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Letter to the Editor

Density of wavelet frames ☆

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Abstract

Density conditions for wavelet systems with arbitrary sampling points to be frames are studied. We show that for a wavelet system generated by admissible functions with irregular affine lattices to be a frame, the sampling points must have a positive lower affine Beurling density. The same is true for wavelet systems with arbitrary sampling points and nice generating functions.

Keywords: Wavelet frames; Affine Beurling densities; Affine lattices

1. Introduction and main results

Density is a useful concept in the study of frames. For example, it was shown in [9,13] that $\{e^{i2\pi\lambda_n\omega}\colon\ n\in\mathbb{Z}\}$ forms a frame for $L^2([-1/2,1/2])$ provided $\{\lambda_n\colon\ n\in\mathbb{Z}\}$ is separable and possesses a lower Beurling density greater than 1. In [3], Christensen, Deng and Heil studied the density of Gabor frames and proved that for a Gabor system $\{e^{i2\pi b_n x}g(x-a_n)\colon\ n\in\mathbb{Z}\}$ to be a frame for $L^2(\mathbb{R})$, the time–frequency parameters (a_n,b_n) must have a finite upper Beurling density and possess a lower Beurling density no less than 1. See also [1,2,8] for more examples.

For the case of wavelet systems, one can consider similar problems. In [7,11,12,16–18], it was shown that for a wavelet system with arbitrary sampling points to be a frame for $L^2(\mathbb{R})$, the sampling points must be relatively uniformly discrete, or equivalently, they must have a finite upper affine Beurling density.

For the lower affine Beurling density, however, there is no general result. In [19], the authors studied density conditions for irregular multi-generated wavelet systems of the form

$$\left\{\tau(s_{\ell,j},s_{\ell,j}t_{\ell,k})\psi_\ell\colon\ j\in\mathbb{J}_\ell,\ k\in\mathbb{K}_\ell,\ 1\leqslant\ell\leqslant r\right\}$$

to be frames, where r is a fixed positive integer, $\psi_\ell\in L^2(\mathbb{R})$, $s_{\ell,j}>0$, $t_{\ell,k}\in\mathbb{R}$, $\mathbb{J}_\ell,\mathbb{K}_\ell\subset\mathbb{Z}$, and

$$\big(\tau(s,t)\psi\big)(x):=s^{-1/2}\psi\big((x-t)/s\big).$$

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For convenience, we call a sequence of sampling points of the form $\{(s_{\ell,j}, s_{\ell,j}t_{\ell,k})\colon\ j \in \mathbb{J}_\ell,\ k \in \mathbb{K}_\ell,\ 1 \leqslant \ell \leqslant r\}$ an irregular affine lattice. Even for this special case, it is not clear whether the sampling points have a positive lower affine Beurling density for the wavelet system to be a frame.

On the other hand, Example 2.1 in [19] shows that the (one-dimensional) lower Beurling density of the translation parameters $\{t_{\ell,k}\colon\ k \in \mathbb{K}_\ell,\ 1 \leqslant \ell \leqslant r\}$ may be zero even if the corresponding wavelet system forms a frame. Specifically, let

$$\hat{\psi}(\omega) = \begin{cases} |\omega|^{1/4}(1 - |2\pi\omega|), & |\omega| \leqslant 1/2\pi, \\ 0, & \text{otherwise} \end{cases}$$

and $\{t_k\colon\ k \in \mathbb{Z}\}$ be a rearrangement of $\{k \in \mathbb{Z}\colon\ k \leqslant 2 \text{ or } 2^{2l} \leqslant k \leqslant 2^{2l+1} \text{ for some } l \geqslant 1\}$ such that $t_k \leqslant t_{k+1}$, here $\hat{\psi}(\omega) = \int_{\mathbb{R}} \psi(x)e^{-i2\pi x\omega}\,\mathrm{d}x$ is the Fourier transform of ψ. Then $\{\tau(2^j, 2^j t_k)\psi\colon\ j,k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$. Obviously, $\sup_k(t_{k+1} - t_k) = +\infty$. This seems to suggest that the lower affine Beurling density of the sampling points might be zero. Fortunately, when the "affine density" is considered, it is also positive in this example. In fact, it is true for every wavelet frame generated by admissible functions with irregular affine lattices. Before stating our results, we introduce some notations.

The group action in $\mathcal{G} := \{(s,t)\colon\ s > 0,\ t \in \mathbb{R}\}$ is defined by

$$(a,b)(a',b') = (aa', b + ab').$$

For any $(x,y) \in \mathcal{G}$, its (a,b)-neighborhood is defined by $Q_{a,b}(x,y) = (x,y)V$ with $V = [a^{-1/2}, a^{1/2}] \times [-b/2, b/2]$, $a > 1, b > 0$. It is easy to check that

$$Q_{a,b}(x,y) = \left[a^{-1/2}x, a^{1/2}x\right] \times \left[y - \frac{bx}{2}, y + \frac{bx}{2}\right].$$

Let $\Gamma = \{(x_n, y_n)\colon\ n \in \Lambda\}$ be a sequence of elements of $\mathcal{G}$.

