

## Gabor analysis, Noncommutative Tori and Feichtinger's algebra

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*"Often in mathematics, understanding comes from generalization,  
instead of considering the object per se when one tries to  
find the concepts which embody the power of the object."*

Alain Connes

### 1. Abstract

We point out a connection between Gabor analysis and noncommutative analysis. Especially, the strong Morita equivalence of noncommutative tori appears as underlying setting for Gabor analysis, since the construction of equivalence bimodules for noncommutative tori has a natural formulation in the notions of Gabor analysis. As an application we show that Feichtinger's algebra is such an equivalence bimodule. Furthermore, we present Connes's construction of projective modules for noncommutative tori and the relevance of a generalization of Wiener's lemma for twisted convolution by Gröchenig and Leinert. Finally we indicate an approach to the biorthogonality relation of Wexler-Raz on the existence of dual atoms of a Gabor frame operator based on results about Morita equivalence.

## 2. Introduction

By definition, Morita equivalent algebras  $\mathcal{A}$  and  $\mathcal{B}$  have equivalent categories of (left) modules. The main theorem of algebraic Morita theory states that Morita equivalences are implemented by a  $(\mathcal{A}, \mathcal{B})$ -bimodule  $X$ . In [Mor58] Morita proved the fundamental theorem, showing that this bimodule is invertible if and only if the bimodule is projective and finitely generated as a left  $\mathcal{A}$ -module and as a right  $\mathcal{B}$ -module and  $\mathcal{A} \rightarrow \text{End}_{\mathcal{B}}(X)$  and  $\mathcal{B} \rightarrow \text{End}_{\mathcal{A}}(X)$  are algebra isomorphisms. For further discussion on the Morita equivalence of algebras and the relevant modules and functors we refer to the recent paper [BW04].

In the seminal papers [Rief74a,Rief74b,Rief76] Rieffel developed the notion of *strong Morita Equivalence*, which is an extension of Morita equivalence of algebras to the setting of  $C^*$ -algebras. The relevant category of modules over a  $C^*$ -algebra, to be preserved under strong Morita equivalence, is that of Hilbert spaces on which the  $C^*$ -algebra acts through bounded operators. More precisely, for a given  $C^*$ -algebra  $\mathcal{A}$ , we consider the category  $\text{Herm}(\mathcal{A})$  whose objects are pairs  $(\mathcal{H}, \rho)$ , where  $\mathcal{H}$  is a Hilbert space and  $\rho : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  is a nondegenerate  $*$ -homomorphism of algebras, and morphisms are bounded intertwiners. Since we are dealing with more elaborate modules, it is natural that a bimodule giving rise to a functor  $\text{Herm}(\mathcal{B}) \rightarrow \text{Herm}(\mathcal{A})$  should be equipped with extra structure. If  $(\mathcal{H}, \rho) \in \text{Herm}(\mathcal{B})$  and  ${}_{\mathcal{A}}V_{\mathcal{B}}$  is an  $(\mathcal{A}, \mathcal{B})$ -bimodule, then if  $V$  itself is equipped with an inner product  $\langle \cdot, \cdot \rangle_{\mathcal{B}}$  with values in  $\mathcal{B}$ . More precisely, let  $V$  be a right  $\mathcal{B}$ -module. Then a  $\mathcal{B}$ -valued inner product  $\langle \cdot, \cdot \rangle_{\mathcal{B}}$  on  $V$  is a  $\mathbb{C}$ -sesquilinear pairing  $V \times V \rightarrow \mathcal{B}$  (linear in the second argument) such that, for all  $f_1, f_2 \in V$  and  $T \in \mathcal{B}$ , we have

- (1)  $\langle f_1, f_2 \rangle_{\mathcal{B}} = \langle f_2, f_1 \rangle_{\mathcal{B}}^*$
- (2)  $\langle f_1, f_2 T \rangle_{\mathcal{B}} = \langle f_1, f_2 \rangle_{\mathcal{B}} T$
- (3)  $\langle f_1, f_1 \rangle_{\mathcal{B}} > 0$  if  $f_1 \neq 0$ .

Inner products on left modules are defined analogously, but linearity is required in the first argument. One can show that  $\|f\|_{\mathcal{B}} := \|\langle f, f \rangle_{\mathcal{B}}\|^{1/2}$  is a norm in  $V$ . A (right) **Hilbert  $\mathcal{B}$ -module** is a (right)  $\mathcal{B}$ -module  $V$  together with a  $\mathcal{B}$ -valued inner product  $\langle \cdot, \cdot \rangle_{\mathcal{B}}$  so that  $V$  is complete with respect to  $\|\cdot\|_{\mathcal{B}}$ . Our treatment of strong Morita equivalence is largely based on a recent paper of Bursztyn and Weinstein about the connection of Poisson

geometry and noncommutative geometry, [BW04].

In the study of wavelets Rieffel/Packer and Wood have recently used Hilbert  $C^*$ -modules, see [PR03,PR04,W04]. In the context of Gabor analysis we want to mention the work of Casazza/Lammers [CL03], but their work is unrelated to our investigations.

The main goal of this paper is that Rieffel's results on the strong Morita equivalence of noncommutative tori have a natural interpretation in terms of Gabor analysis. Recall, that

$$T_x f(t) = f(t - x)$$

denotes a *translation* by  $x \in \mathbb{R}^d$  for  $f$  in  $L^2(\mathbb{R}^d)$  and

$$M_\omega f(t) = e^{2\pi i \omega t} f(t)$$

is a *modulation* by  $\omega \in \widehat{\mathbb{R}}^d$ . More precisely, we show – following Rieffel – [Rief88] that the  $C^*$ -algebra  $C^*(\Lambda)$  generated by time-frequency shifts  $\pi(\lambda) = M_\omega T_x$  for  $\lambda = (x, \omega)$  in a lattice  $\Lambda$  of the time-frequency plane  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  is Morita equivalent to the  $C^*$ -algebra of time-frequency shifts  $C^*(\Lambda^0)$  generated by  $\pi(\lambda^0)$  for  $\lambda^0$  in the *adjoint lattice*  $\Lambda^0$ , see Section 2 for the definition of the adjoint lattice.

Our main result is that *Feichtinger's algebra*  $S_0(\mathbb{R}^d)$  is a bimodule for the  $C^*$ -algebras  $(C^*(\Lambda), C^*(\Lambda^0))$  or an *equivalence* bimodule as called by Rieffel.

Rieffel used Schwartz's space  $\mathcal{S}(\mathbb{R}^d)$  in his construction of a bimodule for the  $C^*$ -algebras  $(C^*(\Lambda), C^*(\Lambda^0))$ . Therefore, our investigations are a further realization of a general strategy of Feichtinger, that consists in using  $S_0(\mathbb{R}^d)$  is a good substitute for  $\mathcal{S}(\mathbb{R}^d)$  in many situations.

As examples we state Feichtinger's seminal paper [F81], Reiter's work on Weil's construction of the metaplectic group [Rei89] and Gröchenig/Leinert's result on the irrational case conjecture in Gabor analysis [GL04]. The present paper may be seen as a companion to [GL04], which provides some reasons for the relevance of noncommutative tori in Gabor analysis. On the other hand we show that Rieffel obtained the Janssen representation of Gabor frame operators, and the Fundamental Identity of Gabor analysis for functions in the Schwartz-Bruhat space  $\mathcal{S}(G)$  for a locally compact abelian group  $G$  and a closed subgroup  $D$  in  $G \times \widehat{G}$  already in 1988. More

concretely, Rieffel observed that

$$\langle f, g \rangle_\Lambda = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda) \quad (1)$$

is a  $C^*(\Lambda)$ -valued inner product for  $f, g$  in  $\mathcal{S}(\mathbb{R}^d)$ .

It comes quite unexpectedly that Rieffel's work on Morita equivalence for noncommutative tori deals with the same mathematical structure as Janssen's work on Gabor frame operators, [Jan95]. The main goal of this paper is an exploration of this observation.

The paper is organized as follows: In Section 3 we discuss operator algebras of time-frequency shifts for a separated set and we show that there is a correspondence between symmetry of the separated set in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  and the structure of the corresponding operator algebra. In this context we review some well-known results on projective representations and we define the adjoint set of an arbitrary separated set in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  and discuss its relation to the structure of the commutant of the operator algebra.

In Section 4 we define noncommutative tori and Feichtinger's algebra. In this context we show that (1) is a Hilbert  $C^*(\Lambda)$ -valued inner product for  $f, g$  in  $S_0(\mathbb{R}^d)$  using functorial properties of  $S_0(\mathbb{R}^d)$ . Furthermore we introduce the Short-Time Fourier Transform and its properties for functions in Feichtinger's algebra  $S_0(\mathbb{R}^d)$ .

