Evolution of semiclassical Wigner function (the higher dimensional case)

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Introduction

• The limit Wigner measure of a WKB function satisfies a simple transport equation in phase space and is well suited for capturing oscillations at scale of order $O(\epsilon)$, but it fails, for instance, to provide the correct amplitude on caustics where different scales appear.

• We define the semi-classical Wigner function of an $N$–dimensional WKB function, as a suitable formal approximation of its scaled Wigner function.
The semi-classical Wigner function is an oscillatory integral that provides an $\epsilon$–dependent regularization of the limit Wigner measure, it obeys a transport–dispersive evolution law in phase–space, and it is well defined at least on simple caustics.
Geometrical optics: WKB method-1

- Schrödinger equation and oscillatory initial data

\[ i\epsilon \psi_t^\epsilon(x, t) = -\frac{\epsilon^2}{2} \Delta \psi^\epsilon(x, t) + V(x)\psi^\epsilon(x, t), \quad x \in \mathbb{R}^N, \ t > 0 \]

with WKB initial data

\[ \psi^\epsilon(x, 0) = \psi_0^\epsilon(x) = A_0(x) \exp\left(\frac{iS_0(x)}{\epsilon}\right) \]

- WKB method looks for solutions

\[ \psi^\epsilon(x, t) = A(x, t) \exp\left(\frac{iS(x, t)}{\epsilon}\right) \]
where $S$ and $A$ solve the eikonal

$$S_t + \frac{1}{2} |\nabla_x S|^2 + V(x) = 0, \quad S(x, 0) = S_0(x),$$

and the transport equation

$$(A^2)_t + \nabla_x \cdot (A^2 \nabla_x S) = 0, \quad A(x, 0) = A_0(x)$$
Geometrical optics: Rays

For integrating the eikonal and the transport equations we consider the rays

\[ \ddot{x}(q, t) = \dot{k}, \quad \dot{k}(q, t) = S(x(\dot{x}(q, t)), t) \]

solving the Hamiltonian system

\[ \frac{d}{dt} \dot{x} = \dot{k}, \quad \frac{d}{dt} k = -\nabla V(x), \]

with initial data

\[ \dot{x}(0) = q, \quad \dot{k}(0) = \nabla S_0(q) \]

Along the rays the eikonal is written as an ODE

\[ \frac{dS}{dt} = S_t + |\nabla_x S|^2 = \frac{|\dot{k}|^2}{2} - V, \quad S(x, 0) = S_0(q) \]
**Remark:** Since the eikonal is a nonlinear equation, it has, in general, a smooth solution only up to some finite time $t_c$, when the rays cross each other and singularities are developed that are called *caustics*.

At a caustic point the Jacobian $J(q, t)$ of the ray map $q \mapsto \bar{x}(q, t)$ is zero, and the amplitude becomes infinite since

$$A(x, t) = A_0(q)J^{-1/2}(q, t)$$
The Wigner transform

Consider the Wigner transform $W^\epsilon(x, k) := W^\epsilon[\psi^\epsilon(x)]$
(Wigner-1932, Gerard et. al-1997)

$$W^\epsilon(x, k) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i k \cdot y} \psi^\epsilon(x + \frac{\epsilon y}{2}) \overline{\psi^\epsilon(x - \frac{\epsilon y}{2})} dy, \ (x, k) \in \mathbb{R}^N \times \mathbb{R}^N$$

which has the basic properties

$$\int_{\mathbb{R}^N} W^\epsilon(x, k) dk = |\psi^\epsilon(x)|^2$$

and for $\psi^\epsilon(x) = A(x) \exp(iS(x)/\epsilon)$ (Lions & Paul-1993)

$$W^\epsilon(x, k) \to A^2(x) \delta(k - \nabla S(x)) \ , \ \text{weakly as } \epsilon \to 0$$
The Wigner transform \( f^\epsilon(x, k, t) = W^\epsilon[\psi^\epsilon(x, t)] \) of the solution \( \psi^\epsilon \) of the Schrödinger equation satisfies the Wigner equation

\[
f^\epsilon_t(x, k, t) + k \cdot \nabla_x f^\epsilon(x, k, t) + \mathcal{Z}_\epsilon f^\epsilon(x, k, t) = 0,
\]

where the operator \( \mathcal{Z}_\epsilon \) is defined by the convolution with respect to the momentum \( k \),

