

Excess of Gabor Frames

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Deutsche Zusammenfassung

Ein wichtiger Bestandteil unserer Zeit ist die Telekommunikation, das automatisierte Senden und Erhalten von Daten. Dieser Bereich gewinnt in der heutigen Gesellschaft immer mehr an Bedeutung, wodurch eine immer größere Zunahme der versendeten Datenmenge anzunehmen ist. Ziel ist es, das Übertragen der Daten möglichst schnell und fehlerfrei abzuwickeln.

Auf Grund der unterschiedlichen Beschaffenheit des Übertragungsmediums oder dessen Unvorherbestimmtheit kann es zu Datenverlust kommen, wodurch die Rekonstruktion der Daten nötig wird. Es werden dazu Verfahren verwendet, die jene Menge an Daten rasch bewältigen können.

Die Gabor Theorie, benannt nach Dennis Gábor, ermöglicht die Darstellung eines komplexen Signals in der Zeit-Frequenz Ebene. Das ist vergleichbar mit der Darstellung eines Musikstücks durch eine Partitur. Hören wir die Musik, nehmen wir zu jedem Zeitpunkt die gespielten und an unser Ohr gesandten Frequenzen wahr. Wollen wir das Signal diskretisieren - zu Papier bringen - können wir in einer eindeutigen Schreibweise zu jedem Zeitpunkt die Tonhöhe festhalten.

Diese Arbeit beschäftigt sich mit der Frage, wieviele Elemente eines *frames* entbärllich sind, um immer noch einen *frame* zu erhalten und daher weggelassen werden können. Grob gesprochen entspricht ein *frame* einer Familie von Vektoren in einem Hilbert Raum, die so vielzählig ist, dass kein Element des Hilbert Raumes auf all diese Vektoren senkrecht steht.

Als Grundlage dienten in erster Linie die Veröffentlichungen von R. Balan, P.G. Casazza, C. Heil und Z. Landau, *Deficits and excesses of frames* und *Excess of Gabor frames*. Um die Ergebnisse der unterschiedlichen Literaturquellen für den Leser verständlich zu formulieren und darauf aufbauen zu können, war es notwendig, auf eine einheitliche Notation zu achten.

Im ersten Kapitel werden grundlegende Begriffe erläutert, wie die Fouriertransformation, die Translations- und Modulationsoperatoren und die Kurzzeitfouriertransformation, als auch Werkzeuge der Linearen Algebra, die Singulärwertzerlegung und die Pseudoinverse, beschrieben.

Einen theoretischen Teil bilden das zweite und dritte Kapitel. Ersteres gibt einen Überblick zu dem Gebiet der *frames* und stellt deren Eigenschaften vor. Es wird gezeigt, dass der Verlust eines Elements eines *frames* entweder wieder einen

frame oder eine unvollständige Menge hinterläßt. Diese Aussage motiviert uns zur Formulierung des Begriffs *excess*, der die Anzahl der Elemente beschreibt, die, wenn sie weggelassen werden, den selben Raum aufspannen. Dies ist der zentrale Begriff des theoretischen Teils. Es werden vollständige Folgen und *frames* mit unendlichem *excess* untersucht, wobei die Annahmen immer mehr verallgemeinert werden können.

Kapitel 3 gewährt einen Einblick in die Gabor Theorie. Sämtliche zuvor bei *frames* erlangten Erkenntnisse werden nun auf die besondere Struktur der Gabor *frames* angewandt. Es wird gezeigt, dass jedes Gabor System, welches ein redundanter *frame* für sein abgeschlossenes lineares Erzeugnis ist, eine unendliche Teilmenge besitzt, die weggelassen werden kann und dennoch ein *frame* für den selben Raum übrig bleibt. Diese Aussage kann auf rationell verwandete Gabor Multisysteme ausdehnt werden. Und auch diese Aussagen lassen sich noch weiterentwickeln.

Kapitel 4 widmet sich ganz der praktischen Seite. Experimente werden mit Hilfe der Software MATLAB gemacht und hier erläutert. Dabei spielt unter anderem die Überlegung, wieviele Elemente eines Gabor *frames* weggelassen werden können, um immer noch einen numerisch stabilen *frame* zu erhalten, eine wichtige Rolle.

Chapter 1

Basics of Time-Frequency Analysis

We start by giving an introduction to fundamental results and tools used in time-frequency analysis.

But actually what is time-frequency analysis? The idea of time-frequency analysis is describing the decomposition of a signal in its frequencies at every point in time.

We meet this process quite often in our environment and we are used to do it ourselves. Listening to music would be an obvious example. In doing so the human inner ear performs time-frequency analysis. So if someone plays the piano the sound reaches the ear as variations in air pressure. In the cochlea this signal is decomposed into a sum of short oscillations with different frequency, length and amplitude arising at different time. For us it is possible to hear, when, which tone pitch and how hard different keys get struck. Therefore, the intention of time-frequency analysis can be seen as imitating the ear.

The ideal time-frequency representation would give immediate information about the frequencies occurring at any time. But no mathematical model has the ability to satisfy this condition. The reason lies in the uncertainty principle which says roughly that a function f and its Fourier transform \hat{f} cannot be supported on arbitrarily small sets. For more details see [3] p.26.

1.1 Notation

A generic countable index set will be represented by I . $|E|$ denotes the cardinality of a set E . Let X be a Banach space and let $\mathcal{F} = \{f_i\}_{i \in I}$ be a sequence of elements of X .

The finite linear span of \mathcal{F} is given by $\text{span}(\mathcal{F})$, and $\overline{\text{span}}(\mathcal{F})$ denotes the closure (in the norm-topology of X) of $\text{span}(\mathcal{F})$.

$l^2(I)$ is the space of square summable sequences on I .

$L^1(\mathbb{R})$ is the Banach space of measurable functions $f : \mathbb{R} \mapsto \mathbb{C}$ for which $\int_{\mathbb{R}} |f(x)| dx < \infty$ with norm

$$\|f\|_1 := \int_{\mathbb{R}} |f(x)| dx.$$

$L^2(\mathbb{R})$ is the Hilbert space of all complex-valued, square-integrable functions f on \mathbb{R} with norm

$$\|f\|_2 := \sqrt{\left(\int_{\mathbb{R}} |f(x)|^2 dx\right)}.$$

The inner product of $f, g \in L^2(\mathbb{R})$ is

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) \overline{g(x)} dx.$$

The operator norm is given by

$$\|A\|_{\text{Op}} := \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

1.2 Basic definitions and tools

The Fourier transform

The **Fourier transform** of a function $f \in L^1(\mathbb{R}^d)$ is defined as

$$\hat{f}(\omega) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \omega} dx. \quad (1.1)$$

Applying the Fourier transform we get information about the composition of the frequency spectrum of the signal $f(x)$. $\hat{f}(\omega)$ describes the content of oscillations with frequency ω .

However, the time-information gets lost when applying the Fourier transform. In some cases this turns out to be a big disadvantage. For this reason we want to find another solution, which will yield us to the short-time Fourier transform.

Lemma 1.1 (Riemann-Lebesgue). *If $f \in L^1(\mathbb{R}^d)$, then \hat{f} is uniformly continuous and $\lim_{|\omega| \rightarrow \infty} |\hat{f}(\omega)| = 0$, hence bounded. In fact, $\|\hat{f}\|_{\infty} \leq \|f\|_1$.*

$C_0(\mathbb{R}^d)$ denotes the Banach space of continuous functions vanishing at infinity. So Lemma 1.1 can be interpreted as following mapping

$$\mathcal{F} : L^1(\mathbb{R}^d) \rightarrow C_0(\mathbb{R}^d).$$

Theorem 1.2 (Plancherel). *If $f \in L^1 \cap L^2(\mathbb{R}^d)$ then*

$$\|f\|_2 = \|\hat{f}\|_2.$$

As a consequence \mathcal{F} extends in a unique way to a unitary operator on $L^2(\mathbb{R}^d)$ and satisfies **Parseval's formula**

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$$

Plancherel's theorem can be interpreted that the Fourier transform preserves the energy of a signal, that means $\|f\|_2^2 = \|\hat{f}\|_2^2$.

Theorem 1.3 (Hausdorff-Young). *Let $1 \leq p \leq 2$ and let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\mathcal{F} : L^p(\mathbb{R}^d) \rightarrow L^{p'}(\mathbb{R}^d)$ and $\|\hat{f}\|_{p'} \leq \|f\|_p$.*

Translation and modulation

For $x, \omega \in \mathbb{R}^d$

$$T_x f(t) := f(t - x) \tag{1.2}$$

and

$$M_\omega f(t) := e^{2\pi i \omega \cdot t} f(t) \tag{1.3}$$

T_x describes a **translation** by x also called a time shift and M_ω a **modulation** by ω . So operators of the form $T_x M_\omega$ or $M_\omega T_x$ are known as **time-frequency shifts**. They satisfy the commutation relations

$$\begin{aligned} T_x M_\omega f(x) &= (M_\omega f)(t - x) \\ &= e^{2\pi i \omega \cdot (t-x)} f(t - x) \\ &= e^{2\pi i \omega \cdot t} \cdot e^{-2\pi i \omega \cdot x} f(t - x) \\ &= e^{-2\pi i x \cdot \omega} M_\omega T_x f(x). \end{aligned}$$

So we have

$$T_x M_\omega = e^{-2\pi i x \cdot \omega} M_\omega T_x. \tag{1.4}$$

The modulation and translation operators are quite important building blocks and form the basis in frame and Gabor theory.

Next, we describe further properties of these operators.

Time-frequency shifts are isometries on L^p for each $1 \leq p \leq \infty$, that is,

$$\|T_x M_\omega f\|_p = \|f\|_p.$$

Moreover, we have

$$\widehat{(T_x f)} = M_{-x} \hat{f}$$

and

$$\widehat{(M_\omega f)} = T_\omega \hat{f}.$$

The combination of the last two properties yields to

$$\widehat{(T_x M_\omega f)} = M_{-x} T_\omega \hat{f} = e^{2\pi i x \cdot \omega} T_\omega M_{-x} \hat{f}.$$

The convolution

The convolution of two functions $f, g \in L^1(\mathbb{R}^d)$ is the function $f * g$ defined by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y) g(x - y) dy.$$

It satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

and

$$\widehat{(f * g)} = \hat{f} \cdot \hat{g}.$$

Involution and reflection

The involution $*$ is defined by

$$f^*(x) := \overline{f(-x)}$$

and the reflection operator \mathcal{I} by

$$\mathcal{I}f(x) := f(-x).$$

So we obtain

$$\widehat{f^*} = \overline{\hat{f}}$$

and

$$\widehat{\mathcal{I}f} = \mathcal{I}\hat{f}.$$

Now the convolution can be written as

$$(f * g)(x) = \langle f, T_x g^* \rangle.$$

And we have

$$(f * g)^* = g^* * f^*.$$

The inversion formula

The inversion formula shows that we can express f as a superposition of pure frequencies $e^{2\pi i x \cdot \omega}$.

Theorem 1.4. *If $f \in L^1(\mathbb{R}^d) \cap C^0(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R}^d)$, then*

$$\forall x \in \mathbb{R}^d : f(x) = \int_{\mathbb{R}^d} \hat{f}(\omega) e^{2\pi i x \cdot \omega} d\omega. \quad (1.5)$$

Or to put it in other words,

$$\mathcal{F}^{-1} = \mathcal{IF}.$$

It can be shown that the inversion formula and Plancherel's theorem are equivalent. For further details see [3] p.11.

The short-time Fourier transform

The motivation of the short-time Fourier transform, shortened STFT, is that we would like to obtain what frequencies appear at what times. To solve this problem we restrict f to an interval. This means that we multiply the signal f with a window function g , which is constant on a small interval and decays fast and smoothly to zero outside the interval. Then we take the Fourier transform of this product.

Definition 1.2.1. Fix a function $g \in L^2(\mathbb{R}) \setminus \{0\}$. The **short-time Fourier transform**, also called the **continuous Gabor transform** of a function $f \in L^2(\mathbb{R})$ with respect to the window function g is given by

$$V_g f(x, \omega) = \int_{-\infty}^{\infty} f(t) \overline{g(t-x)} e^{-2\pi i t \omega} dt, \text{ for } x, \omega \in \mathbb{R} \quad (1.6)$$

We can write (1.6) in terms of the modulation and translation operators,

$$V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle$$

Results and tools from linear algebra

One useful concept from linear algebra is the definition of the pseudoinverse.

Definition 1.2.2. Let $A \in \mathbb{C}^{m \times n}$ and $r = \text{rank}(A)$. Then there exist unitary matrices $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$ and a matrix $\Sigma \in \mathbb{C}^{m \times n}$ of the form

$$\Sigma = \begin{pmatrix} \text{diag}(\sigma_1, \dots, \sigma_r) & 0_{r, n-r} \\ 0_{m-r, r} & 0_{m-r, n-r} \end{pmatrix}$$

such that

$$A = U \Sigma V^*.$$

By the notation 0_{kl} we mean the zero matrix of dimension $k \times l$. The values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ are unique and known as the **singular values** of A . The representation $U \Sigma V^*$ is called the **singular value decomposition** of A . The matrices U and V are not necessarily unique.

In a sense, the singular value decomposition helps us to understand what the linear function corresponding to the matrix A really does. The first step is a coordinate transformation. V^*x is a vector of the coefficients of $x \in \mathbb{C}^n$ in the coordinate system given by the columns of V . Σ resizes the axes by the factors given in the main diagonal of Σ and transforms the n -dimensional input into an m -dimensional output vector. U changes the basis in \mathbb{C}^m . The first r columns of V are basis vectors for the row space of A , while the first r columns of U are basis vectors for the column space of A . Thus one can view A as a function that maps the row space onto the column space of A . All elements of \mathbb{C}^n that lie outside the row space of A are orthogonally projected onto the row space first.

Definition 1.2.3. For a matrix $A \in \mathbb{C}^{m \times n}$ with the singular value decomposition $A = U\Sigma V^*$ we define the **pseudoinverse** A^+ as

$$A^+ = V\Sigma^+U^* \in \mathbb{C}^{n \times m}$$

where

$$\Sigma^+ = \begin{pmatrix} \text{diag}(\frac{1}{\sigma_1}, \dots, \frac{1}{\sigma_r}) & 0_{r, m-r} \\ 0_{n-r, r} & 0_{n-r, m-r} \end{pmatrix} \in \mathbb{C}^{n \times m}.$$

In geometrical terms, the following steps are performed when calculating A^+y for $y \in \mathbb{C}^m$: first, y is projected onto the column space of A . Then a coordinate transformation according to the first r columns of U is applied. The axes are scaled by the factors given in the main diagonal of Σ^+ , i.e. either 0 or the reciprocal values of the singular values of A . Finally, another coordinate transformation defined by the first r columns of V results in an element of the row space of A . This process seems to be the exact inversion of the above mentioned steps for the calculation of Ax for some $x \in \mathbb{C}^n$. However, A^+ equals A^{-1} only if A is invertible. In this case Σ^+ is the exact inverse of Σ . Otherwise at most one of the two relations $AA^+ = I$ and $A^+A = I$ holds.

The pseudo-inverse can be also described in a different way. We consider the given $m \times n$ matrix A as a linear mapping of \mathbb{C}^n to \mathbb{C}^m . A is not necessarily injective, but if we restrict the mapping A to the orthogonal complement of the kernel N_A , we receive an injective linear mapping

$$\tilde{A} : N_A^\perp \rightarrow \mathbb{C}^m.$$

A and \tilde{A} have the same range, $R_{\tilde{A}} = R_A$. Therefore, \tilde{A} seen as mapping from N_A^\perp to R_A has an inverse,

$$(\tilde{A})^{-1} : R_A \rightarrow N_A^\perp.$$

We can extend $(\tilde{A})^{-1}$ to an operator $A^+ : \mathbb{C}^m \rightarrow \mathbb{C}^n$ by defining

$$A^+(y + z) = (\tilde{A})^{-1}y \text{ if } y \in R_A, z \in R_A^\perp.$$

So we obtain

$$AA^+x = x, \forall x \in R_A. \tag{1.7}$$

Theorem 1.5. *Let A be an $n \times n$ matrix. Given $y \in R_A$, the equation $Ax = y$ has a unique solution of minimal norm, namely $x = A^+y$.*

Proof: Because of (1.7) we know that $x := A^+y$ is a solution to the equation $Ax = y$. All solutions have the form $x = A^+y + z$, where $z \in N_A$. Since $A^+y \in N_A^\perp$, the norm of the general solution is given by

$$\|x\|^2 = \|A^+y + z\|^2 = \|A^+y\|^2 + \|z\|^2,$$

which is minimal when $z = 0$.

□

The lattice

Operating with Gabor frames in the next chapter will show the necessity of defining a set as lattice. So we give here a general definition.

Definition 1.2.4. A **lattice** $\Lambda \subseteq \mathbb{R}^d$ is a discrete subgroup of \mathbb{R}^d of the form $\Lambda = AZ^d$, where A is an invertible $d \times d$ -matrix over \mathbb{R} . The **volume** of Λ is defined as $\text{vol}(\Lambda) = |\det A| = |A[0, 1]^d|$. The **dual lattice** is defined as $\Lambda^\perp = (A^{-1})^T Z^d$.

The Gauss function

The Gauss function plays an important role in Fourier Analysis. It has the desirable property to minimize the uncertainty principle inequality. So this function is particularly suitable for getting used as elementary signal, which later we will refer to as atom.

Definition 1.2.5. The **Gaussian** with parameter $a > 0$ is the function

$$g_a(x) := e^{-ax^2}, x \in \mathbb{R}. \quad (1.8)$$

Furthermore, the Gaussian is up to constants invariant under the Fourier transform \mathcal{F} . Or, in other words, the Gaussian is eigenfunction of the Fourier transform \mathcal{F} .

Theorem 1.6. *The function $g_\pi(x) = e^{-\pi x^2}$ satisfies $\mathcal{F}g_\pi = g_\pi$.*

Proof: The Fourier transform is given by

$$\begin{aligned} \hat{g}_\pi(\omega) &= \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i \omega x} dx \\ &= \int_{\mathbb{R}} e^{-\pi(x+i\omega)^2} e^{-\pi\omega^2} dx \\ &= e^{-\pi\omega^2} \int_{\mathbb{R}} e^{-\pi(x+i\omega)^2} dx. \end{aligned}$$

We set $h(x, \omega) := e^{-\pi(x+i\omega)^2}$. So we have

$$\frac{\partial h}{\partial \omega}(x, \omega) = -2\pi i(x + i\omega)e^{-\pi(x+i\omega)^2}$$

which is uniformly integrable with respect to x over bounded ranges of ω . Hence it is possible to switch the order of integration and differentiation when we differentiate the parameter integral $H(\omega) = \int_{\mathbb{R}} h(x, \omega) dx$.

$$\begin{aligned} \frac{dH}{d\omega}(\omega) &= \int_{\mathbb{R}} \frac{\partial h}{\partial \omega}(x, \omega) dx \\ &= \int_{\mathbb{R}} -2\pi i(x + i\omega)e^{-\pi(x+i\omega)^2} dx \\ &= \left[ie^{-\pi(x+i\omega)^2} \right]_{-\infty}^{\infty} \\ &= 0 \quad \text{for all } \omega \in \mathbb{R} \end{aligned}$$

Consequently $H(\omega)$ is a constant function and therefore $H(\omega) \equiv H(0)$. For calculating $H(0)$, we consider

$$\begin{aligned} (H(0))^2 &= \int_{\mathbb{R}} e^{-\pi x^2} dx \int_{\mathbb{R}} e^{-\pi y^2} dy \\ &= \iint_{\mathbb{R} \times \mathbb{R}} e^{-\pi(x^2+y^2)} dx dy \\ &= \iint_{\mathbb{R}^+ \times [0, 2\pi]} e^{-\pi r^2} r dr d\varphi \\ &= \int_0^{\infty} 2\pi r e^{-\pi r^2} dr \\ &= \left[-e^{-\pi r^2} \right]_0^{\infty} = 1. \end{aligned}$$

Because $h(x, 0) \geq 0$, we know that $H(0) \geq 0$ and hence $H(0) = 1$, which results to $\hat{g}_{\pi}(\omega) = e^{-\pi\omega^2}$ as desired.