In Section 5 we prove our main theorem, by showing that Feichtinger's algebra is a bimodule with respect to the pair of  $C^*$ -algebras  $C^*(\Lambda)$  and  $C^*(\Lambda^0)$ . We discuss the notion of strong Morita equivalence in this setting and its relation to Gabor expansions, Gabor frame operators,  $\Lambda$ -invariant operators for a lattice  $\Lambda$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . In the proof of Rieffel's associativity condition we observe its equivalence to the Fundamental Identity of Gabor Analysis and the existence of Janssen representation for Gabor frame operators, [Rief88]. Our presentation follows [Rief88], where he used the Poisson's summation formula for symplectic Fourier transform with respect to a lattice and its adjoint lattice in the proof of the Fundamental Identity of Gabor Analysis. Furthermore, we present the Gröchenig/Leinert [GL04] result on Wiener's lemma for twisted convolution in the context of Connes' construction of projective modules over  $C^*(\Lambda)$ .

In Section 6 we apply our results on strong Morita equivalence of the  $C^*$ -algebras  $(C^*(\Lambda), C^*(\Lambda^0))$  to the structure of Gabor frames. Especially we derive the Wexler-Raz biorthogonality principle for the existence of dual

windows for Gabor frames from the relation between traces on the Morita equivalent  $C^*$ -algebras on  $C^*(\Lambda)$  and  $C^*(\Lambda^0)$ , respectively.

### 3. Operator Algebras of Time-Frequency Shifts

The most general framework in Gabor analysis builds reconstruction formulas for functions on  $\mathbb{R}^d$  from a set of time-frequency shifts  $\mathcal{G} = \{\pi(X_j) : X_j \in A\}$  for a countable subset  $A = \{X_j = (x_j, \omega_j) : j \in J\}$  with  $\inf_{j,k} |X_j - X_k| > \delta > 0$  in the time-frequency plane  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . We will refer to this setting as *non-uniform Gabor systems*.

First we derive some results about the structure of the operator algebra of a non-uniform Gabor system  $\mathcal{G}$  from general principles. First the *commutant*  $\mathcal{G}'$  of all bounded operators on  $L^2(\mathbb{R}^d)$  commuting with every operator in  $\mathcal{A}$  is a unital Banach algebra of time-frequency shifts with respect to operator composition and the operator norm, because  $\mathcal{A}$  is a subset of unitary (bounded) operators on  $L^2(\mathbb{R}^d)$ . Furthermore, if  $\mathcal{G}$  is generated by a set  $A$  which is symmetric with respect to the origin, then  $\mathcal{G}$  is invariant under taking adjoints, moreover  $\mathcal{G}$  is actually a group of time-frequency shifts acting on  $L^2(\mathbb{R}^d)$ . We have the following chains between  $\mathcal{G}$  and its  $n$ -th commutant  $\mathcal{G}^{(n)}$ :

$$\begin{aligned} \mathcal{G} \subset \mathcal{G}'' = \mathcal{G}^{(4)} = \dots = \\ \mathcal{G}' = \mathcal{G}^{(3)} = \mathcal{G}^{(5)} = \dots \end{aligned}$$

From now on, let our set of time-frequency shifts  $A$  be a lattice  $\Lambda$  of  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  then  $\mathcal{G} = \mathcal{G}''$ . The associated Gabor system is called to be *regular*. These observations indicate a strong relationship between the symmetry of the set  $A$  and the structure of the corresponding operator algebra  $\mathcal{G}$  and its commutant  $\mathcal{G}'$ . In the following we discuss the connection between the set  $A$  of points in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  and the set  $A^0$  of points, which generate the commutant  $\mathcal{G}'$ . This investigation yields a beautiful structure of regular Gabor systems, which is the very reason for the existence of all of our subsequent results. Therefore, we treat the relation between  $A$  and  $A^0$  more concretely.

Our presentation of the relation between  $A$  and  $A^0$  relies on the properties of projective representations of the time-frequency plane  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . Later we will need some results about bicharacters and 2-cocycles associated to our projective representations. Therefore, we now recall their definitions

and some of their basic properties. For a similar treatment of projective representation, see [DV04].

Let  $G$  be a locally compact abelian group and  $\mathbb{T}$  the multiplicative group of complex numbers of absolute value 1. A map  $\beta$  of  $G \times G$  with values in  $\mathbb{T}$  is a *multiplier* or (2-) *cocycle* if it satisfies for all  $x, y, z \in G$ :

$$\begin{aligned}\beta(x, 0) &= \beta(0, x) = 1, \\ \beta(x + y, z)\beta(x, y) &= \beta(x, y + z)\beta(y, z).\end{aligned}$$

Two cocycles  $\beta$  and  $\beta'$  are called *equivalent* or *cohomologous* if there is a Borel map  $c$  of  $G$  into  $\mathbb{T}$ , such that for all  $x, y \in G$

$$\beta'(x, y) = \beta(x, y) \frac{c(x+y)}{c(x)c(y)}$$

A *projective representation*  $\pi$  of  $G$  is a map of  $G$  into the unitary group of a Hilbert space  $\mathcal{H}$  such that for a cocycle  $\beta$

$$\pi(x)\pi(y) = \beta(x, y)\pi(x + y), \quad \pi(0) = 1.$$

The map  $X = (x, \omega) \mapsto \pi(x, \omega) = M_\omega T_x$  for  $X \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$  into  $\mathcal{U}(L^2(\mathbb{R}^d))$  is a projective representation of the time-frequency plane with cocycle  $\beta'(X, Y) = e^{2\pi i y \cdot \omega}$  for  $X = (x, \omega)$  and  $Y = (y, \eta)$ .

This projective representation of the time-frequency plane is equivalent to a projective representation  $\pi$  with the symplectic bicharacter  $\beta(X, Y) = e^{2\pi i(y\omega - x\eta)}$  via the map  $c : (x, \omega) \mapsto e^{2\pi i x \cdot \omega}$  for  $X = (x, \omega)$  and  $Y = (y, \eta)$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ .

Recall that a *bicharacter* of a locally compact abelian group  $G$  is continuous map  $b$  of  $G \times G$  into  $\mathbb{T}$ , which is a character in each argument. Any such  $b$  induces a morphism  $\gamma = \gamma_b$  of  $G$  into  $\widehat{G}$  by

$$\langle x, \gamma_b(y) \rangle = b(x, y).$$

A bicharacter is called *antisymmetric* if it satisfies for all  $x, y \in G$

$$b(x, y)b(y, x) = 1, \quad b(x, x) = 1$$

and *symplectic* if it is alternating and  $\gamma_b$  is an isomorphism.

In the study of projective representations  $\pi$  of  $G$  with cocycle  $\beta$  symplectic bicharacters of  $G$  appear naturally in the commutation rule

$$\pi(x)\pi(y)\pi(x)^{-1}\pi(y)^{-1} = b(x, y)I \tag{2}$$

with  $b(x, y) = \frac{\beta(x, y)}{\beta(y, x)}$  for  $x, y \in G$ .

For our projective representation by time-frequency shifts of  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  we recover the bicharacter  $\rho(X, Y) = e^{2\pi i(y\omega - \eta x)}$  for  $X = (x, \omega)$  and  $Y = (y, \eta)$ . Due to the close relation of the projective representation of  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  by time-frequency shifts  $\pi(x, \omega) = M_\omega T_x$  with Heisenberg's commutation relation the bicharacter  $\rho(X, Y) = e^{2\pi i(y\omega - \eta x)}$  is called the *Heisenberg cocycle* for  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ .

The commutation relation (2) motivates the following definition.

**Definition 3..1:** Let  $A$  be a subset of  $G$  and  $b$  a bicharacter of  $G$ . Then the *adjoint set*  $A_b^0$  of  $A$  with respect to  $b$  is given by

$$A_b^0 = \{x \in G : b(x, a) = 1 \text{ for all } a \in A\}.$$

The adjoint of a set  $A$  is a closed subgroup of  $G$  by continuity of the character  $b$ . Furthermore a subgroup  $A$  of  $G$  is called to be *isotropic* for  $b$  if  $b|_{A \times A} \equiv 1$ , or equivalently  $A \subset A_b^0$ . We call a group  $A$  of  $G$  *maximal isotropic* for  $b$  if  $A = A_b^0$ , which are again closed subgroups of  $G$ .

We are now in position to answer our original question on the relation between a set  $A$  generating a non-uniform Gabor system  $\mathcal{G}$  and the set  $A^0$  generating the commutant  $\mathcal{G}'$ .

$$\begin{aligned} A \subset A_b^{00} = A_b^{(4)} = \dots = \\ A_b^0 = A_b^{(3)} = A_b^{(5)} = \dots, \end{aligned}$$

where  $A_b^{(n)}$  denotes the  $n$ th adjoint of  $A$ .

The above chains of relations are the set analogous of (2) and therefore, the map  $A \mapsto A_b^{(0)}$  is the desired correspondence.

For the projective representation of  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  by time-frequency shifts  $\pi(x, \omega) = M_\omega T_x$  with the Heisenberg cocycle  $\rho$  we obtain the well-known *adjoint group*  $\Lambda^0$  of a regular Gabor system generated by a lattice  $\Lambda$  of  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . In this case the maximal isotropic lattice of  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  is the standard (von Neumann) lattice  $\mathbb{Z}^d \times \mathbb{Z}^d$ .