(i) Γ is called (p,q)-uniformly discrete if $\mu(Q_{p,q}(x_n,y_n) \cap Q_{p,q}(x_m,y_m)) = 0$, $n \neq m$, where $\mathrm{d}\mu = (1/s^2)\,\mathrm{d}s\,\mathrm{d}t$ is the left-invariant measure on $\mathcal{G}$.
(ii) Γ is called relatively uniformly discrete if it is a finite union of uniformly discrete sequences.

It is easy to see that for any $a > 1, b > 0$, $\{(a^j, a^j bk)\colon\ j,k \in \mathbb{Z}\}$ is (a,b)-uniformly discrete. We denote $\lfloor x \rfloor = \max\{n \in \mathbb{Z}\colon\ n \leqslant x\}$ and $\lceil x \rceil = \min\{n \in \mathbb{Z}\colon\ n \geqslant x\}$, and $\#E$ denotes the cardinality of a sequence or a set E.

$C_c^\infty(\mathbb{R})$ is the set of all functions which are compactly supported and infinite times differentiable.

We call a function $\psi \in L^2(\mathbb{R})$ admissible if

$$C_\psi := \int_{-\infty}^{+\infty} \frac{1}{|\xi|}\left|\hat{\psi}(\xi)\right|^2 \mathrm{d}\xi < +\infty.$$

Let ν be the weighted counting measure defined by $\nu(E) = \sum_{(s,t)\in E} s$, where $E \subset \mathcal{G}$ is a discrete set.

For any sequence $\Gamma \subset \mathcal{G}$, its lower and upper affine Beurling density are defined by

$$D^-(\Gamma) = \varliminf_{a\to+\infty} \varliminf_{b\to+\infty} \inf_{(x,y)\in\mathcal{G}} \frac{\nu(\Gamma \cap Q_{a,b}(x,y))}{\iint_{Q_{a,b}(x,y)} s\,\mathrm{d}\mu} = \varliminf_{a\to+\infty} \varliminf_{b\to+\infty} \inf_{(x,y)\in\mathcal{G}} \frac{\nu(\Gamma \cap Q_{a,b}(x,y))}{bx\ln a}$$

and

$$D^+(\Gamma) = \varlimsup_{a\to+\infty} \varlimsup_{b\to+\infty} \sup_{(x,y)\in\mathcal{G}} \frac{\nu(\Gamma \cap Q_{a,b}(x,y))}{\iint_{Q_{a,b}(x,y)} s\,\mathrm{d}\mu} = \varlimsup_{a\to+\infty} \varlimsup_{b\to+\infty} \sup_{(x,y)\in\mathcal{G}} \frac{\nu(\Gamma \cap Q_{a,b}(x,y))}{bx\ln a},$$

respectively. For the regular case, i.e., $\Gamma = \{(a^j, a^j bk)\colon\ j,k \in \mathbb{Z}\}$, one can check that $D^-(\Gamma) = D^+(\Gamma) = 1/(b\ln a)$. We refer to [18, Example 3.1] for details.

We note that a similar density concept has been established in [5,14,15] and both lead to the same density (up to a constant) when the regular case is considered. However, it is not clear whether they coincide in general since they use different areas to count the point number.

For wavelet frames with irregular affine lattices, we have the following.

Theorem 1.1. *Let $\psi_\ell \in L^2(\mathbb{R})$ be admissible, S_ℓ and T_ℓ be real sequences, and S_ℓ consist of positive numbers, $1 \leqslant \ell \leqslant r$. If $\bigcup_{\ell=1}^r \{\tau(s,st)\psi_\ell\colon\ s \in S_\ell,\ t \in T_\ell\}$ is a frame for $L^2(\mathbb{R})$, then the lower affine Beurling density of $\bigcup_{\ell=1}^r \{(s,st)\colon\ s \in S_\ell,\ t \in T_\ell\}$ is positive.*

For wavelet frames with arbitrary sampling points, it was shown in [18] that the sampling points must have a positive lower affine Beurling density whenever ψ_ℓ, ψ'_ℓ, $X\psi'_\ell$, and $X\psi''_\ell$ are admissible. In this paper, we show that the admissibility of ψ_ℓ and $X\psi'_\ell$ is sufficient.

Theorem 1.2. *Let $\Gamma_\ell \subset \mathcal{G}$ be sequences, $1 \leqslant \ell \leqslant r$, and $\{\tau(s,t)\psi_\ell\colon\ (s,t) \in \Gamma_\ell,\ 1 \leqslant \ell \leqslant r\}$ be a frame for $L^2(\mathbb{R})$. If $\psi_\ell(x)$ is local absolutely continuous and $\psi_\ell(x), x\psi'_\ell(x)$ are admissible, then*

$$D^-\left(\bigcup_{1\leqslant \ell\leqslant r} \Gamma_\ell\right) > 0.$$

Remark 1. It is known that there are critical densities for Gabor frames [3] and Fourier frames [9,13], respectively. One may ask if the same is true for wavelet frames. Specifically, is there a positive constant D_c such that for any wavelet frame of the form $\{\tau(s,t)\psi_\ell\colon\ (s,t) \in \Gamma_\ell,\ 1 \leqslant \ell \leqslant r\}$, we have $D^-(\bigcup_{1\leqslant \ell\leqslant r} \Gamma_\ell) > D_c$?