□

The Frobenius norm

In numerical calculations computing the norm is an essential part. For this purpose we use the Frobenius norm, which is a matrix norm.

Definition 1.2.6. Let $A \in \mathbb{C}^{m \times n}$, $A = (a_{ij})_{\substack{i=1, \dots, m \\ j=1, \dots, n}}$. The **Frobenius norm**, also called **Hilbert-Schmidt norm**, of A is defined as

$$\|A\|_{\text{Fro}} = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2}.$$

This norm treats the matrix A as a long vector. It is also equal to the square root of the matrix trace of $A \cdot A^*$, where A^* is the conjugate transpose, this is

$$\|A\|_{\text{Fro}} = \sqrt{\text{trace}(AA^*)}.$$

Furthermore, we get by applying the Cauchy-Schwarz' inequality

$$\begin{aligned} \|Ax\|_2 &= \sqrt{\sum_{j=1}^m \left| \sum_{k=1}^n a_{jk}x_k \right|^2} = \sqrt{\sum_{j=1}^m |\langle z_j, x \rangle|^2} \\ &\leq \sqrt{\sum_{j=1}^m \|z_j\|_2^2} \|x\|_2 = \|x\|_2 \sqrt{\sum_{j=1}^m \sum_{k=1}^n |a_{jk}|^2} \\ &= \|x\|_2 \|A\|_{\text{Fro}}. \end{aligned}$$

And it follows that $\|A\|_{\text{Op}} \leq \|A\|_{\text{Fro}}$.

Chapter 2

Frames

In the following sections we will explore the notion of frames and the definition of the excess. We start with the intention of frames and the advantages of using them, followed by important properties. Afterwards, Theorem 2.5 gives us an important aspect: it states that the removal of a vector from a frame leaves either a frame or an incomplete set. The outcome of this is the question of how many elements can be omitted from a frame and still leave a frame. This considerations lead us to the notion of excess.

2.1 An introduction to Frames

Initially we will give a motivation why using frames. In Banach spaces we are often searching for a basis, because having one is of great advantage. This means possessing a fixed set of vectors g_n such that any vector f in the space can be written $f = \sum c_n g_n$ for some unique choice of scalars c_n . In most cases we know that a basis exists, but in some cases existence is not sufficient. The desired basis should be easily generated, hence we place the demand on the scalars to be computed without big effort.

In a Hilbert space (such as $L^2(\mathbb{R})$) we know that it possesses an orthonormal basis, a set of vectors that in addition to being a basis is mutually orthogonal. The requirements of orthogonality and the basis property are stringent conditions for finding one.

Frames are a generalization of orthonormal bases. By giving up the requirements of orthogonality and uniqueness of decomposition we get more freedom in the choice of the e_n , but we still retain good control of the behavior of the c_n and the sum.

A frame can be seen as a collection of vectors in a Hilbert space H , which is so comprehensive that there can be found no $x \in H$ which is perpendicular to all the vectors in the collection.

Definition 2.1.1. A basis $\{e_n\}_{n \in I}$ is called **orthonormal basis**, if e_n and e_m are orthogonal for all $n, m \in I$ and $\|e_n\| = 1$ for all $n \in I$.

Let H be a separable Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. In every separable Hilbert space it is always possible to find an orthonormal basis. [7] p.231

If $\{e_n\}$ is an orthonormal basis for H then **Parseval's equation** holds:

$$\forall x \in H : \quad \|x\|^2 = \sum_{n \in I} |\langle x, e_n \rangle|^2.$$

Furthermore, every x in H can be written as

$$\forall x \in H : \quad x = \sum_{n \in I} \langle x, e_n \rangle e_n,$$

which is equivalent to Parseval's equation. For more details we refer to [7] p.230.

If we want to construct such an orthonormal basis we can use the Gram-Schmidt orthogonalisation routine for example.

However, as mentioned before we need the e_n to be easily and numerically stable generated. But the Gram-Schmidt orthogonalisation routine is numerically fairly instable. So our intention is an alternative for orthonormal bases and we achieve it in the notion of frames.

Definition 2.1.2. Let H be a separable Hilbert space and I a countable index set. A sequence $\mathcal{F} = \{f_i\}_{i \in I}$ of elements of H is a **frame** for H if there exist constants $A, B > 0$ such that

$$\forall h \in H, A\|h\|^2 \leq \sum_{i \in I} |\langle h, f_i \rangle|^2 \leq B\|h\|^2. \quad (2.1)$$

The numbers A, B are called *lower* and *upper frame bounds*, respectively. The largest A and the smallest B for which equation (2.1) holds are the *optimal* frame bounds.

Frames were first introduced by Duffin and Schaeffer in the context of nonharmonic Fourier series [10].

So the definition shows that each frame \mathcal{F} is complete in H , this means the finite linear span of \mathcal{F} is dense in H , or

$$\overline{\text{span}}\{f_i\}_{i \in I} = H. \quad (2.2)$$

Another property is that a frame provides basis-like representations of the elements of H . So there exist vectors \tilde{f}_i such that

$$\forall h \in H, h = \sum_{i \in I} \langle h, f_i \rangle \tilde{f}_i = \sum_{i \in I} \langle h, \tilde{f}_i \rangle f_i, \quad (2.3)$$

with unconditional convergence of these series.

In general a frame needs not to be a basis, and the representations in (2.3) need not be unique. Frames which are no basis are called *overcomplete*, this means there exist proper subsets of the frame which are complete.

Definition 2.1.3. A sequence $\mathcal{F} = \{f_i\}_{i \in I}$ is **complete** in X if $\overline{\text{span}}(\mathcal{F}) = X$.

Definition 2.1.4. A sequence $\mathcal{F} = \{f_i\}_{i \in I}$ in a separable Hilbert space H is a **Bessel sequence** if there exists a constant $B > 0$ such that

$$\forall h \in H, \sum_{i \in I} |\langle h, f_i \rangle|^2 \leq B \|h\|^2. \quad (2.4)$$

Definition 2.1.5. The **analysis operator** T is defined by

$$T : H \rightarrow l^2(I)$$

$$h \mapsto \{\langle h, f_i \rangle\}_{i \in I}$$

and the **synthesis operator** T^* defined by

$$T^* : l^2(I) \rightarrow H$$

$$c \mapsto \sum_{i \in I} c_i f_i.$$

The **frame operator** S is defined by

$$S = T^*T.$$

The operators T and T^* are everywhere-defined, bounded operators, each adjoint to the other. The frame operator is a positive, continuously invertible mapping of H onto itself.

We can write (2.1)

$$\langle Af, f \rangle \leq \langle Sf, f \rangle \leq \langle Bf, f \rangle$$

and by applying the inequality relation $\forall f \in H : X \leq Y \Leftrightarrow \langle Xf, f \rangle \leq \langle Yf, f \rangle$ we get

$$AI \leq S \leq BI.$$

In numerical procedures the condition number is basically a measure of stability of a matrix, which is defined as the ratio between the largest and the smallest eigenvalue. In the case of the frame operator, these eigenvalues correspond to the optimal frame bounds in the following way: The optimal lower frame bound is the smallest eigenvalue of S , and the optimal upper frame bound is the largest eigenvalue. This is an important aspect and we will use this consideration later when we describe

experiments.

If $c \in l^2(I)$, then the series $\sum c_i f_i$ defining T^*c converges unconditionally in the norm of H .

That $\overline{\text{span}(\mathcal{F})} = \overline{\text{ran}T^*}$ follows from $\text{span}(\mathcal{F}) \subset \text{ran}T^* \subset \overline{\text{span}(\mathcal{F})}$. The elements of a Bessel sequence are uniformly bounded above in norm, specifically, $\|f_i\|^2 \leq B$ for each $i \in I$.

Frames are special cases of Bessel sequences. An advantage of frames is the existence of a *dual frame* $\{\tilde{f}_i\}_{i \in I}$ such that the frame expansion in (2.3) holds. In general, this is not true for Bessel sequences.

Definition 2.1.6. The *standard dual frame* is given by $\tilde{f}_i = S^{-1}f_i$, where S is the frame operator.

Proposition 2.1. For a sequence $\mathcal{F} = \{f_i\}_{i \in I}$ in a separable Hilbert space H , the following statements are true:

- i. S is invertible and $B^{-1}I \leq S^{-1} \leq A^{-1}I$.
- ii. $\{S^{-1}f_i\}$ is a frame with bounds A^{-1}, B^{-1} .
- iii. Every $f \in H$ can be written $f = \sum \langle f, S^{-1}f_i \rangle f_i = \sum \langle f, f_i \rangle S^{-1}f_i$.

The proof is shown in [8].

Definition 2.1.7. A frame is **tight** if $A = B$. A frame is **normalized tight** if $A = B = 1$.

$\mathcal{F} = \{f_i\}_{i \in I}$ is a tight frame if and only if $S = AI$.

Definition 2.1.8. A frame is called **exact** if it is no longer a frame when any one of its elements is removed, otherwise it is called to be **inexact**.

Because S is a positive operator, it has a positive square root $S^{1/2}$. Furthermore, $S^{-1/2}$ is a bounded, continuously invertible operator and $\{S^{-1/2}f_i\}_{i \in I}$ is a normalized tight frame for H . For this reason every frame is equivalent to a normalized tight frame.

Next, we define Riesz sequences and Riesz bases. Afterwards some important relationships between frames and Riesz bases are given.

Definition 2.1.9. A **Riesz sequence** is a sequence $\mathcal{F} = \{f_i\}_{i \in I}$ for which there exist $A, B > 0$ such that

$$\forall c \in l^2(I), A \sum_{i \in I} |c_i|^2 \leq \left\| \sum_{i \in I} c_i f_i \right\|^2 \leq B \sum_{i \in I} |c_i|^2.$$

If a Riesz sequence is complete then it is called a **Riesz basis** for H .

Proposition 2.2 (Riesz basis and frames). *For a sequence $\mathcal{F} = \{f_i\}_{i \in I}$ in a separable Hilbert space H , the following statements are equivalent:*

- i. $\{f_i\}_{i \in I}$ is both a frame and a basis for H .*
- ii. $\{f_i\}_{i \in I}$ is a Riesz basis for H .*
- iii. $\{f_i\}_{i \in I}$ is an exact frame for H .*

Proof: ii. \Rightarrow iii.

Removing an arbitrary element leaves a family, which is not complete, and therefore not a frame.

iii. \Rightarrow ii.

If $\mathcal{F} = \{f_i\}_{i \in I}$ is an exact frame, it is a basis by [4] Proposition 5.4.8. Let as usual $\{\delta_i\}_{i \in I}$ be the canonical basis for $l^2(\mathbb{N})$. The synthesis operator $T : l^2 \rightarrow H$ associated with $\mathcal{F} = \{f_i\}_{i \in I}$ is bounded and surjective by [4] Theorem 5.5.1, and $T\delta_i = f_i$. It is enough to prove that T is invertible, which follows from $\mathcal{F} = \{f_i\}_{i \in I}$ being a basis, and so our proof is complete.

For the remaining proof we refer to [14] or [4].

□

All Riesz bases are frames. If $\mathcal{F} = \{f_i\}_{i \in I}$ is a Riesz basis, then for each $h \in H$ the frame expansion given in (2.3) is unique. A frame is a Schauder basis for H if and only if it is a Riesz basis for H .

As we have seen in Proposition 2.1 iii. the frame coefficients $\{\langle f, S^{-1}f_i \rangle\}_{i \in I}$ lead to a representation of the given $f \in H$. If we have an overcomplete frame, there exist other coefficients $\{c_i\}_{i \in I} \in l^2(\mathbb{N})$ for which $f = \sum_{i=1}^{\infty} c_i f_i$. The frame coefficients $\{\langle f, S^{-1}f_i \rangle\}_{i \in I}$ have minimal l^2 -norm among all sequences representing f .

Additionally we will use the next proposition to prove Theorem 2.5.

Proposition 2.3. *Given a frame $\mathcal{F} = \{f_i\}_{i \in I}$ and given $f \in H$, let $a_i = \langle f, S^{-1}f_i \rangle$, so $f = \sum a_i f_i$. If it is possible to find other scalars c_i such that $f = \sum c_i f_i$, then*

$$\sum |c_i|^2 = \sum |a_i|^2 + \sum |a_i - c_i|^2.$$

The interested reader can look in [8] for the proof.

Using Theorem 1.5 and Proposition 2.3 we can formulate an explicit expression for the pseudo-inverse of the synthesis operator:

Theorem 2.4. *Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a frame with synthesis operator T and frame operator S . Then*

$$T^+ f = \{\langle f, S^{-1}f_i \rangle\}_{i \in I}.$$

The following theorem describes the removal of one element of the frame. This statement was presented in [10].

Theorem 2.5. *The removal of a vector f_j from a frame $\mathcal{F} = \{f_i\}_{i \in I}$ for H leaves either a frame or an incomplete set. In Particular,*

$$\begin{aligned} & \text{if } \langle f_j, S^{-1}f_j \rangle \neq 1, \text{ then } \{f_i\}_{i \neq j} \text{ is a frame,} \\ & \text{if } \langle f_j, S^{-1}f_j \rangle = 1, \text{ then } \{f_i\}_{i \neq j} \text{ is incomplete.} \end{aligned}$$

Proof: Choose $j \in \mathbb{N}$ arbitrarily. Because of the frame decomposition,

$$f_j = \sum_{k=1}^{\infty} \langle f_j, S^{-1}f_k \rangle f_k.$$

We define $a_k := \langle f_j, S^{-1}f_k \rangle$, so $f_j = \sum_{k=1}^{\infty} a_k f_k$. Clearly, we also have $f_j = \sum_{k=1}^{\infty} \delta_{j,k} f_k$, so Proposition 2.3 yields the following relation between $\delta_{j,k}$ and a_k :

$$\begin{aligned} 1 = \sum_{k=1}^{\infty} |\delta_{j,k}|^2 &= \sum_{k=1}^{\infty} |a_k|^2 + \sum_{k=1}^{\infty} |a_k - \delta_{j,k}|^2 \\ &= |a_j|^2 + \sum_{k \neq j} |a_k|^2 + |a_j - 1|^2 + \sum_{k \neq j} |a_k|^2. \end{aligned}$$

We consider the cases $a_j = 1$ and $a_j \neq 1$ separately.

First, assume that $a_j = 1$. From the above formula, $\sum_{k \neq j} |a_k|^2 = 0$, so that

$$a_k = \langle S^{-1}f_j, f_k \rangle = 0 \forall k \neq j.$$

This denotes $S^{-1}f_j$ is orthogonal to f_k for every $k \neq j$. Because $a_j = \langle S^{-1}f_j, f_j \rangle = 1$ we know that $S^{-1}f_j \neq 0$. So we have found a non-zero element $S^{-1}f_j$ which is orthogonal to $\{f_k\}_{k \neq j}$, therefore $\{f_k\}_{k \neq j}$ is incomplete.

We now assume that $a_j \neq 1$. Then $f_j = \frac{1}{1-a_j} \sum_{k \neq j} a_k f_k$. For any $f \in H$, Cauchy-Schwarz' inequality gives

$$\begin{aligned} |\langle f, f_j \rangle|^2 &= \left| \frac{1}{1-a_j} \sum_{k \neq j} a_k \langle f, f_k \rangle \right|^2 \\ &\leq \frac{1}{|1-a_j|^2} \sum_{k \neq j} |a_k|^2 \sum_{k \neq j} |\langle f, f_k \rangle|^2 \\ &= C \sum_{k \neq j} |\langle f, f_k \rangle|^2, \end{aligned}$$

where $C = \frac{1}{|1-a_j|^2} \sum_{k \neq j} |a_k|^2$. Let A denote a lower frame bound for $\mathcal{F} = \{f_i\}_{i \in I}$. Then

$$\begin{aligned} A\|f\|^2 &\leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \\ &= \sum_{k \neq j} |\langle f, f_k \rangle|^2 + |\langle f, f_j \rangle|^2 \\ &\leq (1+C) \sum_{k \neq j} |\langle f, f_k \rangle|^2, \end{aligned}$$

showing that $\{f_k\}_{k \neq j}$ satisfies the lower frame condition with lower bound $\frac{A}{1+C}$. Thus $\{f_k\}_{k \neq j}$ also satisfies the upper frame condition.

□

2.2 Deficit and excess of frames

In a sense, a frame might contain more elements than necessary for being a basis. In that case a frame is said to be **overcomplete** or **redundant**.

Motivated by Theorem 2.5 we face the question how many elements and which elements can be removed and still leave a frame. Therefore we introduce the definition of the deficit and the excess.

Definition 2.2.1. *Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a sequence in a separable Banach space X . The **deficit** of \mathcal{F} is*

$$d(\mathcal{F}) = \inf\{|\mathcal{G}| : \mathcal{G} \subset X \text{ and } \overline{\text{span}}(\mathcal{F} \cup \mathcal{G}) = X\}.$$

*The **excess** of \mathcal{F} is*

$$e(\mathcal{F}) = \sup\{|\mathcal{G}| : \mathcal{G} \subset \mathcal{F} \text{ and } \overline{\text{span}}(\mathcal{F} \setminus \mathcal{G}) = \overline{\text{span}}(\mathcal{F})\}.$$

The deficit of a sequence is the smallest number of elements of the set which need to be added so that the closure of the linear span of the sequence is still the whole Hilbert space X . Therefore the deficit stands for the number of elements, which would be needed to span the Hilbert space X .

If we have two unit vectors in \mathbb{R}^3 the deficit is one, since without a third unit vector we cannot span \mathbb{R}^3 . So the deficit of a frame for a Hilbert space H is zero, because of (2.2).

The excess of a sequence is the greatest number of elements such that we get the same result whether we take the closure of the linear span of the sequence without these elements or from the whole sequence. Thus the excess describes the number of elements, which can be removed and the remaining elements still span the same space as all elements together did before.

If we have five vectors which span \mathbb{R}^3 , the excess is two, since we can remove two of the vectors and the remaining elements still span \mathbb{R}^3 . Apparently, we must be aware to choose this vectors correct, or we might loose one dimension. Having a Riesz sequence in a Hilbert space H , the excess is zero, since a Riesz sequence is not redundant.

These considerations lead us to the next propositions, which give a connection between Riesz bases and excess.