Let us mention that the operator algebra  $\mathcal{G} = \{\pi(\lambda) : \lambda \in \Lambda\}$  is a commutative group of time-frequency shifts if and only if  $\Lambda$  is isotropic, i.e.  $\Lambda \subset \Lambda^0$ .

It is an important fact that one can interpret the Heisenberg cocycle as follows:  $\rho$  is a character of  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ , and every character of  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  is of the form

$$X \mapsto \rho(X, X') \quad \text{for some } X' \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d. \quad (3)$$

This induces an isomorphism between  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  and its dual group  $\widehat{\mathbb{R}^d \times \widehat{\mathbb{R}}^d}$ .

If  $\Lambda$  is a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ , then every character of  $\Lambda$  extends to a character of  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  and therefore every character of  $\Lambda$  is of the form

$$\lambda \mapsto \rho(\lambda, Y), \quad \lambda \in \Lambda \quad (4)$$

for some  $Y \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ , where  $Y$  needs not to be unique. The homomorphism from  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  to  $\widehat{\Lambda}$  has as kernel the *adjoint lattice*

$$\Lambda^0 = \{Y \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d \mid \rho(\lambda, Y) = 1 \text{ for all } \lambda \in \Lambda\}. \quad (5)$$

As a consequence we have, that the adjoint set of a lattice has the structure of a lattice. The skew-bicharacter  $\rho$  of  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  gives a Fourier transform  $\widehat{F}^s$  on the time-frequency plane. We call

$$\begin{aligned} \widehat{F}^s(Y) &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \rho(Y, X) F(X) dX \\ &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} e^{2\pi i(y \cdot \omega - x \cdot \eta)} F(x, \omega) dx d\omega \end{aligned}$$

the *symplectic Fourier transform* of  $F \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ , since it is induced by the symplectic form  $\Omega$  on  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . The symplectic Fourier transform will be essential in our proof of the Fundamental Identity of Gabor Analysis.

#### 4. Noncommutative Tori and Feichtinger's Algebra

A *noncommutative 2d-torus*  $\mathcal{A}_\Theta$  is the universal  $C^*$ -algebra generated by  $2d$  unitaries  $U_1, \dots, U_{2d}$  subject to the commutation relations

$$U_j U_k = e^{2\pi i \theta_{jk}} U_k U_j, \quad k, j = 1, \dots, 2d$$

for a skew symmetric matrix  $\Theta = (\theta_{jk})$  with real entries.

We regard  $\Theta$  as a real skew-bilinear form on  $\mathbb{Z}^{2d}$ , with entries given by  $\Theta(e_j, e_k) = \theta_{jk}$ . Then a noncommutative  $2d$ -torus  $\mathcal{A}_\Theta$  is the twisted group  $C^*$ -algebra  $C^*(\mathbb{Z}, \beta)$ , where  $\beta : \mathbb{Z}^{2d} \times \mathbb{Z}^{2d} \rightarrow \mathbb{T}$  is a 2-cocycle such that

$$\beta(\lambda, \lambda') \overline{\beta(\lambda', \lambda)} = e^{2\pi i \Theta(\lambda, \lambda')} \quad \text{for } \lambda, \lambda' \in \mathbb{Z}^{2d}.$$

We remark that a noncommutative 2-torus  $A_\theta$  is often called *rotation algebra*. The commutation rules for the two unitary operators  $U$  and  $V$  generating  $A_\theta$  read as

$$UV = e^{2\pi i\theta}VU,$$

for a real number  $\theta$ . Let  $\theta = \alpha\beta$  then the  $C^*$ -algebra generated by time-frequency shifts  $\pi(\alpha k, \beta l) = M_{\beta l}T_{\alpha k}$  for  $k, l \in \mathbb{Z}^d$  is a representation of  $\mathcal{A}_\theta$  on  $L^2(\mathbb{R}^d)$ .

In this paper we want to treat the general case of  $2d$ -noncommutative tori or equivalently  $C^*$ -algebras  $C^*(\Lambda, \beta)$  of time-frequency shifts  $\pi(\lambda)$  for a lattice  $\Lambda$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  with

$$\pi(\lambda)\pi(\lambda') = \beta(\lambda, \lambda')\pi(\lambda + \lambda'), \quad \lambda, \lambda' \in \Lambda.$$

Therefore, an element of  $C^*(\Lambda, \beta)$  is given by

$$\sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda)$$

for a bounded complex-valued sequence  $\mathbf{a} = (a_\lambda)_{\lambda \in \Lambda}$ .

Moreover, this representation is **faithful** on  $L^2(\mathbb{R}^d)$ , which is of great importance in our proofs. One consequence, is that it is sufficient to establish statements for a dense subspace of  $L^2(\mathbb{R}^d)$ . For an operator algebraic proof see [Rief88] and in [GL04] a proof is given using time-frequency methods.

The choice of a sequence spaces on  $\Lambda$  induces on the noncommutative torus an additional structure. The space  $\mathcal{S}(\Lambda)$  of sequences on  $\Lambda$  with decay faster than the inverse of any polynomial yields a *smooth* structure on the algebra of functions on  $C^*(\Lambda, \beta)$ , i.e.

$$\mathcal{A}_\Lambda^\infty = \{A \in \mathcal{B}(L^2(\mathbb{R}^d)) : A = \sum_{\lambda} a_\lambda \pi(\lambda), \mathbf{a} = (a_\lambda)_{\lambda \in \Lambda} \in \mathcal{S}(\Lambda)\}.$$

In the present paper we introduce another structure on  $C^*(\Lambda, \beta)$ . Namely,

$$\mathcal{A}_\Lambda^1 = \{A \in \mathcal{B}(L^2(\mathbb{R}^d)) : A = \sum_{\lambda} a_\lambda \pi(\lambda), \mathbf{a} = (a_\lambda)_{\lambda \in \Lambda} \in \ell^1(\Lambda)\}. \quad (6)$$

The commutation rules for the time-frequency shifts  $\pi(\lambda)$  give  $C^*(\Lambda, \beta)$  the following structure:

- (1) Let  $A_1 = \sum_{\lambda \in \Lambda} a_1(\lambda)\pi(\lambda)$  and  $A_2 = \sum_{\lambda \in \Lambda} a_2(\lambda)\pi(\lambda)$  for  $\mathbf{a}_1, \mathbf{a}_2 \in \ell^1(\Lambda)$  then the product of  $A_1$  and  $A_2$  is given by

$$A_1 \cdot A_2 = \sum_{\lambda \in \Lambda} \mathbf{a}_1 \natural_{\Lambda} \mathbf{a}_2(\lambda)\pi(\lambda),$$

where

$$\mathbf{a}_1 \natural_{\Lambda} \mathbf{a}_2(\lambda) = \sum_{\mu \in \Lambda} a_1(\mu)a_2(\lambda - \mu)\beta(\mu, \lambda - \mu)$$

denotes *twisted convolution* of  $\mathbf{a}_1$  and  $\mathbf{a}_2$  and  $\mathbf{a}_1 \natural_{\Lambda} \mathbf{a}_2(\lambda)$  is again in  $\ell^1(\Lambda)$ .

- (2) Let  $A = \sum_{\lambda \in \Lambda} a(\lambda)\pi(\lambda)$  for  $\mathbf{a} \in \ell^1(\Lambda)$  then involution

$$A^* = \sum_{\lambda \in \Lambda} a(\lambda)^*\pi(\lambda)$$

induces an involution on  $\ell^1(\Lambda)$ :

$$a(\lambda)^* = \beta(\lambda, \lambda)\overline{a(-\lambda)}. \quad (7)$$

Therefore, we have that

**Proposition 4.1:**  $(\ell^1(\Lambda), \natural_{\Lambda}, *)$  is an involutive Banach algebra.

For the construction of a Hilbert  $C^*(\Lambda, \beta)$ -module  $V$  we are looking for a time-frequency homogenous Banach space, with properties similar to  $\mathcal{S}(\mathbb{R}^d)$ . In his seminal paper [F81] Feichtinger has introduced such a space, nowadays called *Feichtinger's algebra* and denoted by  $S_0(\mathbb{R}^d)$ .

Hence, we recall the definition of Feichtinger's algebra and some of its basic properties. For a detailed discussion we refer the reader to [F81, FK98, Gr01]. There are many characterizations of Feichtinger's algebra  $S_0(\mathbb{R}^d)$ . The connection between Feichtinger's algebra and Rieffel's work is most transparent in terms of time-frequency analysis.

