Trace Ideal Criteria for Hankel Operators and Commutators

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I. Introduction and Summary

We present conditions for Hankel operators and related operators defined on the Hardy space $H^2(\mathbb{R})$ to be in the Schatten ideals $\mathcal{S}_p$, $p \geq 1$. The criterion is that the symbol have derivative in an appropriate Bergman space (i.e., that the symbol be in a certain Besov space). Our results are analogous to and offer a new approach to recent results of V. V. Peller involving Hankel operators defined on the Hardy space of the circle.

Background, definitions, and the statement of the main theorem are in Section 2. The proof of the theorem is in Section 3. It is based on Fourier transform calculations and the molecular decomposition of functions in Bergman spaces. Variations on the main theorem are given in Section 4. These include trace ideal criteria for various types of commutators and analogous results for operators defined using weighted projections. We also obtain conditions under which weighted projections are trace ideal perturbations of ordinary projections. In the final section, we mention a few open questions.

II. Preliminaries

A. Function spaces. We denote by $H^2$ the Hardy space consisting of those functions $F$ which are holomorphic on the upper half plane $U$ and for which

$$\|F\|_2^2 = \sup_{y > 0} \int_{-\infty}^{\infty} |F(x + iy)|^2 dx$$

is finite. We will identify such an $F$ with its boundary values. The Fourier transform, $\hat{F}$, of such a function is square integrable on the line and vanishes on the negative half line. Thus we will regard $\hat{F}$ as an element of $L^2(0, \infty)$. The Fourier transform is a unitary map of $H^2$ onto $L^2(0, \infty)$. We denote the inverse Fourier transform of $g$ by $\check{g}$. For further discussion of these issues, see [16].

For $p, r$ with $0 < p < \infty$, $-1/2 < r$, we define the Bergman space $A^{p,r}$ to be the space of functions $F$ holomorphic in $U$ for which

Indiana University Mathematics Journal ©, Vol. 31, No. 6 (1982)
\[ \|F\|_{p,r}^p = \int_U \left| F(x + iy) \right|^p y^{-2r} \, dx \, dy \]

is finite. These spaces are discussed in detail in [5]. We now recall the decomposition theorem from [5]. Let \( d(\cdot, \cdot) \) denote the hyperbolic distance on \( U \). Let \( \eta \) be a positive number. A sequence of points \( \{z_i\} \) in \( U \) is called an \( \eta \)-lattice if for all \( i, j, i \neq j, d(z_i, z_j) \geq \eta/100 \) and if also for all \( z \) in \( U \), \( \inf \{d(z,z_i)\} < \eta \).

(For example, the set \( \{n2^n + i2^n : n,m \in Z\} \) is an \( \eta \)-lattice for appropriate \( \eta \).) For \( z, \xi \) in \( U \) we denote by \( B(z,\xi) \) the Bergman kernel; \( B(z,\xi) = (z - \xi)^{-2} \). (We are ignoring certain inessential multiplicative constants in our definition of \( B \) and in various Fourier transform calculations.) A special case of Theorem 2 of [5] is

**Theorem.** Let \( p, d, \varepsilon \) be given; \( p > 0, d > 1/2p, \varepsilon > d \max(-1, p - 2) \). There is an \( \eta_0 = \eta_0(p,d,\varepsilon) \) such that if \( \eta < \eta_0 \) and the points \( \{z_i\} \) form an \( \eta \)-lattice then every function \( F \) in \( A^{p,dp-1} \) can be written as a series

\[ F(z) = \sum \lambda_i \frac{B(z,\xi_i)^{d+\varepsilon}}{B(\xi_i,\xi_i)^{d+\varepsilon}} \]

with scalars \( \lambda_i \) which satisfy \( \sum |\lambda_i|^p \leq c(\eta, p, d, \varepsilon) \|F\|_{p, dp-1}^p \). Conversely, if \( \sum |\lambda_i|^p \) is finite then \( F(z) \) defined by (2.1) is in \( A^{p, dp-1} \) and

\[ \|F\|_{p, dp-1}^p \leq c(\eta, p, d, \varepsilon) \sum |\lambda_i|^p. \]

Integration of order one-half (that is, the map of \( f \) to \( (t^{-1/2} \hat{f}) \) ) is a unitary map from \( A^{2,0} \) to \( H^2 \) (this is verified by calculating Fourier transforms). This map induces an analogous decomposition theorem for \( F \) in \( H^2 \): Given \( \varepsilon > 0 \) one can write

\[ F(z) = \sum \lambda_i \frac{B(z,\xi_i)^{3/4+\varepsilon}}{B(\xi_i,\xi_i)^{1/2+\varepsilon}} \]

with \( \sum |\lambda_i|^2 \) comparable to \( \|F\|^2 \).