Proposition 2.6. *A frame has zero excess if and only if it is an exact frame.*

Proof: (\Rightarrow) Because of Proposition 2.7 $\{f_i\}_{i \in I}$ is a Riesz basis. And after Proposition 2.2 follows that $\{f_i\}_{i \in I}$ is an exact frame.
(\Leftarrow) Follows from Definition 2.1.8. □

Proposition 2.7. *A frame $\mathcal{F} = \{f_i\}_{i \in I} \in H$ has zero excess, if and only if it is a Riesz basis for H .*

Proof: We use Proposition 2.6 and Proposition 2.2 and the statement follows. □

The next lemma will show that if we have finite nested subsets which can be omitted from a sequence \mathcal{F} and leave a complete set, it will be possible to construct an infinite subset that can be removed and still leave a complete set.

Lemma 2.8. *Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a sequence in a Banach space X , and assume that there exists a subsequence $\{g_n\}_{n \in \mathbb{N}}$ such that $\mathcal{F} \setminus \{g_1, g_2, \dots, g_n\}$ is complete in X for each $n \in \mathbb{N}$. Then there exists an infinite subsequence \mathcal{G} of g_n such that $\mathcal{F} \setminus \mathcal{G}$ is complete in X .*

Proof: Set $M = \mathcal{F} \setminus \{g_n\}_{n \in \mathbb{N}}$. Moreover let $k_1 = 1$. As per assumption $M \cup \{g_n\}_{n=2}^\infty = \mathcal{F} \setminus \{g_1\}$ is complete, thus there exists $k_2 > k_1$ such that

$$\text{dist}(g_{k_1}, \overline{\text{span}}(M \cup \{g_n\}_{n=2}^{k_2-1})) < \frac{1}{2},$$

where $\text{dist}(x, Y) = \inf\{\|x - y\| : y \in Y\}$ is the distance from a vector x to a subset Y of X . Because $M \cup \{g_n\}_{n=k_2+1}^\infty = \mathcal{F} \setminus \{g_1, \dots, g_{k_2}\}$ is complete, there exists $k_3 > k_2$ such that both

$$\text{dist}(g_{k_1}, \overline{\text{span}}(M \cup \{g_n\}_{n=k_2+1}^{k_3-1})) < \frac{1}{3}$$

and

$$\text{dist}(g_{k_2}, \overline{\text{span}}(M \cup \{g_n\}_{n=k_2+1}^{k_3-1})) < \frac{1}{3}.$$

Proceeding this process we can find $k_1 < k_2 < \dots$ such that for each $\ell \in \mathbb{N}$ we have

$$\text{dist}(g_{k_j}, \overline{\text{span}}(M \cup \{g_n\}_{n=k_\ell+1}^{k_{\ell+1}-1})) < \frac{1}{\ell+1}, j = 1, \dots, \ell. \quad (2.5)$$

Let $\mathcal{G} = \{g_{k_j}\}_{j=1}^\infty$. We claim that $\mathcal{F} \setminus \mathcal{G}$ is complete. Since \mathcal{F} is complete, it suffices to show that

$$\forall j \in \mathbb{N}, \text{dist}(g_{k_j}, \overline{\text{span}}(\mathcal{F} \setminus \mathcal{G})) = 0. \quad (2.6)$$

Because of $M \cup \{g_n\}_{n=k_\ell+1}^{k_{\ell+1}-1} \subset \mathcal{F} \setminus \mathcal{G}$, we can follow from (2.5) that for all $\ell \geq j$,

$$\text{dist}(g_{k_j}, \overline{\text{span}}(\mathcal{F} \setminus \mathcal{G})) \leq \text{dist}(g_{k_j}, \overline{\text{span}}(M \cup \{g_n\}_{n=k_\ell+1}^{k_{\ell+1}-1})) < \frac{1}{\ell + 1}.$$

So our assumption holds. □

What does it mean, when we assume that a sequence $\mathcal{F} = \{f_i\}_{i \in I}$ has infinite excess? Being aware of Definition 2.2.1, the supremum is not finite. This means it is always possible to find a suitable sequence \mathcal{G} which satisfies $\overline{\text{span}}(\mathcal{F} \setminus \mathcal{G}) = \overline{\text{span}}(\mathcal{F})$. But this does not necessarily imply the existence of a sequence \mathcal{G} with an infinite number of elements.

In the next theorem the property of nestedness is removed. Instead of this property we have sequences with infinite excess. In every such sequence there exists an infinite subsequence that can be deleted and still leave a complete set.

For the next proof some notation skills are required. Let S be a subspace of a Banach space X , then $\dim(S)$ denotes the dimension of a subspace S . The codimension of S is $\text{codim}(S) = \dim(T)$ where T is any algebraic complement of S , that means any subspace such that $S + T = X$ and $S \cap T = \{0\}$. The codimension of S is independent of the choice of subspace T .

Theorem 2.9. *Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a complete sequence in a Banach space X with infinite excess. Then there exists an infinite subsequence \mathcal{G} of \mathcal{F} such that $\mathcal{F} \setminus \mathcal{G}$ is complete in X .*

Proof: First we will prove that there must exist a subsequence $\{g_n\}_{n \in \mathbb{N}}$ of \mathcal{F} such that $\mathcal{F} \setminus \{g_1, \dots, g_n\}$ is complete in X for each $n \in \mathbb{N}$. If this claim holds, the result then follows immediately from Lemma 2.8.

Assume, that no such infinite subsequence existed, then there would exist at least one maximal finite subset $G = \{g_1, \dots, g_n\}$ of \mathcal{F} such that $\mathcal{F} \setminus G$ is complete, because \mathcal{F} has infinite excess. By the same reason there must also exist a finite subset $H = \{h_1, \dots, h_m\}$ of \mathcal{F} with $m \geq 2n$ such that $\mathcal{F} \setminus H$ is complete. Since G is maximal, we cannot have $G \subset H$. Therefore $G \cap H$ contains at most $n - 1$ elements and $H \setminus G$ contains at least $n + 1$ elements.

Let $E = \mathcal{F} \setminus (G \cup H)$. Since $E \cup (G \setminus H) = \mathcal{F} \setminus H$ and $E \cup (H \setminus G) = \mathcal{F} \setminus G$ are both complete, we have that

$$\overline{\text{span}}(E) + \text{span}(G \setminus H) = X \tag{2.7}$$

and

$$\overline{\text{span}}(E) + \text{span}(H \setminus G) = X. \quad (2.8)$$

It follows from (2.7) that

$$\text{codim}(\overline{\text{span}}(E)) \leq |G \setminus H| \leq n. \quad (2.9)$$

Combining this with (2.8) implies that $\text{span}(H \setminus G)$ contains an algebraic complement of $\overline{\text{span}}(E)$ of dimension at most n . Because of $|H \setminus G| \geq n + 1$, at least one element $h \in H \setminus G$ must lie in the closed span of the union of E and the remaining elements of $H \setminus G$. But then $E \cup (H \setminus (G \cup \{h\})) = \mathcal{F} \setminus (G \cup \{h\})$ is complete. This statement is contradictory to the maximality of G .

□

Examples can be found in [1] Example 5.1, where we have a frame \mathcal{F} in a Hilbert space H from which we delete a single element $f_{j_k}^{n_k}$. So we get a subsequence $\mathcal{F} \setminus \{f_{j_k}^{n_k}\}$ which is also a frame. But there exists no single positive number used as a common lower frame bound for all of the subframes $\mathcal{F} \setminus \{f_{j_k}^{n_k}\}$.

However, if we assume that $\mathcal{F} = \{f_i\}_{i \in I}$ is a frame such that there exists an infinite subsequence $\mathcal{G} = \{g_n\}_{n \in \mathbb{N}}$, so that $\mathcal{F} \setminus \mathcal{G}$ is a frame for H with lower frame bound L . Then $\mathcal{F} \setminus \{g_n\}$ is a frame for H for each fixed n with lower frame bound L , because $\mathcal{F} \setminus \mathcal{G} \subset \mathcal{F} \setminus \{g_n\} \subset \mathcal{F}$. Consequently, we will change the assumption and request the existence of an infinite subsequence described before with lower frame bound L .

That means the property of existence of such a sequence $g_{n \in \mathbb{N}}$ with uniform lower frame bound for each $\mathcal{F} \setminus g_n$ is a necessary condition for removing infinitely many elements from a frame and still being a frame. Therefore, we want to show that this condition is both sufficient and necessary.

Hence we will show that if such a g_n exists, then we can find an infinite subsequence $\mathcal{G}_\epsilon = g_{n_k}$ $k \in \mathbb{N}$ such that $\mathcal{F} \setminus \mathcal{G}_\epsilon$ is a frame with lower frame bound $L - \epsilon$. Our next theorem will represent the special case of normalized tight frames.

Theorem 2.10. *Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a normalized tight frame for a Hilbert space H , and assume that there exists an infinite subsequence $\mathcal{G} = \{g_n\}_{n \in \mathbb{N}}$ of \mathcal{F} such that for each n , $\mathcal{F} \setminus \{g_n\}$ is complete in H (and hence a frame). If there exists a single constant $L > 0$ that is a lower frame bound for each frame $\mathcal{F} \setminus \{g_n\}$, then for every $0 < \epsilon < L$ there exists an infinite subsequence \mathcal{G}_ϵ of \mathcal{G} such that $\mathcal{F} \setminus \mathcal{G}_\epsilon$ is a frame for H with lower frame bound $L - \epsilon$.*

Proof: Because we have a normalized tight frame, the frame bound $A = B = 1$. Furthermore the frame operator S for $\mathcal{F} = \{f_i\}_{i \in I}$ is the identity. That is,

$$\forall f \in H, f = Sf = \sum_{i \in I} \langle f, f_i \rangle f_i.$$

And we have the given property that, for each $n \in \mathbb{N}$, $\mathcal{F} \setminus g_n$ is a frame with lower frame bound L . Let S_n be the frame operator for $\mathcal{F} \setminus g_n$, that is

$$S_n f = \sum_{i \in I} \langle f, f_i \rangle f_i - \langle f, g_n \rangle g_n = f - \langle f, g_n \rangle g_n.$$

And because of

$$\langle S_n f, f \rangle = \|f\|^2 - |\langle f, g_n \rangle|^2 \geq \|f\|^2 - \|f\|^2 \|g_n\|^2 = (1 - \|g_n\|^2) \|f\|^2,$$

we have that $1 - \|g_n\|^2$ is a lower frame bound for $\mathcal{F} \setminus g_n$, and by considering the element $f = g_n$ we see that it is the optimal lower frame bound for $\mathcal{F} \setminus g_n$, and if we look at the element $f = g_n$ we recognize that this is the optimal lower frame bound for $\mathcal{F} \setminus g_n$. So we must have

$$\forall n \in \mathbb{N}, L \leq 1 - \|g_n\|^2.$$

Because $g_{n \in \mathbb{N}}$ is a subset of the frame \mathcal{F} , we have $\sum_k |\langle g_n, g_k \rangle|^2 \leq \|g_n\|^2 < \infty$. Hence,

$$\forall n \in \mathbb{N}, \lim_{k \rightarrow \infty} \langle g_n, g_k \rangle = 0.$$

Since this, we are able to extract a subsequence $\mathcal{G}_\varepsilon = g_{n_k, k \in \mathbb{N}}$ for which this condition holds

$$\sum_{j, k \in \mathbb{N}, k \neq j} |\langle g_{n_k}, g_{n_j} \rangle| < \varepsilon.$$

We claim that $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ is a frame for H with lower frame bound $L - \varepsilon$. So we observe the operator

$$Rf = \sum_{k=1}^{\infty} \langle f, g_{n_k} \rangle g_{n_k}.$$

Since \mathcal{G}_ε is a subset of the frame \mathcal{F} , Rf is a bounded operator.

$$\begin{aligned} \|Rf\|^2 &= \left\langle \sum_{k=1}^{\infty} \langle f, g_{n_k} \rangle g_{n_k}, \sum_{j=1}^{\infty} \langle f, g_{n_j} \rangle g_{n_j} \right\rangle \\ &= \sum_{k=1}^{\infty} |\langle f, g_{n_k} \rangle|^2 \|g_{n_k}\|^2 + \sum_{j, k \in \mathbb{N}, k \neq j} \langle f, g_{n_k} \rangle \langle g_{n_j}, f \rangle \langle g_{n_k}, g_{n_j} \rangle \\ &\leq (\sup_{k \in \mathbb{N}} \|g_{n_k}\|^2) \langle Rf, f \rangle + \|f\|^2 (\sup_{k \in \mathbb{N}} \|g_{n_k}\|^2) \left(\sum_{j, k \in \mathbb{N}, k \neq j} |\langle g_{n_k}, g_{n_j} \rangle| \right) \\ &\leq (1 - L) \|Rf\| \|f\| + \|f\|^2 (1 - L) \varepsilon. \end{aligned}$$

From this we conclude that $\|R\| \leq 1 - L + \varepsilon$, and then we have

$$\sum_{i \in I} |\langle f, f_i \rangle|^2 - \sum_{k=1}^{\infty} |\langle f, g_{n_k} \rangle|^2 = \|f\|^2 - \langle Rf, f \rangle \geq (L - \varepsilon) \|f\|^2.$$

So $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ is a frame with lower frame bound $L - \varepsilon$.

□

The following theorem is a generalization of Theorem 2.10 for non-tight frames. The proof is quite technical and needs some results that exceed the scope of this thesis, so we refer the reader to [1].

Theorem 2.11. *Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a frame for a Hilbert space H , and assume that there exists an infinite subsequence $\mathcal{G} = \{g_n\}_{n \in \mathbb{N}}$ of \mathcal{F} such that for each n , $\mathcal{F} \setminus \{g_n\}$ is complete in H (and hence a frame). If there exists a single constant $L > 0$ that is lower frame bound for each frame $\mathcal{F} \setminus \{g_n\}$, then for every $0 < \varepsilon < L$ there exists an infinite subsequence \mathcal{G}_ε of (\mathcal{G}) such that $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ is a frame for H with lower frame bound $L - \varepsilon$.*

Chapter 3

Gabor Frames

Finally we turn to Gabor frames. The elements of a Gabor frame are shifted and modulated versions of a window function g . The first section contains fundamental results like necessary and sufficient conditions for $\{T_{ak}M_{bn}g\}_{k,n \in \mathbb{Z}^d}$ being a frame. Furthermore, the special structure of Gabor frames helps us to formulate theorems about the excess of Gabor frames.

3.1 An introduction to Gabor frames

Two classes of operators on $L^2(\mathbb{R}^d)$ lay the foundations for the theory of Gabor analysis in $L^2(\mathbb{R}^d)$. These are the translation and the modulation operator as defined in (1.2) and (1.3). Related to the STFT we use a window function g too. The decisive difference is that this atom g is shifted along a lattice, which is defined in Definition 1.2.4.

Definition 3.1.1. *Given a non-zero window function $g \in L^2(\mathbb{R}^d)$ and lattice parameters $a, b > 0$, the set of time-frequency shifts*

$$\mathcal{G}(g, a, b) = \{T_{ak}M_{bn}g\}_{k,n \in \mathbb{Z}^d}$$

*is called a **Gabor system**.*

*If $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R}^d)$, it is called a **Gabor frame** or **Weyl-Heisenberg frame**.*

Sometimes we will also use for $\mathcal{G}(g, a, b)$ the notation of $\mathcal{G}(g, \Lambda)$. The building block g is also called Gabor atom.

The frame operator for a Gabor frame $\mathcal{G}(g, a, b)$ is

$$\forall f \in L^2(\mathbb{R}^d) : Sf = \sum_{k,n \in \mathbb{Z}^d} \langle f, T_{ak}M_{bn}g \rangle T_{ak}M_{bn}g$$

Furthermore the frame operator $\mathcal{G}(g, a, b)$ commutes with translation by a and modulation by b .

$$\begin{aligned}
ST_{ak}M_{bn}f &= \sum_{k',n' \in \mathbb{Z}} \langle T_{ak}M_{bn}f, T_{ak'}M_{bn'}g \rangle T_{ak'}M_{bn'}g \\
&= \sum_{k',n' \in \mathbb{Z}} \langle f, M_{-bn}T_{a(k'-k)}M_{bn'}g \rangle T_{ak'}M_{bn'}g \\
&\stackrel{(1.4)}{=} \sum_{k',n' \in \mathbb{Z}} \langle f, e^{-2\pi ia(k'-k)bn}T_{a(k'-k)}M_{b(n'-n)}g \rangle T_{ak'}M_{bn'}g
\end{aligned}$$

We change the variables $k' \rightarrow k' + k$, $n' \rightarrow n' + n$ and use the commutator relation (1.4) again,

$$\begin{aligned}
&= \sum_{k',n' \in \mathbb{Z}} e^{2\pi ia k' bn} \langle f, T_{ak'}M_{bn'}g \rangle T_{a(k+k')}M_{b(n'+n)}g \\
&= \sum_{k',n' \in \mathbb{Z}} e^{2\pi ia k' bn} \langle f, T_{ak'}M_{bn'}g \rangle e^{-2\pi ia k' bn} T_{ak}M_{bn}T_{ak'}M_{bn'}g \\
&= T_{ak}M_{bn}Sf
\end{aligned}$$

So we obtain for all $f \in L^2(\mathbb{R}^d)$

$$ST_{ak}M_{bn} = T_{ak}M_{bn}S \quad (3.1)$$

The next theorem gives us some important information about the dual Gabor frame and the tight Gabor frame.

Theorem 3.1. *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given, and assume that $\{T_{ak}M_{bn}g\}_{k,n \in \mathbb{Z}}$ is a Gabor frame. Then the canonical dual also has the Gabor structure and is given by $\{T_{ak}M_{bn}S^{-1}g\}_{k,n \in \mathbb{Z}}$. The canonical tight frame associated with $\{T_{ak}M_{bn}g\}_{k,n \in \mathbb{Z}}$ is $\{T_{ak}M_{bn}S^{-1/2}g\}_{k,n \in \mathbb{Z}}$.*

A proof is given in [4].

To compute the dual of a Gabor frame it is not necessary to have $\{S^{-1}T_{ak}M_{bn}g\}$, instead it is enough to calculate $S^{-1}g$ and then apply the modulation and translation operators.

One central question is to find out whether or not a Gabor system generated from a fixed Gabor atom g along a lattice Λ is a frame in $L^2(\mathbb{R})$. Theorem 3.2 gives us one of the most fundamental results. First we have a restriction whereby we do not obtain Gabor frames $\{T_{ak}M_{bn}g\}_{k,n \in \mathbb{Z}}$ for $L^2(\mathbb{R})$. And the second statement gives a relation between the product ab and Riesz bases already having a frame.

Even considering only time-frequency shifts of the type $\{T_{ak}M_{bn}g\}_{k,n \in \mathbb{Z}}$, it turns out to be very difficult to find the exact range of parameters a, b for which $\{T_{ak}M_{bn}g\}_{k,n \in \mathbb{Z}}$ is a frame for a given function $g \in L^2(\mathbb{R})$. Finding a sufficient condition how to obtain Gabor frames is solved for the Gaussians in Theorem 3.3.