Time-frequency representations contain information about the content of time and frequency in a signal  $f$ . For our further investigations we restrict our considerations to the *Short Time Fourier Transform* (STFT). The STFT of a function  $f$  with respect to a window function  $g$  is defined for  $f, g \in L^2(\mathbb{R}^d)$  as

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi it \cdot \omega} dt, \quad (x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d \quad (8)$$

or equivalently as

$$V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle, \quad \text{for } (x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d. \quad (9)$$

Feichtinger's algebra  $S_0(\mathbb{R}^d)$  is defined as follows

$$S_0(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \mid \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_\varphi f(x, \omega)| dx d\omega < \infty\}, \quad (10)$$

where  $\varphi(x) = 2^{-d/4} e^{-\pi x \cdot x}$  is a Gaussian and its norm is defined by

$$\|f\|_{S_0} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_\varphi f(x, \omega)| dx d\omega.$$

Any other non-zero Schwartz function  $g \in \mathcal{S}(\mathbb{R}^d)$ , instead of the Gaussian  $\varphi$ , yields the same space and an equivalent norm for  $S_0(\mathbb{R}^d)$ .

**Remark 4.2:** Feichtinger's algebra is a particular example of a class of Banach spaces, the so-called *modulation spaces*, which Feichtinger defined via integrability and decay conditions on the STFT over  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  (cf. [F03]). They have been recognized as the correct class of function spaces for questions in time-frequency analysis, especially Gabor analysis, [FG89a, FG97].

**Theorem 1:**  $S_0(\mathbb{R}^d)$  is a Banach algebra under pointwise multiplication.

By this definition of  $S_0(\mathbb{R}^d)$  elementary properties of the STFT yield invariance properties of  $S_0(\mathbb{R}^d)$ . In the following lemma we state two well-known facts about STFT.

**Lemma 2:** Let  $f, g \in L^2(\mathbb{R}^d)$  and  $(u, \eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . Then

(1) *Covariance Property of the STFT*

$$V_g(\pi(u, \eta)f)(x, \omega) = e^{2\pi i u \cdot (\omega - \eta)} V_g f(x - u, \omega - \eta).$$

(2)

$$V_g f(x, \omega) = e^{-2\pi i x \cdot \omega} V_{\hat{g}} \hat{f}(\omega, -x).$$

**Proof:**

(1) The covariance property of the STFT is a consequence of the of the commutation relation

$$T_x M_\omega = e^{-2\pi i x \cdot \omega} M_\omega T_x, \quad (x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

furthermore one has by definition of the STFT:

$$\begin{aligned} V_g(\pi(u, \eta)f)(x, \omega) &= \langle M_\eta T_u f, M_\omega T_x g \rangle \\ &= \langle f, T_{-u} M_{-\eta} M_\omega T_x g \rangle \\ &= e^{2\pi i u \cdot (\omega - \eta)} V_g f(x - u, \omega - \eta). \end{aligned}$$

- (2) The formula expresses, that for the STFT the Fourier transform yields a rotation of the time-frequency plane by an angle of  $90^\circ$ . Its another manifestation of the fact that STFT contains information of  $f$  and  $\hat{f}$   $\square$

Many properties of Feichtinger's algebra  $S_0(\mathbb{R}^d)$  are elementary consequences of properties of the STFT. The following theorem may be seen as a realization of this principle, where we show translation invariance and Fourier invariance of  $S_0(\mathbb{R}^d)$  from Lemma (2).

**Theorem 3:** Let  $f \in S_0(\mathbb{R}^d)$  and  $(u, \eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . Then

- (1)  $\pi(u, \eta)f \in S_0(\mathbb{R}^d)$  and  $\|f\|_{S_0} = \|\pi(u, \eta)f\|_{S_0}$ .  
(2)  $\hat{f} \in S_0(\mathbb{R}^d)$  and  $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$ .

**Proof:**

- (1) By definition of  $S_0(\mathbb{R}^d)$  we have by the invariance of the Gaussian  $\varphi$  under Fourier transform that

$$\begin{aligned} \|f\|_{S_0} &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_\varphi(M_\eta T_u f)(x, \omega)| dx d\omega \\ &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_\varphi f(x - u, \omega - \eta)| dx d\omega. \end{aligned}$$

- (2) For the invariance under the Fourier transform we use (2) of Lemma (2), and that the definition of  $S_0(\mathbb{R}^d)$  is independent of the window  $g \in \mathcal{S}(\mathbb{R}^d)$  and that different windows yield equivalent norms for  $S_0(\mathbb{R}^d)$ :

$$\begin{aligned} \|\hat{f}\|_{S_0(\mathbb{R}^d)} &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_\varphi \hat{f}(x, \omega)| dx d\omega \\ &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_{\hat{\varphi}} \hat{f}(x, \omega)| dx d\omega \\ &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_\varphi f(-\omega, x)| dx d\omega \\ &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_\varphi f(x, \omega)| dx d\omega = \|f\|_{S_0}. \end{aligned}$$

□

In our treatment of Rieffel's associativity condition for  $(C^*(\Lambda), C^*(\Lambda^0))$  we shall need the following properties of  $S_0(\mathbb{R}^d)$ .

**Theorem 4:** Let  $f, g$  in  $S_0(\mathbb{R}^d)$  then

$$V_g f \in S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d).$$

For a proof of this statement we refer the reader to [FK98], but it also follows from the functorial properties and minimality of  $S_0(\mathbb{R}^d)$  in [?].

Let  $F$  be a function on the time-frequency plane  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  then the *sampling operator* for a lattice  $\Lambda$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  is defined as follows

$$\mathbf{R}_\Lambda : F \mapsto (F(\lambda))_{\lambda \in \Lambda}.$$

**Remark 4..3:** We will also write occasionally  $F|_\Lambda$  instead of  $\mathbf{R}_\Lambda F$ .

**Theorem 5:** Let  $\Lambda$  be a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  and  $F \in S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$  then

$$\mathbf{R}_\Lambda F \in S_0(\Lambda) = \ell^1(\Lambda) \tag{11}$$

and  $\mathbf{R}_\Lambda$  is bounded on  $S_0(\mathbb{R}^d)$ .

As a consequence of the last theorem we obtain:

**Corollary 4..4:** Let  $f, g \in S_0(\mathbb{R}^d)$  then  $V_g f|_\Lambda \in \ell^1(\Lambda)$ , i.e.

$$\sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle| < \infty. \tag{12}$$

**Remark 4..5:** For  $\Lambda$  discrete  $S_0(\Lambda) = \ell^1(\Lambda)$ .

We now define a left action of  $C^*(\Lambda, \beta)$  on  $S_0(\mathbb{R}^d)$  by a *Gabor expansion* for the window  $g$  and the lattice  $\Lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ :

$$\mathbf{a}g = \sum_{\lambda \in \Lambda} a(\lambda)\pi(\lambda)g, \quad \mathbf{a} = (a_\lambda) \in S_0(\Lambda, \beta).$$

The invariance of  $S_0(\mathbb{R}^d)$  under time-frequency shifts implies that this action is well-defined on  $S_0(\mathbb{R}^d)$ .

**Proposition 4.6:** *Let  $\mathbf{a} \in S_0(\Lambda, \beta)$  and  $g \in S_0(\mathbb{R}^d)$ , then*

$$\left\| \sum_{\lambda \in \Lambda} a(\lambda) \pi(\lambda) g \right\|_{S_0} \leq \|\mathbf{a}\|_1 \|g\|_{S_0}.$$

**Proof:**

$$\begin{aligned} \left\| \sum_{\lambda \in \Lambda} a(\lambda) \pi(\lambda) g \right\|_{S_0} &\leq \sum_{\lambda \in \Lambda} |a(\lambda)| \|\pi(\lambda) g\|_{S_0} \\ &= \sum_{\lambda \in \Lambda} |a(\lambda)| \|g\|_{S_0} \\ &= \|\mathbf{a}\|_1 \|g\|_{S_0}. \quad \square \end{aligned}$$

Corollary 4.4 and Proposition 4.6 yield that for  $f, g \in S_0(\mathbb{R}^d)$  the left action of

$$\langle f, g \rangle_\Lambda = V_g f|_\Lambda$$

is well-defined on  $S_0(\mathbb{R}^d)$ . In Gabor analysis the mapping of  $\langle f, g \rangle_\Lambda \mapsto \langle f, g \rangle_\Lambda g$  is called a Gabor type frame operator with window  $g$  and lattice  $\Lambda$ , denoted by

$$S_{g, \Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) g.$$

The above discussion shows that the Gabor type frame operator  $S_{g, \Lambda}$  is continuous on  $S_0(\mathbb{R}^d)$  for  $f, g \in S_0(\mathbb{R}^d)$ . In Section 6 we present some consequences for the reconstruction of square-integrable functions  $f \in L^2(\mathbb{R}^d)$ .

Rieffel's central observation was that

$$\langle f, g \rangle_{\mathcal{A}} = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda), \quad f, g \in \mathcal{S}(\mathbb{R}^d) \quad (13)$$

is a  $C^*$ -valued innerproduct for  $\mathcal{A} = C^*(\Lambda, \beta)$ . In the sequel we prove that (13) defines a  $C^*$ -valued innerproduct for  $C^*(\Lambda, \beta)$  for  $f, g$  in Feichtinger's algebra  $S_0(\mathbb{R}^d)$ .

First we show that (13) is compatible with the action of  $S_0(\Lambda, \beta)$  on  $S_0(\mathbb{R}^d)$ . More precisely, we prove the following proposition.