For later use, we record the following

\[ (B(z,\xi_j)^{t})^\vee(t) = c_i \Gamma^{2t-1} e^{-t \xi_i}, \quad t > 0 \]

\[ (B(z,\xi_j)^{t_1}, B(z,\xi_j)^{t_2}) = c_{t_1,t_2} B(\xi_j,\xi_j)^{t_1+t_2-1/2}. \]

For (2.3) see [16]. (2.4) follows from (2.3) and the fact that the Fourier transform is unitary.

We use the notation \( B(\cdot, \cdot) \) rather than the explicit formula for the kernel to emphasize our suspicion that most of these results will eventually be found to have analogs for the more general Bergman spaces discussed in [5].
B. Hankel operators. Let $k$ be a function defined on the half-line $(0, \infty)$. We define $H_k$, the Hankel operator with symbol $k$, to be the linear operator on $L^2(0, \infty)$ given by

$$(H_k f)(t) = \int_0^\infty k(s + t)f(s)\,ds.$$  

The related operator on $H^2$ which maps $F$ to $(H_k F)^*$ is also often called a Hankel operator. General discussions of Hankel operators are given in the surveys [13] and [1].

C. Trace ideals. Let $A$ be a bounded linear operator on a Hilbert space. $|A|$ is the positive square root of the positive operator $A^*A$. Let $s_1, s_2, \ldots$ be the eigenvalues of $|A|$. The operator $A$ is said to be in the Schatten class $\mathcal{S}_p$, $p \geq 1$ if

$$\|A\|_p = \left(\sum_j s_j^p\right)^{1/p}$$

is finite. These classes are also called trace ideals. The operators in $\mathcal{S}_1$ are also called trace class operators or nuclear operators. The operators in $\mathcal{S}_2$ are also called Hilbert-Schmidt operators. The general theory of these classes is presented in [9] and [15].

Each $\mathcal{S}_p$ is a Banach space with the norm given by (2.5) and is a two-sided ideal in the space of all bounded operators. If we denote the operator norm by $\|\cdot\|$ and the $\mathcal{S}_p$ norm by $\|\cdot\|_p$ then

$$\|XAY\|_p \leq \|X\| \|A\|_p \|Y\|.$$  

For each $p$ the set of finite rank operators is dense in $\mathcal{S}_p$ and $\mathcal{S}_p$ is contained in the ideal of all compact operators. Finally, if $\{\varphi_n\}$ is any orthonormal set then

$$\sum_n |\langle A \varphi_n, \varphi_n \rangle|^p \leq \|A\|_p^p.$$  

(See page 94 of [9].)

D. The main theorem.

Theorem 1. Let $H_k$ be the Hankel operator with symbol $k$. Let $p \geq 1$ be given. There is a constant $\eta_p$ which depends only on $p$ so that if $\eta$, $n$, $\theta$, and $\varepsilon$ are constants which satisfy

$$\eta_p > \eta, \quad n > \frac{1}{p}, \quad \theta > \frac{n}{2} (p - 1), \quad 2 + 2\varepsilon > \frac{1}{p}$$

and if $\{\zeta_i\}$ is any $\eta$–lattice then the following are equivalent
(a) $H_k$ is in $S_p$, 
(b) $D^n k$ is in $A_p^{np/2-1}$, 
(c) $k(z) = \sum \lambda_i (B(z, \zeta_i) / B(\zeta_i, \zeta_i))^{3/4}$ for scalars which satisfy $\sum |\lambda_i| < \infty$. 
If any of these hold then it is also true that  
\[ \sum_i |(H_k B(z, \zeta_i))^{3/4+\epsilon}, B(z, -\zeta_i)^{3/4+\epsilon}) B(\zeta_i, \zeta_i)^{-1-2\epsilon}|^p < \infty. \]
If (d) holds for all $\eta$-lattices then (a), (b), and (c) also hold.

Finally, $\|H_k\|_p, \|D^n k\|_{p, np/2-1}, \inf \left\{ \left( \sum |\lambda_i|^{3/4} \right)^{1/p} \right\}$; all representations of $k$ as in (c) are equivalent quantities with constants of equivalence which can be chosen to depend only on the data in (2.8).

**Notes.** (1) $D^n k$ denotes the $n$th derivative of $k$. We allow fractional values of $n$ (with $D^n$ defined via Fourier transform). (2) The equivalence of (a) and (b) for the analogous operators defined on $\ell^2(\mathbb{Z}^+)$ is a result of Peller [12]. The equivalence of (a), (b), and (c) for $p = 1$ is proved in [5], earlier related results were obtained by Howland [10].

### III. Proof of Theorem 1

We write $H$ for $H_k$. We suppose that the $\eta$-lattice is picked and fixed.