Theorem 3.2. *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$. Then the following holds:*

- i. *If $ab > 1$, then $\{T_{ak}M_{bn}g\}_{k,n \in \mathbb{Z}}$ is not a frame for $L^2(\mathbb{R})$.*
- ii. *If $\{T_{ak}M_{bn}g\}_{k,n \in \mathbb{Z}}$ is a frame, then $ab = 1$ if and only if $\{T_{ak}M_{bn}g\}_{k,n \in \mathbb{Z}}$ is a Riesz basis.*

Theorem 3.3. *Let $a, b > 0$ and consider the window function $g(x) = e^{-x^2}$. Then the Gabor system $\{T_{ak}M_{bn}g\}_{k,n \in \mathbb{Z}}$ is a frame if and only if $ab < 1$.*

3.2 Deficit and excess of Gabor frames

Next, we will pursue the idea of Gabor multisystems. A Gabor multisystem is the union of r Gabor systems with different window functions and different lattice constants.

Definition 3.2.1. *Given are Gabor systems $\mathcal{G}(g^1, \Lambda_1), \dots, \mathcal{G}(g^r, \Lambda_r)$. The union of such Gabor systems is called a **Gabor multisystem** \mathcal{F} ,*

$$\mathcal{F} = \mathcal{G}(g^1, \dots, g^r; \Lambda_1, \dots, \Lambda_r) = \mathcal{G}(g^1, \Lambda_1) \cup \dots \cup \mathcal{G}(g^r, \Lambda_r)$$

So we might get a combination of coarse meshed and small meshed lattices and different Gabor atoms. Using such a Gabor multisystem would help to represent the characteristics of a given signal in more detail.

Now, provided with the basics of Gabor frames and their properties we start to consider the excess and the deficit of Gabor frames.

The next lemma follows from elementary calculations. Additionally we require the following definition, which provides us with further structural assumptions on Bessel sequences.

Definition 3.2.2. *Let \mathcal{F} be a Bessel sequence in a Hilbert space H with associated analysis operator $T : H \rightarrow l^2(I)$. If there exists a pair (Q, L) of bounded operators $Q : H \rightarrow H$ and $L : l^2(I) \rightarrow l^2(I)$ such that*

$$LT = TQ,$$

*then we call (Q, L) an **intertwining pair** of operators for \mathcal{F} .*

Lemma 3.4. *Let $V_a : l^2(\mathbb{Z}^2) \rightarrow l^2(\mathbb{Z}^2)$, $V_a c := \{e^{-2\pi iam} c_{m,n-1}\}_{m,n \in \mathbb{Z}}$ and $U : l^2(\mathbb{Z}^2) \rightarrow l^2(\mathbb{Z}^2)$, $U c := \{c_{m-1,n}\}_{m,n \in \mathbb{Z}}$. Then the following statements hold.*

- i. *T_a, V_a, M_b and U have no point spectrum if $a, b \neq 0$.*
- ii. *If $\mathcal{G}(g, a, b)$ is a Bessel sequence, then (T_a, V_{ab}) and (M_b, U) are each intertwining pairs of operators for $\mathcal{G}(g, a, b)$.*

Using Theorem 4.4 from [1] the next proposition follows immediately and we receive statements about the deficit and excess of Gabor systems.

Corollary 3.5. *Let $g \in L^2(\mathbb{R}^d)$ and $\alpha, \beta > 0$ be such that $\mathcal{G}(g, a, b)$ is a Bessel sequence in $L^2(\mathbb{R}^d)$. Then the following statements hold:*

- i. $\text{span}(\mathcal{G}(g, \Lambda))$ is either $\{0\}$ or is an infinite-dimensional subspace of $L^2(\mathbb{R}^d)$.
- ii. The deficit of $\mathcal{G}(g, \Lambda)$ is either zero or infinite.
- iii. If $\mathcal{G}(g, \Lambda)$ is a frame for its closed linear span, then its excess is either zero or infinite.

Our intention is to apply and combine theorems of Section 2.2 and Corollary 3.5 iii. So we would be able to say that infinitely many elements can be deleted from any overcomplete Gabor frame still leave a frame.

In contrast to the final theorems in Section 2.2 the parameters are subject to an additional restriction that states that the parameters are rationally related

Definition 3.2.3. *An r -tuple of numbers (a_1, \dots, a_r) is **rationally related**, if there are r integers k_1, \dots, k_r such that $k_1 a_1 = \dots = k_r a_r$.*

Theorem 3.6. *Let $\mathcal{F} = \mathcal{G}(g^1, \dots, g^r; \Lambda_1, \dots, \Lambda_r)$ be a Gabor multisystem that is an overcomplete frame for its closed linear span \mathcal{H} in $L^2(\mathbb{R})$. If either (a_1, \dots, a_r) or (b_1, \dots, b_r) are rationally related, then there exists an infinite subset \mathcal{G} of \mathcal{F} such that $\mathcal{F} \setminus \mathcal{G}$ is a frame for \mathcal{H} .*

Proof: First we assume that (a_1, \dots, a_r) are rationally related, so there exist k_1, \dots, k_r such that $a = a_1 k_1 = \dots = a_r k_r$. Because \mathcal{F} is an overcomplete frame, there can be found one element $\{T_{a_i m_0} M_{b_i n_0} g^i\}_{k, n \in \mathbb{Z}}$ such that $\mathcal{F} \setminus \{T_{a_i m_0} M_{b_i n_0} g^i\}_{k, n \in \mathbb{Z}}$ is a frame for \mathcal{H} .

Further we have for all $m, n, p \in \mathbb{Z}$ and $j = 1, \dots, r$

$$T_{ap} T_{a_j m} M_{b_j n} g^j = T_{a_j k_j p} T_{a_j m} M_{b_j n} g^j = T_{a_j (k_j p + m)} M_{b_j n} g^j.$$

This means for each j that T_{ap} simply permutes the elements of \mathcal{F} .

Therefore,

$$T_{ap}(\mathcal{F} \setminus \{T_{a_i m_0} M_{b_i n_0} g^i\}) = \mathcal{F} \setminus \{T_{a_i (k_i p + m_0)} M_{b_i n_0} g^i\}, \quad p \in \mathbb{Z}. \quad (3.2)$$

T_{ap} is a unitary operator mapping \mathcal{H} onto itself, from this it follows that each of the subsequences in (3.2) is a frame for \mathcal{H} , all with the same frame bounds. Finally, the result follows from Theorem 2.11.

Otherwise, if (b_1, \dots, b_r) are rationally related, there exist l_1, \dots, l_r such that $b = b_1 l_1 \dots b_r l_r$. Applying M_{bp} we get

$$\begin{aligned} M_{bp} T_{a_j m} M_{b_j n} g^j &= M_{b_j l_j p} T_{a_j m} M_{b_j n} g^j \\ &= e^{-2\pi i a_j m b_j n} M_{b_j (l_j p + n)} T_{a_j m} g^j \\ &= e^{2\pi i b_j l_j p} T_{a_j m} M_{b_j (l_j p + n)} g^j \end{aligned}$$

Again, M_{bp} permutes the elements up to a constant. With the same arguments as in the case below the assumption follows.

□

Now we have several results of when there does exist an infinite subset $J \subset I$ such that $\{f_i\}_{i \in I \setminus J}$ is a frame. Our next goal is to show, assuming some slight restrictions on the generating atom g and on the set Λ , that any Gabor frame, which is not a Riesz basis has infinite excess, moreover that we still receive a frame after removing an infinite subset.

For the next theorem we require following definition.

Definition 3.2.4. *Let Λ be a sequence of points in \mathbb{R}^{2d} . The **lower and upper Beurling densities** of Λ are,*

$$D^-(\Lambda) = \liminf_{r \rightarrow \infty} \inf_{z \in \mathbb{R}^{2d}} \frac{|\lambda \cap Q(r, z)|}{r^{2d}}$$

and

$$D^+(\Lambda) = \limsup_{r \rightarrow \infty} \sup_{z \in \mathbb{R}^{2d}} \frac{|\lambda \cap Q(r, z)|}{r^{2d}},$$

respectively, where $Q(r, z)$ is the cube centered at $z = (z_1, \dots, z_{2d}) \in \mathbb{R}^{2d}$ with side lengths r , this means

$$Q(r, z) = \prod_{i=1}^{2d} \left[z_i - \frac{r}{2}, z_i + \frac{r}{2} \right].$$

In case $D^+(\Lambda) = D^-(\Lambda)$, we say that Λ has uniform Beurling density

$$D(\Lambda) = D^+(\Lambda) = D^-(\Lambda).$$

The Beurling densities of a sequence Λ can be interpreted as measure of the average number of elements of Λ lying inside sets of unit measure.

The following theorem gives us an important result of Ramanathan and Steger which is extended in [5].

Theorem 3.7 (Density Theorem). *If $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$, then $1 \leq D^-(\Lambda) \leq D^+(\Lambda) < \infty$.*

If $\mathcal{G}(g, \Lambda)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then $D^-(\Lambda) = D^+(\Lambda) = 1$.

In the proof is shown that each Gabor frame satisfies a certain homogeneous approximation property (HAP).

The term of modulation spaces will be used next, so we give here a short description. For a detailed introduction to modulation spaces see [3]. The elements in M^p correspond to a certain amount of joint localization in both time and frequency. Further,

we have that M^p is dense in L^2 for $p < 2$, and that $M^2 = L^2$. The modulation space norms quantify the time-frequency content of a function or distribution, and appear naturally in mathematical problems involving time-frequency shifts.

For our purposes, the following special case of unweighted modulation spaces will be sufficient. Let $G(x) = 2^{d/4}e^{-\pi x^2}$ be the Gaussian function, normalized so that $\|G\|_2 = 1$. Then for $1 \leq p \leq \infty$ the modulation space M^p consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^p} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\langle f, M_\omega T_x G \rangle|^p dx d\omega \right)^{1/p} < \infty,$$

with the usual adjustment if $p = \infty$.

In the next theorem we will use the set $I(r, z)$, which is the intersection of Λ with the cube $Q(r, z)$ centered at z with side lengths r .

Theorem 3.8. *Let $\mathcal{G}(g, a, b)$ be a Gabor frame for $L^2(\mathbb{R}^d)$. If $g \in \bigcup_{1 \leq p < 2} M^p$, then*

$$\liminf_{r \rightarrow \infty} \inf_{z \in \mathbb{R}^{2d}} \frac{1}{|I(r, z)|} \sum_{\lambda \in I(r, z)} \langle g_\lambda, \tilde{g}_\lambda \rangle \leq \frac{1}{D^+(\Lambda)}.$$

Consequently, if $D^+(\Lambda) > 1$ then there exists an infinite subset J of Λ such that $\mathcal{G}(g, \Lambda \setminus J)$ is a frame for $L^2(\mathbb{R}^d)$.

In this theorem we have a mild restriction on g and the assumption that $D^+(\Lambda) > 1$. Then we have a fundamental connection between the density of the Gabor frame and a particular term which directly yields to the excess of that frame. And in the proof, we have again the HAP playing an important part. Therefore, the excess of the Gabor frame is infinite and further, there exists an infinite subset that can be omitted and we still have a frame. This result can be extended to the case of Gabor multisystems. For a detailed proof see [2].

Note that if $\mathcal{G}(g, \Lambda)$ is an overcomplete frame and Λ is a lattice in \mathbb{R}^{2d} , then necessarily $D^+(\Lambda) > 1$, since a lattice is the image of \mathbb{Z}^{2d} under an invertible linear transform.

Finally, we extend the results of Theorem 3.6 and receive our desired answer. Here, we need to require that the system is a frame for its closed span and not for the entire space. Then our frame has infinite excess and when we remove an infinite subset, we still obtain a frame for the entire space.

Theorem 3.9. *Let $\mathcal{F} = \mathcal{G}(g^1, \dots, g^r; \Lambda_1, \dots, \Lambda_r)$ be a Gabor multisystem. If \mathcal{F} is an overcomplete frame for its closed span H in $L^2(\mathbb{R}^d)$, then this frame has infinite excess and there exists an infinite subset of \mathcal{F} that can be removed yet leave a frame for H . In fact, this subset can be taken to have the form $\{T_{\alpha_k n_j} g_k\}_{j=1}^\infty$, i.e., translates of one of the generators g_k .*

It is possible to lift the restriction that either (a_1, \dots, a_r) or (b_1, \dots, b_r) are rationally related. A detailed proof is given in [2].

Chapter 4

Experiments

In this chapter we show the results of numerical calculations based on the theoretical concepts introduced earlier. Our Gabor frames will no longer be based on a general Hilbert space but on a finite dimensional complex vector space. We will sometimes refer to the elements of this vector space as signals and to the dimension of the vector space as the signal length. Thus we emphasize the relation to audio signal processing. We will use MATLAB to perform necessary calculations.

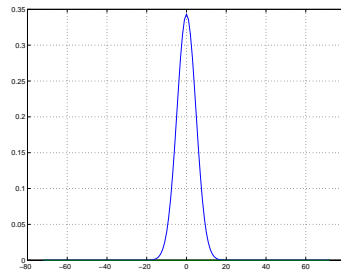
4.1 Implementation in MATLAB

The software MATLAB is a high-level language and interactive environment that makes the implementation of computationally intensive tasks particularly easier than with conventional programming languages such as C, C++, and Fortran.

Throughout this chapter we will use the following variables:

n	The signal length
a	The time interval between two adjacent frame elements
b	The frequency interval between two adjacent frame elements
g	The window function
G	The finite dimensional analysis operator
S	The Gabor frame operator

a and b must be divisors of n . G corresponds to the analysis operator T defined in Definition 2.1.5. G is a matrix with $\frac{n}{a} \cdot \frac{n}{b}$ rows and n columns. Each row represents one element of the Gabor frame associated to G . That is why we will use G to refer to the Gabor frame itself.

Figure 4.1: Window function g

g is a vector of complex numbers that comprise the unmodulated window function centered at time zero. As window function we use the Gauss function, so g is a discretized Gaussian as given in Figure 4.1. We shift and modulate this atom to a certain point in the time-frequency plane to construct one row of G . By running through all the points of the lattice with intervals a and b we get the complete analysis matrix.

In this way we stuff three dimensions into the planar (and thus two-dimensional) structure of a matrix. The column index stands for only one quantity, the time t_s that runs in steps of length 1 through all the n complex numbers that each signal vector consists of. The row index on the other hand is really a combination of two distinct quantities: a frequency index f_l and a time index t_l . These indices together describe where the associated frame element is localized in the time-frequency plane. Running through all the rows of the matrix first advances f_l by b until it reaches the value n . At that point f_l is set back to zero and t_l is advanced by a etc.

Figure 4.2: Construction of the Gabor Matrix, $n = 144$, $a = 12$, $b = 9$

The Figure 4.2 represents sections of the matrix G . The different colors display the entries of the matrix. The colorbar yields from dark blue for 0 to dark red for the value 0.35. Furthermore we recognize the row indices (t_l, f_l) . So the first

row contains the atom at time and frequency 0, the second row the atom at time 0 but modulated with frequency 9. Sixteen rows are always identical, because the plot shows the absolute values from G . The whole matrix G and its adjoint G^* are shown in Figure 4.3.

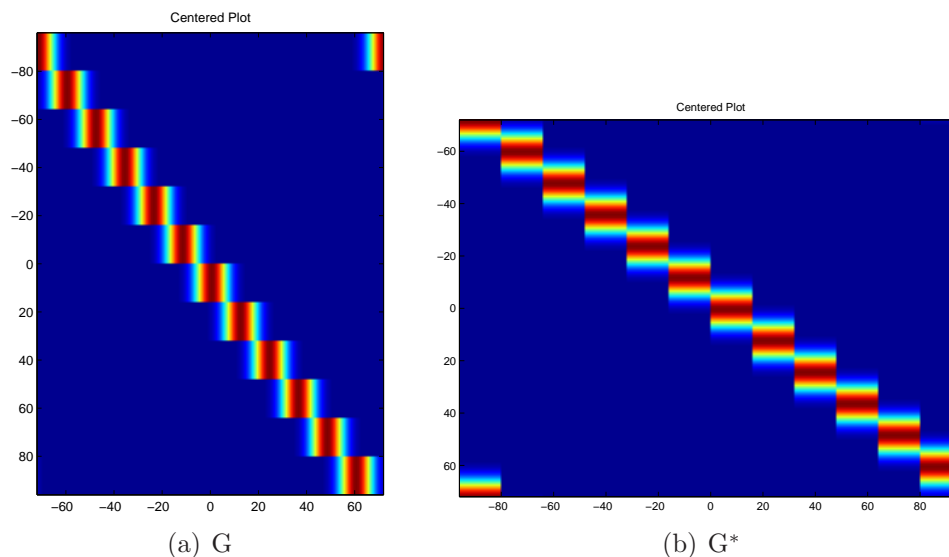


Figure 4.3: Gabor matrix G and G^*

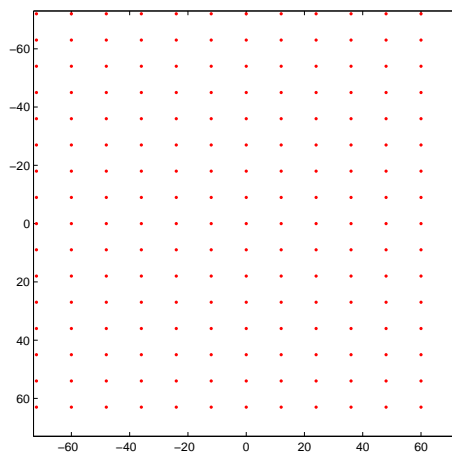


Figure 4.4: Lattice, $n = 144$, $a = 12$, $b = 9$

The time-frequency plane is shown in Figure 4.4 which represents the lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ with the lattice constants \mathbf{a} and \mathbf{b} . In MATLAB the lattice is given by the $n \times n$ matrix \mathbf{xpo} . The entries of this matrix are zeros except the lattice points.

The lattice points appear at a certain time and a certain frequency. This time and frequency steps are given by \mathbf{a} and \mathbf{b} . So we have in the matrix ones, when there is a lattice point. To visualize the lattice as done in Figure 4.4 we use the routine `spyc tob`.

The Gabor matrix \mathbf{G} is produced in MATLAB with

$$\mathbf{G} = \text{gabbas}(\mathbf{g}, \mathbf{xpo}); \quad (4.1)$$

The function `gabbas` shifts the window function \mathbf{g} along the lattice \mathbf{xpo} and so generates the Gabor frame \mathbf{G} .

The Gabor frame operator \mathbf{S} is defined as in Definition 2.1.5. And so

$$\mathbf{S} = \mathbf{G}' * \mathbf{G}; \quad (4.2)$$

represents the code in MATLAB, where \mathbf{G}' is the adjoint matrix in MATLAB notation. The Image 4.5 shows the Gabor frame operator \mathbf{S} .

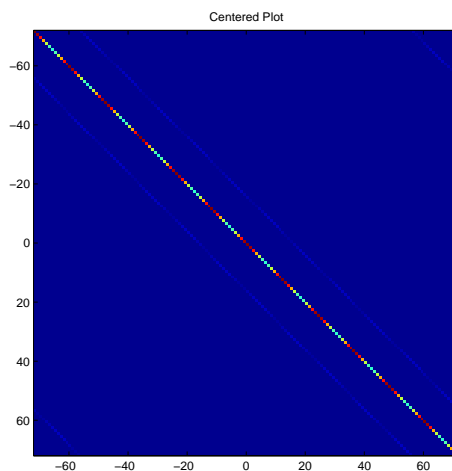


Figure 4.5: Gabor frame operator \mathbf{S}

The Gabor frame operator \mathbf{S} has certain structural properties. This structure is described by the Walnut representation, [3] or [14]. All nonzero elements of \mathbf{S} live on a equidistant and \mathbf{b} -periodic diagonals. Thus, we receive the structure given in Figure 4.5 of the diagonal matrix and its secondary diagonals.