The answer is negative.

As pointed out in [4], if $\{\tau(a^j, a^j bk)\psi\colon\ j,k \in \mathbb{Z}\}$ forms a frame, then so does $\{\tau(a^j, a^j b^\# k)\psi^\#\colon\ j,k \in \mathbb{Z}\}$ with $\psi^\#(x) = (b/b^\#)^{1/2}\psi(bx/b^\#)$ for any $b^\# > 0$. Now $D^+(\{(a^j, a^j b^\# k)\colon\ j,k \in \mathbb{Z}\}) = 1/(b^\# \ln a)$ can be arbitrarily small since $b^\#$ is arbitrary.

Observe that $C_{\psi^\#} = C_\psi \cdot b^\#/b$. We have

$$C_{\psi^\#} D^+\big(\big\{\big(a^j, a^j b^\# k\big)\colon\ j,k \in \mathbb{Z}\big\}\big) = \frac{C_\psi}{b \ln a} = C_\psi D^+\big(\big\{\big(a^j, a^j bk\big)\colon\ j,k \in \mathbb{Z}\big\}\big).$$

This suggests that $C_\psi D^+(\Gamma)$ could have a positive lower bound. Unfortunately, it is not the case. In fact, we can make $C_\psi D^+(\Gamma)$ arbitrary small by substituting $\varepsilon\psi$ for ψ with an arbitrary small ε.

Moreover, even if the norm of ψ is taken into account, a positive lower bound does not exist, either. Specifically, we have the following.

Theorem 1.3. *Let $\mathcal{X}$ be the set of all pairs of (ψ, Γ), for which $\psi \in L^2(\mathbb{R})$, $\Gamma \subset \mathcal{G}$ is a sequence, and $\{\tau(s,t)\psi\colon\ (s,t) \in \Gamma\}$ is a frame for $L^2(\mathbb{R})$. Then we have*

$$\inf_{(\psi,\Gamma)\in\mathcal{X}} \frac{1}{\|\psi\|_2^2} C_\psi D^+(\Gamma) = 0.$$

Remark 2. For a regular wavelet frame $\{\tau(s,t)\psi\colon\ (s,t) \in \Gamma\}$ with $\Gamma = \{(a^j, a^j bk)\colon\ j,k \in \mathbb{Z}\}$, it was shown in [4, Theorem 3.3.1] that

$$2A \leqslant D(\Gamma) C_\psi \leqslant 2B,$$

where A and B are the lower and upper frame bounds, respectively. For the relationship between the density, frame bounds and the admissibility constant with more general sampling points, we refer to [10].

2. Proofs of the main results

For fixed $\psi \in L^2(\mathbb{R})$, the continuous wavelet transform of a function $f \in L^2(\mathbb{R})$ is defined by

$$(W_\psi f)(s,t) = \int_{-\infty}^{+\infty} f(x)|s|^{-1/2}\overline{\psi\left(\frac{x-t}{s}\right)}\,\mathrm{d}x.$$

If ψ is admissible, then

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\frac{1}{s^2}\left|(W_\psi f)(s,t)\right|^2\,\mathrm{d}s\,\mathrm{d}t<+\infty,\quad \forall f\in L^2(\mathbb{R}).$$

Lemma 2.1. Let $\{(s_n,t_n)\colon n\in\Lambda\}\subset\mathcal{G}$ be an (a,b)-uniformly discrete sequence. Suppose that $h(s,t)\geqslant 0$ is a measurable function defined on $\mathcal{G}$. Then for any $p>a^3$ and $q>b$,

$$\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)}\iint_{E_n}h(s,t)\,\mathrm{d}t\,\mathrm{d}s\leqslant\iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)}4\left(1+\frac{a^{3/2}}{s}\right)h(s,t)\,\mathrm{d}t\,\mathrm{d}s,$$

where $E_n=[s_na^{-1/2},s_na^{1/2}]\times[t_n-b/2,t_n+b/2]$.

Proof. Put $\Gamma=\{(s_n,t_n)\colon n\in\Lambda\}$. For any $j\in\mathbb{Z}$, let $\Gamma_j=\{(s,t)\in\Gamma\colon a^{j-1/2}\leqslant s<a^{j+1/2}\}$. We can write

$$\Gamma_j=\left\{(s_{j,k},t_{j,k})\colon k\in\Lambda_j\right\},$$

where $\Lambda_j\subset\mathbb{Z}$. Without loss of generality, we assume that $t_{j,k}\leqslant t_{j,k+1}$. Let

$$E_{j,k}=\left[s_{j,k}a^{-1/2},s_{j,k}a^{1/2}\right]\times\left[t_{j,k}-\frac{b}{2},t_{j,k}+\frac{b}{2}\right].$$

Since Γ_j is (a,b)-uniformly discrete and $s_{j,k}a^{-1/2}<a^j\leqslant s_{j,k}a^{1/2}$, we have

$$t_{j,k}-\frac{b}{2}s_{j,k}\geqslant t_{j,k'}+\frac{b}{2}s_{j,k'},\quad \forall k,k'\in\Lambda_j,\ k>k'.$$

