mathbb{Z}^d, \mathfrak{h}, \star)$ . Consequently, the non-commutative Wiener algebra

$$\mathcal{A}^1(\Lambda) = \{A \in \mathcal{B}(L^2(\mathbb{R}^d)) : \sum_{\lambda \in \Lambda} a(\lambda)\pi(\lambda) \text{ , } \|\mathbf{a}\|_1 < \infty\}.$$

is inverse-closed in the enveloping  $C^*$ -algebra of  $\mathcal{A}^1(\Lambda)$  which is the non-commutative torus  $C^*(\Lambda, c)$ .

**Theorem 10.2.9.**  *$\mathcal{A}^1(\Lambda)$  is inverse-closed in  $C^*(\mathcal{A})$ .*

**Corollary 10.2.10.**  $\mathcal{A}^1(\Lambda)$  is stable under holomorphic calculus.

Gröchenig and Leinert's main result is that the Banach algebra  $(\ell^1, \natural, *)$  is inverse-closed in  $C^*(\Lambda, \chi)$ , see the preceding section for the definition. The preceding results in this section imply that

**Theorem 10.2.11** (Gröchenig-Leinert).  $(\ell^1, \natural, *)$  is closed under holomorphic calculus.

We call a subalgebra  $\mathcal{B}$  of a unital Banach algebra  $\mathcal{A}$  is **stable under holomorphic functional calculus** of  $A$  if it satisfies:

1.  $\mathcal{B}$  is complete under some locally convex topology finer than the topology of  $\mathcal{A}$ ;
2. if  $B \in \mathcal{B}$  and  $f(B)$  is defined by the Dunford integral, then  $f(B) \in \mathcal{B}$ .

In [Con80] Connes coined the notion of **pre- $C^*$ -algebra** which is a subalgebra of a  $C^*$ -algebra that is stable under holomorphic functional calculus.

The most well-known example of a pre- $C^*$ -algebra is the Schwartz class  $\mathcal{S}(\mathbb{R})$  whose  $C^*$ -completion is  $\mathbf{C}_0(\mathbb{R})$ .

**Lemma 10.2.12.** Any  $\star$ -homomorphism from a pre- $C^*$ -algebra into a  $C^*$ -algebra  $\mathcal{B}$  extends to a morphism from the  $C^*$ -algebraic completion of  $\mathcal{A}$  into  $\mathcal{B}$ .

*Proof.* We assume that  $\mathcal{A}$  and  $\mathcal{B}$  are unital  $C^*$ -algebras the non-unital case follows by standard arguments. Let  $\psi$  be a unital  $\star$ -homomorphism from  $\mathcal{A}$  into  $\mathcal{B}$ . Then  $\psi$  is norm-decreasing  $\|\psi(A)\| \leq \|A\|$  and  $\psi$  shrinks spectra. Therefore,  $\psi$  extends by continuity to its completion  $\mathcal{A}$  in the  $C^*$ -norm. This extension is a morphism  $\tilde{\psi}$  from  $\mathcal{A}$  to  $\mathcal{B}$  that is again norm-decreasing.  $\square$

In non-commutative geometry all pre- $C^*$ -algebras are Fréchet algebras like  $C^\infty(\mathbb{R})$  and  $\mathcal{S}(\mathbb{R})$  which arise as smooth elements of a strongly continuous action.

**Lemma 10.2.13.** Let  $\mathcal{A}$  be a  $C^*$ -algebra,  $G$  a Lie group and  $\rho$  a strongly continuous action from  $G$  into the automorphisms  $\text{Aut}(\mathcal{A})$  of  $\mathcal{A}$ . The dense subalgebra  $\mathcal{A}^\infty$  of smooth elements under this action is a Fréchet pre- $C^*$ -algebra.

*Proof.* Now, for each  $A \in \mathcal{A}$  the map  $t \mapsto \rho_t(A)$  is continuous and  $\mathcal{A}^\infty$  consists of those  $A \in \mathcal{A}$  such that this map is smooth. Since for any  $f \in \text{Hol}(A)$  we have  $f(\rho_t(A)) = \rho_t(f(A))$ , the algebra  $\mathcal{A}^\infty$  is closed under holomorphic functional calculus in  $\mathcal{A}$ .  $\square$

The presentation is largely inspired by a lecture course held at the university of Vienna by Gröchenig in Spring 2005.

## 10.3 Topological Stable Rank of Involutive Banach Algebras

Rieffel was led to the introduction of the topological stable rank by the following theorem.

**Theorem 10.3.1.** *Let  $M$  be a compact space. Then the dimension of  $V$  is the least integer  $d$  such that every continuous function from  $X$  into  $\mathbb{R}^{d+1}$  can be approximated arbitrarily closely by functions which do not contain the origin in their range.*

Now, a map from  $M$  to  $\mathbb{R}^{d+1}$  is just an  $(d+1)$ -tuple of elements of real-valued functions, and the condition that  $f$  miss the origin is just the condition that the  $f_i$  nowhere all take the value 0 simultaneously. But, if we let  $C(M)$  denote the Banach algebra of real-valued continuous functions on  $M$ , this last condition is equivalent by the Stone-Weierstraß Theorem to the condition that the ideal in  $C(M)$  generated by all the  $f_j$  together be  $C(M)$  itself. Thus we can rephrase the above theorem as follows.

**Theorem 10.3.2.** *Let  $M$  be a compact space. Then the dimension of  $V$  is the least integer  $d$  such that every  $(d+1)$ -tuple of elements of  $C(M)$  can be approximated arbitrarily closely by  $(d+1)$ -tuples which generate  $C(M)$  as an ideal.*

It is obvious that one can generalize this to define the dimension of any real commutative Banach algebra  $\mathcal{A}$  simply by replacing  $C(M)$  by  $\mathcal{A}$ . For non-commutative Banach algebras we must specify whether one is generating a left ideal or right ideal, and so to distinguish between a left dimension and a right dimension.

We denote the set of  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  in a unital ring  $\mathcal{A}$  which generate  $\mathcal{A}$  as a left ideal by  $\text{Lg}_n(\mathcal{A})$ .

**Definition 10.3.3.** *Let  $\mathcal{A}$  be a unital Banach algebra. By the **left topological rank** of  $\mathcal{A}$  we will call the least integer  $n$  such that  $\text{Lg}_n(\mathcal{A})$  is dense in  $\mathcal{A}^n$ . If there exists no such integer then we set the left topological rank to be  $\infty$ . We will denote the left topological rank of  $\mathcal{A}$  by  $\text{ltsr}(\mathcal{A})$ .*

**Remark 10.3.4.** *In analogous manner we define the **right topological rank**  $\text{rtsr}(\mathcal{A})$  of a Banach algebra  $\mathcal{A}$  and we denote the set of all  $n$ -tuples which generate  $\mathcal{A}$  as a right ideal by  $\text{Rg}_n(\mathcal{A})$ .*

As a first elementary result we give a condition which implies the equivalence of the right and the left topological stable rank of a Banach algebra. If for a Banach algebra  $\mathcal{A}$  the left and the right topological rank coincide then we called it the **topological rank** of  $\mathcal{A}$  and denote it by  $\text{tsr}(\mathcal{A})$ .

**Proposition 10.3.5.** *If  $\mathcal{A}$  is a Banach algebra with continuous involution, then  $\text{ltsr}(\mathcal{A}) = \text{rtsr}(\mathcal{A})$ .*

The following theorem states one reason for the computation of the topological stable rank of a unital Banach algebra.

**Proposition 10.3.6.** *Let  $\mathcal{A}$  be a unital Banach algebra. Then the following are equivalent:*

1.  $\text{tsr}(\mathcal{A}) = 1$ ;
2. *The set of invertible elements of  $\mathcal{A}$  is dense in  $\mathcal{A}$ .*

*Proof.* One direction is elementary. Namely, if the invertible elements are dense, then the set  $\text{Lg}_1(\mathcal{A})$  of left invertible elements are also dense.

Conversely, suppose that  $\text{Lg}_1(\mathcal{A})$  is dense in  $\mathcal{A}$ . Let  $A$  be a left invertible element with left inverse  $B$ . Now, we look for a left invertible element  $C$  such that  $\|B - C\| < \|A\|^{-1}$ . Then

$$\|CA - I\| = \|(C - B)A\| \leq \|C - B\| \|A\| < 1.$$

Therefore  $CA$  is invertible and  $C$  is right invertible with right inverse  $A(CA)^{-1}$ . Since  $C$  is also left invertible, we have that  $C$  is invertible. This implies the invertibility of  $A$  with inverse  $(CA)^{-1}C$ . Thus  $\text{GL}(\mathcal{A}) = \text{Lg}_1(\mathcal{A})$  is dense in  $\mathcal{A}$ .  $\square$

As examples we state two classes of  $C^*$ -algebras which have topological stable rank one, i.e minimal topological stable rank.

1. If  $\mathcal{A}$  is an AF- $C^*$  algebra then  $\text{tsr}(\mathcal{A}) = 1$ .
2. Let  $\mathcal{A}$  be a  $C^*$ -algebra and let  $\mathcal{K}(\mathcal{H})$  be the algebra of compact operators on a Hilbert space. Then  $\text{tsr}(\mathcal{A} \otimes \mathcal{K}(\mathcal{H})) = 1$  if and only if  $\text{tsr}(\mathcal{A}) = 1$ .

What about Banach algebras with maximal topological stable rank?

1. The algebra of bounded operators  $\mathcal{B}(\mathcal{H})$  on a Hilbert space  $\mathcal{H}$  has the topological stable rank  $\text{tsr}(\mathcal{B}(\mathcal{H})) = \infty$ .
2. Let  $\mathcal{A}$  be the Cuntz algebra  $\mathcal{O}_n$ . Then  $\text{tsr}(\mathcal{O}_n) = \infty$ .

**Proposition 10.3.7.** *Let  $\mathcal{A}$  be a commutative unital Banach algebra. Then  $\text{tsr}(\mathcal{A}) \leq n$  if and only if for every multiplicative linear functional  $\chi : \mathcal{A} \rightarrow \mathbb{C}$ , the induced map has the following property: If  $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{A}^n$ ,  $\gamma_n(\mathbf{a}) \in \mathbb{C}^n \setminus \{0\}$  then for every  $\varepsilon > 0$ , there exists  $\mathbf{a}' \in \text{Lg}_n(\mathcal{A})$  such that  $\|\mathbf{a} - \mathbf{a}'\| < \varepsilon$  and  $\gamma_n(\mathbf{a}) = \gamma_n(\mathbf{a}')$ .*

**Definition 10.3.8.** *Let  $V$  be a left  $\mathcal{A}$ -module and we fix an integer  $k$ . Then we consider the  **$k$ -generator space**  $\text{Gen}_k(V)$  which consists of elements  $(v_j)$  of  $V^k$  such that*

$$\mathcal{A}v_1 + \dots + \mathcal{A}v_k = V. \tag{10.5}$$

**Remark 10.3.9.** *We can consider  $\mathcal{A}$  as a left  $\mathcal{A}$ -module, which we denote by  ${}_{\mathcal{A}}\mathcal{A}$ . In this case the  $k$ -generator space  $\text{Gen}_k(V)$  is just  $\text{Lg}_k(V)$ .*

The next proposition will be of some interest in our discussion of multi-window Gabor frames in the setting of Hilbert  $C^*$ -modules.

**Proposition 10.3.10.** *Let  $V$  be a projective left  $\mathcal{A}$ -module. Then  $Gen_k(V)$  is an open subset of  $V^k$  for every  $k$ .*

*Proof.* From the identification of  $V^k$  with  $Hom_{\mathcal{A}}(\mathcal{A}^k, V)$  then  $Gen_k(V)$  corresponds to the surjections. But in a projective module every surjection has a right inverse and the set of right invertible elements is open in  $Hom_{\mathcal{A}}(\mathcal{A}^k, V)$ .  $\square$

The great importance of projective modules over  $C^*$ -algebras in non-commutative geometry suggest an extension of the stable rank of Banach algebras to modules. In the following all Banach algebras have a unit and projective modules are finitely generated.

**Definition 10.3.11.** *Let  $V$  be a left  $\mathcal{A}$ -module. Then the **Bass stable rank** of  $V$  is the smallest integer  $n$  such that  $f \otimes g$  is a surjective left  $\mathcal{A}$ -module homomorphism from  $\mathcal{A}^n \otimes \mathcal{A}$  onto  $V$ , then there is a homomorphism  $h$  from  $\mathcal{A}^n$  to  $\mathcal{A}$  such that the homomorphism  $f + gh$  from  $\mathcal{A}^n$  to  $\mathcal{A}$  is surjective. In this case we denote this as  $Bsr(V) = n$  and otherwise we set  $Bsr(V) = \infty$ .*

**Remark 10.3.12.** *Under the identification of  $\mathcal{A}$  with  ${}_{\mathcal{A}}\mathcal{A}$  as a left  $\mathcal{A}$ -module. Then  $Bsr(V) = lBsr(V)$ .*

**Definition 10.3.13.** *Let  $\mathcal{A}$  be a Banach algebra and let  $V$  be a  $\mathcal{A}$ -module. Then we call the smallest integer  $k$  such that  $Gen_k(V)$  is dense in  $V^k$  the **topological stable rank** of  $V$  and denote it by  $tsr(V) = k$ .*

For any Banach  $\mathcal{A}$ -module  $V$  we denote  $Hom_{\mathcal{A}}(V, {}_{\mathcal{A}}\mathcal{A})$  by  $V'$  the **dual** of  $V$ . By means of the right action of  $\mathcal{A}$  on itself  $V'$  becomes a right Banach  $\mathcal{A}$ -module. By definition  $V'$  can be identified with  $\mathcal{A}$  as a right  $\mathcal{A}$ -module, i.e.  $V' = \mathcal{A}_{\mathcal{A}}$ .