First note that by the decomposition theorem, (b) is satisfied for some $n > 1/p$ if and only if it is satisfied for all such $n$. Term-by-term integration and differentiation yields the equivalence of a sum of the form given in (2.1) with a representation of the type given in (c). Thus (b) is equivalent to (c).

We now show that (a) implies (d). Write $f_i$ for the individual building blocks in (2.2). That is,

\[ f_i(z) = B(z, \zeta_i)^{3/4+\epsilon} B(\zeta_i, \zeta_i)^{-1/2-\epsilon}. \]

The points obtained by reflecting the numbers $\{\zeta_i\}$ in the $y$-axis also forms an $\eta$-lattice. Denote the corresponding functions by $f_i^*$,

\[ f_i^*(\zeta) = B(z, -\zeta_i)^{3/4+\epsilon} B(-\zeta_i, -\zeta_i)^{-1/2-\epsilon}. \]

(Note: $B(z, \zeta) = B(-\zeta, -z)$. Let $T$ be the linear map from sequences to functions on $(0, \infty)$ given by $T((\alpha_i)) = \sum \alpha_i f_i$. The representation theorem summarized in (2.2) shows that $T$ is a bounded map from $\ell^2$ to $L^2(0, \infty)$. Similarly the map $S$ given by $S((\alpha_i)) = \sum \alpha_i f_i^*$ is bounded. Let $b_i$ denote the $i$th standard basis element of $\ell^2$; $(b_i)_j = \delta_{ij}$. The quantity to be estimated is

\[ \sum |\langle HTb_i, Sb_i^* \rangle|^p = \sum |\langle S^*HTb_i, b_i \rangle|^p. \]
By (2.7), this is dominated by \( \|S*HT\|_p^p \). (2.6) is valid even if the three operators are not all defined on the same space. Thus, since \( S \) and \( T \) are bounded, we may dominate this last quantity by \( \|H\|_p^p \). Thus (d) is established.

We now suppose (d) holds for any \( \eta \)-lattice. We will establish (b). Write \( \zeta_j = x_j + iy_j \). Using (2.3) we find \( f_j(t) = y_j^{1+\varepsilon} t^{1/2+\varepsilon} e^{-i\zeta_j t} \). Thus, using the definitions we can write \( \langle H f_j, \hat{f}_j \rangle \) as

\[
\int_0^\infty \int_0^\infty k(s + t) y_j^{2+2\varepsilon} t^{1/2+\varepsilon} s^{1/2+\varepsilon} e^{-i\zeta_j s} e^{i\zeta_j t} ds dt.
\]

We replace the integration variables \( s \) and \( t \) by new variables \( t \) and \( w = s + t \) and evaluate the \( t \) integral explicitly. This produces

\[
c y_j^{2+2\varepsilon} \int_0^\infty k(w) w^{2+2\varepsilon} e^{-i\zeta_j w} dw
\]

which by definition equals \( c y_j^{2+2\varepsilon} (D^{2+2\varepsilon} k)(\zeta_j) \). Thus the sum in (d) is

\[
\sum_j |D^{2+2\varepsilon} k(\zeta_j)|^p (y_j^{2+2\varepsilon})^p.
\]

This sum is, roughly, a Riemann sum for

\[
\int \int |D^{2+2\varepsilon} k|^p y^{2(\rho + \rho \varepsilon - 1)} dy dx
\]

which is the integral we wish to estimate. To actually obtain the estimate we pick a covering of \( U \) by sets \( D_i \) of the sort described in Lemma 2.4 of [5]. Roughly \( D_i \) is a covering by disjoint sets which are almost balls of radius \( c\eta \). Pick \( \zeta_i^+ \) in \( D_i \) at which \( |D^{2+2\varepsilon} k| \) attains its maximum. \( \{\zeta_i^+\} \) will also be a \( c\eta \)-lattice and (3.1) for this new lattice will be a majorant for an upper sum for (3.2). Thus the integral in (3.2) is finite.

We now show that (c) implies (a). We obtain this by complex interpolation of operators. The most obvious limit at \( p = \infty \) of the spaces in (c) is the Bloch space. (See Theorem 2' of [5] for a description of the Bloch space in this form.) However the condition that \( k \) be in the Bloch space is not sufficient to insure that \( H_k \) is bounded. On the other hand, it is not clear how to use the condition which insures that \( H_k \) be bounded (the condition is that \( k \) is in BMO) as an endpoint for an interpolation argument which has \( A^{1,0} \) as the other endpoint. For this reason, we consider operators a bit more general than Hankel operators. For complex \( \alpha, \beta \) and \( k \) defined on \( (0, \infty) \) define \( H_k^{\alpha,\beta} \) by

\[
(H_k^{\alpha,\beta} f)(t) = \int_0^\infty s^\alpha t^\alpha (s + t)^{-\beta} k(s + t) f(t) dt.
\]