Hence

$$t_{j,k}-t_{j,k'}\geqslant\frac{b}{2}(s_{j,k}+s_{j,k'})\geqslant a^{j-1/2}b.$$

Therefore, we can split Λ_j into at most $N_j:=\lceil a^{-j+1/2}\rceil$ subsets $\Lambda_{j,\ell}$, $1\leqslant\ell\leqslant N_j$, such that

$$t_{j,k}-t_{j,k'}\geqslant N_ja^{j-1/2}b\geqslant b,\quad k,k'\in\Lambda_{j,\ell},\ k\neq k'.$$

Note that $N_j\leqslant a^{-j+1/2}+1$. We have

$$\begin{aligned}\sum_{|j|\geqslant\frac{\ln p-\ln a}{2\ln a},\ k\in\Lambda_j}\iint_{E_{j,k}}h(s,t)\,\mathrm{d}t\,\mathrm{d}s&=\sum_{|j|\geqslant\frac{\ln p-\ln a}{2\ln a}}\sum_{\ell=1}^{N_j}\sum_{k\in\Lambda_{j,\ell}}\int_{s_{j,k}a^{-1/2}}^{s_{j,k}a^{1/2}}\int_{t_{j,k}-b/2}^{t_{j,k}+b/2}h(s,t)\,\mathrm{d}t\,\mathrm{d}s\\&\leqslant\sum_{|j|\geqslant\frac{\ln p-\ln a}{2\ln a}}\int_{a^{j-1}}^{a^{j+1}}\int_{-\infty}^{+\infty}N_jh(s,t)\,\mathrm{d}t\,\mathrm{d}s\\&\leqslant\sum_{|j|\geqslant\frac{\ln p-\ln a}{2\ln a}}\int_{a^{j-1}}^{a^{j+1}}\int_{-\infty}^{+\infty}\left(1+\frac{a^{3/2}}{s}\right)h(s,t)\,\mathrm{d}t\,\mathrm{d}s\\&\leqslant\iint_{\substack{s\notin[(p/a^3)^{-1/2},(p/a^3)^{1/2}]\\ t\in\mathbb{R}}}2\left(1+\frac{a^{3/2}}{s}\right)h(s,t)\,\mathrm{d}t\,\mathrm{d}s.\end{aligned}$$

Similarly we can prove that

$$\sum_{\substack{|t_{j,k}|\geqslant q/2\\ j\in\mathbb{Z},\ k\in\Lambda_j}}\iint_{E_{j,k}}h(s,t)\,\mathrm{d}t\,\mathrm{d}s\leqslant\iint_{\substack{s>0\\ |t|>(q-b)/2}}2\left(1+\frac{a^{3/2}}{s}\right)h(s,t)\,\mathrm{d}t\,\mathrm{d}s.$$

On the other hand, if $s_{j,k} \notin [p^{-1/2}, p^{1/2}]$, then either $a^{j+1/2} > p^{1/2}$ or $a^{j-1/2} < p^{-1/2}$. Hence $|j| \geqslant (\ln p - \ln a)/(2\ln a)$. It follows that

$$\begin{aligned}\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \iint_{E_n} h(s,t)\,\mathrm{d}t\,\mathrm{d}s &\leqslant \sum_{|j|\geqslant \frac{\ln p-\ln a}{2\ln a},\ k\in\Lambda_j} \iint_{E_{j,k}} h(s,t)\,\mathrm{d}t\,\mathrm{d}s + \sum_{|t_{j,k}|\geqslant q/2} \iint_{E_{j,k}} h(s,t)\,\mathrm{d}t\,\mathrm{d}s \\ &\leqslant \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} 4\left(1+\frac{a^{3/2}}{s}\right) h(s,t)\,\mathrm{d}t\,\mathrm{d}s. \qquad \square\end{aligned}$$

The following lemma is a consequence of Wirtinger's inequality [6].

Lemma 2.2. If $f(x)$ is absolutely continuous on $[a,b]$, $f, f' \in L^2[a,b]$ and there is some $c \in [a,b]$ such that $f(c)=0$, then $\int_a^b |f(x)|^2\,\mathrm{d}x \leqslant 4(b-a)^2/\pi^2 \cdot \int_a^b |f'(x)|^2\,\mathrm{d}x$.

Lemma 2.3. Let $\{(s_n,t_n)\colon n\in\Lambda\} \subset \mathcal{G}$ be an (a,b)-uniformly discrete sequence. Suppose that $f \in C_c^\infty(\mathbb{R})$, $\psi \in L^2(\mathbb{R})$ is locally absolutely continuous and $\psi(x)$ and $x\psi'(x)$ are admissible. Then we have

$$\begin{aligned}\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \left|(W_\psi f)(s_n,t_n)\right|^2 \leqslant C_{a,b} \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} \left(\frac{1}{s}+\frac{1}{s^2}\right)\Big(&\left|(W_\psi f)(s,t)\right|^2 + \left|(W_{\tilde\psi} f)(s,t)\right|^2 \\ &+ \left|(W_\psi f')(s,t)\right|^2 + \left|(W_{\tilde\psi} f')(s,t)\right|^2\Big)\,\mathrm{d}t\,\mathrm{d}s, \quad f\in L^2(\mathbb{R}),\end{aligned}$$

where $\tilde\psi(x) = \psi(x) + x\psi'(x)$ and $C_{a,b}$ is a constant.