We make the observation that  $V$  is a finitely generated projective module if and only if  $V'$  is a finitely generated projective module.

**Definition 10.3.14.** *If  $V$  is a left Banach  $\mathcal{A}$ -module, then we call  $tsr(V')$  the **dual topological stable rank**  $dtsr(V)$ .*

**Proposition 10.3.15.** *Let  $V$  be a finitely generated projective  $\mathcal{A}$ -module. Then  $dtsr(V)$  is the least integer  $k$  such the left-invertible elements of  $Hom_{\mathcal{A}}(V, \mathcal{A}^k)$  are dense.*

**Theorem 10.3.16.** *Let  $V$  be a finitely generated projective left  $\mathcal{A}$ -module. Then  $Bsr(V) = dtsr(V)$ .*

**Theorem 10.3.17.** *If  $\mathcal{A}$  is a  $C^*$ -algebra then for every finitely generated projective left  $\mathcal{A}$ -module  $V$  we have  $dtsr(V) = tsr(V)$ .*

**Definition 10.3.18.** *A projective module  $V$  is called to be **stable free** if  $V \otimes \mathcal{A}^m \cong \mathcal{A}^n$  for some integers  $m$  and  $n$ .*

**Lemma 10.3.19.** *Let  $v \in \mathcal{A}^n$ . Then  $v \in Rg_n(\mathcal{A})$  if and only if  $\mathcal{A}v$  is a free direct summand with basis  $v$  of the left  $\mathcal{A}$ -module  $\mathcal{A}^n$ .*

# 11 Morita Equivalence in Gabor Analysis

## 11.1 Multi-window Gabor Frames

Recall, a *multi-window Gabor system* consists of a finite union of Gabor systems  $\mathcal{G}(g_i, \Lambda)$  for a lattice  $\Lambda$  and functions  $g_i \in L^2(\mathbb{R}^d)$  for  $i = 1, \dots, r$ . We therefore define a *multi-window Gabor frame operator*  $S$  of  $\bigcup_{j=1}^r \mathcal{G}(g_j, \Lambda)$  as sum of the Gabor frame operators

$$S_{g_j, \Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g_j \rangle \pi(\lambda) g_j, \quad j = 1, \dots, r$$

is invertible on  $L^2(\mathbb{R}^d)$ .

Since  $S = \sum_{j=1}^r S_{g_j, \Lambda}$  is a finite sum of Gabor frame operators it inherits the nice properties from the building blocks. In other words  $S$  is a positive selfadjoint operator on  $L^2(\mathbb{R}^d)$ , i.e. there are positive constants  $A, B$  such that

$$A \|f\|^2 \leq \sum_{j=1}^r \langle S_{g_j, \Lambda} f, f \rangle \leq B \|f\|^2. \quad (11.1)$$

Furthermore  $S$  commutes with time-frequency shifts  $\pi(\lambda) \in \Lambda$  since for each  $S_{g_j, \Lambda}$  we have

$$\pi(\lambda) S_{g_j, \Lambda} = S_{g_j, \Lambda} \pi(\lambda), \quad \text{for all } \lambda \in \Lambda. \quad (11.2)$$

Our multi-window Gabor system yields the following decomposition of  $f \in L^2(\mathbb{R}^d)$ :

$$f = \sum_{j=1}^r \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g_j \rangle \pi(\lambda) S^{-1} g_j \quad (11.3)$$

$$= \sum_{j=1}^r \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) S^{-1} g_j \rangle \pi(\lambda) g_j. \quad (11.4)$$

The reconstruction of  $f$  relies on the knowledge of the *dual windows*  $\gamma_j = S^{-1} g_j$ .

We now draw some consequences from our main result on the Morita equivalence of  $C^*(\Lambda, c)$  and  $C^*(\Lambda^\circ, \bar{c})$ , Theorem 9.3.13. The discussion relies on the notion of frames in Hilbert  $C^*$ -modules in Chapter 8 and the notion of topological stable rank of a Banach algebra in Chapter 10.

Since  $\Lambda$  is a discrete group  $C^*(\Lambda, c)$  and  $C^*(\Lambda^\circ, \bar{c})$  are unital  $C^*$ -algebras. We are in the setting of the following theorem.

**Theorem 11.1.1.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be both unital  $C^*$ -algebras with identity elements. If  $V$  is a  $\mathcal{A} - \mathcal{B}$ -equivalence bimodule, then  $V$  is a projective right  $\mathcal{B}$ -module, and a projective left  $\mathcal{A}$ -module. Furthermore  $\mathcal{A}$  will be the  $C^*$ -algebra of compact operators of the  $\mathcal{B}$ -module  $V$ .*

See proof of Proposition 2.1 in [Rief81a] for one line of reasoning. Therefore, we have shown that our main result is equivalent to the following:

**Theorem 11.1.2.** *Let  $\Lambda$  be a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ , let  $\mathcal{A} = C^*(\Lambda, c)$ , and let  $V$  denote the left  $\mathcal{A}$ -module obtained by completing  $M^1(\mathbb{R}^d)$  as described earlier. Then  $M^1(\mathbb{R}^d)$  is a finitely generated projective  $\mathcal{A}$ -module whose  $C^*$ -algebra of compact operators is  $C^*(\Lambda^\circ, \bar{c})$ , acting as described earlier.*

Recall that the finite-rank operators of  $C^*(\Lambda, c)$ -module  $M^1(\mathbb{R}^d)$  are just multi-window Gabor frames. It may come as no surprise that they are of relevance in the description of  $M^1(\mathbb{R}^d)$  as projective module.

Following Rieffel and Frank-Larson we know that we can find elements  $g_1, \dots, g_n$  in the projective  $\mathcal{A} = C^*(\Lambda, c)$ -module such that

$$\sum_{i=1}^n \langle g_i, g_i \rangle_{\mathcal{B}} = 1_{\mathcal{B}}, \quad (11.5)$$

and thus so that  $(\langle g_i, g_j \rangle_{\mathcal{A}})_{i,j}$  is a projection  $P$  in  $M_n(\mathcal{A})$ , the algebra of  $n \times n$  matrices over  $\mathcal{A}$ . Then the mapping

$$f \mapsto \langle g_j, f \rangle_{\mathcal{A}} \quad (11.6)$$

is an isomorphism of  $M^1(\mathbb{R}^d)$  onto  $P(\mathcal{A}^n)$ , with inverse given (on  $P(\mathcal{A}^n)$ ) by

$$(A_i) \mapsto \sum_{i=1}^n g_i A_i. \quad (11.7)$$

What does this mean in terms of Gabor frame operators? First of all Equation 11.5 is by the associativity condition, i.e. Janssen's representation, equivalent to

$$\begin{aligned} f = f \cdot 1_{\mathcal{B}} &= \sum_{i=1}^n \langle g_i, g_i \rangle_{\mathcal{B}} f \\ &= \sum_{i=1}^n \sum_{\lambda^\circ \in \Lambda^\circ} \langle g_i, \pi(\lambda^\circ) g_i \rangle \pi(\lambda^\circ) g_i \\ &= \sum_{i=1}^n \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g_i \rangle \pi(\lambda) g_i. \end{aligned}$$

In other words the structure of a finitely generated projective  $\mathcal{A}$ -module yields the existence of a **tight multi-window Gabor frame**. Since the Banach algebra  $\ell^1(\Lambda, c)$  is a dense subalgebra of  $C^*(\Lambda, c)$  the norm of an element in  $C^*(\Lambda, c)$  is dominated by the  $\ell^1$ -norm. Consequently the constructed modular tight frame of the Hilbert  $C^*(\Lambda, c)$ -module  $M^1(\mathbb{R}^d)$  is a **BANACH FRAME** in the sense of Gröchenig. We summarize the preceding observations in the following theorem:

**Theorem 11.1.3.** *Suppose  $\{g_j\}_{j=1}^n$  is a modular frame for the Hilbert  $C^*(\Lambda, c)$ -module  $M^1(\mathbb{R}^d)$ . Then  $\mathcal{G}(g_j, \Lambda)$  is a multi-window Gabor Banach frame and extends to a multi-window Gabor frame of  $L^2(\mathbb{R}^d)$ .*

This generalizes the results of Dörfler, Feichtinger and Gröchenig in [DFG04] where they constructed multi-window Gabor frames for  $\Lambda = \mathbb{Z}^{2d}$ . Their results relied on the properties of the Zak transform, mainly that the Zak transform of a function in  $M^1(\mathbb{R}^d)$  is continuous. Since  $C^*(\mathbb{Z}^{2d}, c)$  is commutative for  $c \equiv 1$ . Their arguments show that the topological stable rank of  $C^*(\Lambda, c)$  is connected with the cardinality of the multi-window Gabor system. Especially, the topological stable rank of  $C^*(\Lambda, c)$  is for most cocycles  $c$  equal to 2 which implies that in most cases we have to use a two-window Gabor frame operator for the decomposition of  $f \in L^2(\mathbb{R}^d)$ .

Another consequence relies on the observation Theorem 8.4.2 that for a finitely generated projective  $\mathcal{A}$ -module every generating set is a modular frame. Consequently the set of generating sets and of modular sets coincide. But in Chapter 10 we stated a result, Proposition 10.3.10, by Rieffel that the set of generators of a finitely generated projective module is *open*, [Rief83]. In other words,

**Theorem 11.1.4.** *The set of all multi-window Gabor frames  $\mathcal{G}(g_1, \dots, g_n, \Lambda)$  with  $g_1, \dots, g_n \in M^1(\mathbb{R}^d)$  is open.*

## 11.2 Connes's Construction of Projective Hilbert $C^*$ -modules

In his work [Con80] on inverse-closed subalgebras of non-commutative tori Connes introduced a method to construct projective modules for the inverse-closed subalgebras. His main result reads as follows.

**Theorem 11.2.1.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra, let  $V$  be projective right  $\mathcal{A}$ -module with  $\mathcal{A}$ -valued inner-product, and let  $\mathcal{B}$  the  $C^*$ -algebra of  $\mathcal{A}$ -compact operators. Then  $\mathcal{B}$  is a  $C^*$ -algebra, and  $V$  has a corresponding  $\mathcal{B}$ -valued inner-product. Let  $\mathcal{A}_0$  and  $\mathcal{B}_0$  be dense  $\star$ -algebras of  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, containing the identity elements, and let  $V_0$  be a dense subspace of  $V$  which is closed under the actions of  $\mathcal{A}_0$  and  $\mathcal{B}_0$ , and such that the restrictions to  $V_0$  of the inner-products have values in  $\mathcal{A}_0$  and  $\mathcal{B}_0$ , respectively. If  $\mathcal{B}_0$  has the property that any element of  $\mathcal{B}_0$  which is invertible in  $\mathcal{B}$  has its inverse in  $\mathcal{B}_0$ , then  $V_0$  is a projective right  $\mathcal{A}_0$ -module.*

*Proof.* Since  $V$  is finitely generated and projective, there are finite collections  $\{g_1, \dots, g_n\}$  and  $\{\gamma_1, \dots, \gamma_n\}$  of elements of  $V$  such that

$$\sum_{j=1}^n \langle g_j, \gamma_j \rangle_{\mathcal{B}} = 1_{\mathcal{B}}.$$

Since  $V_0$  is dense, we can approximate the  $g_j$ 's and  $\gamma_j$ 's closely enough that the corresponding sum of inner-products, which is an element of  $\mathcal{B}_0$ , will be invertible in  $\mathcal{B}$ . By hypothesis its inverse is in  $\mathcal{B}_0$ , and so, multiplying the sum of by the inverse, we find that  $1_{\mathcal{B}}$  is expressed as the sum of inner-products of elements of  $V_0$ . It follows from [Rief82] that  $V_0$  is a projective  $\mathcal{A}$ -module. Let now the  $g_j$ 's and  $\gamma_j$ 's be as above except in  $V_0$ . Then for any  $f \in V$

$$f = 1_{\mathcal{B}}f = \sum_{i=1}^n \langle g_i, \gamma_i \rangle_{\mathcal{B}} f = \sum_{i=1}^n g_i \langle \gamma_i, f \rangle_{\mathcal{B}}. \quad (11.8)$$

This gives the desired result.  $\square$

In this section we restrict our attention to the lattice  $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ . Therefore the  $C^*(\Lambda, c)$  is just  $C^*(\mathbb{Z}^{2d}, c)$  for the specific cocycle  $c_{\theta}((k, l), (m, n)) = e^{-2\pi i(\theta l \cdot m)}$  where  $\theta = \alpha\beta$ . We apply Connes's construction to  $\mathcal{A}_0$  given by the non-commutative Wiener algebra

$$\mathcal{A}^1(\mathbb{Z}^{2d}) = \{A \in \mathcal{B}(\mathcal{H}) : A = \sum_{k,l} a(k, l)\pi(\alpha k, \beta l), \mathbf{a} \in \ell^1(\mathbb{Z}^{2d})\}$$

and  $\mathcal{A} = C^*(\mathbb{Z}^{2d}, c_{\theta})$ . Furthermore, we take

$$\langle f, g \rangle_{\mathcal{A}} = \sum_{k,l \in \mathbb{Z}^d} \langle f, \pi(\alpha k, \beta l) \rangle \pi(\alpha k, \beta l), \mathbf{a} \in \ell^1(\mathbb{Z}^{2d}).$$

In an analogous manner we define the corresponding objects over  $\frac{1}{\beta}\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d$  according to our discussion in Section 9.3 which we denote by  $\mathcal{B}^1(\mathbb{Z}^{2d})$ . Then the fundamental result of Gröchenig and Leinert

**Theorem 11.2.2** (Gröchenig-Leinert).  $\mathcal{A}^1(\mathbb{Z}^{2d})$  is inverse-closed in  $C^*(\mathbb{Z}^{2d}, c_{\theta})$ .

allows an application of Connes construction.