**Proposition 4.7:** *Let  $f, g \in S_0(\mathbb{R}^d)$  and let  $\mathbf{a} \in S_0(\Lambda, \beta)$ . Then*

$$\langle \mathbf{a} f, g \rangle_\Lambda = \mathbf{a} \sharp_\Lambda \langle f, g \rangle_\Lambda.$$

**Proof:** For  $\lambda \in \Lambda$  we have

$$\begin{aligned}
 \langle \mathbf{a}f, \pi(\lambda)g \rangle &= \sum_{\lambda' \in \Lambda} a(\lambda') \langle \pi(\lambda')f, \pi(\lambda)g \rangle \\
 &= \sum_{\lambda' \in \Lambda} a(\lambda') \langle f, \pi^*(\lambda')\pi(\lambda)g \rangle \\
 &= \sum_{\lambda' \in \Lambda} a(\lambda') \langle f, \pi(\lambda - \lambda')g \rangle \beta(\lambda', \lambda - \lambda') \\
 &= \mathbf{a} \sharp_{\Lambda} \langle f, \pi(\lambda)g \rangle,
 \end{aligned}$$

since  $\pi(\lambda')\pi(\lambda - \lambda') = \beta(\lambda', \lambda - \lambda')\pi(\lambda)$ .  $\square$

We now come to the statement of one of our main theorems.

**Theorem 6:** Feichtinger's algebra  $S_0(\mathbb{R}^d)$  is a left Hilbert  $C^*(\Lambda, \beta)$ -module with respect to the inner product, given for  $f, g$  in  $S_0(\mathbb{R}^d)$  by

$$\langle f, g \rangle_{\mathcal{A}} = \sum_{\lambda \in \Lambda} V_g f(\lambda) \pi(\lambda).$$

The proof of Theorem 6 is postponed now and will be given after the discussion of the **Fundamental Identity of Gabor Analysis**, because the positivity of the innerproduct is a direct consequence of FIGA. In Section 5. we derive FIGA from an identity for products of STFT by an application of Poisson's summation formula for the symplectic Fourier transform, which requires the adjoint lattice  $\Lambda^0$  of the lattice  $\Lambda$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . In Section 5. we study the structure of  $C^*(\Lambda)$  and define a  $C^*(\Lambda^0)$ -valued innerproduct. The proof of Rieffel's associativity condition is an elementary reformulation of the FIGA. Therefore, Section 5. is the natural place for our presentation of the FIGA.

At the end of this section we present generalizations of some notions of Hilbert spaces to Hilbert  $C^*$ -modules.

Let  $V$  be a Hilbert  $\mathcal{A}$ -module then a *Hilbert module map* from  $V$  to  $V$  is a linear map  $T : V \rightarrow V$  that respects the module action:  $T(\mathbf{a}f) = \mathbf{a}(T(f))$  for  $\mathbf{a} \in \mathcal{A}$  and  $f \in V$ . The adjoint of an operator on a Hilbert space plays a central role in the study of operators on Hilbert spaces and of operator algebras, such as  $C^*$ -algebras or von Neumann algebras of operators. The following definition gives a generalization of adjoints to Hilbert  $\mathcal{A}$ -module.

**Definition 4..8:** Let  $V$  be a Hilbert  $\mathcal{A}$ -module. A map  $T : V \rightarrow V$  is *adjointable* if there exists a map  $T^* : V \rightarrow V$  satisfying

$$\langle Tf, g \rangle_{\mathcal{A}} = \langle f, T^*g \rangle_{\mathcal{A}}$$

for all  $f, g$  in  $\mathcal{A}$ . The map  $T^*$  is called the *adjoint* of  $T$ . We denote the set of all adjointable maps by  $\mathcal{L}(V)$  and the set of all bounded module maps in  $V$  by  $\mathcal{B}(V)$ .

An elementary consequence of the definitions is the following facts about adjointable maps.

- (1) Let  $T$  be in  $\mathcal{L}(V)$ , then its adjoint is unique and adjointable with  $T^{**} = T$ .
- (2) Let  $T, S$  be in  $\mathcal{L}(V)$ , then  $ST \in \mathcal{L}(V)$  with  $(ST)^* = T^*S^*$ .
- (3)  $\mathcal{L}(V)$  equipped with the operator norm  $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$  is a  $C^*$ -algebra.
- (4)  $\mathcal{B}(V)$  equipped with the operator norm  $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$  is a Banach algebra.

In the case of  $S_0(\mathbb{R}^d)$  as  $C^*(\Lambda, \beta)$ -module the adjointable maps are those operators  $T : S_0(\mathbb{R}^d) \rightarrow S_0(\mathbb{R}^d)$  where  $T^*$  commutes with all time-frequency shifts  $\{\pi(\lambda) : \lambda \in \Lambda\}$ . By definition of the  $C^*(\Lambda, \beta)$ -innerproduct we have

$$\begin{aligned} \langle Tf, g \rangle_{\mathcal{A}} &= \sum_{\lambda \in \Lambda} \langle Tf, \pi(\lambda)g \rangle \pi(\lambda) \\ &= \sum_{\lambda \in \Lambda} \langle f, T^* \pi(\lambda)g \rangle \pi(\lambda) \\ &= \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)T^*g \rangle \pi(\lambda) \\ &= \langle f, T^*g \rangle_{\mathcal{A}}. \end{aligned}$$

In [FK98] Feichtinger and Kozek treated selfadjoint operators on  $S_0(\mathbb{R}^d)$ , which commute with  $\{\pi(\lambda) : \lambda \in \Lambda\}$ . They called those operators  $\Lambda$ -invariant. The set of all selfadjoint adjointable operators of  $\mathcal{L}(S_0(\mathbb{R}^d))$  is an ideal in  $\mathcal{L}(S_0(\mathbb{R}^d))$ .

The notion of finite rank operators and of compact module operators is of great relevance in the construction of Morita equivalences between  $C^*$ -algebras, see Section 5..

**Definition 4.9:** Let  $f, g$  be elements of a Hilbert  $\mathcal{A}$ -module  $V$ . Then a **rank one operator**  $K_{f,g} : V \rightarrow V$  is defined by

$$K_{f,g}h := \langle f, h \rangle_{\mathcal{A}} g.$$

The set of **compact Hilbert module operators** on  $V$  is the closed subspace of  $\mathcal{L}(V)$  generated by the closure of the rank one maps  $K_{f,g}$ . We denote the set of compact Hilbert module operators by  $\mathcal{K}(V) = \overline{\{K_{f,g} : f, g \in V\}}$ .

**Remark 4.10:** A compact Hilbert module operator is not necessarily a compact operator on  $V$ , but for Hilbert  $\mathbb{C}$ -modules  $\mathcal{H}$  the notion specializes to the definition of a compact operator on  $\mathcal{H}$ .

The following proposition gives some elementary facts about compact Hilbert module operators.

**Proposition 4.11:** *Let  $f, g \in V$  and  $T \in \mathcal{L}(V)$ . Then one has*

- (1)  $K_{f,g}$  is adjointable and  $K_{f,g}^* = K_{g,f}$ .
- (2)  $TK_{f,g} = K_{Tf,g}$ .
- (3)  $K_{f,g}T = K_{f,T^*g}$ .
- (4)  $\|K_{f,g}\| \leq \|f\| \|g\|$ .

A direct consequence of the previous observations is the following statement.

**Proposition 4.12:** *Let  $V$  be a Hilbert  $\mathcal{A}$ -module,  $\mathcal{K}(V)$  is a closed ideal in  $\mathcal{L}(V)$ .*

Now we investigate the set of rank one module operators for our Hilbert  $C^*(\Lambda, \beta)$  module  $S_0(\mathbb{R}^d)$ . By definition a rank one module operator is given by

$$\begin{aligned} K_{g,f}\gamma &= \langle f, \gamma \rangle_{\mathcal{A}} g \\ &= \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g = S_{g,\gamma,\Lambda}f, \end{aligned}$$

for  $f, g, \gamma \in S_0(\mathbb{R}^d)$ . The operator  $S_{g,\gamma,\Lambda}$  is called a Gabor frame operator with analysis window  $\gamma$  and synthesis window  $g$  for a lattice  $\Lambda$ .

Therefore, a finite rank module operator is a finite sum of Gabor frame operators, a so-called *multi-window* Gabor frame operator. Furthermore, a rank one module operator  $S_{g,\gamma,\Lambda}$  is an adjointable operator, i.e. it is a  $\Lambda$ -invariant operator. This elementary fact has far reaching consequences, see Section 6.

### 5. Feichtinger's Algebra as Bimodule for $C^*(\Lambda)$ and $C^*(\Lambda^0)$

In Section 3 we discussed the relation between an operator algebra of time-frequency shifts generated by a lattice  $\Lambda$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ . In this section we continue the discussion in the light of Morita equivalence of  $C^*$ -algebras.