We receive the dual atom \mathbf{gd} by applying the inverse frame operator $\mathbf{gd} = \mathbf{S}^{-1} \mathbf{g}$

which yields reconstruction as in Proposition 2.1 iii.

The dual Gabor frame is achieved by shifting the dual window function \mathbf{gd} along the lattice \mathbf{xpo} , so we have

$$\mathbf{GD} = \text{gabbas}(\mathbf{gd}, \mathbf{xpo}); \quad (4.3)$$

and

$$\mathbf{GT} = \text{gabbas}(\mathbf{gt}, \mathbf{xpo}); \quad (4.4)$$

to get the tight Gabor frame.

So now we are equipped with all the necessary variables and functions, we will use further.

4.2 Experiments

Now we want to analyze the effect of removing atoms from the time-frequency plane. During this section we will consider six different cases of removing atoms from the lattice as shown in Figure 4.4.

At first we want to observe the loss of one atom. Then the cases where two atoms are omitted follow. There the interesting point of view is that we take two atoms with two remaining atoms between them and then two adjacent ones. Moreover a lattice without nine adjacent atoms is observed. And finally we look at the effects of losing a whole row and a whole column. We will append numbers to all the variables to refer to the case that these variables apply to.

lattice	removed elements	code in MATLAB
\mathbf{xpo}	none	
$\mathbf{xp1}$	one atom	$\mathbf{xp1} = \mathbf{xpo}; \mathbf{xp1}(1,1) = 0;$
$\mathbf{xp2}$	two non-adjacent atoms	$\mathbf{xp2} = \mathbf{xpo}; \mathbf{xp2}(28,1) = 0;$
$\mathbf{xp3}$	two adjacent atoms	$\mathbf{xp3} = \mathbf{xpo}; \mathbf{xp3}(10,1)=0;$
$\mathbf{xp4}$	nine adjacent atoms	$\mathbf{xp4} = \mathbf{xpo} .* (1 - \text{fltrecs}(n,n,12,12));$
$\mathbf{xp5}$	one row	$\mathbf{xp5} = \mathbf{xpo}; \mathbf{xp5}(1,:) = 0;$
$\mathbf{xp6}$	one column	$\mathbf{xp6} = \mathbf{xpo}; \mathbf{xp6}(:,1) = 0;$

This raises the following questions:

- How will the loss of elements influence the Gabor matrix \mathbf{G} , the frame operator \mathbf{S} , the eigenvalues of \mathbf{S} and thus the frame bounds?

- What can we say about the dual Gabor frame?
- How many elements can be removed and still leave a frame?
- Are there any quantities that tell us if we can remove elements at all?

We will answer these questions in the next sections with the aid of experiments in MATLAB.

4.2.1 Gabor matrix and Gabor frame operator

So we first look at the Gabor matrices. We obtain the Gabor matrices appendant to their respective lattice by applying the function `gabbas` to this lattice. The command `imgc` produces a centered plot of a matrix. This function gives a quick overview of the entries of a matrix by encoding their absolute values as different colors ranging from blue for the lowest absolute value to red for the highest absolute value appearing in the matrix.

Turning to Figure 4.3 we recognize the Gabor matrix \mathbf{G} with the complete lattice underlying. Figure 4.6 displays the matrices $\mathbf{G1}$ to $\mathbf{G6}$. They are obtained by applying the function `gabbas` their corresponding lattice `xp1` to `xp6`. Since we lose some of our atoms, the dimension of the matrices $\mathbf{G1}$ to $\mathbf{G6}$ changes. The numbers of columns remain unchanged, whereas the numbers of rows correspond to the numbers of atoms left.

Therefore, in Figure 4.6 (a) we have 191 rows, since we removed the first one. 190 rows are shown in (b) and (c). The two rows we have omitted were just removed on different places. In (d) 9 rows are removed, so we have 183 rows remaining. In case of (e) and (f) 180 rows and 176 rows respectively remain in the matrices.

4.2.2 Eigenvalues

First we have to make sure that $\mathbf{G1}$ to $\mathbf{G6}$ are frames (or not). Therefore we have to pay attention to the eigenvalues of the frame operator. Since we are in a finite-dimensional complex vector space the Cauchy-Schwarz inequality ensures that the upper frame condition is satisfied

$$\forall f \in \mathbb{C}^n : \sum_{k=1}^n |\langle f, f_k \rangle|^2 \leq \sum_{k=1}^n \|f_k\|^2 \|f\|^2.$$

The frame bound $\sum_{k=1}^n \|f_k\|^2$ must not be the optimal bound. Since the lowest eigenvalue serves as lower frame bound, we have to find out if any eigenvalue is 0.

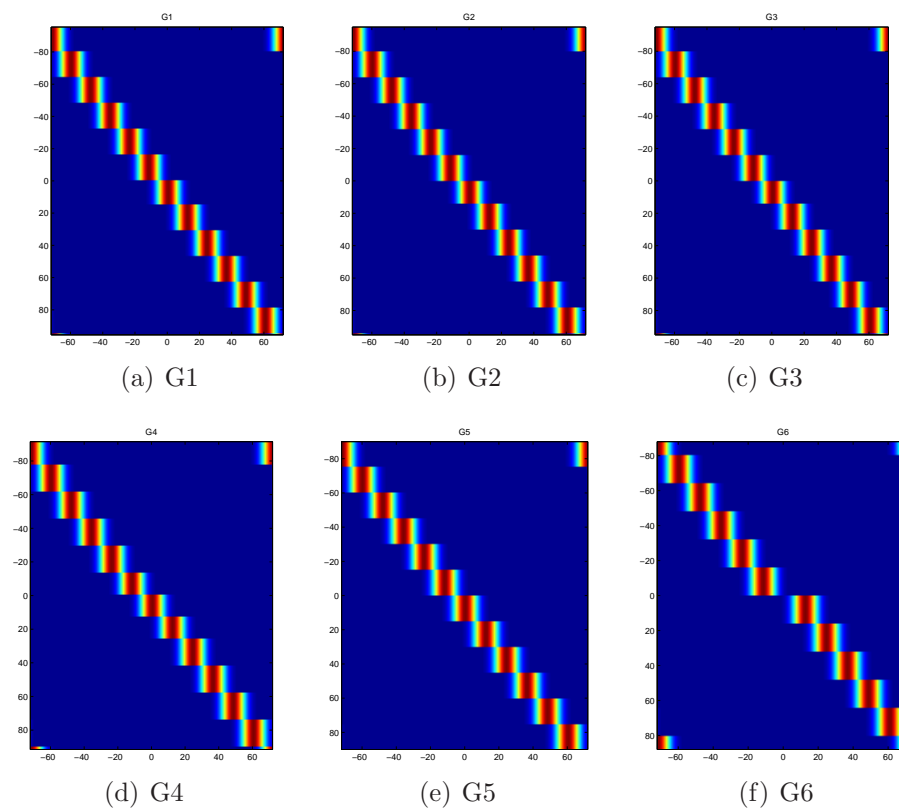


Figure 4.6: Remaining matrices, when removing rows from the Gabor matrix \mathbf{G} , $n = 144$, $a = 12$, $b = 9$

The eigenvalues get calculated in MATLAB with

$$[V, \mathbf{eigs}] = \mathbf{eigsort}(S); \quad . \quad (4.5)$$

The function `eigsort` is a further development of the MATLAB standard routine `eig`. `[V,D] = eig(A)` returns a diagonal matrix `D` of `A`'s eigenvalues and a matrix `V` whose columns are the corresponding eigenvectors, so that $A * V = V * D$. In the function `eigsort` the eigenvalues get sorted and we obtain them in the vector `eigs` whereas the matrix `V` shows the respective eigenvectors.

<code>eigs</code>	0.7603	0.7603	0.7603	0.7603	0.7603	0.7603	0.7603	0.7603
<code>eigs1</code>	0.7603	0.7603	0.7603	0.7603	0.7603	0.7603	0.7603	0.2887
<code>eigs2</code>	0.7603	0.7603	0.7603	0.7603	0.7603	0.7603	0.3304	0.2546
<code>eigs3</code>	0.7603	0.7603	0.7603	0.7603	0.7603	0.7603	0.4071	0.0886
<code>eigs4</code>	0.6108	0.3765	0.1890	0.0733	0.0203	0.0036	0.0004	$1.6242 \cdot 10^{-5}$
<code>eigs5</code>	0.3781	0.2018	0.2018	0.0878	0.0878	0.0305	0.0305	0.0142
<code>eigs6</code>	0.1014	0.0399	0.0399	0.0117	0.0117	0.0024	0.0024	0.0007

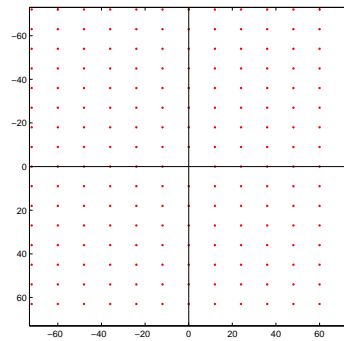
Table 4.1: The smallest eight eigenvalues of `S` to `S6`

The lowest eight eigenvalues are listed in Table 4.1. Removing one atom affects only the lowest eigenvalue but leaves still a frame. In case of omitting two atoms we observe differences. When we remove two non-adjacent atoms, the least eigenvalue is greater than the one where we remove two adjacent atoms. Concerning the second last eigenvalue we observe the opposite effect. So removing nine atoms at once leads to the worst result.

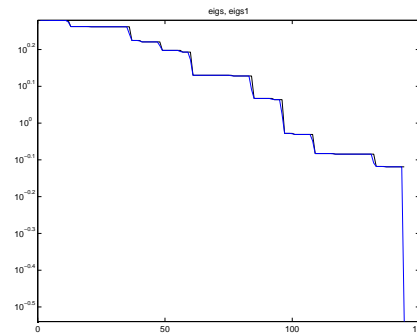
If we are missing a row, we have lost $\frac{n}{a} = 12$ elements and discuss a missing column, we sustain a loss of $\frac{n}{b} = 16$ elements. Therefore in the latter case more elements get removed. Nevertheless the least eigenvalue in both cases is greater than the last one of the vector `eigs4`.

All eigenvalues are displayed in Figure 4.7. In the graphic we applied the MATLAB command `semilogy`, which plots data as logarithmic scales for the y-axis. We used this function to keep all values well visible and to emphasize they are all unequally zero.

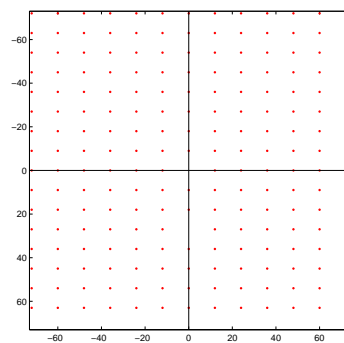
Each figure on the right is build of two graphs. The black one is composed of the eigenvalues of `S` and the blue gives the eigenvalues of the frame operator corresponding to the lattices which is mapped on the left hand.



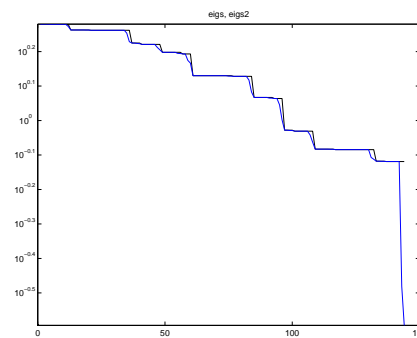
(a) Lattice xp1



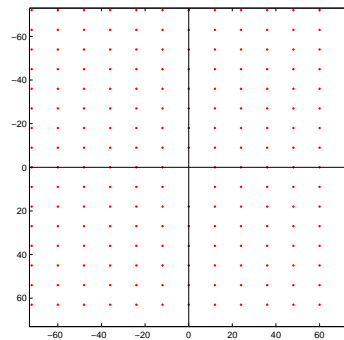
(b) Eigenvalues of S1



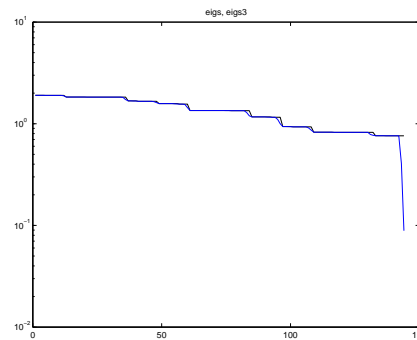
(c) Lattice xp2



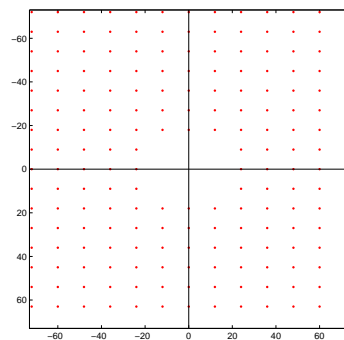
(d) Eigenvalues of S2



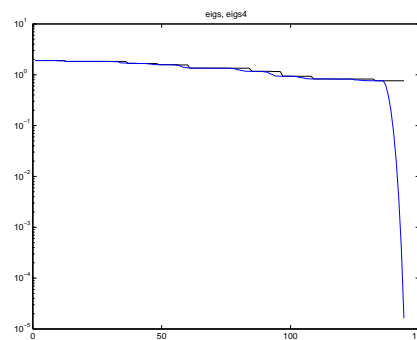
(e) Lattice xp3



(f) Eigenvalues of S3



(g) Lattice xp4



(h) Eigenvalues of S4

Figure 4.7: Eigenvalues belonging to a respective lattice

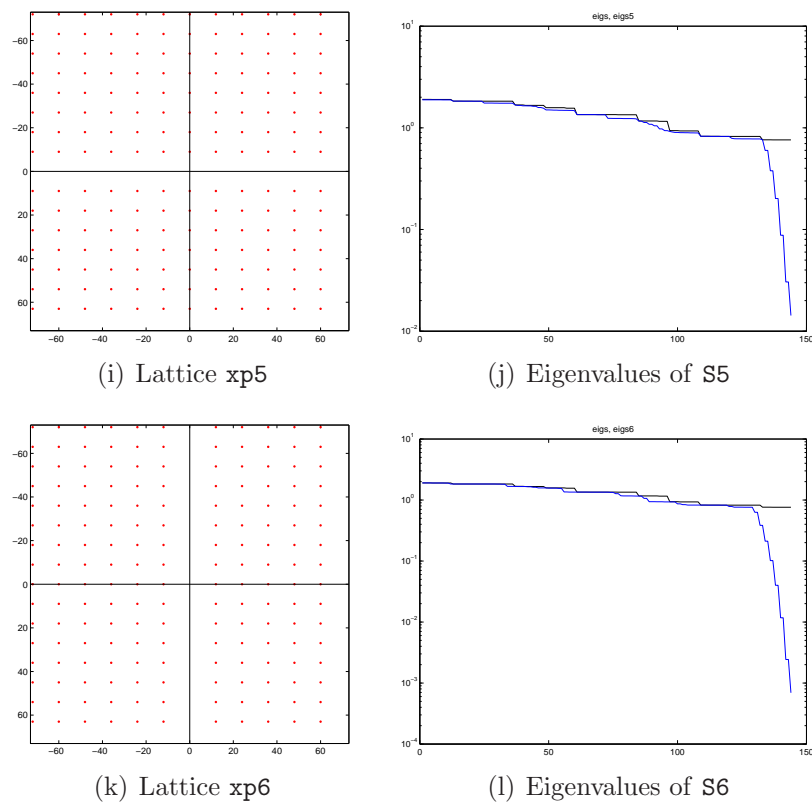


Figure 4.7: Eigenvalues belonging to a respective lattice

In Figure 4.7 (a) the lattice `xp1` is shown. Although we removed the first atom, we have a hole in the center of the plot. That is because the MATLAB command `spyctob` we used returns a centered plot. (b) gives the sorted eigenvalues of \mathbf{S} and $\mathbf{S1}$. Comparing them we recognize that the blue graph decreases faster than the black one. This decrease is due to the removed atom. In (c) we see the lattice with two removed atoms: the first and the fourth one were omitted. Further, (e) shows the loss of two adjacent, the first and the second atoms. (g) presents a lattice without nine atoms. The cases of a removed row and column are given in (i) and (k) respectively. The behavior described for (b) increases in (d),(f),(h),(j) and (l). So the more elements we remove the more decreases the graph of the eigenvalues.

Therefore, the consequence of removing elements on the eigenvalues of their frame operator is directly shown in this pictures. There is a strong influence on the lower frame bounds depending on how many atoms and on which position are omitted. These results indicate a slice of the spectrum.

4.2.3 Lattice constants and redundancy

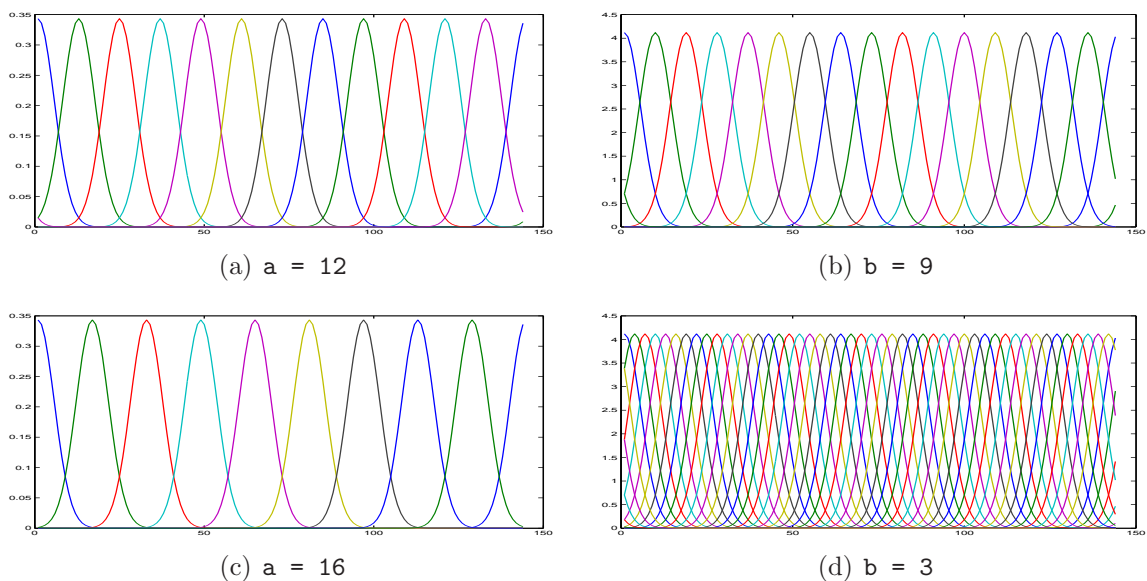


Figure 4.8: Overlapping, plotted with `tr1bas`

A great difference occur when we change the lattice constants. Till now we have worked with $n = 144$, $a = 12$ and $b = 9$. If we choose $a = 16$ and $b = 9$ instead, we obtain a wider meshed lattice. In this case removing one atom causes the lowest eigenvalue being 0. The opposite is achieved, if we take $a = 4$ and $b = 3$.

To work out the details of this topic we make use of the MATLAB function `tr1bas`.

Having a wider meshed lattice means that the atoms are less overlapped. In that case leaving one gives much more worse results.