**Theorem 11.2.3.**  $M^1(\mathbb{R}^d)$  is a finitely generated projective right  $\mathcal{A}^1(\mathbb{Z}^{2d})$ -module.

**Corollary 11.2.4.** The involutive Banach algebras  $\mathcal{A}^1(\mathbb{Z}^{2d})$  and  $\mathcal{B}^1(\mathbb{Z}^{2d})$  are Morita equivalent.

Due to a comment of Rieffel this seems to be the first non-trivial result of Morita equivalent involutive Banach algebras.

In the rest of this section we show that the result of Gröchenig and Leinert implies Connes's original result on the spectral-invariance of

$$\mathcal{A}^{\infty}(\mathbb{Z}^{2d}) = \{A \in \mathcal{B}(\mathcal{H}) : A = \sum_{k,l} a(k, l)\pi(\alpha k, \beta l), \mathbf{a} \in \mathcal{S}(\mathbb{Z}^{2d})\}$$

in  $C^*(\mathbb{Z}^{2d}, c_{\theta})$ . Recall that  $\mathcal{S}(\mathbb{Z}^{2d})$  is the space of all rapidly decreasing sequences on  $\mathbb{Z}^{2d}$  with the following family of seminorms  $\|\mathbf{a}\|_s = \sum_{k,l \in \mathbb{Z}^d} |a(k, l)|(1 + |k| + |l|)^s$ . In Chapter 5 we pointed out the importance of derivations on  $C^*(\mathbb{R}^{2d}, c)$  for the

construction of "good" function spaces. We go for a similiar treatment in the case of the group  $C^*$ -algebra of  $\ell^1(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c_\theta)$ . But instead of non-commutative analoga of differentiation operators we define non-commutative analoga of difference operators on  $\mathbb{Z}^{2d}$  by

$$\delta_1\left(\sum_{k,l \in \mathbb{Z}^d} a(k,l)M_{\omega l}T_{\alpha k}\right) = 2\pi i\alpha \sum_{k,l \in \mathbb{Z}^d} ka(k,l)M_{\omega l}T_{\alpha k}$$

and

$$\delta_2\left(\sum_{k,l \in \mathbb{Z}^d} a(k,l)M_{\omega l}T_{\alpha k}\right) = 2\pi i\beta \sum_{k,l \in \mathbb{Z}^d} la(k,l)M_{\omega l}T_{\alpha k}.$$

Consequently,  $\delta_1, \delta_2$  are the non-commutative analoga of taking differences in the horizontal directions, vertical directions of  $A = \sum_{k,l \in \mathbb{Z}^d} a(k,l)\pi(\alpha k, \beta l)$ . As in the continuous case  $\delta_1$  and  $\delta_2$  are unbounded derivations on  $L^2(\mathbb{R}^d)$ , i.e.

$$\delta_j(AB) = \delta_j(A)B + A\delta_j(B), \quad A, B \in C^*(\mathbb{Z}^{2d}, c_\theta).$$

Their domains  $\text{dom}(\delta_1), \text{dom}(\delta_2)$  are obviously the sets of those  $A$  such that  $\|\delta_1\|_{\text{op}} < \infty$  and  $\|\delta_2\|_{\text{op}} < \infty$ . But we look for the domain of  $\delta := 1 + (2\pi i)^{-1}(\delta_1 + \delta_2)$  given by

$$\delta(A) = \sum_{k,l \in \mathbb{Z}^d} (1 + \alpha k + \beta l)a(k,l)\pi(\alpha k, \beta l).$$

Then the domain of  $\delta$  is precisely the space

$$\text{dom}(\delta) = \left\{A \in \mathcal{B}(L^2(\mathbb{R}^d)) : A = \sum_{k,l \in \mathbb{Z}^d} |a(k,l)|\pi(\alpha k, \beta l), \quad \mathbf{a} \in \ell_{v_s}^1(\mathbb{Z}^{2d})\right\},$$

where  $\ell_{v_s}^1(\mathbb{Z}^{2d})$  consists of all sequences on  $\mathbb{Z}^{2d}$  such that

$$\|\mathbf{a}\|_s = \sum_{k,l} (1 + \alpha|k| + \beta|l|)|a(k,l)| < \infty.$$

Now the space of **smooth elements**  $\mathcal{S}(\delta)$  of the derivation  $\delta$  is  $\cap_{s \geq 0} \ell_{v_s}^1(\mathbb{Z}^{2d})$ .

In [Jan95] the subalgebra of  $C^*(\mathbb{Z}^{2d}, c_\theta)$  given by

$$\mathcal{A}^{1,s}(\mathbb{Z}^{2d}) = \left\{A \in \mathcal{B}(\mathcal{H}) : A = \sum_{k,l} a(k,l)\pi(\alpha k, \beta l), \quad \mathbf{a} \in \ell_{v_s}^1(\mathbb{Z}^{2d})\right\}$$

appeared for the first time in Gabor analysis but Connes implicitly treated it as the levels of his smooth subalgebra  $\mathcal{A}^\infty(\mathbb{Z}^{2d})$  of  $C^*(\mathbb{Z}^{2d}, c_\theta)$ . Janssen proved that

**Theorem 11.2.5.**  $\mathcal{A}^{1,s}(\mathbb{Z}^{2d})$  is an involutive Banach algebra with  $\|A\|_{1,s} = \sum_{k,l \in \mathbb{Z}^d} (1 + |k| + |l|)^s |a(k,l)|$ .

Later Gröchenig and Leinert proved in [GL04] the following:

**Theorem 11.2.6.** For any  $s \geq 0$  the involutive Banach algebra  $\mathcal{A}^{1,s}(\mathbb{Z}^{2d})$  is inverse-closed in  $C^*(\mathbb{Z}^{2d}, c_\theta)$ .

In [GL04] they acutally showed that their argument is valid for any submultiplicative weight  $v$  that satisfies the (GRS)-condition:

1.  $v(k + m, l + n) \leq v(k, l)v(m, n)$  for all  $(k, l), (m, n) \in \mathbb{Z}^{2d}$ ,
2.  $\lim_{n \rightarrow \infty} v(nk, nl)^{1/n} = 1$  for all  $(k, l) \in \mathbb{Z}^{2d}$ .

In a subsequent work Gröchenig and Leinert proved that the above conditions on the weight characterize all inverse-closed weighted involutive Banach subalgebras of the non-commutative torus  $C^*(\mathbb{Z}^{2d}, c_\theta)$ .

**Theorem 11.2.7** (Connes).  $\mathcal{A}^\infty(\mathbb{Z}^{2d})$  is a spectral invariant subalgebra of  $C^*(\mathbb{Z}^{2d}, c_\theta)$ .

*Proof.* Since  $\mathcal{A}^\infty(\mathbb{Z}^{2d}) = \bigcap_{s \geq 0} \mathcal{A}^{1,s}(\mathbb{Z}^{2d})$  we use Theorem 10.1.12 that the inverse-closedness of each Banach algebra yields the spectral invariance of the Fréchet algebra  $\mathcal{A}^\infty(\mathbb{Z}^{2d})$ .  $\square$

Janssen rediscovered this theorem in his work on the quality of Gabor frames, [Jan95]. His proof does not use any of the aforementioned results on spectral invariance of Fréchet algebras or inverse-closedness of Banach algebras. His approach relies on various adhoc statements on the spectral radius of elements in  $\mathcal{A}^\infty(\mathbb{Z}^{2d})$  which he derives in an ingenious way by hard analysis.

# 12 Projections in Operator Algebras

The twisted group  $C^*$ -algebra of  $\mathbb{R}^{2d}$  offers a very intriguing interpretation of the Stone-von Neumann Theorem. In Chapter 5 we presented von Neumann's proof in the light of operator algebras. This interpretation was Rieffel's starting point to develop the notion of Morita equivalence of  $C^*$ -algebras, [Rief72]. Later he adapted this point of view to recast Mackey's imprimitivity theorem, the utmost generalization of Stone-von Neumann's theorem, in terms of Morita equivalence between transformation group  $C^*$ -algebras [Rief72a, Rief79]. In this chapter we describe a method for the construction of projections in Morita equivalent  $C^*$ -algebras and von Neumann algebras. One of our main results is an application of Rieffel's method to our pair  $(C^*(\Lambda, c), C^*(\Lambda^\circ, \bar{c}))$  of Morita equivalent twisted group  $C^*$ -algebras.

Subsequently we describe Rieffel's method in the case of von Neumann algebras. As most important consequence we give a proof of the **density theorem of Gabor analysis** for a general lattice  $\Lambda$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ . Since Daubechies's first announcement of the density theorem in [Daub90] many researchers have discussed different approaches to this fundamental fact in Gabor analysis. Many contributions were designed to avoid an explicit use of operator algebras and many expressed their satisfaction about circumventing Rieffel's result from [Rief81b]. Others claimed that Rieffel's method in [Rief82] is not applicable in the general case, e.g. [Bek04]. During the work on connections between Gabor analysis and non-commutative geometry we reached the contrary point of view since in [Rief88] all relevant facts are stated to derive the density theorem in full generality. In [Bek04] Bekka presented a very technical approach to the general density theorem which does not offer any new insight on Gabor frames. We hope that the reader finds our proof more enlightening despite it is based on the construction of an equivalence bimodule.

## 12.1 Projections in $C^*$ -algebras

In contrast to von Neumann algebras  $C^*$ -algebras do not necessarily contain non-trivial projections such as the continuous functions  $C(\mathbb{T}^2)$  since the torus  $\mathbb{T}^2$  is connected. On the contrary, the non-commutative tori  $\mathbb{T}_\theta$  contains a large family of non-trivial projections. Rieffel in [Rief81a] showed that, for a given  $\theta$  irrational and for all  $\alpha\mathbb{Z} \times \mathbb{Z}\theta \cap [0, 1]$ , there exists a projection  $P_\alpha$  in  $\mathbb{T}_\theta$ , with  $\text{Tr}(P_\alpha) = \alpha$ . In the final section of [Rief81a] presented a different approach to the construction of projections which he called a curiosity. We show that this curious method is the first appearance of a *tight Gabor frame* which were independently described by Daubechies, Grossmann and Meyer in [DGM86].

Recall that, if  $\mathcal{A}$  is a  $C^*$ -algebra and  $p$  is a projection in  $\mathcal{A}$ , then the sub- $C^*$ -algebra  $p\mathcal{A}p$  is called the **cut-down** of  $\mathcal{A}$  by  $p$ .

**Theorem 12.1.1.** *Suppose that two unital  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  are Morita equivalent through a  $\mathcal{A} - \mathcal{B}$  equivalence bimodule  $V$ . Then for  $f \in V$ ,  $p := \langle f, f \rangle_{\mathcal{A}}$  is a projection in  $\mathcal{A}$  if and only if  $\langle f, f \rangle_{\mathcal{B}} = 1_{\mathcal{B}}$ .*

*Proof.* Suppose  $f\langle f, f \rangle_{\mathcal{B}} = f$ . Then the associativity condition  $f\langle f, f \rangle_{\mathcal{B}} = \langle f, f \rangle_{\mathcal{A}}f$  gives that  $\langle f, f \rangle_{\mathcal{A}}^2 = \langle f, f \rangle_{\mathcal{A}}$  is idempotent, and so it is a projection since it is self-adjoint.

Conversely, suppose that  $\langle f, f \rangle_{\mathcal{A}}$  is a projection in  $\mathcal{A}$ . Then

$$\langle f\langle f, f \rangle_{\mathcal{B}} - f, f\langle f, f \rangle_{\mathcal{B}} - f \rangle_{\mathcal{A}} = 0,$$

so that  $f\langle f, f \rangle_{\mathcal{B}} = f$ . □

This method for the construction of projections was introduced by Rieffel in [Rief74b, Rief81a]. Moreover, we have an explicit involutive isomorphism of  $\mathcal{B}$  onto  $p\mathcal{A}p$  induced by  $f$ :

$$B \mapsto \langle f, B \cdot f \rangle_{\mathcal{A}}.$$

The inverse of this map is simply  $p\mathcal{A}p \mapsto \langle f \cdot A, f \rangle_{\mathcal{B}}$ .

Furthermore, Rieffel stated a necessary condition for an element  $f$  in an equivalence-bimodule  $V$  to be a projection.