Proof. Since $(W_\psi f)(s,t) = \langle f(\cdot+t), s^{-1/2}\psi(\cdot/s)\rangle$, it is easy to check that

$$\begin{aligned}&\frac{\partial}{\partial t}(W_\psi f)(s,t) = (W_\psi f')(s,t),\\ &\frac{\partial}{\partial s} s^{-1/2}(W_\psi f)(s,t) = -s^{-3/2}(W_{\tilde\psi} f)(s,t).\end{aligned}$$

Put $E_n = [s_n a^{-1/2}, s_n a^{1/2}] \times [t_n - b/2, t_n + b/2]$. We have

$$\begin{aligned}&\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \iint_{E_n} \frac{1}{s}\left|(W_\psi f)(s,t) - (W_\psi f)(s,t_n)\right|^2 \mathrm{d}t\,\mathrm{d}s \\ &\quad= \sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \int_{s_n a^{-1/2}}^{s_n a^{1/2}} \frac{1}{s} \int_{t_n-b/2}^{t_n+b/2} \left|(W_\psi f)(s,t) - (W_\psi f)(s,t_n)\right|^2 \mathrm{d}t\,\mathrm{d}s \\ &\quad\leqslant \sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \int_{s_n a^{-1/2}}^{s_n a^{1/2}} \frac{1}{s}\cdot\frac{4b^2}{\pi^2} \int_{t_n-b/2}^{t_n+b/2} \left|(W_\psi f')(s,t)\right|^2 \mathrm{d}t\,\mathrm{d}s \quad \text{(Lemma 2.2)} \\ &\quad= \sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \frac{4b^2}{\pi^2} \iint_{E_n} \frac{1}{s}\left|(W_\psi f')(s,t)\right|^2 \mathrm{d}t\,\mathrm{d}s \\ &\quad\leqslant \frac{4b^2}{\pi^2} \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} 4\left(\frac{1}{s}+\frac{a^{3/2}}{s^2}\right)\left|(W_\psi f')(s,t)\right|^2 \mathrm{d}t\,\mathrm{d}s, \qquad (2.1)\end{aligned}$$

where Lemma 2.1 is used in the last step. Using this lemma again, we get

$$\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \iint_{E_n} \frac{1}{s}\left|(W_\psi f)(s,t)\right|^2 \mathrm{d}t\,\mathrm{d}s \leqslant \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} 4\left(\frac{1}{s}+\frac{a^{3/2}}{s^2}\right)\left|(W_\psi f)(s,t)\right|^2 \mathrm{d}t\,\mathrm{d}s. \tag{2.2}$$

By the triangle inequality, we have

$$\begin{aligned}
&\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \iint_{E_n} \frac{1}{s}\left|(W_\psi f)(s,t_n)\right|^2 \mathrm{d}t\,\mathrm{d}s \\
&\quad\leqslant M_{a,b} \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} \left(\frac{1}{s}+\frac{1}{s^2}\right)\left(\left|(W_\psi f)(s,t)\right|^2+\left|(W_\psi f')(s,t)\right|^2\right)\mathrm{d}t\,\mathrm{d}s,
\end{aligned} \tag{2.3}$$

where $M_{a,b}$ is a constants. Similarly we can prove that

$$\begin{aligned}
&\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \iint_{E_n} \left|s^{-1/2}(W_\psi f)(s,t_n)-s_n^{-1/2}(W_\psi f)(s_n,t_n)\right|^2 \mathrm{d}t\,\mathrm{d}s \\
&\quad\leqslant \sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \frac{4s_n^2(a-1)^2}{\pi^2 a} \iint_{E_n} \frac{1}{s^3}\left|(W_{\tilde\psi} f)(s,t_n)\right|^2 \mathrm{d}t\,\mathrm{d}s \\
&\quad\leqslant \sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \frac{4(a-1)^2}{\pi^2} \iint_{E_n} \frac{1}{s}\left|(W_{\tilde\psi} f)(s,t_n)\right|^2 \mathrm{d}t\,\mathrm{d}s \\
&\quad\leqslant M'_{a,b} \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} \left(\frac{1}{s}+\frac{1}{s^2}\right)\left(\left|(W_{\tilde\psi} f)(s,t)\right|^2+\left|(W_{\tilde\psi} f')(s,t)\right|^2\right)\mathrm{d}t\,\mathrm{d}s,
\end{aligned} \tag{2.4}$$

where (2.3) is used in the last step. Putting (2.3) and (2.4) together, we get

$$\begin{aligned}
&\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \frac{(a-1)b}{a^{1/2}}\left|(W_\psi f)(s_n,t_n)\right|^2 = \iint_{E_n} \left|s_n^{-1/2}(W_\psi f)(s_n,t_n)\right|^2 \mathrm{d}t\,\mathrm{d}s \\
&\quad= \sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \iint_{E_n} \left|s^{-1/2}(W_\psi f)(s,t_n)-\left(s^{-1/2}(W_\psi f)(s,t_n)-s_n^{-1/2}(W_\psi f)(s_n,t_n)\right)\right|^2 \mathrm{d}t\,\mathrm{d}s \\
&\quad\leqslant M''_{a,b} \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} \left(\frac{1}{s}+\frac{1}{s^2}\right)\left(\left|(W_{\tilde\psi} f)(s,t)\right|^2+\left|(W_{\tilde\psi} f')(s,t)\right|^2\right. \\
&\qquad\left.+\left|(W_\psi f)(s,t)\right|^2+\left|(W_\psi f')(s,t)\right|^2\right)\mathrm{d}t\,\mathrm{d}s. \qquad \square
\end{aligned}$$