This consideration yields to the redundancy of frames, which is given by $\frac{n}{ab}$. By removing one atom the adjacent atoms can set off a loss. Or in other words, the neighbors take up the job of the missing atoms. Having a small meshed lattice obviously implies a high redundancy.

If we take $a = 12$ and $b = 9$, redundancy is $\text{red} = 1.333$. So this confirms our statement that work-sharing between neighbors really works. Further, $a = 16$ and $b = 9$ gives $\text{red} = 1$. Therefore compensating a hole does not work. Whereas, $a = 4$ and $b = 3$ turns out being a tight frame with $\text{red} = 12$.

These different constants can be seen in Figure 4.2.3, where we applied the command `trlbas` as mentioned before. In Figure 4.2.3 we fix a frequency, e.g. frequency 0, so the x-axis plots the signal length. Therefore we receive a cross section in the time-frequency plane with shifted Gauss functions. The overlapping is influenced by lattice parameters. In case of (a) we see the shifted elements in the row with frequency 0 and with lattice parameter $a = 12$. In (c) we set $a = 16$. In case of (b) and (d) we fix the point in time 0 and get the unmodulated Gaussians with lattice parameter $b = 9$ and $b = 3$ respectively.

4.2.4 Condition number

Another issue is the condition number of S and $S1 - S6$.

	S	S1	S2	S3	S4	S5	S6
cond(x)	2.5041	6.5961	7.4790	21.4780	$1.1720 \cdot 10^{+5}$	132.3828	$2.7550 \cdot 10^{+3}$

Table 4.2: Condition numbers of S to $S6$

Since we need the inversion of the frame operator for computing the dual atom, we are interested in a numerically stable operator. In Figure 4.9 we can see the frame operators and in Table 4.2.4 the appendant condition numbers. The more elements we remove the higher the condition number gets. The numerical instability grows.

Also we can find two different lattices having the same redundancy, but different condition numbers, which depends on high lattice constants, [15]. Therefore, being aware of the condition numbers is necessary.

In Figure 4.9 we see $S1 - S6$. The structure of the diagonal matrix and its secondary diagonals gets disturbed by additional entries. Therefore, we have nonzero

entries not only living on the diagonals. These entries occur due to the omitted elements. In case of (f) we receive a hole in the diagonals.

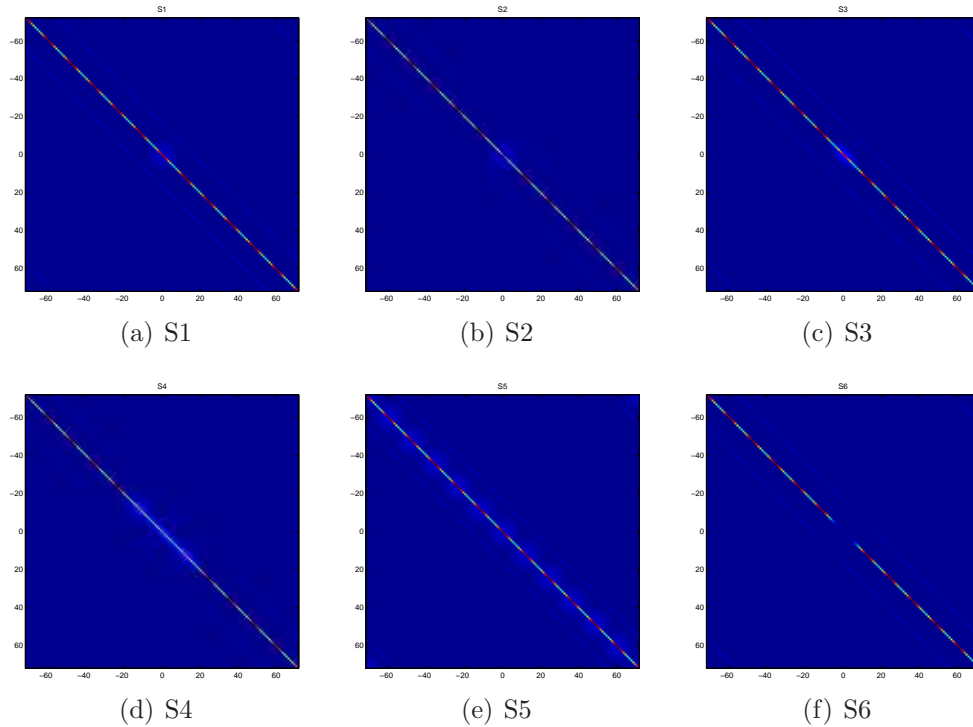


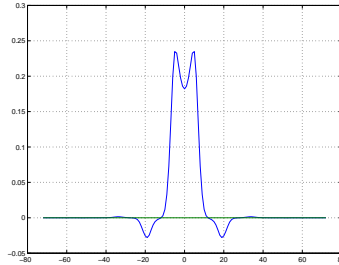
Figure 4.9: Frame operators, $n = 144$, $a = 12$, $b = 9$

4.2.5 Varying the window function

After changing lattice constants, one naturally wants to see the effect of varying the window function g . The dual atom would be a obvious choice.

The results of the smallest eight eigenvalues and the condition numbers are listed below. The dual atom gd , as shown in Figure 4.10, is shifted along the lattices generating GD1 – GD6. Graphical representation of these matrices shows a structure similar to Figure 4.6.

A comparison of the results of the smallest eigenvalues of S and SD implies that the values of $eigsd$ lie below those of $eigs$. This is also true for the other cases. Furthermore, considering the condition numbers shows that the frame operators using gd as window function have a higher numerical instability.

Figure 4.10: Window function gd

eigsd	0.5258	0.5258	0.5258	0.5258	0.5252	0.5252	0.5252	0.5252
eigsd1	0.5258	0.5258	0.5258	0.5254	0.5252	0.5252	0.5252	0.1820
eigsd2	0.5258	0.5258	0.5254	0.5253	0.5252	0.5252	0.1918	0.1722
eigsd3	0.5258	0.5258	0.5254	0.5254	0.5252	0.5252	0.3903	0.0422
eigsd4	0.5247	0.3826	0.1431	0.0453	0.0092	0.0015	0.0003	$7.8037 \cdot 10^{-5}$
eigsd5	0.2742	0.1495	0.1495	0.0574	0.0574	0.0173	0.0173	0.0076
eigsd6	0.0943	0.0235	0.0235	0.0047	0.0047	0.0008	0.0008	0.0002

Table 4.3: The smallest eight eigenvalues of SD to SD6

	SD	SD1	SD2	SD3	SD4	SD5	SD6
cond(x)	2.5041	7.2269	7.6366	31.1921	$1.6853 \cdot 10^{+5}$	172.8710	$6.7267 \cdot 10^{+3}$

Table 4.4: Condition numbers of SD to SD6

4.2.6 Dualisation line by line

In addition to shifting the dual window \mathbf{gd} along the respective lattice we have the aim to calculate the dual frame from $\mathbf{G1}$ to $\mathbf{G6}$. This can be achieved by applying the pseudo-inverse to the self-adjoint frame matrix. Another possibility is to build dual atoms row by row. In case where we omit atoms we want to compare both results and show the differences if any occur.

The command `pinv` is a standard MATLAB routine and calculates the pseudo inverse of a matrix. The function `rowdual(m,M)` computes the dual atom of \mathbf{m} with regard to the matrix \mathbf{M} . The dependence on a matrix \mathbf{M} in our case is very important, since the dual atom of some atom \mathbf{g} is given by $\mathbf{gd} = \mathbf{inv}(\mathbf{S}) * \mathbf{g}$, whereas $\mathbf{S} = \mathbf{G}' * \mathbf{G}$. Therefore it is necessary to have two arguments for the function `rowdual`.

We observe the cases where a row and a column are removed and start defining $\mathbf{A5} = \mathbf{pinv}(\mathbf{G5}')$. To calculate dualisation line by line we take each row of $\mathbf{G5}$ and apply the function `rowdual`. The obtained vector is now filled into the matrix $\mathbf{G5D}$.

The procedure described above is given below in terms of MATLAB code. The routine in case of $\mathbf{G6}$ works in an analogous way except for the loop counters due to different matrix dimensions.

```
...
A5=pinv(G5');
G5D=zeros(180,144);
for i=1:180
    m=G5(i,:);
    n=rowdual(m,G5);

    for k=1:144
        G5D(i,k)=n(1,k);
    end;
end;
...
```

For comparing the results the difference between \mathbf{GD} and \mathbf{A} , $\mathbf{G5D}$ and $\mathbf{A5}$, $\mathbf{G6D}$ and $\mathbf{A6}$ is calculated. Those matrices are shown in Figure 4.11 where we can see negligibly small entries.

Additionally the Frobenius norm is analyzed and the values are given in Table 4.5. A comparison of the results shows numerical differences.

The dual Gabor frames of our six considered cases are given in Figure 4.12. The blocks which build the diagonal of the matrix seem to have about the same value.

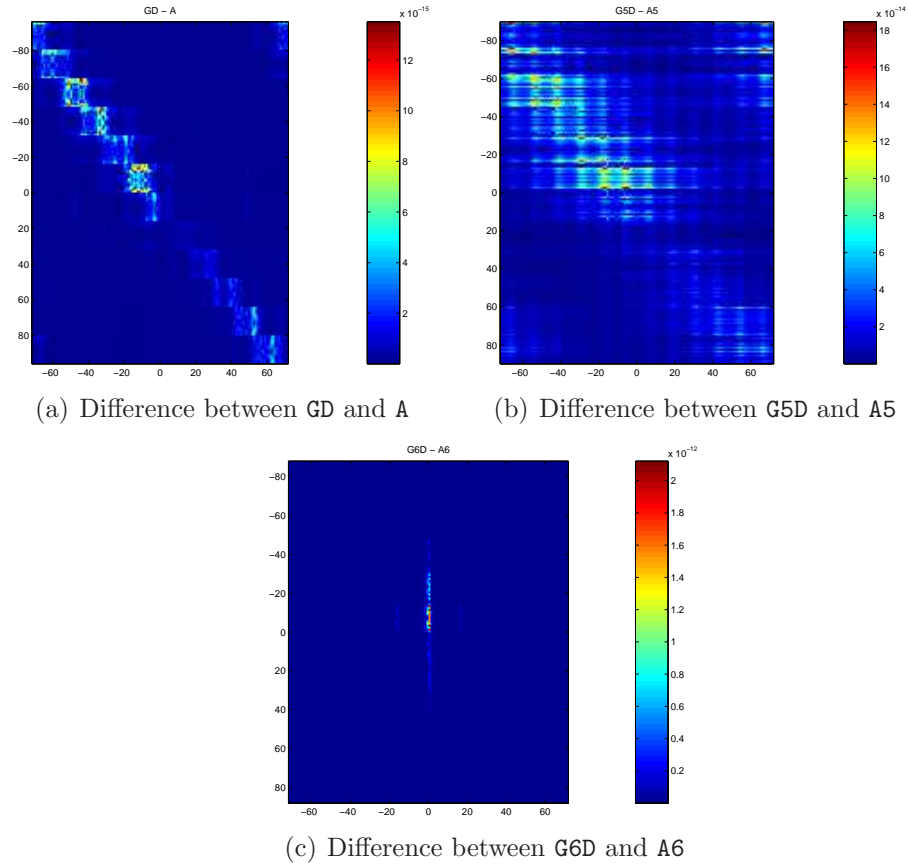


Figure 4.11: Difference between dualisation line by line and calculating the pseudo-inverse

<code>norm(GD - A, 'fro')</code>	<code>norm(G5D - A5, 'fro')</code>	<code>norm(G6D - A6, 'fro')</code>
$1.6755 \cdot 10^{-13}$	$3.8651 \cdot 10^{-12}$	$7.0843 \cdot 10^{-12}$

Table 4.5: Comparing dualisation line by line and applying the pseudoinverse by using the Frobenius norm

Due to the loss of elements there might be big differences. To ensure this assumption we want to build the difference between the \mathbf{GD} and the modified dual frames. Therefore we have to be aware of the matrix dimensions. At the position where we now miss a row, due to lost elements, we insert a row with zero entries. This enables us to calculate the difference as shown in Figure 4.13.

In Figure 4.13 (a) the differences are located around the first column in a symmetric way. That is because of the structure of the window function \mathbf{gd} given in Figure 4.10. The entries in the first row have the highest values, because we calculate the difference between the first row of \mathbf{GD} and a row with zero entries. The rows remaining unchanged indicate the atoms in the lattice which do not get affected of the removed atom. Apparently, since we removed only one atom, the rows which get affected are located in the same, previous or next column and correspond to the atoms near the hole.

In (b) the highest values are located in the first and the fourth row, since we have removed the corresponding atoms. Obviously, more differences are received, hence omitting two elements effects more atoms. Looking at (c) - (f) those effects increase. In (c) we removed the first and the second atom, therefore the entries of the first and the second row are clear. In comparison to (b) the structure of influenced atoms is different. Nine atoms are omitted in (d) and we miss a row and a column from the lattice in (e) and (f), respectively.

The corresponding frame operators are shown in Figure 4.14. Related to Figure 4.9 the diagonal structure is disturbed by additional non zero entries. In (d) and (f) we still have non zero entries in the diagonals, although the graphical representation does not show them as they are very small compared to the additional non zero entries.

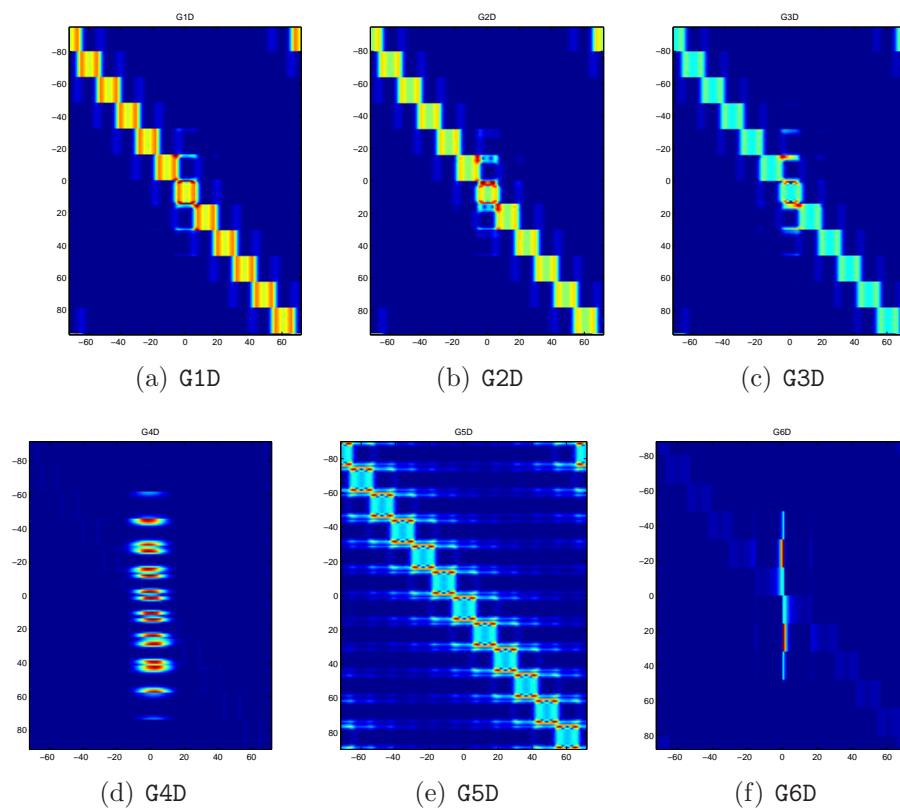


Figure 4.12: Remaining matrices, when removing rows from the dual Gabor matrix, $n = 144$, $a = 12$, $b = 9$

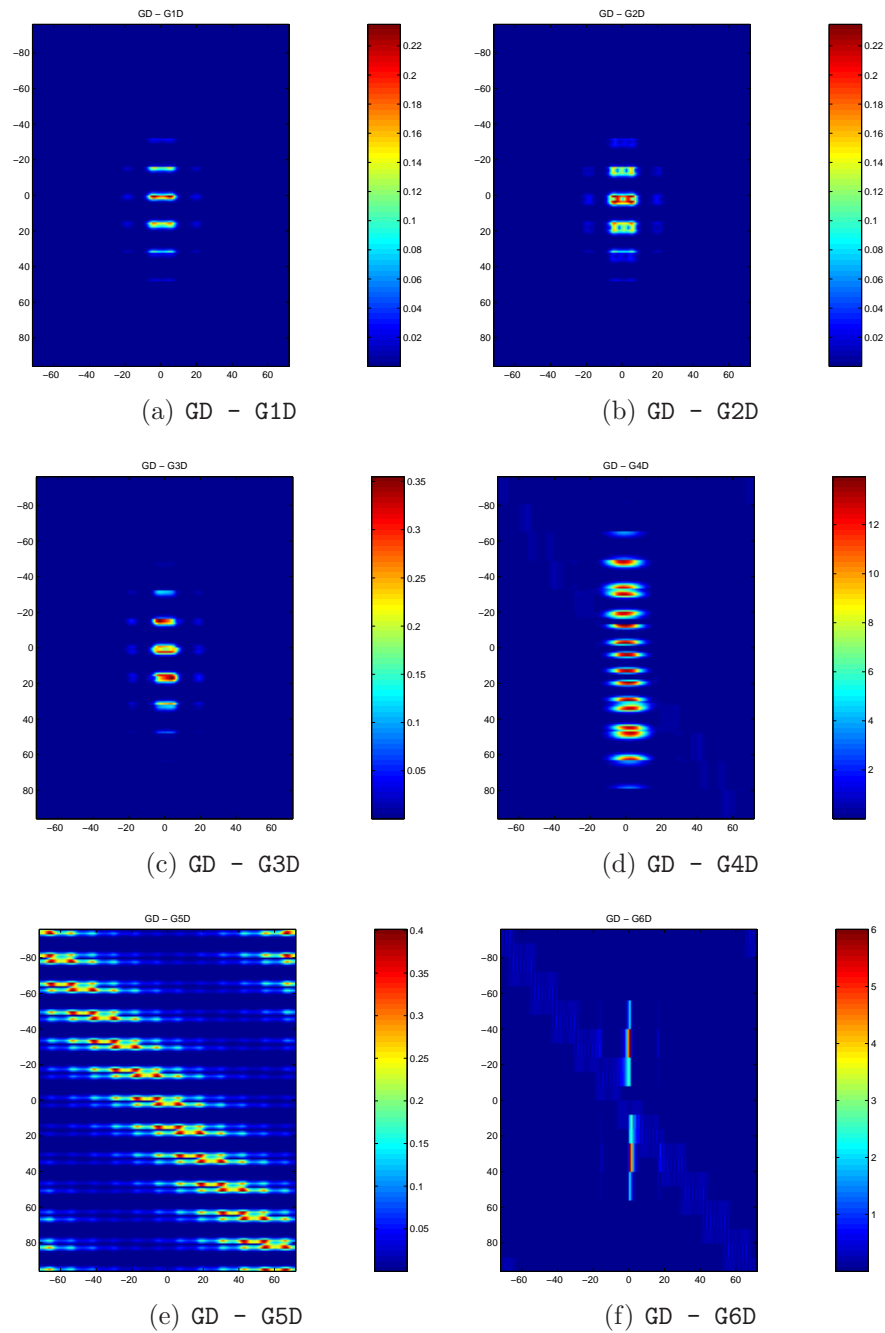
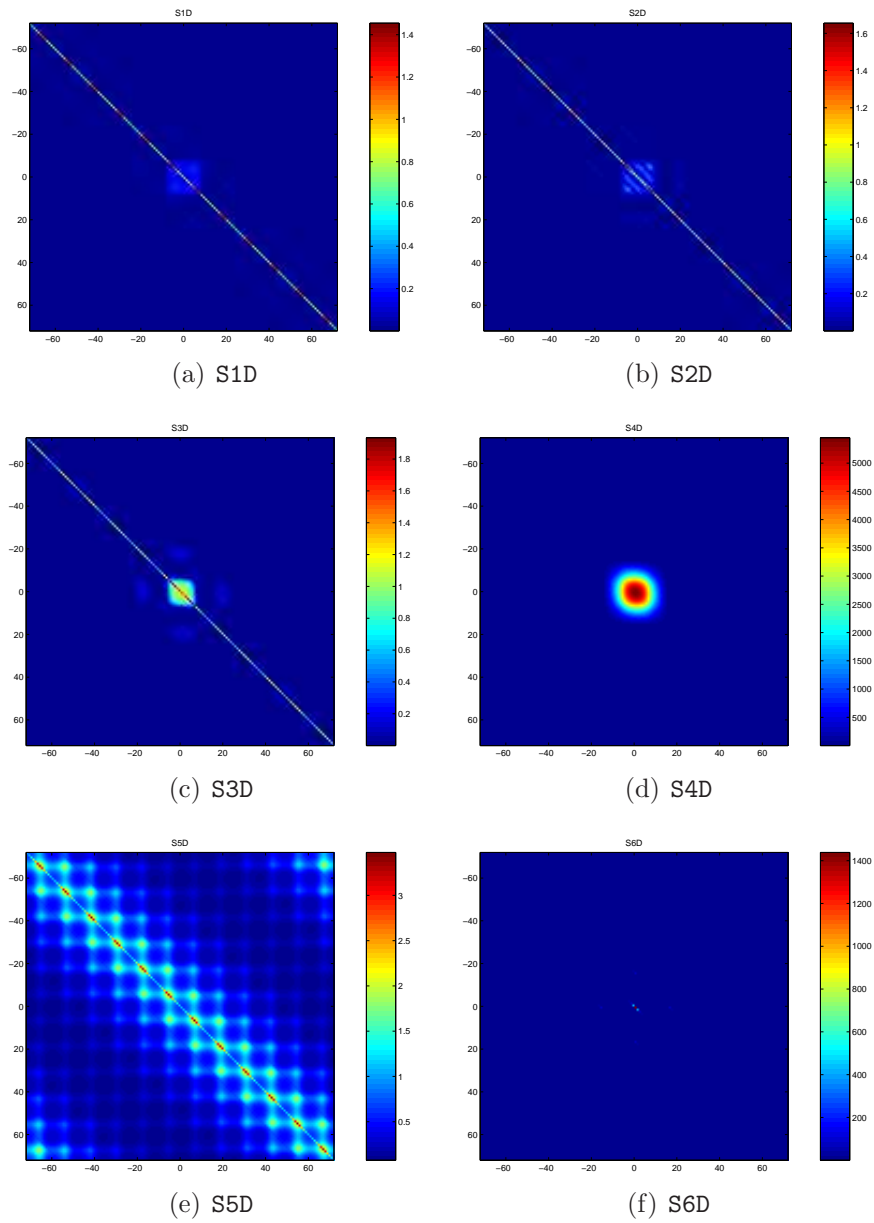