**Theorem 12.1.2.** *Suppose given a bimodule  ${}_{\mathcal{A}}V_{\mathcal{B}}$  for the  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ . If an element  $f \in {}_{\mathcal{A}}V_{\mathcal{B}}$  admits an invertible  $\star$ -invariant square root  $\langle f, f \rangle_{\mathcal{B}}^{1/2}$ , then the element  $h := f\langle f, f \rangle_{\mathcal{B}}^{-1/2}$  satisfies  $h\langle h, h \rangle_{\mathcal{B}} = h$ .*

The proof consists of the following elementary computation

$$\langle f\langle f, f \rangle_{\mathcal{B}}^{-1/2}, f\langle f, f \rangle_{\mathcal{B}}^{-1/2} \rangle_{\mathcal{B}} = \langle f, f \rangle_{\mathcal{B}}^{-1} \langle f, f \rangle_{\mathcal{B}}^{-1/2} f = f.$$

If we take  $\mathcal{A} = C^*(\Lambda, c)$  and  $\mathcal{B} = C^*(\Lambda^\circ, \bar{c})$ , then the preceding construction has an intrinsic interpretation in terms of Gabor frames. Let  $\mathcal{G}(g, \Lambda)$  be a Gabor system with  $g \in M^1(\mathbb{R}^d)$ , then the frame operator  $S_{g, g, \Lambda}$  is just  $\langle g, g \rangle_{\mathcal{A}}$ . Rieffel's condition that  $\langle g, g \rangle_{\mathcal{A}}$  is invertible, thus means that  $\mathcal{G}(g, \Lambda)$  is a frame for  $L^2(\mathbb{R}^d)$ . Furthermore, the element  $h_0 = \langle g, g \rangle_{\mathcal{A}}^{-1/2}g$  is the so-called **canonical tight atom**.

**Theorem 12.1.3.** *If  $\mathcal{G}(g, \Lambda)$  is a Gabor frame for  $L^2(\mathbb{R}^d)$ , then the canonical tight Gabor atom  $h_0$  gives a projection  $p = \langle h_0, h_0 \rangle_{\mathcal{B}}$  in  $C^*(\Lambda^\circ, \bar{c})$ .*

Operator algebraists use this method to prove deep results on the structure of non-commutative tori. In his famous paper [Rief81a] Rieffel constructed projections in the irrational rotation algebra by this method. He actually called this construction of a tight Gabor frame a curiosity. Instead he presented an explicit example

of a function  $f$  which generates a projection. This alternative method is very technically and does not offer some insight in the structure of the projection but it has found its place in every textbook on non-commutative tori. Many years later Boca brought the attention of operator algebraists to this curious way of constructing projections [Boc99]. He applied Rieffel's construction to the Gabor frame  $\mathcal{G}(\varphi_0, \mathbb{Z} \times \theta\mathbb{Z})$ , where  $\varphi_0 = e^{-t^2}$  is the standard Gaussian. Then he showed that  $\varphi_0$  generates a Gabor frame if  $\theta \leq 0.0948$ . Boca was apparently not aware of the relation to Gabor analysis, since he proved the invertibility of the Gabor frame operator by hard analysis. His computations involve theta functions and many clever tricks to estimate their sums over the lattice  $\mathbb{Z} \times \theta\mathbb{Z}$ . He definitely was not aware of the connection between non-commutative tori and Gabor analysis, otherwise he would state the theorem of Lyubarski and Seip. By a clever use of complex analysis Lyubarski and Seip independently proved that the Gaussian  $\varphi_0$  generates a Gabor frame for  $\mathbb{Z} \times \theta\mathbb{Z}$  if and only if  $\theta \leq 1$ .

On the other hand Boca constructs automorphisms in  $C^*(\mathbb{Z} \times \theta\mathbb{Z})$  using Rieffel's method. His construction depends on the covariance properties of the STFT, see [Boc99]. Walters followed the same line of thought in his work on the structure of  $C^*(\Lambda, c)$ , see [Wal03, Wal04a, Wal04b]. In a series of papers Manin generalized Boca's construction to arbitrary locally compact abelian groups [Man04a, Man04b]. Furthermore, he took this results as starting point for the introduction of non-commutative analoga of theta functions, so-called **quantum theta's**. Independently, Schwarz defined **theta vectors** another non-commutative analog of theta functions [Sch01]. The relation between quantum theta's and theta vector's was an open question for quite some time. Manin observed that some quantum thetas are just  $C^*(\Lambda, c)$ -valued inner-products for certain Gaussians like theta vectors in [Man04a]. But he does not give any precise conditions that force the equivalence of both objects. After we had realized that Gabor frame operators are Rieffel's  $C^*(\Lambda, c)$ -inner products we were able to clarify the picture. In our terminology Schwarz studied Gabor frames generated by generalized Gaussians on  $\alpha\mathbb{Z} \times \beta\mathbb{Z}$ . In his proof he rediscovered the Zak transform and its power for the study of Gabor frames if  $\alpha$  and  $\beta$  are rational numbers, see [Lu05b].

## 12.2 The Density Theorem

### 12.2.1 Basic Definitions

Let  $\mathcal{H}$  be a Hilbert space. Then a **von Neumann algebra**  $\mathcal{M}$  is meant to be an unital algebra, which is closed under involution  $A \mapsto A^*$ , and is closed in the strong operator topology (=topology of pointwise convergence on  $\mathcal{H}$ ). Let  $\mathcal{A}$  be a set of operators. Then by the **commutant** of  $\mathcal{A}$  we mean the set of all operators on  $\mathcal{H}$  which commute with every element of  $\mathcal{A}$ . We denote the commutant of  $\mathcal{A}$  by  $\mathcal{A}'$ . Commutants are automatically strongly closed. A fundamental theorem of von Neumann asserts that the commutant of the the commutant of a  $\star$ -algebra is the same as the strong closure. Therefore the **double commutant**  $[\mathcal{A}']'$  is called the

von Neumann algebra generated by  $\mathcal{A}$ .

The spectral theorem shows that an abelian von Neumann algebra is isomorphic to  $L^\infty(X, \mu)$  for some  $\sigma$ -finite measure space  $(X, \mu)$ .

A von Neumann algebra  $\mathcal{M}$  is said to be a **factor** if

$$\mathcal{M} \cap \mathcal{M}' = \{\lambda I_{\mathcal{H}} : \lambda \in \mathbb{C}\}.$$

This means exactly that the center of  $\mathcal{M}$  consists of scalar multiples of the identity operator. Therefore, factors are the most non-commutative von Neumann algebras. By Schur's theorem the algebra of all bounded operators on  $\mathcal{H}$  is a factor.

Von Neumann and Murray showed that one can define a **dimension function**, which is unique up to a normalizing scalar, on projections of a factor  $\mathcal{M}$ , which extends to a **trace function** on the factor  $\mathcal{M}$ . A factor  $\mathcal{M}$  is said to be of **type**  $I_\infty$  if this dimension function has the value  $\infty$ , e.g.  $\mathcal{B}(\mathcal{H})$  is a type I. A factor  $\mathcal{M}$  is said to be of **type**  $II_1$  if this dimension function takes on exactly the values  $[0, 1]$  otherwise  $\mathcal{M}$  is called to be of **type**  $II_\infty$ .

Of course not all von Neumann algebras are factors, and there is a vast literature describing the structures and decomposition of von Neumann algebras, [Dix77, Ta01a, Ta01b, Ta01c].

We digress our discussion and recall the definition of a coupling constant and other notions of relevance for an understanding of Rieffel's results.

1. Two projections  $p$  and  $q$  in a von Neumann algebra  $\mathcal{M}$  are said to be **equivalent** if there exists an element  $u \in \mathcal{M}$  with  $u^*u = p$  and  $uu^* = q$ . We denote two equivalent projections by  $p \approx q$ .
2. A projection is called **finite** if there is no proper projection  $q$  equivalent to  $p$ .
3. A von Neumann algebra is called **finite** if its identity is finite.

It is of great relevance that finite von Neumann algebras possess a trace. Furthermore, let  $\mathcal{A}$  be a subspace of a Hilbert space  $\mathcal{H}$  then the orthogonal projection of  $\mathcal{H}$  onto  $\mathcal{A}$  is an element of the commutant  $\mathcal{A}'$ .

**Theorem 12.2.1.** *For a von Neumann algebra  $\mathcal{M}$ , the following conditions are equivalent:*

1.  $\mathcal{M}$  is finite.
2. There exists a linear map  $C$  from  $\mathcal{M}$  onto the center  $\mathcal{C}$  with the properties
  - a)  $C(A^*A) = C(AA^*) \geq 0$ ,
  - b)  $C(XA) = X \cdot C(A)$ ,  $X \in \mathcal{C}$ ,  $A \in \mathcal{M}$ ,
  - c)  $C(I) = 1$ ,

d)  $C(A^*A) \neq 0$  for every non-zero  $A$  in  $\mathcal{M}$ .

If this is the case, then  $C$  is unique and  $\sigma$ -weakly continuous.

The linear map  $C$  is called the **center valued trace** and will be denoted by  $C_{\mathcal{M}}$ . Now we introduce the important notion of the coupling constant.

**Definition 12.2.2.** Let  $\mathcal{M}$  be a finite von Neumann algebra with finite commutant  $\mathcal{M}'$ . If  $C$  and  $C'$  are both the center valued trace of  $\mathcal{M}$  and  $\mathcal{M}'$ , respectively, then the element  $C$  of  $\widehat{\mathcal{C}}_+$  of the above theorem is called the **coupling function** between  $\mathcal{M}$  and  $\mathcal{M}'$ , and denoted by  $C_{\mathcal{M}}$ . If  $\mathcal{M}$  is a factor then  $C_{\mathcal{M}}$  is a positive number, which is called the **coupling constant**.

**Theorem 12.2.3.** Let  $\mathcal{M}$  be a finite von Neumann algebra, then the center-valued trace  $C_{\mathcal{M}}$  is norm continuous.

**Proposition 12.2.4.** If  $\mathcal{M}$  is a finite von Neumann algebra with finite commutant  $\mathcal{M}'$ , then we have the following

$$C_{\mathcal{M}'} = C_{\mathcal{M}}^{-1}. \quad (12.1)$$

A vector  $g \in \mathcal{H}$  is called **cyclic** if the closed subspace  $\mathcal{M}g$  is dense in  $\mathcal{H}$ . A vector  $g \in \mathcal{H}$  is called **separating** if  $Ag = 0$  for all  $A \in \mathcal{M}$  implies  $g = 0$ . The following theorem relates the existence of cyclic and separating vectors with the range of the coupling constant.

**Theorem 12.2.5.** Let  $\mathcal{M}$  be a finite von Neumann algebra with finite commutant  $\mathcal{M}'$ .

1. There exists a cyclic vector if and only if  $C_{\mathcal{M}} \leq 1$ .
2. There exists a separating vector if and only if  $C_{\mathcal{M}} \geq 1$ .

Now we discuss the structure of von Neumann algebras generated by time-frequency shifts. As a primer we treat the von Neumann algebra  $\mathcal{M}(\mathbb{R}^d)$  generated by  $\{\pi(x, \omega) : (x, \omega) \in \mathbb{R}^{2d}\}$ . Later we move on to consider von Neumann algebras  $\mathcal{M}(\Lambda)$  generated by  $\{\pi(\lambda) : \lambda \in \Lambda\}$  for a lattice  $\Lambda$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ . Remember the increasing complexity we had to deal with when we restricted our square-integrable projective representation  $\{\pi(x, \omega) : (x, \omega) \in \mathbb{R}^{2d}\}$  to a lattice  $\Lambda \in \mathbb{R}^{2d}$ . A similar phenomenon arises in the discussion of  $\mathcal{M}(\Lambda)$ .

The following result is well-known, see [Ta01b].

**Theorem 12.2.6.** The von Neumann algebra  $\mathcal{M}(\mathbb{R}^{2d})$  is the factor  $\mathcal{B}(L^2(\mathbb{R}^d))$  of all bounded operators on  $L^2(\mathbb{R}^d)$ .

Acually, let  $G$  be a locally compact abelian group,  $U_x$  a unitary representation of  $G$  and  $V_\omega$  a unitary representation of  $\widehat{G}$ . Then the von Neumann algebra generated by the projective representation of  $U_x V_\omega$  is isomorphic to the von Neumann algebra  $\mathcal{M}(\mathbb{R}^{2d})$ .