**Lemma 2.** Suppose \( \alpha \) is complex, \( \Re \alpha > -1/2 \). The map from \( k \) to \( H_k^{\alpha,2+2\varepsilon} \) is a continuous map of \( A^{1,0} \) to \( \mathcal{F}_1 \).
Proof. We wish to decompose $k$ using the decomposition theorem. However, we first note that differentiation of purely imaginary order is an isomorphism of $A^{1.0}$. Hence we can obtain a decomposition (2.1) with $\varepsilon$ replaced by $\varepsilon + i \theta$, $\theta$ real. We decompose $k$ according to the decomposition theorem, that is, we write $k$ in the form (2.1) with $p = 1$, $d = 1$, $\varepsilon = -1/2 + \alpha$. We then use (2.3) and obtain

$$k(t) = \sum \lambda_i y_i^{1+2\alpha} t^{2+2\alpha} e^{-i\alpha t}.$$ 

Let $T_i$ be the operator generated by the single term

$$k_i = y_i^{1+2\alpha} t^{2+2\alpha} e^{-i\alpha t}.$$ 

By direct calculation, $T_i f$ is given by $T_i f = \langle f, u_i \rangle v_i$ with

$$u_i(t) = y_i^{1/2+\alpha} t^{i \alpha} e^{i \alpha t} \quad \text{and} \quad v_i(t) = y_i^{1/2+\alpha} t^{i \alpha} e^{-i \alpha t}.$$ 

For a one-dimensional operator such as $T_i$, we can compute the $\mathcal{F}_1$ norm directly, $\|T_i\|_1 = \|u_i\|_2 \|v_i\|_2$. But, by direct calculation, $\|u_i\|_2 = \|v_i\|_2 = O(1)$. Thus $\|T_i\|_1 = O(1)$, and the operator of interest is an absolutely convergent series in $\mathcal{F}_1$.

**Lemma 3.** Let $\varepsilon > 0$ be given. Let $\text{Bloch}_\varepsilon$ be the space of holomorphic functions $F$ on $U$ which satisfy $\|F\| = \sup_{y > 0} |y^{-\varepsilon} D^\alpha F(x + iy)| = c < \infty$. If $\alpha$ is a complex number, $\Re \alpha > 0$ then the map from $k$ to $H_k^{\alpha,2\alpha}$ is a continuous map from $\text{Bloch}_\varepsilon$ to the space of bounded linear maps on $L^2(0,\infty)$.

**Note.** For $\alpha$ real this is essentially the fact that a Hankel operator on a Bergman space is bounded if the symbol is in the Bloch space. (See Section 5 of [7].) For more on the space $\text{Bloch}_\varepsilon$ see [5] and [8].

**Proof.** Pick $f$, $g$ in $L^2(0,\infty)$. We estimate $|\langle H_k^{\alpha,2\alpha} f, g \rangle|$.

$$\langle H_k^{\alpha,2\alpha} f, g \rangle = \int_0^\infty \int_0^\infty s^\alpha t^{\alpha} (s + t)^{-2\alpha} k(s + t) f(t) g(s) ds dt$$

$$= \int_0^\infty \int_0^u (u - t)^\alpha t^{\alpha} u^{-2\alpha} k(u) f(u - t) g(t) dt du$$

$$= \int_0^\infty u^{-2\alpha} k(u) ((s^\alpha f) * (s^\alpha g))(u) du.$$ 

The Fourier transform is a unitary map from $A^{2,r}$ to $L^2((0,\infty), t^{-1-2r} dt)$. Hence, if we denote the inner product on $A^{2,r}$ by $\langle \cdot, \cdot \rangle_r$ and select $r = \Re \alpha - 1/2$ and $\Im \alpha = a$ we find

$$\langle H_k^{\alpha,2\alpha} f, g \rangle = \langle D^{-2i\alpha} k, ((s^\alpha f) * (s^\alpha g))^r \rangle,$$

$$= \langle D^{-2i\alpha} k, (s^\alpha f) (s^\alpha g)^r \rangle_r.$$ 

The product $(s^\alpha f)^r (s^\alpha g)^r$ is in $A^{1,r}$ with norm.
Thus it suffices to show that $\ell (F) = \langle D^{-2\alpha} k, F \rangle$, is a bounded linear functional on $A^{1\alpha}$. This is insured by the hypotheses on $k$. (See, for example, Theorem 2' of [5] and the discussion following that theorem. The operator $D^{-2\alpha}$ can be "moved," $\ell (F) = \langle k, D^{2\alpha} F \rangle$. The lemma is proved.