\[
\mathcal{Z}_\epsilon f(x, k, t) = f(x, k, t) \ast \left( \frac{i}{(2\pi)^N \epsilon} \int_{-\infty}^{\infty} \exp(-i k \cdot y) \left( V(x + \frac{\epsilon}{2} y) - V(x - \frac{\epsilon}{2} y) \right) \right) dy.
\]
Assuming that the potential $V$ is smooth enough, we can expand $V(x \pm \frac{\epsilon}{2} y)$ into Taylor series, so that we can rewrite the Wigner equation as a transport dispersive equation

$$f_t^\epsilon + \mathbf{k} \cdot \nabla_x f^\epsilon - \nabla V \cdot \nabla_k f^\epsilon = \sum_{m \geq 1} c_m \epsilon^{2m} \sum_{|\alpha| = 2m+1} D^\alpha V D_k^\alpha f^\epsilon,$$

which sheds some light to the combined (transport-dispersive) mechanism of the evolution of $f^\epsilon$ near the Lagransian manifold $\Lambda_t = \{ k = \nabla S(x, t) \}.$
The limit Wigner equation

In the limit $\epsilon \to 0$, the Wigner equation reduces formally to the Liouville equation for $f^0$ (dispersion disappears!!)

$$f_t^0(x, k, t) + k \cdot \nabla_x f^0(x, k, t) - \nabla V(x) \cdot \nabla_k f^0(x, k, t) = 0$$

with initial data

$$f_0^0(x, k, t = 0) = A_0^2(x)\delta(k - \nabla S_0(x))$$

For $0 < t < T$, $T < t_c = \text{time up to which the solution of the eikonal is single-valued},$

$$f^0(x, k, t) = A^2(x, t)\delta(k - \nabla_x S(x, t))$$
We rewrite the Wigner integral

\[ W^\epsilon(x, k) = \frac{1}{(\epsilon\pi)^N} \int_{\mathbb{R}^N} e^{-\frac{2i}{\epsilon}k \cdot \sigma} \psi^\epsilon(x + \sigma) \overline{\psi^\epsilon(x - \sigma)} d\sigma \]

in the form

\[ W^\epsilon(x, k) = \frac{1}{(\epsilon\pi)^N} \int_{\mathbb{R}^N} D(x, \sigma) e^{i \frac{F(x, k; \sigma)}{\epsilon}} d\sigma \]

where

\[ D(x; \sigma) := A(x + \sigma)A(x - \sigma) \]

\[ F(x, k; \sigma) := S(x + \sigma) - S(x - \sigma) - 2k \cdot \sigma \] (Wigner phase)
The stationary points of the Wigner phase are found from

\[ \nabla_\sigma F = \nabla_x S(x + \sigma) + \nabla_x S(x - \sigma) - 2k. \]

1. The stationary points of \( F \) always come in pairs \( \pm \sigma_0(x, k) \), and as \( k \to \nabla S(x) \), there always exist stationary points which coalesce to \( \sigma_0(x, k) \to 0 \).

2. In particular, when \( k = \nabla S(x) \), then \( \sigma_0 = 0 \) is always a degenerate critical point.
Remark 1: The structure of the critical set of $F$ can be quite complicated (even in the case $N = 1$), but we expect that, modulo highly oscillatory terms that tend weakly to zero as $\epsilon \to 0$, the main contribution to the asymptotics of $W^\epsilon$ comes from points close to the Lagrangian manifold, that is, from $k \approx \nabla S(x)$.

This is in agreement with the fact that as $\epsilon \to 0$ then $W^\epsilon \to W^0 = A^2(x)\delta(k - \nabla S(x))$, is concentrated on the Lagrangian manifold.
Critical points of the Wigner phase-3

**Remark 2:** Let us consider the projection identity

\[ |\psi^\epsilon(x)|^2 = \int_{\mathbb{R}^N} W^\epsilon(x, k) dk = \frac{1}{(\epsilon\pi)^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} D(x, \sigma) e^{i \frac{F(x, k; \sigma)}{\epsilon}} d\sigma d\mathbf{k} \]

The main contribution in the calculation of the last integral (over \(\mathbb{R}^N_\sigma \times \mathbb{R}^N_k\)) comes from the points \((\sigma, k)\) at which \(\nabla_{\sigma,k} F = 0\). These are easily seen to be \(\sigma = 0, k = \nabla S(x)\). At each \(x\) these critical points are always non-degenerate.