*Proof.* Let  $\mathcal{FL}^1(\mathbb{R}^d)$  be the Fourier algebra of  $\mathbb{R}^d$ . Then we identify  $\mathcal{FL}^1(\mathbb{R}^d)$  and  $L^1(\mathbb{R}^d)$  by the non-symmetric pairing  $\mathcal{I}f \in L^1(\mathbb{R}^d)$  and  $\mathcal{F}f \in \mathcal{FL}^1(\mathbb{R}^d)$ , where  $\mathcal{I}f(t) = f(-t)$ . For each  $f \in L^1(\mathbb{R}^d)$

$$\tilde{\rho}(f) = \int_{\mathbb{R}^d} f(t)M_\omega dx$$

is a  $\star$ -representation of  $L^1(\mathbb{R}^d)$  which can be extended to the enveloping  $C^*$ -algebra  $C_0(\mathbb{R}^d)$ . The representation  $\tilde{\rho}(f)$  is just the multiplication of  $f \in C_0(\mathbb{R}^d)$  on  $L^2(\mathbb{R}^d)$ . Hence the von Neumann algebra generated  $\mathcal{M}$  by  $\{\tilde{\rho}(f) : f \in C_0(\mathbb{R}^d)\}$  is the multiplication algebra  $L^\infty(\mathbb{R}^d)$  on  $L^2(\mathbb{R}^d)$  which is maximal abelian. The relation

$$T_x \tilde{\rho}(f) T_{-x} = M_\omega(T_x f), \quad x \in \mathbb{R}^d, f \in L^\infty(\mathbb{R}^d)$$

implies that the operators commuting with all  $\{T_x : x \in \mathbb{R}^d\}$  are only scalars. Therefore,  $\{\pi(x, \omega) : (x, \omega) \in \mathbb{R}^{2d}\}' = \mathbb{C}$ , i.e.  $\mathcal{M}(\mathbb{R}^{2d})$  is irreducible and generates the factor  $\mathcal{B}(L^2(\mathbb{R}^d))$ .  $\square$

Consequently,  $\mathcal{M}(\mathbb{R}^{2d})$  is too large for many purposes, e.g. as observables in quantum mechanics. We just remark that the  $C^*(\mathbb{R}^{2d})$ , i.e. the norm-closure of  $\{\pi(x, \omega) : (x, \omega) \in \mathbb{R}^{2d}\}$  has a fascinating structure. Look for *CCR*-algebras in the literature to get a glimpse of the dramatic differences between  $\mathcal{M}(\Lambda)$  and  $C^*(\Lambda)$ .

In Chapter 9 we treated the  $C^*$ -algebras generated by  $\{\pi(\lambda) : \lambda \in \Lambda\}$  for a lattice  $\Lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}^d}$ . There we unveiled an intimate relation between such  $C^*$ -algebras and objects in Gabor analysis. Therefore our hope is satisfied that the von Neumann algebras  $\mathcal{M}(\Lambda)$  offer another connection with Gabor analysis.

We recall the framework of Chapter 9. Let  $\Lambda$  be a lattice in  $\mathbb{R}^{2d}$  and  $\Lambda^\circ$  the adjoint lattice. Then we treat the relation of the von Neumann algebras  $\mathcal{M}_\Lambda$  and  $\mathcal{M}_{\Lambda^\circ}$ . These von Neumann algebras were first considered by Connes and Takesaki in their work on factors of type *III* in [CT77]. Consequently, these are highly non-trivial objects. Connes and Takesaki showed that  $\mathcal{M}_\Lambda$  and  $\mathcal{M}_{\Lambda^\circ}$  are each others commutant.

**Theorem 12.2.7** (Connes-Takesaki). *Let  $\Lambda$  be a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ . Then the von Neumann algebras  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$  are each others commutant.*

In [Rief88] a proof of this fact is deduced from the Morita equivalence of  $C^*(\Lambda, c)$  and  $C^*(\Lambda^\circ, \bar{c})$ . Namely, we let  $\mathcal{A}_0$  and  $\mathcal{B}_0$  the involutive algebras  $\ell^1(\Lambda, c)$  and  $\ell^1(\Lambda^\circ, \bar{c})$  with their actions on  $M^1(\mathbb{R}^d)$ , and if we equip  $M^1(\mathbb{R}^d)$  with the ordinary inner-product from  $L^2(\mathbb{R}^{2d})$ , then  $M^1(\mathbb{R}^d)$  becomes a Hilbert  $\mathcal{A} - \mathcal{B}$ -bi-rigged space, see [Rief76a]. Then an application of Theorem 1.9 of [Rief76a] yields, that  $\mathcal{A}$  and  $\mathcal{B}$  are each other's commutant, as algebras of operators on  $L^2(\mathbb{R}^{2d})$ .

Now, these commutants are also generated by the corresponding projective representations of  $\Lambda$  and  $\Lambda^\circ$ . Therefore the Theorem of Connes and Takesaki is proved. Furthermore, we derived its equivalence to

**Theorem 12.2.8.** *Let  $\pi$  be the projective representation of  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$  on  $L^2(\mathbb{R}^{2d})$ . If  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$  denote the von Neumann algebras generated by  $\pi(\Lambda)$  and  $\pi(\Lambda^\circ)$ , then  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$  are each other's commutant.*

Rieffel relates this result of Connes-Takesaki to the construction of an equivalence bimodule for the von Neumann algebras  $\mathcal{A}$  and  $\mathcal{B}$  generated by  $\ell^1(\Lambda, c)$  and  $\ell^1(\Lambda^\circ, \bar{c})$ , respectively. The von Neumann algebra  $\mathcal{A}$  is isomorphic to the left regular representation generated by  $\ell^1(\Lambda, c)$  with respect to its finite trace. In analogous manner we consider the von Neumann algebra  $\mathcal{B}$  generated by  $\ell^1(\Lambda^\circ, \bar{c})$ .

We proceed with the discussion of the von Neumann algebras  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$ .

**Proposition 12.2.9.** *The von Neumann algebras  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$  are faithful finite von Neumann algebras, respectively.*

Since  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$  are finite von Neumann algebras we can ask for the coupling constant between  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$ . It is well-known that the coupling constant relates the center-valued traces of  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$ , respectively. Therefore, we firstly determine the common center of  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$ . Before we start our investigations we state one more notation. In the sequel  $\delta_\lambda$  will denote the unitary in  $\mathcal{M}(\Lambda)$  corresponding to the  $\delta$ -function at  $\lambda$ .

**Proposition 12.2.10.** *The common center of  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$  is generated by  $\ell^1(\Lambda \cap \Lambda^\circ)$ .*

**Corollary 12.2.11.** *If  $\Lambda^\circ \subseteq \Lambda$  then the common center of  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$  is generated by  $\ell^1(\Lambda^\circ)$ .*

We will denote  $\Lambda \cap \Lambda^\circ$  by  $\Lambda_c$

*Proof.* An elementary argument shows that  $\ell^1(\Lambda_c)$  is contained in the center  $\mathcal{Z}(\mathcal{A}) \cap \mathcal{Z}(\mathcal{B})$ . The reverse inclusion is easily seen.  $\square$

The proof of the corollary consists of the observation that the stated condition implies that the von Neumann algebras are commutative. In the sequel we set  $\mathcal{A} = \mathcal{M}(\Lambda)$  and  $\mathcal{B} = \mathcal{M}(\Lambda^\circ)$ .

**Theorem 12.2.12.** *The center-valued trace  $C_{\mathcal{A}}$  is given by the restriction of functions in  $M^1(\mathbb{R}^{2d})$  to  $\Lambda_c = \Lambda \cap \Lambda^\circ$ .*

*Proof.* We assume that  $\lambda \in \Lambda$  but not in  $\Lambda_c$ , i.e. there exists a  $\lambda' \in \Lambda$  with  $c(\lambda, \lambda') \neq 1$ . Then

$$C_{\mathcal{A}}(\delta_\lambda) = (T_{\mathcal{A}}(\lambda' \lambda \lambda'^{-1})) = c(\lambda, \lambda') C_{\mathcal{A}}(\lambda')$$

implies that  $C_{\mathcal{A}}(\delta_\lambda) = 0$ . From the norm continuity of  $C_{\mathcal{A}}$  it follows that  $C_{\mathcal{A}}$  on  $\ell^1(\Lambda, c)$  is just the restriction of functions to  $\Lambda_c$ .  $\square$

More explicitly, for  $f, g \in M^1(\mathbb{R}^d)$  the center-valued trace  $C_{\mathcal{A}}$  of  $\langle f, g \rangle_{\mathcal{A}}$  at  $\lambda \in \Lambda_c$  is

$$C_{\mathcal{A}}(\langle f, g \rangle_{\mathcal{A}})(\lambda) = \langle f, \pi(\lambda)g \rangle. \quad (12.2)$$

**Proposition 12.2.13.** *The center-valued trace  $C_{\mathcal{B}}$  of the von Neumann algebra  $\mathcal{B}$  generated by  $L^1(\Lambda^0, \bar{c})$  is the restriction of functions in  $M^1(\mathbb{R}^{2d})$  to  $\Lambda_c$ .*

Therefore, for  $f, g \in M^1(\mathbb{R}^d)$  we have that

$$C_{\mathcal{B}}(\langle f, g \rangle_{\mathcal{B}}) = \langle g, \pi(\lambda)f \rangle. \quad (12.3)$$

The main result of this section is the computation of the coupling constant for  $\mathcal{A}$  and  $\mathcal{B}$ .

**Theorem 12.2.14.** *Let  $\Lambda$  be a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ . Then the coupling constant for  $\mathcal{A}$  and  $\mathcal{B}$  is the scalar operator  $|\Lambda|$ .*

*Proof.* The argument is to determine the operators associated to  $C_{\mathcal{A}}$  and  $C_{\mathcal{B}}$  on  $L^2(\mathbb{R}^d)$ , respectively.

Let  $f, g \in M^1(\mathbb{R}^d)$  and let  $h$  be in  $M^1(\mathbb{R}^d)$ . Then the operator corresponding to  $C_{\mathcal{A}}$  assigns to  $h$

$$C_{\mathcal{A}}(\langle f, g \rangle_{\mathcal{A}}) = |\Lambda|^{-1} \sum_{\lambda \in \Lambda_c} \langle f, \pi(\lambda)g \rangle \pi(\lambda)h.$$

In the same way we consider the operator associated to  $C_{\mathcal{B}}$ .

$$\begin{aligned} h\langle g, f \rangle_{\mathcal{B}} &= \sum_{\lambda \in \Lambda_c} \pi(\lambda)^* h \langle \pi(\lambda)g, f \rangle \\ &= \sum_{\lambda \in \Lambda_c} \pi(\lambda)^* h \langle g, \pi(\lambda)^* f \rangle \\ &= \sum_{\lambda \in \Lambda_c} c(\lambda, \lambda) \pi(-\lambda) h \langle g, c(\lambda, \lambda) \pi(-\lambda) f \rangle \\ &= \sum_{\lambda \in \Lambda} \pi(\lambda) h \langle f, \pi(\lambda) f \rangle. \end{aligned}$$

This implies the following relation between the associated operators of  $C_{\mathcal{A}}$  and  $C_{\mathcal{B}}$ :

$$C_{\mathcal{A}}(\langle f, g \rangle_{\mathcal{A}}) = |\Lambda|^{-1} C_{\mathcal{B}}(\langle g, f \rangle_{\mathcal{B}}),$$

which is the desired assertion.  $\square$

Recently, Packer gave an excellent survey of the use of direct integrals in the study of Gabor frames and wavelets [Pac04]. Despite our approach is based on the fact that the construction of a tight Gabor frame is equivalent to the construction of a projections in  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$ . In [Pac04] the abstract results of Rieffel, [Rief81a], are linked to the construction of tight Gabor frames. Therefore, Packer closed a longstanding gap in the literature on Gabor frames. But Packer treated only Gabor frames of the form  $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ . We remove the restriction of Packer and derived

the density theorem for Gabor frames  $\mathcal{G}(g, \Lambda)$  over a lattice  $\Lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}^d}$ . Since we rely our presentation on the results in [Rief88] where Rieffel implicitly uses Gabor frames and their Janssen representation to the construct projective modules over non-commutative tori.

Now, we attack the following problem: Let  $g$  be a non-zero element of  $L^2(\mathbb{R}^d)$  under which circumstances is the set  $\{\pi(\lambda)g : \lambda \in \Lambda\}$  dense in  $L^2(\mathbb{R}^d)$ ? The answer obviously relies on the lattice  $\Lambda$ . Recall that a Gabor frame  $\mathcal{G}(g, \Lambda)$  is a *normalized tight frame* for  $L^2(\mathbb{R}^d)$  if and only if the following reconstruction formula holds:

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g \quad \text{for all } f \in L^2(\mathbb{R}^d).$$

Consequently our question about the density of  $\{\pi(\lambda)g : \lambda \in \Lambda\}$  is equivalent to the construction of a tight Gabor frame for  $L^2(\mathbb{R}^d)$ . Recalling the definition of a cyclic vector our problem becomes the following. When is a non-zero function  $g$  a cyclic vector for the von Neumann algebra  $\mathcal{M}(\Lambda)$ ? But we have a general criterion for the existence of a cyclic vector in  $\mathcal{M}(\Lambda)$ . A function  $g \in L^2(\mathbb{R}^d)$  is a cyclic vector for  $\mathcal{M}(\Lambda)$  if and only if the coupling constant between  $\mathcal{M}(\Lambda)$  and  $\mathcal{M}(\Lambda^\circ)$  is less than 1. In Theorem 12.2.14 we have shown that this is just the multiplication operator by  $|\Lambda|$ . Therefore we obtained the following theorem:

**Theorem 12.2.15** (Density Theorem). *If for some  $g \in L^2(\mathbb{R}^d)$  the set  $\{\pi(\lambda)g : (\lambda \in \Lambda)\}$  spans  $L^2(\mathbb{R}^d)$ , then  $|\Lambda| \leq 1$ .*

Or equivalently,

**Corollary 12.2.16.** *The Gabor frame  $\mathcal{G}(g, \Lambda)$  is tight in  $L^2(\mathbb{R}^d)$ .*

We asserted the equivalence of the existence of a cyclic vector for  $\mathcal{M}(\Lambda)$  and the construction of a tight Gabor frame. The argument relies on Rieffel's construction of projections in Morita equivalent  $C^*$ -algebras. Since this construction applies equally well to the construction of projections in von Neumann algebras. In Section 12.1 we saw that  $p = \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ)g \rangle \pi(\lambda^\circ)g$  is a projection in  $C^*(\Lambda^\circ, c)$  if and only if  $f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g$ . By definition  $g$  generates a tight Gabor frame. Now every Gabor frame  $G(g, \Lambda)$  gives rise to a tight Gabor frame  $\mathcal{G}(h, \Lambda)$  for  $h = S_{g, \Lambda}^{-1/2}$ . Consequently,

**Theorem 12.2.17.** *If  $G(g, \Lambda)$  is a Gabor frame for  $L^2(\mathbb{R}^d)$ , then  $|\Lambda| \leq 1$ .*

In [Lu05a] we present an extensive discussion of the density theorem. As final remark we want to mention that the density theorem was recently rediscovered by Vershik in the setting of dynamical systems [Ver03].