We now complete the proof of the theorem. If $p = 1$ then the required result is Lemma 2 with $\alpha = 0$. We now suppose $p > 1$. We will use complex interpolation of operators. Let $w = x + iy, 0 \leq x \leq 1$ be the interpolation parameter. Let $\alpha$ and $\beta$ be fixed $-1/2 < \alpha < 0 < \beta$. Let $h(w) = \alpha + w(\beta - \alpha)$ and $j(w) = 2 + 2\alpha + w(2\beta - (2 + 2\alpha))$. For given $k$ let $R(w)(k) = H^p_{\kappa(w),j(w)}$. Thus $R$ is an analytic map from functions to operators on $L^2(0,\infty)$. If $w = iy$, then Re $h(w) = \alpha$, Re $j(w) = 2 + 2\alpha$ and we conclude, by Lemma 2, that the mapping from $k$ in $A^{1\alpha}$ to $R(w)k$ is a continuous map into $\mathcal{F}_1$. For $w = 1 + iy$, Re $h(w) = \beta$, Re $j(w) = 2\beta$ and we conclude, now by Lemma 3, that the map from $k$ to $R(w)k$ is a continuous map from Bloch to bounded operators on $L^2(0,\infty)$.

The interpolation spaces between $A^{1\alpha}$ and Bloch are $A^{p\alpha}$ $1 < p < \infty$. The appropriate interpolation theorem for analytic maps into $\mathcal{F}_p$ spaces is in Section III, 13 of [9]. (In that context, the bounded operators are the end point space, $\mathcal{F}_\infty$.) To obtain the result of interest, we evaluate at $w_0 = -\alpha/(\beta - \alpha)$. The corresponding $p$ is $p = 1 - \alpha/\beta$ (note that we can obtain any $p$); $h(w_0) = 0, j(w_0) = 2/p$. We conclude $H^{1,2/p}_{\kappa,j}$ is in $\mathcal{F}_p$ if $k$ is in $A^{p\alpha}$. However $H^{1,2/p}_{\kappa,j} = H_{D^{-2\alpha}}$. Thus $H_{D^{-2\alpha}}$ is in $\mathcal{F}_p$ if $k$ is in $A^{p\alpha}$. This is the required conclusion.

A final note. We have been informal about the growth restrictions necessary to apply the interpolation theorem. What is at issue are the estimates on $R(w)$ for $|\text{Im } w|$ very large. The only place that $\text{Im } w$ affects the estimates in Lemma 2 or Lemma 3 is through operator norm estimates of the operator $D^{iy}$ (and similar operators) acting on $A^{1\alpha}$. The required estimates for these operators follow, for example, from the analogous estimates on the Hardy spaces $H^p 0 < p < 1$ and the inclusion theorems relating Hardy and Bergman spaces (e.g., Proposition 4.4 of [5]). The proof is complete.

IV. Variations

A. Commutators. Roughly, commutators on the line are sums of Hankel operators and hence Theorem 1 applies to such commutators. Let $b$ be a function defined on $\mathbb{R}$, let $P$ be the projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{R})$ and let $Q = I - P$. Let $M$ be the multiplication operator $Mf = bf$ and let $M_+$ and $M_-$ be the analogously defined operators using $b_+ = P(b)$ and $b_- = Q(b)$. The commutator we consider is $[M,P]$. (For more about this operator and related operators see [7], [10], [17].) The relationship between this operator and Hankel operators is shown by the following (formal) calculation. For $f$ in $L^2, f = Pf + Qf$. 

\[\|(s^a f)(s^b g)\|_{1,r} \leq \|(s^a f)^\#\|_{1/2} \|(s^b g)^\#\|_{1/2} = \|f\|_{1/2} \|g\|_{1/2}.\]
\[ = QMPf - PMQf = QM_Pf - PM_Qf. \]

The operator \( P_{M+Q} \) is essentially the Hankel operator with symbol \( k(t) = \hat{b}_+(t) \). More precisely, if we let \( U \) be the map of \( L^2(0, \infty) \) into \( L^2(-\infty, \infty) \) given by
\[ (Uf)(t) = \begin{cases} f(-t), & t \leq 0 \\ 0, & t > 0 \end{cases} \]
then \( [P_{M+Q}(Uf)^\ast] = H_kf \). This is verified by direct calculation. Because \( U \) is unitary onto its range, the trace ideal properties of \( H_k \) and \( P_{M+Q} \) are the same.

Similarly \( Q_{M-P} \) is essentially the Hankel operator with symbol \( k(t) = \hat{b}_-(t) \).