Figure 4.13: Differences between GD and the dual matrices with removed rows

Figure 4.14: Frame operators S1D to S6D, $n = 144$, $a = 12$, $b = 9$

4.2.7 Reconstruction of signals

Now we are sure that $G_1 - G_6$ are frames and we have already calculated their corresponding dual frame. Therefore we are ready to analyze the effects of signal reconstruction. Using a random signal of length $n = 144$ we apply G for analysis. The next step is synthesis done through applying GD' .

This procedure is used in each of the six considered cases. For comparing the original and the reconstructed signal, \mathbf{sig} and \mathbf{sigrec} , we compute $\text{norm}(\mathbf{sig} - \mathbf{sigrec})/\text{norm}(\mathbf{sig})$.

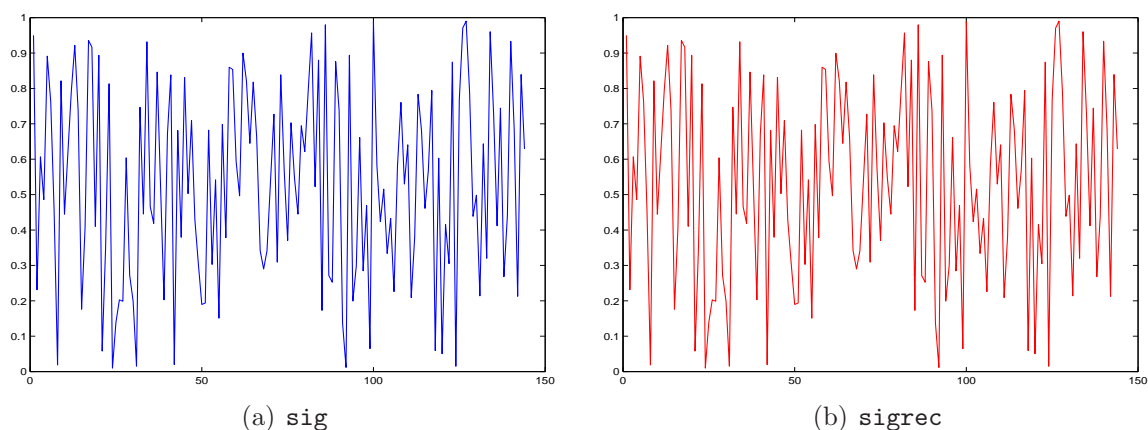


Figure 4.15: Reconstruction of signals with G

x	$\text{norm}(\mathbf{sig} - x)/\text{norm}(\mathbf{sig})$
\mathbf{sigrec}	$2.2743 \cdot 10^{-14}$
$\mathbf{sigrec1}$	$2.9875 \cdot 10^{-15}$
$\mathbf{sigrec2}$	$3.1842 \cdot 10^{-15}$
$\mathbf{sigrec3}$	$3.6189 \cdot 10^{-15}$
$\mathbf{sigrec4}$	$2.5675 \cdot 10^{-14}$
$\mathbf{sigrec5}$	$3.2129 \cdot 10^{-15}$
$\mathbf{sigrec6}$	$1.3326 \cdot 10^{-14}$

Table 4.6: Difference between \mathbf{sig} and \mathbf{sigrec}

The results are given in Figure 4.15 and 4.6. Although we used Gabor matrices where we have removed rows to do analysis and synthesis, the reconstruction of a random signal works.

4.2.8 Dual atoms viewed as linear combination of original atoms

Now we want to work out the details of the procedure that we called work-sharing in neighborhood earlier. Every atom of the dual Gabor matrix can be written as a linear combination of atoms of the Gabor matrix. Therefore, every row of \mathbf{GD} can be written as a linear combination of rows of \mathbf{G} . In the case of removing one atom, the case we will discuss, this is still true, but coefficients will change.

In our experiment we will fix a dual atom near the hole and observe its coefficients. In MATLAB terms, we have $\mathbf{z11} = \mathbf{G1D}(1, :)$, whereas $\mathbf{z11}$ is the fixed atom of $\mathbf{G1D}$. Note that $\mathbf{z11}$ is the first row of $\mathbf{G1D}$, but corresponds to the second row of \mathbf{GD} , since there is no element missing.

Then we set $\mathbf{c11} = \mathbf{G1D} * \mathbf{z11}'$. So the vector $\mathbf{c11}$ presents the coefficients we were looking for.

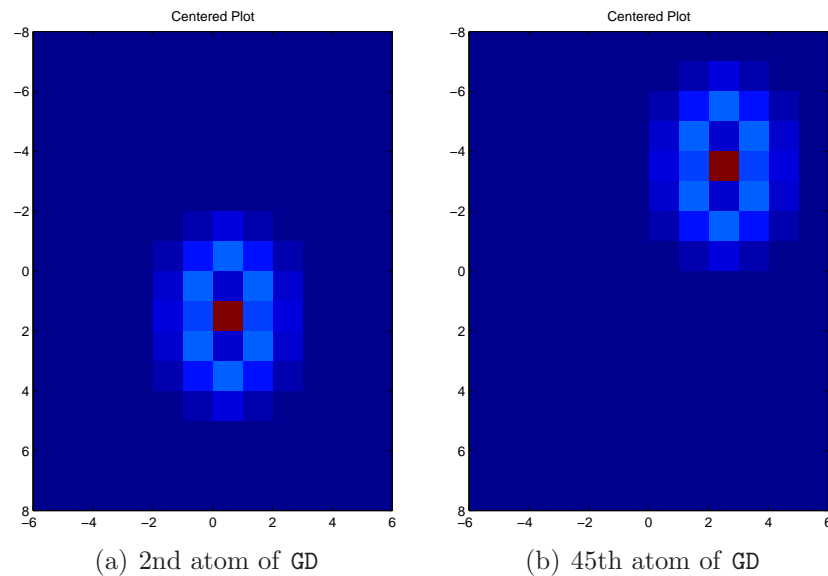


Figure 4.16: Coefficients of original dual atoms

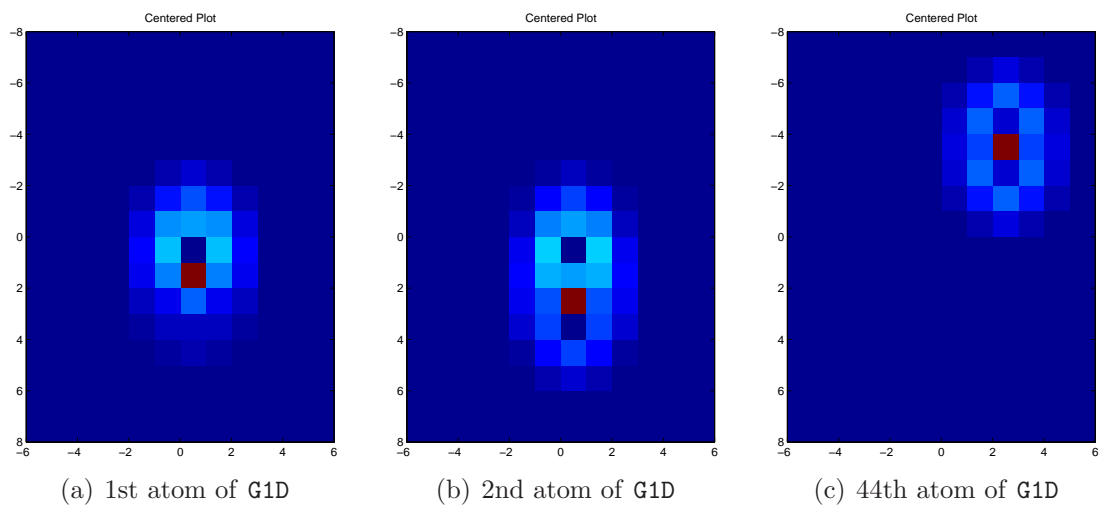


Figure 4.17: Coefficients of dual atoms of G1D

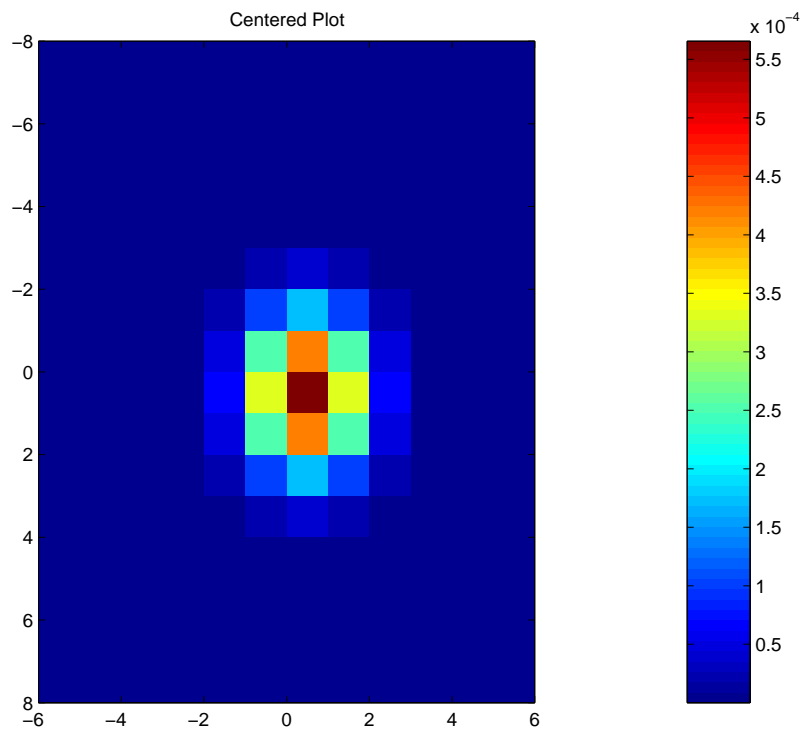


Figure 4.18: Difference between the coefficients of 45th atom of GD and 44th atom of G1D

Every entry in this vector is related to a particular atom in our lattice. These coefficients indicate to what extent the first atom must compensate for the effect of removing its neighboring atom. To show this relation we fill the vector into a matrix according their position in the lattice. In Figure 4.16 the second and the 45th atom are pictured. The coefficients are the same, since we do not miss any atoms. In Figure 4.17 the compensation of the loss is shown by means of the first, second and 44th atom. Be aware that since we omitted the first atom of **GD** the first atom of **G1D** corresponds to the second one of **GD**. In (a) and (b) the coefficients differ from those in Figure 4.16 for compensating the loss of the removed atom.

Furthermore, we compare the 45th atom of **GD** and the 44th atom of **G1D**. The small differences between those coefficients are located around the removed atom too. This implies, that the compensation of the loss decreases, if we are moving away from the hole.

4.2.9 Tight frames

Finally we consider the special case of tight frames. The smallest eight eigenvalues are given in Table 4.7. In the first case all eigenvalues are equal to one, which implies that we have a normalized tight frame. This property is useful to show the slice of the spectrum clearly in Figure 4.19. Considering the condition number in Table 4.8, the numerical instability increases, analog to Section 4.2.4.

eigst	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
eigst1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.2500
eigst2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.2660	0.2340
eigst3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.4361	0.0639
eigst4	0.6875	0.4040	0.1725	0.0587	0.0138	0.0024	0.0004	$1.1715 \cdot 10^{-5}$
eigst5	0.3629	0.1885	0.1885	0.0755	0.0755	0.0241	0.0241	0.0109
eigst6	0.0988	0.0310	0.0310	0.0075	0.0075	0.0014	0.0014	0.0004

Table 4.7: The smallest eight eigenvalues of **ST** to **ST6**

	ST	ST1	ST2	ST3	ST4	ST5	ST6
cond(x)	1.0000	4.0000	4.2731	15.6474	$8.5363 \cdot 10^{+4}$	91.7420	$2.7114 \cdot 10^{+3}$

Table 4.8: Condition numbers of **ST** to **ST6**

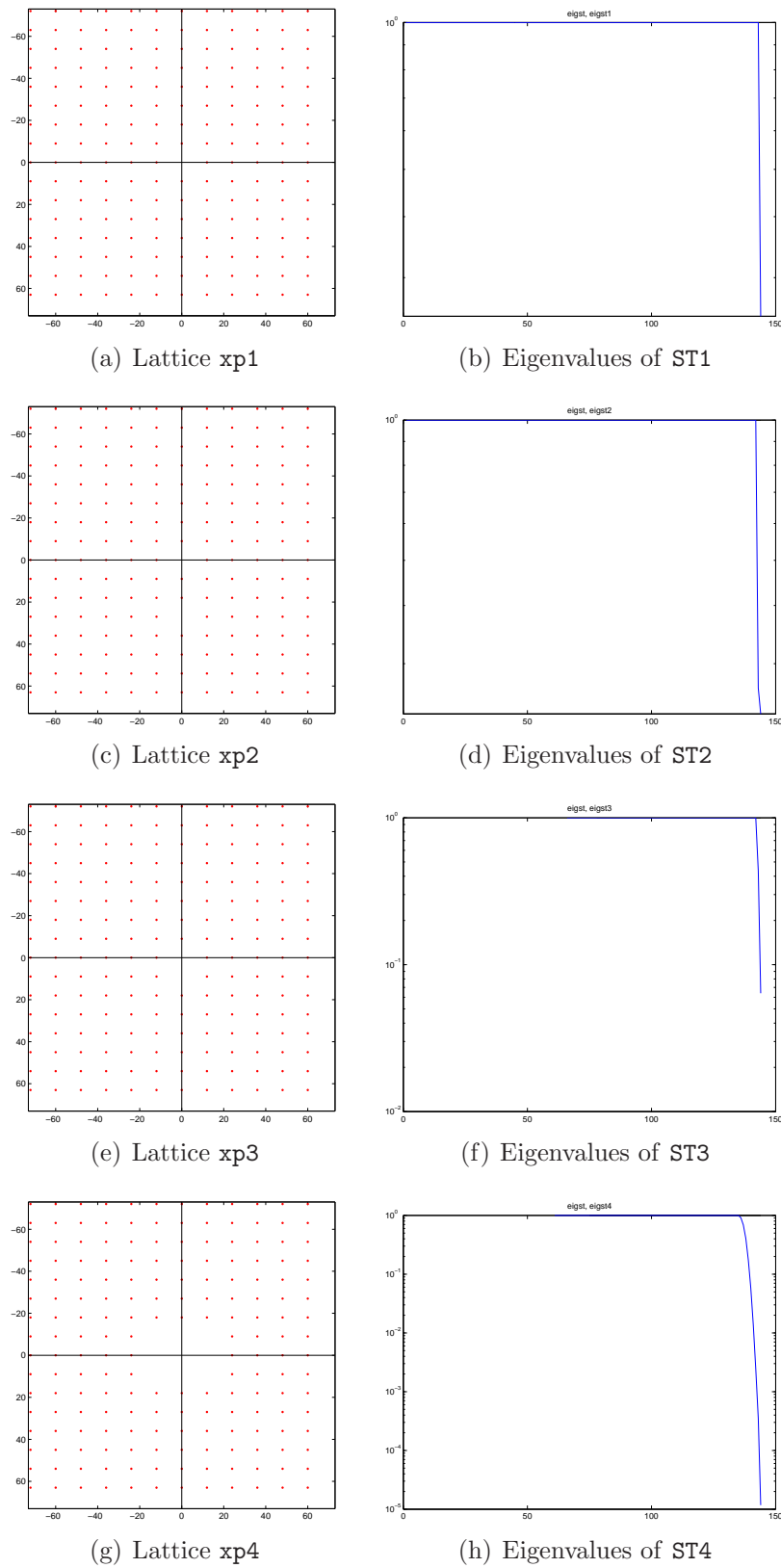


Figure 4.19: Eigenvalues belonging to a respective lattice using the tight window

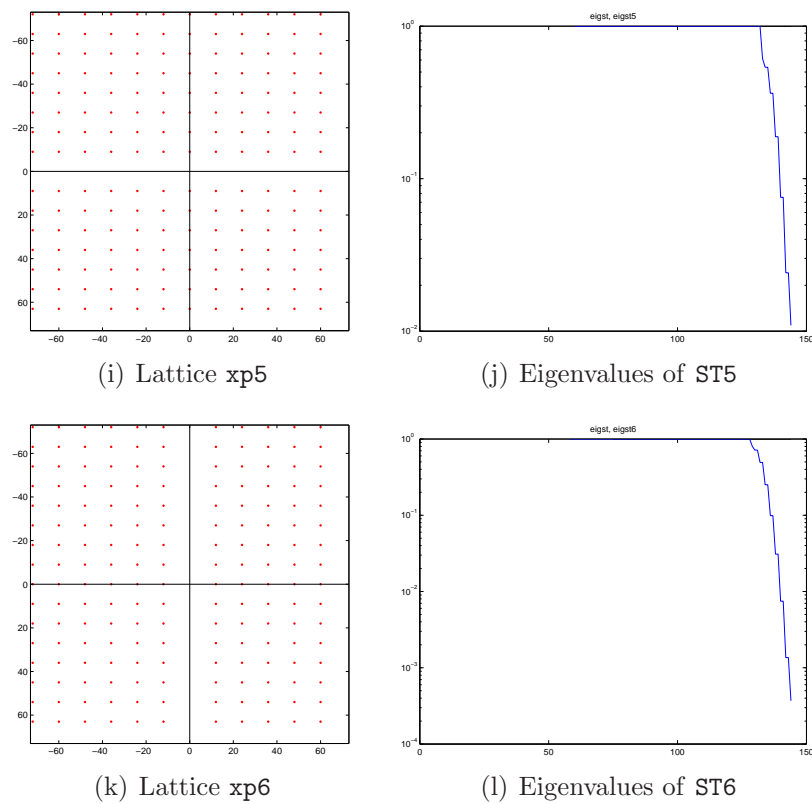


Figure 4.19: Eigenvalues belonging to a respective lattice using the tight window

4.3 Findings

Finally, we give a summary of our experiments.