Proof of Theorem 1.2. Put $\Gamma=\bigcup_{\ell=1}^r \Gamma_\ell$. Let A be the lower frame bound.

By [18, Theorem 3.2], Γ_ℓ is relatively uniformly discrete. Hence we can split Γ_ℓ into N_ℓ uniformly discrete sequences $\Gamma_{\ell,k}$. Therefore, we can find some $a_0>1, b_0>0$ such that $\Gamma_{\ell,k}$ is (a_0,b_0)-uniformly discrete, $1\leqslant \ell\leqslant r$, $1\leqslant k\leqslant N_\ell$.

Take some $f\in L^2(\mathbb{R})\cap C_c^\infty(\mathbb{R})$ such that $f\neq 0$ and f, f' are admissible. Since $(W_\psi f)(s,t)=\langle s^{1/2}f(s\cdot+t),\psi\rangle$, we have $(W_\psi f)(s,t)=\overline{(W_f\psi)}(1/s,-t/s)$. It follows that

$$\begin{aligned}
\int_0^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{s}\left|(W_\psi f)(s,t)\right|^2 \mathrm{d}t\,\mathrm{d}s &= \int_0^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{s}\left|(W_f\psi)\left(\frac{1}{s},-\frac{t}{s}\right)\right|^2 \mathrm{d}t\,\mathrm{d}s = \int_0^{+\infty}\int_{-\infty}^{+\infty} \left|(W_f\psi)\left(\frac{1}{s},t\right)\right|^2 \mathrm{d}t\,\mathrm{d}s \\
&= \int_0^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{s^2}\left|(W_f\psi)(s,t)\right|^2 \mathrm{d}t\,\mathrm{d}s<\infty.
\end{aligned}$$

Similarly, we can prove that $1/s\cdot|(W_\psi f')(s,t)|^2$, $1/s\cdot|(W_{\tilde\psi} f)(s,t)|^2$, and $1/s\cdot|(W_{\tilde\psi} f')(s,t)|^2$ are integrable on $\mathcal{G}$. By Lemma 2.3, there are some $p>1$ and $q>0$ such that for any (a_0,b_0)-uniformly discrete sequence $\Gamma'\subset\mathcal{G}$,

$$\sum_{(s,t)\in\Gamma'\setminus Q_{p,q}(1,0)}\left|(W_\psi f)(s,t)\right|^2\leqslant\frac{A}{2r(N_1+\cdots+N_r)}\|f\|_2^2.$$

On the other hand, it is easy to check that $\{(s/x,(t-y)/x)\colon\ (s,t)\in\Gamma_{\ell,k}\}$ is (a_0,b_0)-uniformly discrete for any $(x,y)\in\mathcal{G}$ since $\Gamma_{\ell,k}$ is, $1\leqslant\ell\leqslant r$, $1\leqslant k\leqslant N_\ell$. Furthermore, $(s/x,(t-y)/x)\notin Q_{p,q}(1,0)$ whenever $(s,t)\notin Q_{p,q}(x,y)$. Hence

$$\begin{aligned}
A\|f\|_2^2=A\left\|x^{-\frac12}f\left(\frac{\cdot-y}{x}\right)\right\|^2&\leqslant\sum_{\ell=1}^r\sum_{(s,t)\in\Gamma_\ell}\left|\left\langle x^{-1/2}f\left(\frac{\cdot-y}{x}\right),s^{-1/2}\psi_\ell\left(\frac{\cdot-t}{s}\right)\right\rangle\right|^2\\
&=\sum_{\ell=1}^r\sum_{(s,t)\in\Gamma_\ell}\left|\left\langle f,\left(\frac{s}{x}\right)^{-1/2}\psi_\ell\left(\frac{\cdot-\frac{t-y}{x}}{\frac{s}{x}}\right)\right\rangle\right|^2\\
&=\sum_{\ell=1}^r\sum_{k=1}^{N_\ell}\sum_{(s,t)\in\Gamma_{\ell,k}\setminus Q_{p,q}(x,y)}\left|(W_{\psi_\ell}f)\left(\frac{s}{x},\frac{t-y}{x}\right)\right|^2\\
&\quad+\sum_{\ell=1}^r\sum_{k=1}^{N_\ell}\sum_{(s,t)\in\Gamma_{\ell,k}\cap Q_{p,q}(x,y)}\left|(W_{\psi_\ell}f)\left(\frac{s}{x},\frac{t-y}{x}\right)\right|^2\\
&\leqslant\frac{A}{2}\|f\|_2^2+\sum_{\ell=1}^r\sum_{k=1}^{N_\ell}\sum_{(s,t)\in\Gamma_{\ell,k}\cap Q_{p,q}(x,y)}\left|(W_{\psi_\ell}f)\left(\frac{s}{x},\frac{t-y}{x}\right)\right|^2.
\end{aligned}$$