**Proposition 4.** The commutator \([M, P]\) is in \( \mathcal{F}_p \) if and only if both \( b_+(z) \) \( b_-(\bar{z}) \) are in the space described by the equivalent conditions (b) and (c) of Theorem 1. Equivalently, letting \( b(z) \) be the harmonic extension of \( b \) to \( U \), \([M, P]\) is in \( \mathcal{F}_p \) if and only if for some \( n \geq 1/p \)
\[ \left( \int \int_U \left| \frac{\partial^p}{\partial y^n} b(x + iy) \right|^p y^{np-2} dy dx \right)^{1/p} \leq c < \infty. \]

**Proof.** The first statement follows from Theorem 1 and the calculations preceding the proposition. The second statement follows from the first and the fact that the operator \( P \) is bounded on the space of functions \( b \) which satisfy (4.1).

**B. Weighted commutators.** We can also obtain results on the commutator of \( M \) with weighted projections. We will assume some familiarity with the theory of weights as described in [2] and [14]. We will ignore the distinction between bounded operators and densely defined operators which satisfy an a priori estimate.

We begin by recalling the \( A_2 \) condition of Muckenhoupt. (We give the condition in a form convenient for our purposes; other, simpler, forms are given in [14].) A non-negative function \( V \) on \( \mathbb{R} \) is said to satisfy the condition \( A_2 \) if
\[ \sup_{(x_0, y_0) \in U} \left( \int_R V(x) \frac{y_0}{(x-x_0)^2 + y_0^2} dx \right) \left( \int_R V^{-1}(x) \frac{y_0}{(x-x_0)^2 + y_0^2} dx \right) = c < \infty. \]
If \( w \) is a positive function and \( w^{-2} \) satisfies \( A_2 \), then the operators \( P_w \) and \( P_{w^{-1}} \) given by \( P_w(f) = w P(w^{-1} f) \) and \( P_{w^{-1}}(f) = w^{-1} P(wf) \) are bounded on \( L^2 \). We are interested in conditions which insure that the commutators \([M, P_w]\) be in \( \mathcal{F}_p \). First we obtain an endpoint result similar to the \( \alpha = 0 \) case of Lemma 2.

**Lemma 5.** In order for \([M, P_w]\) to be in \( \mathcal{F}_1 \), it is sufficient that both \( b''_+(z) \) and \( b''_-(\bar{z}) \) be in \( A^{1,0} \).

**Proof.** We consider the operator \([M_+, P_w]\). The other is similar.
By the decomposition theorem we can write $b_+$ in the form shown in (c) of Theorem 1 with parameter $\theta = 1/2$. It suffices to obtain uniform $\mathcal{F}_1$ estimates for each term in the resulting operator sum. Thus we must estimate the $\mathcal{F}_1$ norm of the operator $T_i$ which sends $f$ to

$$(T_i f)(x) = \int \left( \frac{y_i}{x - \xi_i} - \frac{y_i}{y - \xi_i} \right) \frac{1}{w(x)} \frac{w(y)}{x - y} f(y) dy.$$ 

This can be rewritten as

$$(T_i f)(x) = c \int \frac{1}{(x - \xi_i)(y - \xi_i)} \frac{w(x)}{w(y)} f(y) dy.$$ 

Thus $T_i f = c \langle f, U_i \rangle V_i$ with

$$U_i = \frac{1}{w(x)} \frac{y_i^{1/2}}{x - \xi_i} \quad \text{and} \quad V_i = w(x) \frac{y_i^{1/2}}{x - \xi_i}.$$ 

Thus $\|T_i\|_1 \leq \|U_i\|_2 \|V_i\|_2 = \int U_i(x)^2 dx \int V_i(x)^2 dx$. We now substitute the appropriate expressions for $U_i$ and $V_i$ and use (4.2) at the point $(x_0, y_0) = (x_i, y_i)$. We find $\|T_i\|_1 \leq c$. The lemma is proved.

We now interpolate between this result and Proposition 4. Let $p$ be given $1 < p < \infty$.

**Proposition 6.** The following pair of conditions is sufficient to insure that $[M, P, w]$ is in $\mathcal{F}_p$.

(a) $b_+(z)$ and $b_-(z)$ satisfy condition (b) of Theorem 1

(b) The function $w^{2p}$ satisfies the condition $A_2$.