The adjoint lattice of  $\Lambda$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  was defined as the set of all points  $X = (x, \omega)$  in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  such that  $\rho(\lambda, X) = 1$ , which by the commutation relation of time-frequency shifts (2) is equivalent to

$$\Lambda^0 = \{\lambda^0 \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d : \pi(\lambda)\pi(\lambda^0) = \pi(\lambda^0)\pi(\lambda) \text{ for all } \lambda \in \Lambda\}.$$

Therefore, the set of all bounded operators on  $L^2(\mathbb{R}^d)$  commuting with elements from  $C^*(\Lambda, \beta)$  is the  $C^*$ -algebra generated by time-frequency shifts  $\pi(\lambda^0)$  for  $\lambda^0$  in  $\Lambda^0$ .

In Section 4 we have defined a left action of  $C^*(\Lambda, \beta)$  on  $S_0(\mathbb{R}^d)$ . Now  $S_0(\mathbb{R}^d)$  has the structure of a bimodule, where the right action is induced by the opposite algebra of  $C^*(\Lambda^0, \beta)$ . Following Rieffel in [Rief88],  $C^*(\Lambda^0, \beta)$  can be generated by  $\pi^*(\lambda^0)$  acting on the left on  $S_0(\mathbb{R}^d)$ , which commutes with the right action of  $C^*(\Lambda, \beta)$  on  $S_0(\mathbb{R}^d)$ . Therefore, the opposite algebra of  $C^*(\Lambda, \beta)^{\text{opp}}$  is generated by  $\pi^*(\lambda^0)$  with  $\overline{\beta}(X, Y) = \overline{\beta(X, Y)}$  for  $X = (x, \omega)$  and  $Y = (y, \eta)$  as cocycle, i.e.  $C^*(\Lambda^0, \overline{\beta})$ . By definition the opposite algebra of  $C^*(\Lambda, \beta)$  gives a right action on  $S_0(\mathbb{R}^d)$  by a Gabor expansion with respect to the lattice  $\Lambda^0$

$$g\mathbf{b} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} b(\lambda^0)\pi^*(\lambda^0)g, \quad g \in S_0(\mathbb{R}^d), \mathbf{b} \in S_0(\Lambda^0, \overline{\beta}).$$

Note, that cohomologous cocycles yield isomorphic  $C^*$ -algebras. By a reasoning similar to the one used in Section 4 for the left action  $C^*(\Lambda, \beta)$  we obtain that the right action is well-defined on  $S_0(\mathbb{R}^d)$ .

Before  $S_0(\mathbb{R}^d)$  is given the structure of a right  $C^*(\Lambda^0, \overline{\beta})$ -module we state the Fundamental Identity of Gabor analysis, because it is essential in our construction of the bimodule  $S_0(\mathbb{R}^d)$  for  $C^*(\Lambda, \beta)$  and  $C^*(\Lambda^0, \overline{\beta})$ .

**Theorem 7:** [FIGA] Let  $f_1, g_1, f_2, g_2 \in S_0(\mathbb{R}^d)$ . Then

$$\sum_{\lambda \in \Lambda} V_{g_1} f_1(\lambda) \overline{V_{g_2} f_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_{g_1} g_2(\lambda^0) \overline{V_{f_1} f_2(\lambda^0)}$$

In [Rief88] Rieffel proved FIGA for Schwartz functions  $f_1, f_2, g_1, g_2$  in  $\mathcal{S}(G)$  for an elementary locally compact abelian group  $G$ . In [TO95] Tolmieri and Orr proved a special case of Rieffel's result for functions on  $\mathbb{R}$  in their study of Gabor frames. Later, Janssen continued the work of Tolmieri/Orr and introduced a representation of Gabor frame operators, Janssen's representation [Jan95]. In his proof of the Morita equivalence of  $C^*(\Lambda, \beta)$  and  $C^*(\Lambda^0, \bar{\beta})$  Rieffel had derived Janssen's representation of a Gabor frame operator.

Following Rieffel we use Poisson summation formula for symplectic Fourier transform in the proof of FIGA. The following theorem states the Poisson summation formula for the symplectic Fourier transform, see Section 2 for the definition.

**Theorem 8:** Let  $F \in S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ . Then

$$\sum_{\lambda \in \Lambda} F(\lambda) = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} \widehat{F}^s(\lambda^0) \quad (14)$$

holds pointwise and with absolute convergence of both sums.

**Proof:** [FIGA]

If  $f, g \in S_0(\mathbb{R}^d)$  we have that  $V_g f \in S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ . Then  $F = V_{g_1} f_1 \overline{V_{g_2} f_2}$  is in  $S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ , because  $S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$  is a Banach algebra under multiplication. Poisson's summation formula for  $F$  yields FIGA. Therefore, we compute the symplectic Fourier transform of  $F$ .

$$\begin{aligned} \widehat{F}^s(Y) &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_{g_1} f_1(X) \overline{V_{g_2} f_2(X)} \rho(Y, X) dX \\ &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle \pi(Y) f_1, \pi(Y) \pi(X) g_1 \rangle \overline{\langle f_2, \pi(X) g_2 \rangle} \rho(X, Y) dX \\ &= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle \pi(Y) f_1, \pi(X) \pi(Y) g_1 \rangle \overline{\langle f_2, \pi(X) g_2 \rangle} \rho(X, Y) dX \\ &= \langle f_1, \pi(Y) f_2 \rangle \overline{\langle g_1, \pi(Y) g_2 \rangle}, \end{aligned}$$

where in the last step we used Moyal's formula.  $\square$

As a first application of FIGA we finish the proof of Theorem 6 by showing the positivity of

$$\langle f, f \rangle_{\mathcal{A}} = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)f \rangle \pi(\lambda) \quad (15)$$

as an operator on  $L^2(\mathbb{R}^d)$ .

**Proposition 5.1:** *Let  $f \in S_0(\mathbb{R}^d)$ . Then  $\langle f, f \rangle_{\mathcal{A}}$  is a positive element of  $C^*(\Lambda, \beta)$ .*

**Proof:** The representation of time-frequency shifts of  $C^*(\Lambda, \beta)$  is faithful, therefore, it suffices to establish positivity for a dense subspace of  $L^2(\mathbb{R}^d)$ . Of course we choose  $S_0(\mathbb{R}^d)$  as dense subspace. Let  $g \in S_0(\mathbb{R}^d)$

$$\begin{aligned} \langle \langle f, f \rangle_{\mathcal{A}} g, g \rangle &= \left\langle \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)f \rangle \pi(\lambda)g, g \right\rangle \\ &= \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)f \rangle \overline{\langle g, \pi(\lambda)g \rangle} \\ &= \sum_{\lambda^0 \in \Lambda^0} \langle f, \pi(\lambda^0)g \rangle \overline{\langle f, \pi(\lambda^0)g \rangle} \geq 0. \quad \square \end{aligned}$$

In an analogous manner as in our discussion of Theorem 6 we get that the right action of  $C^*(\Lambda^0, \bar{\beta})$  with properly defined  $\natural_{\Lambda^0}$  and involution  $*$  defines a right Hilbert  $C^*(\Lambda^0, \bar{\beta})$ -module structure on  $S_0(\mathbb{R}^d)$  with respect to the  $\mathcal{B} := C^*(\Lambda^0, \bar{\beta})$ -innerproduct

$$\langle f, g \rangle_{\mathcal{B}} := |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} \langle \pi(\lambda^0)g, f \rangle \pi(\lambda^0), \quad f, g \in S_0(\mathbb{R}^d).$$

Two  $C^*$ -module structures  $(\mathcal{A}, \langle \cdot, \cdot \rangle_{\mathcal{A}})$  and  $(\mathcal{B}, \langle \cdot, \cdot \rangle_{\mathcal{B}})$  on a bimodule  $V$  are compatible if

$$\langle f, g \rangle_{\mathcal{A}} h = f \langle g, h \rangle_{\mathcal{B}}, \quad \text{for all } f, g, h \in V. \quad (16)$$

Some authors call (16) **Rieffel's associativity condition** for  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$  and  $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ .

In our setting Rieffel's associativity condition expresses Janssen's representation of a Gabor frame operator  $S_{g, \gamma}$  for a window  $g, \gamma \in S_0(\mathbb{R}^d)$ .

**Theorem 9:** Let  $\mathcal{A} = C^*(\Lambda, \beta)$  and  $\mathcal{B} = C^*(\Lambda^0, \bar{\beta})$  with the above defined actions and innerproducts  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$  and  $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ , respectively. Then

$$S_{g, \gamma, \Lambda} f = |\Lambda|^{-1} S_{f, \gamma, \Lambda^0} g$$

for all  $f, g, \gamma \in S_0(\mathbb{R}^d)$ .