Therefore,

$$\Gamma\cap Q_{p,q}(x,y)=\bigcup_{\ell=1}^r\bigcup_{k=1}^{N_\ell}\big(\Gamma_{\ell,k}\cap Q_{p,q}(x,y)\big)\neq\emptyset,\quad\forall(x,y)\in\mathcal{G}.$$

For any $a>p$, $b>a^{-1/2}p^{1/2}q$ and $(x,y)\in\mathcal{G}$, we have

$$Q_{p,q}\left(a^{-1/2}xp^{j+1/2},y-\frac{bx}{2}+\left(k+\frac12\right)qa^{-1/2}xp^{j+1/2}\right)\subset Q_{a,b}(x,y),$$

$$0\leqslant j\leqslant\left\lfloor\frac{\ln a}{\ln p}\right\rfloor-1,\ 0\leqslant k\leqslant\frac{b}{qa^{-1/2}p^{j+1/2}}-1.$$

Hence

$$\nu\big(\Gamma\cap Q_{a,b}(x,y)\big)\geqslant\sum_{j=0}^{\lfloor\ln a/\ln p\rfloor-1}\left(\frac{b}{qa^{-1/2}p^{j+1/2}}-1\right)\cdot a^{-1/2}xp^j\geqslant\frac{bx}{p^{1/2}q}\left(\frac{\ln a}{\ln p}-1\right)-a^{-1/2}x\cdot\frac{a-1}{p-1}.$$

Therefore,

$$D^-(\Gamma)=\varliminf_{a\to+\infty}\varliminf_{b\to+\infty}\inf_{(x,y)\in\mathcal{G}}\frac{\nu(\Gamma\cap Q_{a,b}(x,y))}{bx\ln a}\geqslant\frac{1}{p^{1/2}q\ln p}.$$

This completes the proof. □

The following lemma can be proved similarly to Lemma 2.1.

Lemma 2.4. Let $\{(s_j,t_{j,k})\colon\ j\in\Lambda,\ k\in\Lambda_j\}$ and $\{(s_j,0)\colon\ j\in\Lambda\}$ be (a,b)-uniformly discrete sequence. Suppose that $h(s,t)\geqslant0$ is a measurable function defined on $\mathcal{G}$. Then for any $p>a$ and $q>b$,

$$\sum_{(s_j,t_{j,k})\notin Q_{p,q}(1,0)}\iint_{F_{j,k}}h(s,t)\,\mathrm{d}t\,\mathrm{d}s\leqslant\iint_{(s,t)\notin Q_{p/a,p^{-1/2}(q-b)}(1,0)}2\left(1+\frac{a^{1/2}}{s}\right)h(s,t)\,\mathrm{d}t\,\mathrm{d}s,$$

where $F_{j,k} = [s_j a^{-1/2}, s_j a^{1/2}] \times [t_{j,k}/s_j - b/(2s_j), t_{j,k}/s_j + b/(2s_j)]$.

Lemma 2.5. Let $\{(s_j, t_{j,k})\colon\ j \in \Lambda, k \in \Lambda_j\}$ and $\{(s_j, 0)\colon\ j \in \Lambda\}$ be (a,b)-uniformly discrete sequence. Suppose that $\psi \in L^2(\mathbb{R})$, $f \in C_c^\infty(\mathbb{R})$ and f, f', $\tilde{f}$, and $\tilde{f}'$ are admissible, where $\tilde{f}(x) = f(x)/2 + xf'(x)$. Then we have

$$\sum_{(s_j,t_{j,k})\notin Q_{p,q}(1,0)} \left|(W_\psi f)(s_j, t_{j,k})\right|^2 \leqslant C'_{a,b} \iint\limits_{(s,t)\notin Q_{p/a,p^{-1/2}(q-b)}(1,0)} \left(1+\frac{1}{s}\right)\Big(\left|(W_\psi f)(s,st)\right|^2 + \left|(W_\psi f')(s,st)\right|^2 + \left|(W_\psi \tilde{f})(s,st)\right|^2 + \left|(W_\psi \tilde{f}')(s,st)\right|^2\Big)\,\mathrm{d}t\,\mathrm{d}s,$$

where $C'_{a,b}$ is a constant.