**Note.** We have no information to suggest (b) is actually necessary. ($w^2$ must satisfy $A_2$ to start the analysis.)

**Proof.** It suffices to consider the case $b = b_+$. If $w^{2p}$ satisfies (4.2) then by the general theory of weights $w^{2p(1+\varepsilon)}$ satisfies (4.2) for some positive $\varepsilon$. Let $\tilde{p} = p + (p - 1)/\varepsilon$. We now interpolate between the lemma for the weight $w^{2p(1+\varepsilon)}$ and Proposition 4 for $\tilde{p}$ with no weight. That is, for complex $\zeta$ define $h(\zeta) = (-p/(p - 1)) \left( p - 1 + \varepsilon \right) \zeta + p(1 + \varepsilon)$. If $v$ is real then $\text{Re} h(iv) = p(1 + \varepsilon)$ and $\text{Re} h(1 - 1/\tilde{p} + iv) = 0$. Also $h(1 - 1/p) = 1$. For $\zeta$ with $0 \leq \text{Re} \zeta \leq 1 - 1/\tilde{p}$ we define $R(\zeta)$ to be the map from functions which are analytic in $U$ to linear operators on $L^2(R)$ given by $R(\zeta)(b)(f) = w^{b(\zeta)} [M, P] w^{-b(0)} f$. Because $w^{2p(1+\varepsilon)}$ satisfies (4.2) we may apply the previous lemma and conclude that $R(0)(b)$ is in $\mathcal{F}_1$ if $b''$ is in $A^{1.0}$. $R(iv)(b)$ is $R(0)(b)$ conjugated by a unitary operator and hence is also in $\mathcal{F}_1$. $R(1 - 1/\tilde{p})(b)(f) = [M, P](f)$. By Proposition 4, this is in $\mathcal{F}_{\tilde{p}}$ if $b'$ is in $A^{p, \theta/2-1}$. Again, the same estimate holds for
R(1 - 1/p + iv). By complex interpolation, we conclude that $R(1 - 1/p)$ maps those $b$ with $b'$ in $A^{p',p/2-1}$ to operators $W[M,P]W^{-1}$ in $\mathcal{S}_p$. This last statement is the desired conclusion. The proof is complete.

C. Other projections, other commutators. One of the main reasons for studying commutators is that they are, in various senses, building blocks from which other more complicated operators are constructed. Because of this, we can use Proposition 6 as a starting point and obtain trace class criteria for other operators. We now give several examples.

We will say of two operators that one is an $\mathcal{S}_p$ perturbation of the other if the difference belongs to $\mathcal{S}_p$. Such conditions arise, for example, in the theory of bases of Hilbert space (e.g., Section VI, 3 of [9]).

Proposition 7. Let $w$ be a non-negative function on $\mathbb{R}$. Let $p$ be given $1 \leq p < \infty$. Let $M$ be multiplication by $\log w$. The following are equivalent.

(a) $Pw$ is an $\mathcal{S}_p$ perturbation of $P$
(b) $[M,P]$ is in $\mathcal{S}_p$
(c) $(\log w)_+(z)$ and $(\log w)_-(z)$ both satisfy the conditions (b) of Theorem 1.

Proof. The equivalence of (b) and (c) is Proposition 4. The fact that (a) implies (b) is proven the same way as the corresponding proof for bounded operators (page 621 of [7]). We must show (c) implies (a). First note that if $w$ satisfies (c) then $\log w$ has vanishing mean oscillation and hence $w'$ satisfies (4.2) for all real $t$. In particular the operators $P(t) = Pw$, are bounded for $0 \leq t \leq 1$.

$$Pw - P = \int_0^1 \left( \frac{d}{dt} P(t) \right) dt.$$ 

Thus it suffices to show that $(d/dt) P(t)$ is in $\mathcal{S}_p$. $(d/dt) P(t) = [M,P(t)]$. Because $w$ satisfies (4.2) for all $t$, Proposition 6 can be applied and we are done.

The estimates for the higher commutators now follow as in [7]. Let $K_1 = [M,P], K_{n+1} = [M,K_n]$ for $n = 1, 2, \ldots$

Corollary 8. Let $w, M, p$ be as in Proposition 7. If $K_1 = [M,P]$ is in $\mathcal{S}_p$ then for each $n, n = 1, 2, \ldots$ $K_n$ is in $\mathcal{S}_p$.

The operator $Pw$ is essentially the projection $P$ being interpreted on the weighted space $L^2(w^2 dx)$. There is a closely related operator, $Q_w$, the projection in $L^2(w^2 dx)$ interpreted as an operator on $L^2$. That is, let $Q_w$ be the operator which is the self-adjoint projection of $L^2(\mathbb{R},w^2 dx)$ onto the closure in $L^2(\mathbb{R},w^2 dx)$ of the analytic functions. We now regard $Q_w$ as an operator defined on $L^2(\mathbb{R},dx)$. $Q_w$ is a non-self-adjoint projection onto analytic functions. Such operators are considered systematically in [6]. Using the results of [6], we obtain the following corollary of the previous proposition.