# 13 Appendix

## 13.1 Banach Algebras Basics

As a primer we present the definitions and basic properties of Banach algebras and  $C^*$ -algebras. All of the material in this chapter is well-known. Our presentation is taken from the excellent survey [Dav96] by Davidson.

In the seminal paper [Gel41] commutative Banach algebras were introduced by Gelfand. Since their introduction Banach algebras and have become a very important field of mathematical research with a vast literature. The results in [Gel41] is known as Gelfand theory of commutative Banach algebras.

A **Banach algebra** is a complex normed algebra  $\mathcal{A}$  which is complete and satisfies

$$\|AB\| \leq \|A\|\|B\|, \text{ for all } A, B \in \mathcal{A}.$$

The most important example of a commutative Banach algebra is the space  $C_0(X)$  of all continuous functions on a locally compact Hausdorff space  $X$  with pointwise multiplication. Since the function  $1_X \equiv 1$  does not vanish at infinity for non-compact  $X$  the commutative Banach algebra is unital if  $X$  is a compact Hausdorff space.

The **spectrum** of an element  $A$  of a unital Banach algebra  $\mathcal{A}$  is the set

$$\sigma(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\}.$$

The complement of the spectrum is called **resolvent**  $\rho(A)$ , and  $R(\lambda, A) = (\lambda I - A)^{-1}$  is the **resolvent function** of  $A$ .

**Theorem 13.1.1.** *In any unital Banach algebra  $\mathcal{A}$ , the spectrum  $\sigma(A)$  of each  $A$  in  $\mathcal{A}$  is a non-empty compact set; and the resolvent function is analytic on  $\rho(A)$ .*

*Proof.* If  $|\lambda| > \|A\|$ , then  $\|\lambda^{-k}A^k\| \leq (|\lambda|^{-1}\|A\|)^k$  decreases geometrically fast; so the series  $\sum_{k=0}^{\infty} \lambda^{-k-1}A^k$  is norm convergent. The limit is  $(\lambda I - A)^{-1}$  since

$$(\lambda I - A) \sum_{k=0}^n \lambda^{-k-1}A^k = I - \lambda^{-n-2}A^{n+1}$$

which converges to  $I$ . Moreover, this shows that  $R(\lambda, A)$  is analytic, and has a Laurent expansion about the point infinity. Furthermore,

$$\lim_{|\lambda| \rightarrow 0} \|R(\lambda, A)\| \leq \lim_{|\lambda| \rightarrow 0} |\lambda|^{-1}(1 - |\lambda|^{-1}\|A\|)^{-1} = 0.$$

Similarly, if  $\lambda_0 I - A$  is invertible and  $|\lambda - \lambda_0| \leq \|(\lambda_0 I - A)^{-1}\|^{-1}$ , then

$$R(A, \lambda) = \sum_{k \geq 0} (\lambda - \lambda_0)^k (\lambda_0 I - A)^{-k-1}$$

is the Taylor expansion of  $R(\cdot, A)$  in a neighborhood of  $\lambda_0$ . So the resolvent function is analytic on the resolvent of  $A$ . In particular,  $f(R(\cdot, A))$  is an analytic scalar valued function for every continuous linear functional  $f$  on  $\mathcal{A}$ . The resolvent is therefore an open set containing all  $\lambda \in \mathbb{C}$  with  $|\lambda| > \|A\|$ . So  $\sigma(A)$  is a compact subset of  $\{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\}$ .

If  $\sigma(A)$  were empty, then  $R(\cdot, A)$  would be an entire function. By Liouville's Theorem, this leads to the absurd conclusion that  $R(\cdot, A)$  is the constant zero function. Indeed, for every functional  $f$ ,  $f(R(A, \lambda))$  is a scalar entire function vanishing at infinity; whence it is zero. So  $R(\cdot, A) \equiv 0$  by the Hahn-Banach Theorem. Hence the spectrum must be non-empty.  $\square$

The proof suggests the definition of the **spectral radius** of  $A$  by

$$\text{spr}(A) = \sup_{\lambda \in \text{Sp}(A)} |\lambda|.$$

The power series technique of the proof provides the useful fact which was first observed by Neumann.

**Corollary 13.1.2.** *Let  $\mathcal{A}$  be a unital Banach algebra. If  $A \in \mathcal{A}$  with  $\|A\| \leq 1$  then  $(I - A)$  is invertible with inverse  $\sum_{k \geq 0} A^k$ .*

The following elementary functional property of the spectrum is known as the **spectral mapping property**.

**Lemma 13.1.3.** *If  $p$  is a polynomial and  $A \in \mathcal{A}$  is an element of a unital Banach algebra, then  $\text{Sp}(p(A)) = p(\sigma(A))$ .*

The spectral mapping property is essential in the proof of an important fact due to Gelfand for the spectral radius of an element  $A$  in  $\mathcal{A}$ .

**Theorem 13.1.4.** *For each  $A$  in a Banach algebra  $\mathcal{A}$ , the spectral radius is determined by*

$$\text{spr}(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}.$$

*Proof.* The behaviour of the Laurent series

$$(\lambda I - A)^{-1} = \sum_{k \geq 0} \lambda^{-k-1} A^k$$

for  $R(\cdot, A)$  at infinity. Since  $R(\cdot, A)$  is analytic on  $\{\lambda : |\lambda| > \text{spr}(A)\}$ , this series converges absolutely and uniformly for  $\|\lambda\| \geq C \text{spr}(A)$ . In particular, the Taylor series coefficients  $C^{-k-1} \|A^k\|$  converge to 0 for  $C \text{spr}(A)$  which implies that

$$\limsup_{k \rightarrow \infty} \|A^k\|^{1/k} \leq \text{spr}(A).$$

On the other hand, there is a  $\lambda_0$  in  $\sigma(A)$  with  $|\lambda_0| = \text{spr}(A)$ . By the spectral mapping property,  $\lambda_0^k$  lies in the spectrum of  $A^k$ . Hence,

$$\sigma(A) = |\lambda_0^k|^{1/k} \leq \|A^k\|^{1/k} \text{ for all } k \geq 1.$$

Thus,

$$\limsup_{k \rightarrow \infty} \|A^k\|^{1/k} \leq \text{spr}(A) \leq \inf_{k \geq 1} \|A^k\|^{1/k},$$

which shows that the limit exists.  $\square$

The rest of this section is devoted to the Gelfand theory of commutative Banach algebras. A **multiplicative linear functional** on a commutative Banach algebra  $\mathcal{A}$  is a non-zero homomorphism of  $\mathcal{A}$  into  $\mathbb{C}$ . The set  $\mathcal{M}_{\mathcal{A}}$  of all multiplicative linear functionals on  $\mathcal{A}$  is called the **maximal ideal space** of  $\mathcal{A}$ . We shall show that multiplicative linear functionals have norm one. So  $\mathcal{M}_{\mathcal{A}}$  may be endowed with the topology induced by the weak  $\star$ -topology on the Banach space dual of  $\mathcal{A}$ .

**Theorem 13.1.5.** *The multiplicative linear functionals on a unital commutative Banach algebra are continuous and of norm one. The map taking each multiplicative linear functional to its kernel is a bijection onto the set of maximal ideals of  $\mathcal{A}$ .*

*Proof.* Suppose that  $\varphi$  is a multiplicative linear functional and  $A \in \mathcal{A}$  such that  $\|A\| < 1 = \varphi(A)$ . Let  $B = \sum_{k=1}^{\infty} A^k$ . Then since  $A + AB = B$ ,

$$\varphi(B) = \varphi(A) + \varphi(A)\varphi(B) = 1 + \varphi(B)$$

which is absurd. So  $\|\varphi\| \leq 1$ . Since  $\varphi(I) = 1$ , this is an equality.

It follows that  $M = \ker \varphi$  is a closed ideal of codimension 1 in  $\mathcal{A}$ , and thus is maximal. Since  $\varphi$  is determined by  $M$  and the fact that  $\varphi(I) = 1$ , this correspondence is one-to-one. Conversely, if  $M$  is a maximal ideal of  $\mathcal{A}$ , then  $\text{dist}(I, M) = 1$  because the unit ball about  $I$  consists of invertible elements. It follows that the closure of  $M$  still does not contain  $I$ . As this is a larger proper ideal, we deduce that  $M$  is closed. So the quotient algebra  $\mathcal{A}/M$  is a simple commutative Banach algebra. By Banach-Mazur Theorem this quotient is isomorphic to  $\mathbb{C}$ . So the quotient map  $\varphi$  is a continuous homomorphism of  $\mathcal{A}$  onto  $\mathbb{C}$  with kernel  $M$ .  $\square$

**Corollary 13.1.6.** *The maximal ideal space  $\mathcal{M}_{\mathcal{A}}$  of a unital commutative Banach algebra is a compact Hausdorff space. If  $\mathcal{A}$  is commutative but not unital, then  $\mathcal{M}_{\mathcal{A}}$  is locally compact.*

*Proof.* It is clear that a weak- $\star$  limit of a multiplicative linear functionals is again multiplicative. So  $\mathcal{M}_{\mathcal{A}}$  is a weak- $\star$  closed subset of the unit ball of the dual space of  $\mathcal{A}$ . By the Banach-Alaoglu Theorem, it is weak- $\star$  compact and Hausdorff.

In the non-unital case one has to form the unitization of  $\mathcal{A}$ .  $\square$

Now we define the **Gelfand transform**  $\Gamma$  of a commutative Banach algebra  $\mathcal{A}$  into  $C_0(\mathcal{M}_{\mathcal{A}})$  by  $\Gamma(A) = \hat{A}$  where

$$\hat{A}(\varphi) := \varphi(A).$$

**Theorem 13.1.7.** *The Gelfand transform is a contractive algebra homomorphism of a commutative Banach algebra  $\mathcal{A}$  into  $C_0(\mathcal{M}_{\mathcal{A}})$ . The image algebra separates the points of the maximal ideal space  $\mathcal{M}_{\mathcal{A}}$ .*

*Proof.* The functions  $\hat{A}$  are continuous because of the definition of weak- $\star$  topology. The map is contractive since each  $\varphi$  is contractive and the norm of  $C_0(\mathcal{M}_{\mathcal{A}})$  is the sup norm. Finally,  $\Gamma(\mathcal{A})$  separates points of  $\mathcal{M}_{\mathcal{A}}$  because the points correspond to distinct multiplicative linear functionals of  $\mathcal{A}$ .  $\square$

**Corollary 13.1.8.** *In a unital commutative Banach algebra  $\mathcal{A}$ ,  $A$  is invertible if and only if  $\hat{A}$  is invertible, which is precisely when  $\hat{A}$  does not vanish on  $\mathcal{M}_{\mathcal{A}}$ . Thus,*

$$\sigma(A) = \text{Sp}(\hat{A}) = \{\varphi(A) : \varphi \in \mathcal{M}_{\mathcal{A}}\}$$

and  $\|\hat{A}\|_{\infty} = \text{spr}(A)$ .

*Proof.* If  $A$  has an inverse, then since  $\Gamma$  is a homomorphism,  $\Gamma(A^{-1}) = \Gamma(A)^{-1}$ . Conversely, if  $A$  is not invertible, then as in the proof of the Banach-Mazur Theorem, the ideal  $\overline{A\mathcal{A}}$  is proper and thus contained in a maximal ideal space  $M$ . Let  $\varphi$  be the associated multiplicative linear functional. It follows that  $\hat{A}(\varphi) = 0$  and thus  $\hat{A}$  is not invertible in  $C(\mathcal{M}_{\mathcal{A}})$ . It is now immediate that  $\hat{A}$  has the same spectrum as  $A$ , and that this coincides with the range of  $\hat{A}$ . As the norm in  $C(\mathcal{M}_{\mathcal{A}})$  is the supremum norm and the range of  $\hat{A}$  is  $\sigma(A)$ , we conclude that  $\|\hat{A}\|_{\infty} = \sigma(A)$ .  $\square$

The first important application of Gelfand's theory of commutative Banach algebras was a short proof of *Wiener's lemma*.