Proof. Since $(W_\psi f)(s,st) = \langle s^{1/2} f(s(\cdot + t)), \psi\rangle$, we have

$$\frac{\partial}{\partial t}(W_\psi f)(s,st) = s(W_\psi f')(s,st), \qquad \frac{\partial}{\partial s}(W_\psi f)(s,st) = \frac{1}{s}(W_\psi \tilde{f})(s,st).$$

Similar to the proof of Lemma 2.3, we can prove that

$$\sum_{(s_j,t_{j,k})\notin Q_{p,q}(1,0)} \iint\limits_{F_{j,k}} \left|(W_\psi f)(s,st) - (W_\psi f)(s, st_{j,k}/s_j)\right|^2 \mathrm{d}t\,\mathrm{d}s \leqslant \frac{4ab^2}{\pi^2} \iint\limits_{(s,t)\notin Q_{p/a,p^{-1/2}(q-b)}(1,0)} 2\left(1+\frac{a^{1/2}}{s}\right)\left|(W_\psi f')(s,st)\right|^2 \mathrm{d}t\,\mathrm{d}s$$

and

$$\sum_{(s_j,t_{j,k})\notin Q_{p,q}(1,0)} \iint\limits_{F_{j,k}} \left|(W_\psi f)(s, st_{j,k}/s_j) - (W_\psi f)(s_j, t_{j,k})\right|^2 \mathrm{d}t\,\mathrm{d}s \leqslant M'''_{a,b} \iint\limits_{(s,t)\notin Q_{p/a,p^{-1/2}(q-b)}(1,0)} 2\left(1+\frac{a^{1/2}}{s}\right)\Big(\left|(W_\psi \tilde{f})(s,st)\right|^2 + \left|(W_\psi \tilde{f}')(s,st)\right|^2\Big)\,\mathrm{d}t\,\mathrm{d}s.$$

Now the conclusion follows by the triangle inequality and Lemma 2.4. □

Proof of Theorem 1.1. By [19, Theorem 2.1], $S_\ell \times \{0\}$ is relatively uniformly discrete, $1 \leqslant \ell \leqslant r$. Using Lemma 2.5 instead of Lemma 2.3, the conclusion can be proved similarly to Theorem 1.2, which we leave to interested readers. □

Proof of Theorem 1.3. Denote $D_c = \inf_{(\psi,\Gamma)\in\mathcal{X}} C_\psi/\|\psi\|_2^2 \cdot D^+(\Gamma)$.

Fix some $a > 1$ and $0 < \delta, \varepsilon < 1$. Let $b = 1/(2a)$, $\Gamma_0 = \{(a^j, a^j bk)\colon\ j,k \in \mathbb{Z}\}$, and $\hat{\psi} = \chi_{[-a,-a^{1-\varepsilon})} + \delta\chi_{[-a^{1-\varepsilon},-1)} + \delta\chi_{[1,a^{1-\varepsilon})} + \chi_{[a^{1-\varepsilon},a)}$. For any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
\sum_{j,k\in\mathbb{Z}} \left|\left\langle f, \tau\left(a^j, a^j bk\right)\psi\right\rangle\right|^2 &= \sum_{j,k\in\mathbb{Z}} \left|\int\limits_{\mathbb{R}} \hat{f}(\omega) a^{-j/2} \overline{\hat{\psi}\left(a^{-j}\omega\right)} e^{i2\pi a^{-j}bk\omega}\,\mathrm{d}\omega\right|^2 \\
&= \sum_{j,k\in\mathbb{Z}} \left|\int\limits_{-a}^{a} \hat{f}\left(a^j\omega\right) a^{j/2} \overline{\hat{\psi}(\omega)} e^{i2\pi bk\omega}\,\mathrm{d}\omega\right|^2 = \sum_{j\in\mathbb{Z}} 2a \int\limits_{-a}^{a} a^j \left|\hat{f}\left(a^j\omega\right)\hat{\psi}(\omega)\right|^2 \mathrm{d}\omega \\
&= \sum_{j\in\mathbb{Z}} 2a \int\limits_{-a^{j+1}}^{a^{j+1}} \left|\hat{f}(\omega)\hat{\psi}\left(a^{-j}\omega\right)\right|^2 \mathrm{d}\omega = \sum_{j\in\mathbb{Z}} 2a \int\limits_{\mathbb{R}} \left|\hat{f}(\omega)\hat{\psi}\left(a^{-j}\omega\right)\right|^2 \mathrm{d}\omega.
\end{aligned}$$

Since $\delta^2 \leqslant \sum_{j\in\mathbb{Z}} |\hat{\psi}(a^{-j}\omega)|^2 \leqslant 1$, a.e., the above equalities show that $\{\tau(a^j, a^j bk)\psi\colon\ j,k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$. It is easy to check that

$$\|\psi\|_2^2 = 2\delta^2\big(a^{1-\varepsilon} - 1\big) + 2a\big(1 - a^{-\varepsilon}\big) \quad \text{and} \quad C_\psi = 2\big(\delta^2(1-\varepsilon) + \varepsilon\big)\ln a.$$

Hence

$$\frac{C_\psi}{\|\psi\|_2^2} D^+(\Gamma_0) = \frac{2\big(\delta^2(1-\varepsilon)+\varepsilon\big)\ln a}{2\delta^2(a^{1-\varepsilon}-1) + 2a(1-a^{-\varepsilon})} \cdot \frac{1}{b\ln a}.$$

By letting $\delta \to 0$ and $\varepsilon \to 0$ consecutively, we get

$$D_c \leqslant \frac{1}{ab\ln a} = \frac{2}{\ln a}.$$

Since $a > 1$ is arbitrary, we get $D_c \leqslant \lim_{a\to\infty} 2/\ln a = 0$. This completes the proof. □

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