Corollary 9. Suppose $w$ is a non-negative function on $\mathbb{R}$. Let $M$ be multiplication by $\log w$. Let $p$ be given, $1 \leq p < \infty$. If $[M,P]$ is in $\mathcal{S}_p$ then for all $t$ sufficiently small $Q_w$ is an $\mathcal{S}_p$ perturbation of $P$. 
Our final example of an operator which can be analyzed by these methods is a commutator of the type studied by A. P. Calderón. Let \( b \) be a function on \( \mathbb{R} \) and \( B \) its primitive. Let \( M_B \) be multiplication by \( B \) and let \( D \) be differentiation. The operator of interest is \( C_B \) defined by

\[
C_B(f) = [M_B, P]Df.
\]

This operator is known to be bounded if \( b \) has bounded mean oscillation. One proof that \( C_B \) is bounded is based on the fact that, if we define \( H(iy) \) to be the commutator of multiplication by \( D^\alpha b \) with \( P \), then \( C_B \) can be represented as

\[
C_B(f) = c \int_{-\infty}^{\infty} H(iy)(D^\alpha f) \frac{dy}{1 + y^2}
\]

(see Proposition 11 of [3]). Thus any information about the \( H(iy) \) will propagate to \( C_B \).

In particular all the conditions we have considered for \( b \) are preserved by differentiation of imaginary order. Hence.

**Corollary 10.** Let \( p \) be given \( 1 \leq p < \infty \). Let \( B \) be given and suppose \( (B')_+(z) \) and \( (B')_-(\xi) \) satisfy the conditions of (b) of Theorem 1; then \( C_B \) is in \( \mathcal{F}_p \).

**V. Comments**

There are many natural variations on the issues we have considered. We mention three that seem especially intriguing.

**A. \( p < 1 \).** Theorem 1 may be true as stated for \( 0 < p < 1 \). The relationship between parts (b), (c) and (d) of the theorem is valid (using the decomposition theorem). It is still true that these conditions imply (a). The reason for this is that if \( 0 < p < 1 \) then the estimate on individual terms can be combined in a way similar to the case \( p = 1 \). Thus a proof of the model of Lemma 2 works essentially without change. The new difficulty is that (2.7) is not valid for \( p < 1 \) and thus we do not know how to draw conclusions from (a).

Intuitively, part of the content of the theorem for \( p = 1 \) is that the sets \( \{\xi\} \) and \( \{f^n\} \) (described in the proof) nearly diagonalize Hankel operators. A precise quantitative form of this would be quite interesting and would probably suffice to complete Theorem 1 for \( p < 1 \). For example, if we define \( P_f, f = \langle f, f_i \rangle f_i \) and

\[
Q_f = \langle f, f_i^* \rangle f_i^* \text{ then it may be that as operators } \left| \sum P_fHQ_f \right| < C|H| \text{ for some positive } C. \text{ (If this is true then } H \text{ in } \mathcal{F}_p \text{ implies } \sum P_fHQ_f \in \mathcal{F}_p \text{ for any } p.)
\]

**B. Higher dimensions.** Both Hankel operators and the commutators we considered have various generalizations.

We could replace \( (0,\infty) \) with a more general cone. Let \( \Gamma \) be an open convex cone in \( \mathbb{R}^n \) and let \( dV \) be volume measure on \( \mathbb{R}^n \). Let \( k \) be defined on \( \Gamma \). We define the generalized Hankel operator with symbol \( k \) as a map from \( L^2(\Gamma,dV) \) to itself given by
\[(H_kf)(x) = \int_I k(x + y)f(y)dy.\]

If the cone has additional structure then the techniques we have been using can be brought to bear. It would be interesting to know about these operators. Presumably, if \(\Gamma\) is homogeneous and self dual (and hence the results of [5] are available) then a result of the same general form as Theorem 1 is valid.

The situation with commutators for functions of several variables is less clear. For example, we do not know what techniques (involving generalized Hankel operators or other ideas) would be appropriate for the following natural question in the spirit of Proposition 4.

**Question.** Let \(b\) be a function on \(\mathbb{R}^2\). Let \(M_b\) be multiplication by \(b\) and let \(R\) be one of the Riesz transforms on \(L^2(\mathbb{R}^2)\). What are the conditions on \(b\) which are necessary and sufficient for \([M_b,R]\) to be in \(\mathcal{S}_1\)?

**C. Other operators.** In the last section we saw that commutators in \(\mathcal{S}_p\) could be used as starting points from which to show other operators are in \(\mathcal{S}_p\). There are powerful general techniques for generating a very large class of naturally occurring operators from commutators ([3], [4]). It would be interesting to know what additional conclusions could be drawn in general if the commutators so used are assumed to be in \(\mathcal{S}_p\).

(Added notes: (1) The given proof of the implication "(a) implies (b)" in Proposition 7 requires the further hypothesis that \(P_{a+ia}\) be an \(\mathcal{S}_p\) perturbation of \(P\) for all real \(\alpha\). (2) Further results on these issues can be found in [18], [19], and [20].)

**References**


This work was partially supported by National Science Foundation Grant MCS-8002689.

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Received March 2, 1981