**Proof:** As in the proof of positivity of  $\langle f, f \rangle_{\mathcal{A}}$  for  $f \in S_0(\mathbb{R}^d)$  it suffices to show that for all  $\gamma, h \in S_0(\mathbb{R}^d)$

$$\langle S_{g, \gamma, \Lambda} f, h \rangle = |\Lambda|^{-1} \langle S_{f, \gamma, \Lambda^0} g, h \rangle.$$

$$\begin{aligned} \langle \langle f, g \rangle_{\mathcal{A}} \gamma, h \rangle &= \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \overline{\langle h, \pi(\lambda) \gamma \rangle} \\ &\stackrel{FIGA}{=} |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} \langle f, \pi(\lambda^0) h \rangle \overline{\langle g, \pi(\lambda^0) \gamma \rangle} \\ &= \langle f \langle g, \gamma \rangle_{\mathcal{B}}, h \rangle. \quad \square \end{aligned}$$

A Hilbert  $C^*$ -module  $V$  over  $\mathcal{A}$  is called *full* when the collection  $\{\langle f, g \rangle_{\mathcal{A}} : f, g \in V\}$  is dense in  $\mathcal{A}$ .

**Definition 5..2:** Two  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  are **strongly Morita equivalent** if there exists a full Hilbert  $C^*$ -module  $V$  over  $\mathcal{B}$  such that  $\mathcal{B} \simeq \mathcal{K}(V, \mathcal{A})$ .

**Remark 5..3:** We denote by  $\mathcal{K}(V, \mathcal{A})$  the operator closure of the finite linear combinations of finite linear combinations of "rank-one" operators  $K_{f, g}^{\mathcal{A}}$ .

The Morita equivalence of  $C^*(\Lambda, \beta)$  and  $C^*(\Lambda^0, \bar{\beta})$  is a consequence of the following theorem.

**Theorem 10:** Let  $S_0(\mathbb{R}^d)$  be given a bimodule structure as defined above. Let  $\mathcal{A} = C^*(\Lambda, \beta)$  and  $\mathcal{B} = C^*(\Lambda^0, \bar{\beta})$ . Then

- (1)  $\{\langle f, g \rangle_{\mathcal{A}} : f, g \in S_0(\mathbb{R}^d)\}$  is dense in  $\mathcal{A}$ , i.e.  $S_0(\mathbb{R}^d)$  is a full Hilbert  $\mathcal{A}$ -module.
- (2)  $\{\langle f, g \rangle_{\mathcal{B}} : f, g \in S_0(\mathbb{R}^d)\}$  is dense in  $\mathcal{B}$ , i.e.  $S_0(\mathbb{R}^d)$  is a full Hilbert  $\mathcal{B}$ -module.
- (3) For all  $f \in S_0(\mathbb{R}^d)$  and  $A \in \mathcal{A}$ , we have

$$\langle fA, fA \rangle_{\mathcal{A}} \leq \|A\|^2 \langle f, f \rangle_{\mathcal{A}},$$

i.e. boundedness of the right action.

(4) For all  $f \in S_0(\mathbb{R}^d)$  and  $B \in \mathcal{B}$ , we have

$$\langle Bf, Bf \rangle_{\mathcal{B}} \leq \|B\|^2 \langle f, f \rangle_{\mathcal{B}},$$

i.e. boundedness of the left action.

implies that  $S_0(\mathbb{R}^d)$  is an equivalence bimodule  $(\mathcal{A}, \mathcal{B})$  with norm  $\|f\| := \langle f, f \rangle_{\mathcal{A}}^{1/2}$ .

**Proof:** Our proof follows Rieffel's approach, see [Rief88].

- (1) The linear span of the range of  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$  is an ideal in  $\mathcal{A}$ . Then the norm closure  $I$  of this linear span is an ideal in  $\mathcal{A}$ . Furthermore  $I$  is invariant under modulation and because  $\pi(\lambda)$  is a faithful representation of  $\mathcal{A}$ , we get the desired conclusion.
- (2) By similar arguments as in (1).
- (3) It suffices to verify the inequality for a dense subspace of  $L^2(\mathbb{R}^d)$ . Let  $h \in S_0(\mathbb{R}^d)$  and  $A \in C^*(\Lambda, \beta)$ , then

$$\begin{aligned} \langle h \langle Af, Af \rangle_{\mathcal{A}}, h \rangle &= \langle \langle h, Af \rangle_{\mathcal{B}} Af, h \rangle \\ &= \langle Af, \langle Af, h \rangle_{\mathcal{B}} h \rangle \\ &= \langle Af, Af \langle h, h \rangle_{\mathcal{A}} \rangle \\ &= \langle A \langle f \langle h, h \rangle_{\mathcal{A}} \rangle^{1/2}, A \langle f \langle h, h \rangle_{\mathcal{A}} \rangle^{1/2} \rangle \\ &\leq \|A\|^2 \langle f, f \rangle_{\mathcal{A}} \\ &= \|A\|^2 \langle h \langle f, f \rangle_{\mathcal{A}}, h \rangle \end{aligned}$$

holds for all  $f$  in  $S_0(\mathbb{R}^d)$ . A standard density argument yields the desired result.

- (4) By similar arguments as in (3). □

**Corollary 5.4:**  $C^*(\Lambda, \beta)$  and  $C^*(\Lambda^0, \overline{\beta})$  are strongly Morita equivalent.

By definition a *projective module*  $V$  is isomorphic to a direct summand of a free module  $\mathcal{A}^n$  with standard basis  $\{e_j\}$ , i.e. there is a self-adjoint  $n \times n$ -matrix  $P$  with entries in  $\mathcal{A}$  which is a projection, such that  $V = P\mathcal{A}^n$ . Rieffel proved that if  $\mathcal{A}$  and  $\mathcal{B}$  are unital  $C^*$ -algebras and if  $V$  is a  $(\mathcal{B}, \mathcal{A})$ -equivalence bimodule, then  $V$  is a projective right  $\mathcal{B}$ -module, and a projective left  $\mathcal{A}$ -module. Furthermore,  $\mathcal{A}$  is equivalent to the  $C^*$ -algebra  $\mathcal{K}(V, \mathcal{B})$  of compact Hilbert  $\mathcal{B}$ -module operators.

In particular, let  $\mathcal{B} = C^*(\Lambda^0, \overline{\beta})$  and let  $V$  denote the right  $\mathcal{A}$ -module obtained by completing  $S_0(\mathbb{R}^d)$  as described earlier. Then, we have:

**Theorem 11:** Feichtinger's algebra  $S_0(\mathbb{R}^d)$  is a **finitely generated projective**  $\mathcal{B}$ -module and  $\mathcal{K}(S_0(\mathbb{R}^d), \mathcal{B})$  is equivalent to  $C^*(\Lambda, \beta)$ .

In [Rief81a] Rieffel made the observation that finitely generated projective  $C^*$ -modules possess a reconstruction formula in terms of a tight module frame, which is a generalization of the familiar notion of tight frames for Hilbert spaces. In a subsequent paper we discuss the connection between tight module frames for  $C^*(\Lambda, \beta)$  and the characterization of  $S_0(\mathbb{R}^d)$  with multi-window Gabor frames.

## 6. Application to Gabor Analysis: Biorthogonality Relation of Wexler-Raz

Recently, Gabor frames have been applied in various fields of mathematics, electrical engineering and signal analysis, see [FS98,FS03]. In this section we give a first glimpse of the usefulness of Rieffel's work on strong Morita equivalence of  $C^*$ -algebras generated by time-frequency shifts.

Let  $\Lambda$  be a lattice in  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$  and  $g \in L^2(\mathbb{R}^d)$  then a *Gabor system*  $\mathcal{G}(g, \Lambda) := \{\pi(\lambda)g : \lambda \in \Lambda\}$  for a Gabor atom  $g \in L^2(\mathbb{R}^d)$  is a *Gabor frame* if there are finite positive reals  $A, B$  such that

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in L^2(\mathbb{R}^d).$$

This is equivalent to invertibility and boundedness of the *Gabor frame operator*

$$S_{g,\Lambda}f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g, \quad \text{for all } f \in L^2(\mathbb{R}^d).$$

As a consequence of the invertibility of  $S_{g,\Lambda}$  we have the following reconstruction formulas for  $f \in L^2(\mathbb{R}^d)$

$$f = (S_{g,\Lambda})^{-1}S_{g,\Lambda}f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)(S_{g,\Lambda})^{-1}g \quad (17)$$

or

$$f = S_{g,\Lambda}(S_{g,\Lambda})^{-1}f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)(S_{g,\Lambda})^{-1}g \rangle \pi(\lambda)g.$$

The coefficients in reconstruction formulas (17) and (6.) are not unique, because in general time-frequency shifts  $\pi(\lambda)$  and  $\pi(\lambda')$  are not linearly independent for  $\lambda, \lambda' \in \Lambda$ . Therefore, many researchers have investigated the set of all possible dual windows  $\gamma$  such that  $S_{g,\gamma} = I$ . Of special importance is the function  $\gamma_0 := (S_{g,\Lambda})^{-1}g$ , the *canonical dual window*. There are many characterizations of  $\gamma_0$  in the set of all dual windows.