Generating the six different lattices, where different number of elements on different places are removed shall help to consider the consequences. We start to compute the Gabor matrix, we shift the window function along the modified lattices and receive $\mathbf{G1} - \mathbf{G6}$, which turn out being frames, since none of the lowest eigenvalues is equal zero. Removing atoms has effects to the eigenvalues, which we call a slice of the spectrum. Furthermore, we consider the lattice parameters. Changing them a wider or smaller meshed lattice returns, which leads us to the redundancy. Having a wider meshed lattice, which corresponds to low redundancy, and removing one atom causes the lost of being a frame. In the opposite case the adjacent atoms are able to compensate a loss. This thought we will engross later.

Obviously, the numerical instability grows when omitting elements. The structure of the matrix of the frame operators $\mathbf{S1} - \mathbf{S6}$ gets disturbed by additional entries. We apply a different window function \mathbf{gd} and receive the same effects as with \mathbf{g} .

Computing the dual frames we try different ways. Once we use the pseudo-inverse and then, dualisation line by line. In both ways we receive the same result except of numerical small differences. Additional we compute the dual frames $\mathbf{G1D} - \mathbf{G6D}$ and their differences to \mathbf{GD} , which are located round the removed atoms.

We need the dual frames to do signal reconstruction, what is done next. The signal we use is a random generated one. Comparing the norms implies that we can reconstruct the signal in a good way, although we have removed atoms.

Renewing the consideration of compensating the loss of elements, we want to look in detail to the dual atoms viewed as the linear combination of original ones. These coefficients of the first atom of $\mathbf{G1D}$ tell us how much it has to compensate the hole. This is the reason, why we are able to reconstruct signals as long as work-sharing in neighborhood works!

Appendix A

MATLAB Files

A.1 excess3.m

```

% EXCESS3.m - computes the lattices with removed atoms, the analysis
%             operators, the frame operators, the sorted eigenvalues and the figures of
%             the lattices and the the eigenvalues; uses the window function g
%
% Author: Karin Nowak 10/2005
%
%
[V,eigs] = eigsort(S);           %V is the matrix of eigenvectors,
eigs(n-7:n)                     %eigs is the vector of sorted eigenvalues

xp1 = xpo; xp1(1,1) = 0;        %lattice without one element
figure; spyc tob(xp1); plotax;  %plots the lattice

G1= gabbas(g,xp1);             %generates analysis matrix with window g and lattice xp1
S1 = G1'*G1;                   %Frameoperator
[V1,eigs1] = eigsort(S1);
eigs1(n-7:n)
figure; semilogy(eigs,'k'); hold; semilogy(eigs1,'b'); title('eigs, eigs1');

% logarithmic semilogy(...) creates a plot using a base 10 logarithmic
% scale for the y-axis and a linear scale for the x-axis.

xp2 = xp1; xp2(28,1) = 0;      %lattice without two non adjacent elements
figure; spyc tob(xp2); plotax;

G2 = gabbas(g,xp2);
S2 = G2'*G2;
[V2,eigs2] = eigsort(S2);
eigs2(n-7:n)
%plot([eigs',eigs1',eigs2']);
figure; semilogy(eigs,'k'); hold; semilogy(eigs2,'b'); title('eigs, eigs2');

xp3 = xp1; xp3(10,1)=0;        %lattice without two adjacent elements
figure; spyc tob(xp3); plotax;

G3 = gabbas(g,xp3);
S3 = G3'*G3;
[V3,eigs3]= eigsort(S3);
eigs3(n-7:n)
figure; semilogy(eigs,'k'); hold; semilogy(eigs3,'b'); title('eigs, eigs3');

xp4 = xpo .* (1 - fltre cs(n,n,12,12)); %lattice without nine adjacent elements

```

```

figure; spyc Tob(xp4); plotax;

G4= gabbas(g,xp4);
S4 = G4'*G4;
[V4,eigs4] = eigsort(S4);
eigs4(n-7:n)
figure;semilogy(eigs,'k');hold;semilogy(eigs4,'b');title('eigs, eigs4');

xp5 = xpo; xp5(1,:) = 0;          %lattice without a row
figure; spyc Tob(xp5); plotax;

G5 = gabbas(g,xp5);
S5 = G5'*G5;
[V5,eigs5]= eigsort(S5);
eigs5(n-7:n)
figure;semilogy(eigs,'k');hold;semilogy(eigs5,'b');title('eigs, eigs5');

xp6 = xpo; xp6(:,1) = 0;        %lattice without a column
figure; spyc Tob(xp6); plotax;

G6 = gabbas(g,xp6);
S6 = G6'*G6;
[V6,eigs6]= eigsort(S6);
eigs6(n-7:n)
figure;semilogy(eigs,'k');hold;semilogy(eigs6,'b');title('eigs, eigs6');

```

A.2 excessd3.m

```

% EXCESSD3.m - computes the lattices with removed atoms, the analysis
%              operators, the frame operators, the sorted eigenvalues and the figures of
%              the lattices and the the eigenvalues; uses the window function gd
%
% Author: Karin Nowak 10/2005
%
%

SD = GD'*GD;
[Vd,eigsd] = eigsort(SD);eigsd(n-7:n)
xp1 = xpo; xp1(1,1) = 0;figure; spyc Tob(xp1);plotax;
GD1= gabbas(gd,xp1);
SD1 = GD1'*GD1; %Frameoperator
[Vd1,eigsd1] = eigsort(SD1); eigsd1(n-7:n)
figure;semilogy(eigsd,'k');hold;semilogy(eigsd1,'b');title('eigsd, eigsd1');

xp2 = xp1; xp2(28,1) = 0; figure;spyc Tob(xp2); plotax;
GD2 = gabbas(gd,xp2);
SD2 = GD2'*GD2;
[Vd2,eigsd2] = eigsort(SD2); eigsd2(n-7:n)
figure;semilogy(eigsd,'k');hold;semilogy(eigsd2,'b');title('eigsd, eigsd2');

xp3 = xp1; xp3(10,1)=0; figure;spyc Tob(xp3); plotax;
GD3 = gabbas(gd,xp3);
SD3 = GD3'*GD3; [Vd3,eigsd3]= eigsort(SD3); eigsd3(n-7:n)
figure;semilogy(eigsd,'k');hold;semilogy(eigsd3,'b');title('eigsd, eigsd3');

xp4 = xpo .* (1 - fltreccs(n,n,12,12)); figure; spyc Tob(xp4); plotax;
GD4= gabbas(gd,xp4);
SD4 = GD4'*GD4; [Vd4,eigsd4] = eigsort(SD4); eigsd4(n-7:n)
figure;semilogy(eigsd,'k');hold;semilogy(eigsd4,'b');title('eigsd,eigsd4');

xp5 = xpo; xp5(1,:) = 0; figure; spyc Tob(xp5); plotax;
GD5 = gabbas(gd,xp5);
SD5 = GD5'*GD5; [Vd5,eigsd5]= eigsort(SD5);eigsd5(n-7:n)

```

```

figure;semilogy(eigsd,'k');hold;semilogy(eigsd5,'b');title('eigsd, eigsd5');

xp6 = xpo; xp6(:,1) = 0; figure; spycetob(xp6); plotax;
GD6 = gabbas(gd,xp6);
SD6 = GD6'*GD6; [Vd6,eigsd6]= eigsort(SD6);eigsd6(n-7:n)
figure;semilogy(eigsd,'k');hold;semilogy(eigsd6,'b');title('eigsd, eigsd6');

```

A.3 excesst3.m

```

% EXCESST3.m - computes the lattices with removed atoms, the analysis
%               operators, the frame operators, the sortet eigenvalues and the figures of
%               the lattices and the the eigenvalues; uses the window function gt
%
% Author: Karin Nowak 10/2005
%
%

```

```

ST = GT'*GT;
[Vt,eigst] = eigsort(ST);
eigst(n-7:n)

```

```

xp1 = xpo; xp1(1,1) = 0;figure; spycetob(xp1);plotax;
GT1= gabbas(gt,xp1);
ST1 = GT1'*GT1;
[Vt1,eigst1] = eigsort(ST1);eigst1(n-7:n)
figure;semilogy(eigst,'k');hold;semilogy(eigst1,'b');title('eigst, eigst1');

```

```

xp2 = xp1; xp2(28,1) = 0;figure;spycetob(xp2); plotax;
GT2 = gabbas(gt,xp2);
ST2 = GT2'*GT2;
[Vt2,eigst2] = eigsort(ST2);eigst2(n-7:n)
figure;semilogy(eigst,'k');hold;semilogy(eigst2,'b');title('eigst, eigst2');

```

```

xp3 = xp1; xp3(10,1)=0;figure;spycetob(xp3); plotax;
GT3 = gabbas(gt,xp3);
ST3 = GT3'*GT3;
[Vt3,eigst3]= eigsort(ST3);eigst3(n-7:n)
figure;semilogy(eigst,'k');hold;semilogy(eigst3,'b');title('eigst, eigst3');

```

```

xp4 = xpo .* (1 - fltrenc(n,n,12,12));figure; spycetob(xp4); plotax;
GT4= gabbas(gt,xp4);
ST4 = GT4'*GT4;
[Vt4,eigst4] = eigsort(ST4);eigst4(n-7:n)
figure;semilogy(eigst,'k');hold;semilogy(eigst4,'b');title('eigst, eigst4');

```

```

xp5 = xpo; xp5(1,:) = 0;figure; spycetob(xp5); plotax;
GT5 = gabbas(gt,xp5);
ST5 = GT5'*GT5;
[Vt5,eigst5]= eigsort(ST5);eigst5(n-7:n)
figure;semilogy(eigst,'k');hold;semilogy(eigst5,'b');title('eigst, eigst5');

```

```

xp6 = xpo; xp6(:,1) = 0;figure; spycetob(xp6); plotax;
GT6 = gabbas(gt,xp6);
ST6 = GT6'*GT6;
[Vt6,eigst6]= eigsort(ST6);eigst6(n-7:n)
figure;semilogy(eigst,'k');hold;semilogy(eigst6,'b');title('eigst, eigst6');

```

A.4 output.m

```

% OUTPUT.m - returns the graphical representation of G1-G6, S1-S6

```

```

%
% Author: Karin Nowak 10/2005
%
%

figure; imgc(G1);title('G1');
figure; imgc(G2);title('G2');
figure; imgc(G3);title('G3');
figure; imgc(G4);title('G4');
figure; imgc(G5);title('G5');
figure; imgc(G6);title('G6');

figure; imgc(S1);title('S1');
figure; imgc(S2);title('S2');
figure; imgc(S3);title('S3');
figure; imgc(S4);title('S4');
figure; imgc(S5);title('S5');
figure; imgc(S6);title('S6');

```

A.5 trlbas1.m

```

% TRLBAS1.m - uses trlbas.m and returns the images
%
% Author: Karin Nowak 10/2005
%
%
TG=trlbas(g,a); figure;plot(TG')

TGF= trlbas(fft(g),b);figure;plot(abs(TGF'))

```

A.6 rowdual.m

```

% rowdual.m - computes the dual of a row
%
% Usage: rowdual(g,G);
%
% Author: Karin Nowak 10/2005
%
%

function gd = rowdual(g, G);

g = g.'; % actually, we assume input to be row vector;
S = G' * G;
gdc = inv(S) * g;
gd = gdc.';

```

A.7 duale1.m

```

% DUALE1.m - compares dualisation line by line and using the pseudoinverse
%
% Author: Karin Nowak 10/2005
%
%

A=pinv(G');
figure;imgc(A);title('A')
figure;imgc(GD);title('GD')

```

```

AGD = GD - A;
figure;imgc(AGD);title('GD - A')
norm0 = norm(AGD)
norm0f = norm(AGD,'fro')

A5=pinv(G5');
figure;imgc(A5);title('A5')
G5D=zeros(180,144);
for i=1:180 m=G5(i,:);n=rowdual(m,G5); for k=1:144 G5D(i,k)=n(1,k); end; end;
figure;imgc(G5D);title('G5D')
A5G5D = abs(G5D - A5);
figure;imgc(A5G5D);title('G5D - A5');
norm5 = norm(A5G5D)
norm5f = norm(A5G5D,'fro')

A6=pinv(G6');
figure;imgc(A6);title('A6')
G6D=zeros(176,144);
for i=1:176 m=G6(i,:);n=rowdual(m,G6); for k=1:144 G6D(i,k)=n(1,k); end; end;
figure;imgc(G6D);title('G6D')
A6G6D = abs(G6D - A6);
figure;imgc(A6G6D);title('G6D - A6');
norm6 = norm(A6G6D)
norm6f = norm(A6G6D,'fro')

```

A.8 duale2.m

```

% DUALE2.m - computes the duale frames G1D-G6D and their frame operators;
%
% Author: Karin Nowak 10/2005
%
%
G1D=pinv(G1');
figure;imgc(G1D);title('G1D');
G2D=pinv(G2');
figure;imgc(G2D);title('G2D');
G3D=pinv(G3');
figure;imgc(G3D);title('G3D');
G4D=pinv(G4');
figure;imgc(G4D);title('G4D');
G5D=pinv(G5');
figure;imgc(G5D);title('G5D');
G6D=pinv(G6');
figure;imgc(G6D);title('G6D');

S1D = G1D'*G1D;
figure;imgc(S1D);title('S1D');
S2D = G2D'*G2D;
figure;imgc(S2D);title('S2D');
S3D = G3D'*G3D;
figure;imgc(S3D);title('S3D');
S4D = G4D'*G4D;
figure;imgc(S4D);title('S4D');
S5D = G5D'*G5D;
figure;imgc(S5D);title('S5D');
S6D = G6D'*G6D;
figure;imgc(S6D);title('S6D');

```

A.9 diffdual

```
% DIFFDUAL.m - returns the difference of the dual frames to GD
%
% Author: Karin Nowak 10/2005
%
%
DIFF1 = abs(GD-G1D); figure; imgc(DIFF1);title('GD - G1D'); colorbar;

DIFF2 = abs(GD-G2D);
figure;
imgc(DIFF2);title('GD - G2D'); colorbar;

DIFF3 = abs(GD -G3D);
figure;
imgc(DIFF3);title('GD - G3D'); colorbar;

DIFF4 = abs(GD -G4D);
figure;
imgc(DIFF4);title('GD - G4D'); colorbar;

DIFF5 = abs(GD - G5D);
figure;
imgc(DIFF5);title('GD - G5D'); colorbar;

DIFF6 = abs(GD - G6D);
figure;
imgc(DIFF6);title('GD - G6D'); colorbar;
```

A.10 sigrec.m

```
% SIGREC.m - signal reconstruction with G-G6; computes the norm;
%
% Author: Karin Nowak 10/2005
%
%
sig=rand(144,1);
figure;plot(abs(sig));
x= G*sig;
sigrec = GD'*x;
norm(sig-sigrec)/norm(sig)
figure;plot(abs(sigrec),'r');

x1 = G1*sig;
sigrec1 = G1D'*x1;
norm(sig-sigrec1)/norm(sig)

x2 = G2*sig;
sigrec2 = G2D'*x2;
norm(sig-sigrec2)/norm(sig)

x3 = G3 *sig;
sigrec3 = G3D'*x3;
norm(sig-sigrec3)/norm(sig)

x4 = G4 *sig;
sigrec4 = G4D'*x4;
norm(sig-sigrec4)/norm(sig)

x5 = G5 *sig;
sigrec5 = G5D'*x5;
norm(sig-sigrec5)/norm(sig)
```

```
x6 = G6 *sig;
sigrec6 = G6D'*x6;
norm(sig-sigrec6)/norm(sig)
```

A.11 coeff2.m

```
% COEFF2.m - dual atoms given as linear combination of original atoms
%
% Author: Karin Nowak 10/2005
%
%
z1=G1D(1,:);
c1=G1D*z1';
K = zeros(16,12);
i=1;
for j = 1:12
    for k = 1 : 16
        K(k,j) = c1(i,1);
        i=i+1;
    end;
end;
figure; imgc(K);

z11=G1D(1,:);
cc11=G1D*z11';
c11=[0; cc11];
K = zeros(16,12);
i=1;
for j = 1:12
    for k = 1 : 16
        K(k,j) = c11(i,1);
        i=i+1;
    end;
end;
figure; imgc(K);

z12=G1D(2,:);
cc12=G1D*z12';
c12=[0; cc12];
K = zeros(16,12);
i=1;
for j = 1:12
    for k = 1 : 16
        K(k,j) = c12(i,1);
        i=i+1;
    end;
end;
figure; imgc(K);

z144=G1D(44,:);
cc144=G1D*z144';
c144=[0; cc144];
K = zeros(16,12);
i=1;
for j = 1:12
    for k = 1 : 16
        K(k,j) = c144(i,1);
        i=i+1;
    end;
end;
figure;imgc(K);
```

```
z45=GD(45,:);
c45=GD*z45';
K = zeros(16,12);
i=1;
for j = 1:12
    for k = 1 : 16
        K(k,j) = c45(i,1);
        i=i+1;
    end;
end;
figure;imgc(K);
diff=abs(c45-c144);
K = zeros(16,12);
i=1;
for j = 1:12
    for k = 1 : 16
        K(k,j) = diff(i,1);
        i=i+1;
    end;
end;
figure;imgc(K);
```

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