More precisely, let  $\mathcal{W}$  be the set of all absolutely convergent Fourier series  $f(t) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k t}$  with norm  $\|f\|_{\mathcal{W}} = \sum_{k \in \mathbb{Z}} |a_k|$ . Then  $\mathcal{W}$  is a Banach algebra by defining addition, scalar multiplication and multiplication as the corresponding operations on the functions  $f$ . We want to apply Corollary 13.1.8 to determine the inverse elements of  $\mathcal{W}$ . Therefore we have to determine the maximal ideals of  $\mathcal{W}$ . Observe that a Fourier series is generated by the character  $\chi_1(t) = e^{2\pi i t}$  with  $\|\chi_1\|_{\mathcal{W}} = 1$ . Therefore we study only the linear functionals for  $\chi_1$ . Assume  $\chi_1$  is assigned the complex number  $a$  then  $\chi_1^{-1} \mapsto \chi_1$ . But  $|a| \leq \|\chi_1\|_{\mathcal{W}} = 1$  and  $|a^{-1}| \leq \|\chi_1^{-1}\|_{\mathcal{W}} = 1$  yields that  $|a| = 1$ , i.e.  $a = e^{2\pi i t_0}$  for some point  $t_0 \in [0, 1)$ . This implies that  $\sum_{k \in \mathbb{Z}} a_k e^{2\pi i k t}$  is mapped to  $\sum_{k \in \mathbb{Z}} a_k e^{2\pi i k t_0}$ . Therefore, every maximal ideal of  $\mathcal{W}$  is given by some point  $t_0 \in [0, 1)$ . In other words the maximal ideal spaces  $\mathcal{M}_{\mathcal{W}}$  is the space of all Fourier series  $f(t) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k t}$  which vanish at  $t_0$ .

**Theorem 13.1.9** (Wiener's lemma). *If  $f \in \mathcal{W}$  does not vanish on  $[0, 1)$ , then  $1/f(t)$  is again in  $\mathcal{W}$ .*

*Proof.* By assumption  $f$  does not belong to any maximal ideal and therefore by Corollary 13.1.8 it has an inverse in  $\mathcal{W}$ . This inverse coincides with the function  $1/f(t)$ .  $\square$

## 13.2 $C^*$ -algebras Basics

In 1943 Gelfand and Naimark defined the notion of an abstract  $C^*$ -algebras [GN43]. Their definition was somewhat stronger than the presented definition below but Kaplansky proved the redundancy of the axiom system due to Gelfand and Naimark. Since then  $C^*$ -algebras have found an application in a variety of areas in mathematics and physics.

A **Banach  $\star$ -algebra** is a complex Banach algebra  $\mathcal{A}$  with a conjugate linear involution  $\star$  which is an anti-isomorphism. That is, for all  $A, B \in \mathcal{A}$  and  $\lambda \in \mathbb{C}$ ,

$$\begin{aligned} (A + B)^* &= A^* + B^* \\ (\lambda A)^* &= \bar{\lambda} A^* \\ (A)^{**} &= A \\ (AB)^* &= B^* A^*. \end{aligned}$$

We will often call a Banach  $\star$ -algebra an *involutive* Banach algebra and we denote it by  $(\mathcal{A}, \star)$ . The simplest example of a Banach  $\star$ -algebra is the space  $C_0(X)$  of all continuous functions on a locally compact Hausdorff space  $X$  with complex conjugation as the adjoint operation,  $f^*(t) = \overline{f(t)}$ .

The Banach  $\star$ -algebra  $(C_0(X), \star)$  has an additional property,

$$\|\bar{f}f\|_X = \sup_{x \in X} |\overline{f(x)}f(x)| = \sup_{x \in X} |f(x)|^2 = \|f\|_X^2.$$

Another example of an involutive Banach algebra with this additional property is the algebra of bounded operators  $\mathcal{B}(\mathcal{H})$  on a Hilbert space  $\mathcal{H}$  with the usual adjoint operation,

$$\|A^*A\| = \sup_{f, g \in \mathcal{H}} \frac{\langle A^*Af, g \rangle}{\|f\|\|g\|} = \frac{\langle Af, Ag \rangle}{\|f\|\|g\|} = \|A\|^2.$$

The preceding observation led Gelfand and Naimark to the introduction of the notion of a  $C^*$ -algebra which had tremendous impact on the understanding of operators.

A  **$C^*$ -algebra** is a Banach  $\star$ -algebra with the additional norm condition

$$\|A\|^2 = \|A^*A\| \quad \text{for all } A \in \mathcal{A}.$$

When a  $C^*$ -algebra has an identity element  $I$ , then  $\|I\| = 1$ . The framework of  $C^*$ -algebras provides us with a good notion of positivity even, in the non-commutative case.

Following the definitions for operators, we say that an element  $A$  of a  $C^*$ -algebra  $\mathcal{A}$  is **self-adjoint** if  $A^* = A$ ;  $N$  is **normal** if  $N^*N = NN^*$ ; and  $U$  is **unitary** if  $U^*U = I = UU^*$ .

We define  $A$  to be **positive** if  $A$  is self-adjoint and the spectrum  $\sigma(A)$  is contained in the non-negative real line  $[0, \infty)$ .

The norm condition for  $C^*$ -algebra has far reaching consequences on the structure which do not hold for involutive Banach algebras in general, e.g. the adjoint map is isometric on a  $C^*$ -algebra but not a priori on an involutive Banach algebra.

We have already meet another important example of a  $C^*$ -algebra: the  $C^*$ -algebra  $\mathcal{K}(\mathcal{H})$  of compact operators on a separable Hilbert space. Before we treat non-commutative  $C^*$ -algebras we present a characterization of commutative  $C^*$ -algebras.

**Theorem 13.2.1.** *Let  $\mathcal{A}$  be a commutative  $C^*$ -algebra. Then the Gelfand transform is an isometric  $\star$ -isomorphism of  $\mathcal{A}$  onto  $C_0(\mathcal{M}_{\mathcal{A}})$ .*

*Proof.* We treat only the unital case since the non-unital case follows by standard arguments. Let  $\varphi$  be a multiplicative linear functional on  $\mathcal{A}$ . We will show that  $\varphi(A^*) = \overline{\varphi(A)}$ . Suppose first that  $A$  is self-adjoint. Then form a family of unitary elements

$$U_t := e^{itA} = \sum_{k=0}^{\infty} \frac{(itA)^k}{k!} \quad \text{for all } t \in \mathbb{R}.$$

Therefore,

$$1 \geq |\varphi(U_t)| = e^{it\varphi(A)} = e^{-t\text{Im}(\varphi(A))}.$$

As this holds for all real  $t$ , we deduce that  $\varphi(A)$  is real. Now if  $X$  is an arbitrary element of  $\mathcal{A}$ , we write it as  $X = A + iB$  where

$$A = (X + X^*)/2 \quad \text{and} \quad B = (X - X^*)/2i$$

are the self-adjoint elements known as the **real** and **imaginary parts** of  $X$ . We know that  $\varphi(A)$  and  $\varphi(B)$  are real, and thus

$$\varphi(X^*) = \varphi(A) - i\varphi(B) = \overline{\varphi(A) + i\varphi(B)} = \overline{\varphi(X)}.$$

So the Gelfand transform  $\Gamma$  satisfies  $\widehat{A^*} = \widehat{A}^*$ . Therefore  $\Gamma$  is a  $\star$ -homomorphism. For  $A = A^*$ , we have  $\|A\|^2 = \|A^*A\| = \|A^2\|$ . Thus

$$\|\widehat{A}\|_{\infty} = \text{spr}(A) = \lim_{k \rightarrow \infty} \|A^{2^k}\|^{1/2^k} = \lim_{k \rightarrow \infty} (\|A\|^{2^k})^{1/2^k} = \|A\|.$$

So  $\|\widehat{A}\|_{\infty} = \|A\|$ . For a general element  $T$ , we obtain

$$\|T\|^2 = \|T^*T\| = \|\widehat{T^*T}\|_{\infty} = \|\widehat{T^*}\widehat{T}\|_{\infty} = \|\widehat{T}\|_{\infty}^2.$$

Consequently, the Gelfand transform is isometric. Thus the image of  $\mathcal{A}$  under the Gelfand transform is a unital norm closed self-adjoint subalgebra of  $C(\mathcal{M}_{\mathcal{A}})$  which separates points. By the Stone-Weierstraß Theorem,  $\Gamma$  is surjective and hence is a  $\star$ -isomorphism.  $\square$

As a first application we treat the commutative  $C^*$ -algebra  $C^*(N)$  generated by a normal element  $N$ .

**Corollary 13.2.2.** *If  $N$  is a normal element of a unital  $C^*$ -algebra  $\mathcal{A}$ , then  $C^*(N)$  is isometrically  $\star$ -isomorphic to  $C(\text{Sp}(N))$ , the continuous functions on the spectrum of  $N$ , via a map that takes  $N$  to the identity function  $z(t) = t$ .*

*Proof.* Since  $N$  is normal, then  $C^*(N)$  is commutative. It suffices to determine the maximal ideal space  $X$ . Note, that a multiplicative linear functional  $\varphi$  in  $X$  is determined by  $\varphi(N) = \lambda$  for then  $\varphi(p(N, N^*)) = p(\lambda, \bar{\lambda})$  for every polynomial  $p$ . Thus, the map from  $X$  into  $\mathbb{C}$  taking  $\varphi$  to  $\varphi(N)$  is a homomorphism onto  $\widehat{N}(X)$ . From the Gelfand theory,  $\widehat{N}(X) = \text{Sp}(N)$ . This map identifies  $\widehat{N}$  with the identity function  $z$  as desired, this map is an isometric  $\star$ -isomorphism.  $\square$

In our discussion of Gabor frames we have to take the square-root of a positive operator. Here we discuss the **continuous functional calculus** for a normal element. In our treatment of reconstruction formulas for frames in a Hilbert space we will meet an important application of the continuous functional calculus for a selfadjoint operator on a Hilbert space.

**Corollary 13.2.3.** *If  $N$  is a normal element of a unital  $C^*$ -algebra and  $f$  is a continuous function on  $\text{Sp}(N)$ , the operator  $f(N)$  is defined as the inverse of  $f$  under the Gelfand transform of  $C^*(N)$ . This map is an isometric  $\star$ -isomorphism of  $C(\text{Sp}(N))$  onto  $C^*(N)$ . If  $0 \in \text{Sp}(N)$  and  $f(0) = 0$ , then  $f(N)$  lies in the non-unital algebra generated by  $N$  and  $N^*$ .*

*Proof.* The first part is immediate from the preceding corollary. We have

$$\text{Sp}(f(N)) = \text{Sp}(\widehat{f(N)}) = \text{Sp}(f) = f(\sigma(N)).$$

When  $p$  is a polynomial in  $z$  and  $\bar{z}$ , it is immediate from the fact that the functional calculus is a homomorphism that  $p(f(N)) = (p \circ f)(N)$ . The general case follows by approximating the continuous function  $f$  by polynomials.  $\square$

**Corollary 13.2.4.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra.*

1. *If  $N$  is normal, then  $\|N\| = \text{spr}(N)$ .*
2. *If  $A$  is self-adjoint, then  $\sigma(A)$  is real.*
3. *If  $U$  is unitary then  $\text{Sp}(U)$  is contained in the unit circle.*

*Proof.* By Corollary 13.2.2, we have

$$\|N\| = \|\widehat{N}\|_{\text{Sp}(N)} = \|z\|_{\text{Sp}(N)} = \text{spr}(N).$$

If  $A$  is self-adjoint, then  $\overline{\widehat{A}} = \widehat{A}^* = \widehat{A}$  is a real-valued function. Since  $\sigma(A)$  is the range of  $\widehat{A}$ , part (ii) follows. Similarly, if  $U$  is unitary, then

$$|\widehat{U}|^2 = \widehat{U}^* \widehat{U} = \widehat{I} = 1.$$

$\square$

We return to the investigation of general non-commutative  $C^*$ -algebras. Motivated by the notion of positive self-adjoint bounded operators on a Hilbert space.

A self-adjoint element  $A$  of a unital  $C^*$ -algebra is **positive** if  $\sigma(A) \subset [0, \infty)$ .

The positive elements determine an **order** on the self-adjoint elements  $\mathcal{A}_{sa}$  of  $\mathcal{A}$  by setting  $A \leq B$  if  $B - A$  is positive. We begin with another easy consequence of the continuous functional calculus.

**Corollary 13.2.5.** *Each positive element of a  $C^*$ -algebra has a unique positive square root.*

*Proof.* The square root function  $f(x) = \sqrt{x}$  is a continuous function on  $[0, \|A\|]$  which contains  $\sigma(A)$ . Thus  $B := f(A)$  is self-adjoint because  $f$  is real valued; and  $\text{Sp}(B) = f(\sigma(A))$  is contained in the positive real line. So  $B$  is positive. Moreover  $B^2 = f^2(A) = A$ . Suppose that  $C$  is another positive square root of  $A$ . Then by the functional calculus,  $C = f(C^2) = f(A) = B$ .  $\square$

The following lemma contains some useful characterizations of positivity.

**Lemma 13.2.6.** *For a self-adjoint element  $A$  in a  $C^*$ -algebra  $\mathcal{A}$ , the following conditions are equivalent:*

- (1)  $A \geq 0$ ;
- (2)  $A = B^2$  for some self-adjoint element  $B$ ;
- (3)  $\|\lambda I - A\| \leq \lambda$  for all  $\lambda \geq \|A\|$ ;
- (4)  $\|\lambda I - A\| \leq \lambda$  for one  $\lambda \geq \|A\|$ .