The Gabor frame operator  $S_{g,\Lambda}$  commutes with time-frequency shifts  $\{\pi(\lambda) : \lambda \in \Lambda\}$ , therefore, the dual Gabor frame  $\{\pi(\lambda)\gamma_0 : \lambda \in \Lambda\}$  has the structure of a Gabor frame. This observation and (17) for  $(S_{g,\Lambda})^{-1}f$  yields that the inverse frame operator of a frame  $\mathcal{G}(\Lambda, g)$  is given by

$$(S_{g,\Lambda})^{-1}f = S_{\gamma_0,\Lambda}f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma_0 \rangle \pi(\lambda)\gamma_0. \quad (18)$$

Gröchenig and Leinert were motivated by a practical question on the quality of the canonical dual window  $S_{g,\Lambda}^{-1}g$  of a Gabor frame  $\mathcal{G}(g, \Lambda)$  generated by a Gabor atom  $g$  in  $S_0(\mathbb{R}^d)$ . They established that Feichtinger's algebra is a good class of Gabor atoms. Namely,

**Theorem 12:** [Gröchenig-Leinert] Let  $\mathcal{G}(g, \Lambda)$  be a Gabor frame generated by  $g \in S_0(\mathbb{R}^d)$  then the canonical dual window  $\gamma_0 = S_{g,\Lambda}^{-1}g$  is in  $S_0(\mathbb{R}^d)$ .

We refer the reader to [GL04] for a proof of this deep result. In [Jan95] Janssen had proved that for a Gabor frame  $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$  generated by a Schwartz function  $g$  the canonical dual window is also a Schwartz function under the restriction that  $\alpha, \beta \in \mathbb{Q}$ . Janssen had conjectured that his result is also valid for irrational lattice constants  $\alpha, \beta$ . We remark that a resolution of Janssen's conjecture is a corollary of Connes result that  $\mathcal{S}(\mathbb{R}^d)$  is closed under holomorphic functional calculus, [Con81] and [Rief88].

In [WR90] Wexler/Raz characterized the set of all dual atoms with the structure of a Gabor frame for Gabor expansions on finite abelian groups. Their work had been extended to the continuous setting independently by Daubechies, H.L. Landau and Z. Landau in [DLL90], by Janssen in [Jan95] and by Ron and Shen [RS93],[RS97]. In the work on this problem the so-called *Janssen representation* of a Gabor frame operator was introduced in [Jan95]. Also Feichtinger and Zimmermann considered this topic and found the minimal assumptions for the validity of Wexler-Raz's biorthogonality relation and Janssen's representation, [FZ98]. In [FK98] and [FZ98] Feichtinger and his collaborators introduced the notion of the adjoint lat-

tice for elementary locally compact abelian groups, which Rieffel already used in his construction of equivalence bimodules between noncommutative tori, [Rief88]. In this section we derive the result of Wexler-Raz from the Morita equivalence of  $C^*(\Lambda, \beta)$  and  $C^*(\Lambda^0, \bar{\beta})$  and the relation between the canonical traces  $\tau_{\mathcal{A}}$  and  $\tau_{\mathcal{B}}$ , respectively.

One of the early successes of operator algebras was the classification of all commutative  $C^*$ -algebras by Gelfand as the involutive complex-valued continuous functions over a compact space. Riesz's representation theorem for positive linear functionals of involutive complex-valued continuous functions over a compact space  $X$  yields to an extension of the Lebesgue integral. Therefore, integration of continuous functions over a compact space is considered as a trace on a commutative  $C^*$ -algebra. Therefore, traces or states on general  $C^*$ -algebras are the natural framework for non-commutative Radon measure theory.

The existence of canonical traces on  $\mathcal{A} = C^*(\Lambda, \beta)$  and  $\mathcal{B} = C^*(\Lambda^0, \bar{\beta})$  is one of the pleasant properties of noncommutative tori.

First we recall that a *faithful trace*  $\tau_{\mathcal{C}}$  on a  $C^*$ -algebra  $\mathcal{C}$  is a linear functional satisfying

$$\begin{aligned}\tau(I) &= 1, \text{ for the identity operator } I \text{ of } \mathcal{C}, \\ \tau(AB) &= \tau(BA), \text{ for all } A, B \in \mathcal{C}, \\ \tau(A^*A) &> 0 \text{ for all nonzero } A \text{ in } \mathcal{C}.\end{aligned}$$

In the case of  $\mathcal{A}$  a normalized faithful trace  $\tau_{\mathcal{A}}$  is given by

$$\tau_{\mathcal{A}}(\langle f, g \rangle_{\mathcal{A}}) = \langle f, g \rangle, \quad f, g \in S_0(\mathbb{R}^d),$$

and for  $\mathcal{B}$  the canonical trace  $\tau_{\mathcal{B}}$  is normalized by

$$\tau_{\mathcal{B}}(\langle f, g \rangle_{\mathcal{B}}) = |\Lambda|^{-1} \langle f, g \rangle, \quad f, g \in S_0(\mathbb{R}^d),$$

which follows from Morita equivalence of  $\mathcal{A}$  and  $\mathcal{B}$ . This fact can be considered as a noncommutative Poisson summation formula

$$\tau_{\mathcal{A}}(\langle f, g \rangle_{\mathcal{A}}) = |\Lambda|^{-1} \tau_{\mathcal{B}}(\langle f, g \rangle_{\mathcal{B}}). \quad (19)$$

Our restriction in the following theorem to  $g, \gamma \in S_0(\mathbb{R})$  is just for convenience. We refer to Gröchenig's excellent survey [Gr01] of Gabor analysis for the general case of  $g, \gamma \in L^2(\mathbb{R}^d)$ .

**Theorem 13:** [Wexler-Raz] Let  $\mathcal{G}(g, \Lambda)$  be a Gabor system. Then the following statements are equivalent:

- (1)  $S_{g,\gamma} = I$ .
- (2)  $|\Lambda|^{-1} \langle \gamma, \pi(\lambda^0)g \rangle = \delta_{\lambda,0}$ .

**Proof:** (2)  $\Rightarrow$  (1) :

Follows from the fact that the identity of  $\mathcal{A}$  is  $I = \delta_{\lambda,0}$ , where  $\delta_{i,k}$  is the Kronecker delta. Therefore, by assumption

$$\tau_{\mathcal{B}}(S_{g,\gamma}) = \tau_{\mathcal{B}}(I) = \delta_{\lambda,0}$$

and by application of (19) we get

$$\tau_{\mathcal{B}}(I) = |\Lambda|^{-1} \tau_{\mathcal{A}}(\langle g, \gamma \rangle) = |\Lambda|^{-1} \langle \gamma, g \rangle.$$

The implication (1)  $\Rightarrow$  (2) is trivial in the light of Rieffel's associativity condition.  $\square$

**Corollary 6.1:** For  $g, \gamma \in S_0(\mathbb{R}^d)$  dual functions the two Gabor systems  $\mathcal{G}(g, \Lambda^0)$  and  $\mathcal{G}(\gamma, \Lambda^0)$  are biorthogonal to each other on  $L^2(\mathbb{R}^d)$ .

The proof is an elementary reformulation of Theorem 13.

**Corollary 6.2:** A Gabor system  $\mathcal{G}(g, \Lambda)$  is a tight frame if and only if  $\mathcal{G}(\gamma, \Lambda^0)$  is an orthonormal system with frame bound  $A = \|\Lambda\|^{-1} \|g\|^2$ .

The statement is well-known, see [Gr01] for the elementary proof.

**Corollary 6.3:** Let  $g_1, \dots, g_n, \gamma_1, \dots, \gamma_n \in S_0(\mathbb{R}^d)$ , then for the multi-window Gabor frame  $S = \sum_{i=1}^n S_{g_i, \gamma_i}$  the following are equivalent:

- (1)  $S_{g_1, \gamma_1} + \dots + S_{g_n, \gamma_n} = I$ .
- (2)  $|\Lambda|^{-1} (\langle \gamma_1, \pi(\lambda^0)g_1 \rangle + \dots + \langle \gamma_n, \pi(\lambda^0)g_n \rangle) = \delta_{\lambda,0}$ .

The proof follows the same reasoning as for a single Gabor frame.

## 7. Conclusions

In the last decade operator algebra techniques have been of minor interest in Gabor analysis. But in [DLL90],[Jan95] and [GL04] deep results about Gabor frames were obtained with the help of operator algebras. We included

our approach to the Wexler-Raz biorthogonality principle as an indication for the usefulness of Morita equivalence in Gabor analysis. In the following we list some topics, where our approach gives new insight, too.

- (1) The original motivation for our study of Rieffel's results about Morita equivalence was the density result. There are different approaches to this important theorem [DLL90],[FK98] and [Be04], which at the first sight seem unrelated. In [Lu04a] we show that all these approaches cover different aspects of Morita equivalence between  $C^*$ -algebras generated by time-frequency shifts with respect to a lattice in the time-frequency plane.
- (2) Our interpretation of Rieffel's construction of equivalence- bimodules for noncommutative tori in the notions of Gabor analysis enables us to answer the question posed by Manin on the connection between his quantum theta functions and the quantum theta vectors of Schwarz, see [Lu04b].

Other applications of Rieffel' setting yield new results on Feichtinger' conjecture and on the structure of multi-window Gabor frames, which is part of our current research.

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