*Proof.* (1) implies (2) is the preceding Corollary. Assuming (2), there is a self-adjoint element  $B$  such that  $A = f(B)$ , where  $f \in C(\sigma(A))$  is the function  $f(x) = x^2$ . Consequently  $\|f\|_{\sigma(B)} = \|A\|$ ; and thus  $0 \leq f \leq \|A\| \leq \lambda$ . Therefore,  $0 \leq \lambda - f \leq \lambda$ . So

$$\|\lambda I - A\| = \|(c - f)(B)\| = \|c - f\|_{\text{Sp}(B)} \leq \lambda.$$

This establishes (3), which clearly implies (4). Assuming that (4) holds for a particular value of  $\lambda$ , we conclude that

$$\lambda \geq \|\lambda I - A\| = \|(c - z)(A)\| = \|c - z\|_{\sigma(A)}.$$

Thus the identity function is non-negative on  $\sigma(A)$ . That is,  $\sigma(A)$  is contained in  $[0, \infty)$ ; and thus  $A$  is positive.  $\square$

**Corollary 13.2.7.** *If  $A$  and  $B$  are positive elements of  $\mathcal{A}$ , then  $A + B$  is also positive.*

*Proof.* Choose  $r \geq \|A\|$  and  $s \geq \|B\|$ . Then  $r + s \geq \|A + B\|$  and

$$\|(r + s)I - (A + B)\| \leq \|rI - A\| + \|sI - B\| \leq r + s.$$

Therefore  $A + B$  is positive. □

The main result about positivity is the following theorem.

**Theorem 13.2.8.** *If  $A$  belongs to a  $C^*$ -algebra, then  $A^*A$  is positive.*

We refer the reader to [Dav96] for a proof.

### 13.3 Representations of $C^*$ -algebras

This section is devoted to basic properties of  $C^*$ -algebras and the fundamental theorem of Gelfand and Naimark that every  $C^*$ -algebra is isomorphic to a concrete  $C^*$ -algebra of operators. Our presentation is largely of expository style since we want to introduce the basic notions, constructions and theorems of a wide field of mathematics.

A **representation**  $\pi$  of a  $C^*$ -algebra  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$  is a  $\star$ -homomorphism of  $\mathcal{A}$  into  $\mathcal{B}(\mathcal{H})$ . We say that  $\pi$  is **topologically irreducible** if  $\pi(\mathcal{A})$  has no proper closed invariant subspaces. It is called **algebraically irreducible** if it has no proper invariant subspaces that are not necessarily closed. A fundamental theorem shows that these two notions coincide for  $C^*$ -algebras. Therefore we will call  $\pi$  irreducible when this condition holds.

**Theorem 13.3.1.** *Every topologically irreducible representation  $\pi$  of a  $C^*$ -algebra is algebraically irreducible.*

The assertion follows from Kadison's Transitivity Theorem since we have to show that if  $f$  and  $g$  are unit vectors in  $\mathcal{H}$  then there is an element  $A$  in  $\mathcal{A}$  such that  $\pi(A)f = g$ .

If  $S$  is any subset of  $\mathcal{B}(\mathcal{H})$ , then the **commutant** of  $S$  is

$$S' := \{T \in \mathcal{B}(\mathcal{H}) : ST = TS \text{ for all } S \in S\}.$$

There is a characterization of irreducible representations in terms of commutants which is often called **Schur's lemma**.

**Theorem 13.3.2** (Schur's lemma). *Let  $\pi$  be a representation of a  $C^*$ -algebra on a Hilbert space  $\mathcal{H}$ . Then  $\pi(\mathcal{A})$  is irreducible if and only if  $\pi(\mathcal{A})' = \mathbb{C}I$ .*

Since Schur's lemma is one of the key tools in our study of time-frequency analysis we present a proof and an equivalent formulation.

*Proof.* If  $\pi(A)'$  is larger than the scalars, then it contains a non-scalar positive and thus it contains a proper projection  $P$ . Thus  $P\mathcal{H}$  is an invariant subspace for  $\pi(A)$ .

Conversely, suppose that  $\mathcal{M}$  is a proper invariant subspace for  $\pi(\mathcal{H})$ ; and let  $P$  be the orthogonal projection onto  $\mathcal{M}$ . Invariance is expressed algebraically as  $\pi(A)P = P\pi(A)$  for every  $A \in \mathcal{A}$ . However, it then follows that

$$P\pi(A) = (\pi(A^*)P)^* = (P\pi(A)P^*)^* = P\pi(A)P = \pi(A)P$$

for every  $A \in \mathcal{A}$ . Thus  $P$  is a non-scalar operator in  $\pi(A)'$ .  $\square$

Recall that the commutant of  $\mathbb{C}I$  is  $\mathcal{B}(\mathcal{H})$ , Schur's lemma is equivalent to  $\pi(\mathcal{A})'' = \mathcal{B}(\mathcal{H})$ . Another formulation of Schur's lemma relies on the notion of a cyclic vector. A representation  $\pi$  of a  $C^*$ -algebra is **cyclic** if there is a **cyclic vector**  $f$  such that  $\pi(\mathcal{A})f$  is dense in  $\mathcal{H}$ . A representation  $\pi$  of a  $C^*$ -algebra is **non-degenerate** if  $\pi(\mathcal{A})\mathcal{H}$  is dense in  $\mathcal{H}$ .

**Corollary 13.3.3.** *A representation  $\pi$  of a  $C^*$ -algebra is irreducible if and only if  $\pi(\mathcal{A})$  is cyclic.*

*Proof.* Suppose there is a vector  $f$  in  $\mathcal{H}$  such that  $\pi(\mathcal{A})f$  is not dense in  $\mathcal{H}$ , then the projection onto the closure of  $\pi(\mathcal{A})\mathcal{H}$  is an element of  $\pi(\mathcal{A})'$ , therefore  $\pi$  is not irreducible. The converse assertion is trivial.  $\square$

An important notion is that of a state on a  $C^*$ -algebra. Weyl and von Neumann motivated by statistical mechanics introduced states for the  $C^*$ -algebra of compact operators. Later Segal extended the notion of states to a general  $C^*$ -algebra which arose from the observation that Gelfand and Naimark's theorem is based on the construction of certain states on a general  $C^*$ -algebra.

A **positive linear functional** on a  $C^*$ -algebra is a linear functional such that  $f(A) \geq 0$ . A **state** is a positive linear functional of norm 1. The set of all states on  $\mathcal{A}$  is known as the **state space**  $\mathcal{S}(\mathcal{A})$  of  $\mathcal{A}$ . A state is called **pure** if it is an extreme point of  $\mathcal{S}(\mathcal{A})$ .

Positive linear functionals are continuous and for unital  $C^*$ -algebras  $\|\psi\| = f(I)$ . The great importance of positive linear functionals arises from the fact that they allow the definition of a positive semidefinite sesqui-linear form on  $\mathcal{A}$  given by

$$[A, B] := \psi(B^*A).$$

That is,  $[\cdot, \cdot]$  is linear in the first variable and conjugate linear in the second, and  $[A, A] \geq 0$  for all  $A \in \mathcal{A}$ . Hence it satisfies the **Cauchy-Schwarz inequality**

$$|[A, B]|^2 \leq [A, A][B, B]$$

or equivalently

$$|\psi(B^*A)|^2 \leq \psi(A^*A)\psi(B^*B).$$

Every representation  $\pi$  of a  $C^*$ -algebra  $\mathcal{A}$  on a Hilbert space defines a positive linear functional, the so-called **vector state** of  $\pi$ . Let  $f$  be a unit vector in  $\mathcal{H}$ , then

$$\psi(A) := \langle \pi(A)f, f \rangle$$

is positive. Since, if  $A \geq 0$ , then

$$\psi(A) = \langle (\pi(A^{1/2})^2)f, f \rangle = \|\pi(A^{1/2})f\|^2 \geq 0.$$

In the unital case this is a state when  $\|f\| = 1$ . For example, the pure states of the continuous functions  $C_0(X)$  on a locally compact space  $X$  vanishing at infinity are homeomorphic to the points of  $X$ .

There is a correspondence between irreducible representations of a  $C^*$ -algebra  $\mathcal{A}$  and pure states of  $\mathcal{A}$ .

**Theorem 13.3.4.** *Let  $\pi$  be a representation of a  $C^*$ -algebra  $\mathcal{A}$  with a cyclic unit vector  $f$ . Then the state  $\psi(A) := \langle \pi(A)f, f \rangle$  is pure if and only if  $\pi$  is irreducible.*

The key to representing a  $C^*$ -algebra on a Hilbert space is to build representations from states. This important procedure is called the **GNS-construction** named after **Gelfand, Naimark and Segal**.

**Theorem 13.3.5** (GNS-construction). *Let  $f$  be a positive linear functional on  $\mathcal{A}$ . Then there is a representation  $\pi_\psi$  of  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$  and a vector  $f_\psi$  in  $\mathcal{H}$  which is a cyclic vector for  $\pi(\mathcal{A})$  such that  $\|f_\psi\|^2 = \|f\|$  and*

$$\psi(A) = \langle \pi_\psi(A)f_\psi, f_\psi \rangle \text{ for all } A \in \mathcal{A}.$$

Now the key step in constructing states is that every self-adjoint element of  $\mathcal{A}$  gives a pure state on  $\mathcal{A}$ .

**Lemma 13.3.6.** *Let  $A$  be a self-adjoint element of  $\mathcal{A}$ . Then there exists a pure state  $\psi$  on  $\mathcal{A}$  such that  $|\psi(A)| = \|A\|$ .*

As a corollary we get sufficiently many states to represent a  $C^*$ -algebra  $\mathcal{A}$  isometrically as a concrete  $C^*$ -algebra of operators on a Hilbert space  $\mathcal{H}$ .

**Corollary 13.3.7.** *If  $A$  is an element of  $\mathcal{A}$ , then there exists an irreducible representation of  $\mathcal{A}$  and a unit vector  $f$  such that  $\|\pi(A)f\| = \|A\|$ .*

We have all the pieces to state the **Gelfand-Naimark Theorem**.

**Theorem 13.3.8.** *Every abstract  $C^*$ -algebra  $\mathcal{A}$  is isometrically involutive isomorphic to a concrete  $C^*$ -algebra of operators on a Hilbert space  $\mathcal{H}$ . If  $\mathcal{A}$  is separable, then one may take  $\mathcal{H}$  to be separable.*

*Proof.* Take the **universal representation**  $\pi_u$  of  $\mathcal{A}$ . Note that every state on  $\pi_u(\mathcal{A})$  is a vector state in this representation. Then  $\|\pi_u(A)\| = \|A\|$  for every  $A \in \mathcal{A}$  by the previous corollary. When  $\mathcal{A}$  is separable, it suffices to choose one representation  $\pi_n$  for each element  $A_n$  of a countable dense subset of  $\mathcal{A}$  such that  $\|\pi_n(A_n)\| = \|A_n\|$ . Then  $\pi := \sum \otimes_n \pi_n$  works and is separately acting.  $\square$

Recall that the **Jacobson radical**  $\text{rad}(\mathcal{A})$  of a Banach algebra  $\mathcal{A}$  is the intersection of the kernels of all algebraically irreducible representations. An algebra is called **semi-simple** if the Jacobson radical is  $\{0\}$ .

**Corollary 13.3.9.** *A  $C^*$ -algebra is semi-simple.*

*Proof.* Since topologically irreducible representations of a  $C^*$ -algebra are algebraically irreducible. We apply this to vector states  $\pi_\psi$  for  $\psi$  pure. Since the pure states separate points in  $\mathcal{A}$ , the representation

$$\sum_{\psi \text{ pure } \in \mathcal{S}(\mathcal{A})} \pi_\psi$$

is isometric. Hence,

$$\text{rad}(\mathcal{A}) = \bigcap_{\psi \text{ pure } \in \mathcal{S}(\mathcal{A})} \ker \pi_\psi = \{0\}.$$

□

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## Curriculum Vitae

Name: Franz Luef

Address: Brigittenauerlände 174/6/1

A-1200 Vienna, AUSTRIA

Nationality: Austria

Born on 6th of October, 1975 in Vienna

1982-1986: Grammar School in Kirchberg am Wechsel

1986-1990: High School in Kirchberg am Wechsel

1990-1995: Technical high school in Wr. Neustadt

1995-2005: Studies in Mathematics and Physics

Oct. 2001: Master's Degree in Mathematics

Master Thesis on the topic: "Perturbative Oscillation Theorems for Jacobi equations"

Since 1996: Teaching assistant for several mathematics courses at the Department of Mathematics, University of Vienna

Since 2001 working with the group NUHAG (Numerical Harmonic Analysis Group) headed by Prof. Hans G. Feichtinger, University Vienna, Austria

2001: Visiting Philips Research Lab Natlab, Eindhoven, Netherlands,

2003: Conference Signal Analysis and Imaging, Singapore,

2004: Conference Third Viennese Gabor Workshop, Austria,

2005: Guest Researcher at Max-Planck Institute for Mathematics, Bonn, Germany,

2005: Conference Modern Methods in Time-Frequency Analysis, Strobl, Austria

2005: Conference GPOTS, Orlando, USA

2005: Conference FA XI, Dubrovnik, Croatia