Thus in the calculation of \(|\psi^\epsilon(x)|\), the main contribution also comes from the points near the Lagrangian manifold.
Once we restrict attention to points $\mathbf{k} \approx \nabla S(\mathbf{x})$, it is natural to approximate $D$ and $F$ by their Taylor expansion about $\sigma = 0$

$$D(\mathbf{x}; \sigma) = A^2(\mathbf{x}) + O(|\sigma|^2),$$

$$F(\mathbf{x}, \mathbf{k}; \sigma) = 2(\nabla S(\mathbf{x}) - \mathbf{k}) \cdot \sigma + \frac{1}{3} \sum_{i,j,k=1}^{N} \frac{\partial^3 S(\mathbf{x})}{\partial x_i \partial x_j \partial x_k} \sigma_i \sigma_j \sigma_k + O(|\sigma|^5).$$
The semiclassical approximation-2

Keeping only the linear term in the right hand side of $F$, we get the (limit) Wigner measure:

$$
\frac{A^2(x)}{(\epsilon \pi)^N} \int_{\mathbb{R}^N} e^{-\frac{i}{\epsilon} 2(k - \nabla S(x)) \cdot \sigma} d\sigma = A^2(x) \delta(k - \nabla S(x)) =: W^0(x, k),
$$

while keeping the cubic terms we are led to define the semi-classical Wigner function $\tilde{W}^\epsilon$,

$$
\tilde{W}^\epsilon(x, k) = \left( \frac{2}{\epsilon^3} \right)^N A^2(x) P_N \left( -\frac{2}{\epsilon^3} (k - \nabla S(x)), \frac{\partial^3 S(x)}{\partial x_i \partial x_j \partial x_k} \right).
$$
The semiclassical approximation-3

where $P_N$ is the oscillatory integral

$$
P_N(z, C_k) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{i[z^T\sigma + \frac{1}{3} \sigma^T C_k \sigma \sigma_k]} d\sigma
$$

with

$$
C_k = \{c_{ijk}\} = S_{x_i, x_j, x_k} \quad i, j, k = 1, 2 \ldots N
$$

Note that

$$
P_1(z, c) = |c|^{-1/3} \text{Ai}(zc^{-1/3}) \quad \text{and} \quad P_N(z, 0) = \delta(z) \quad \text{for} \quad C_k = 0
$$

Remark: The integral $P_N$ is defined, in general, as a distribution, but it is a smooth function if

$$
\sum_{i=1}^{N} (\sigma^T C_i \sigma)^2 \neq 0, \quad \forall \sigma \in \mathbb{R}^N \setminus \{0\}
$$
Degenerate points
Failure of the last condition is equivalent to the fact that there exists a unit vector $\nu \in \mathbb{R}^N$ such that $\nu^T C_i \nu = 0$ for all $i = 1, \ldots, N$ (degenerate points).

Since $C_i$ is the second derivative matrix of $\frac{\partial S(x)}{\partial x_i}$, this is equivalent to the fact that, for all $i$'s, the surfaces $k_i = \frac{\partial S(x)}{\partial x_i}$ have zero normal curvature in the direction of $\nu \in \mathbb{R}_x^N$. 
The semiclassical approximation-5

It is important to observe that $\tilde{W}^\epsilon$ obeys the basic properties

$$\tilde{W}^\epsilon(x, k) \to A^2(x)\delta(k - \nabla S(x)), \quad \text{as} \quad \epsilon \to 0,$$

and

$$\int_{\mathbb{R}^N} \tilde{W}^\epsilon(x, k) \, dk = A^2(x).$$

Moreover, near non–degenerate points the semi-classical Wigner $\tilde{W}^\epsilon$ is a smooth function that approximates the scaled Wigner $W^\epsilon$, as $\epsilon$ tends to zero.

At degenerate points either the semi-classical Wigner $\tilde{W}^\epsilon$ fails to approximate $W^\epsilon$, or else $W^\epsilon$ itself is a distribution.
Consider the Hamiltonian system

\[ \frac{d}{dt} \hat{x}(q, p, t) = \hat{k}(q, p, t), \quad \frac{d}{dt} \hat{k}(q, p, t) = -\nabla_x V(\hat{x}(q, p, t)) \]

with initial data

\[ \hat{x}(q, p, t = 0) = q \quad \hat{k}(q, p, t = 0) = p \]

The "distance" from the manifold

\[ a(t) = a(q, p, t) = \hat{k}(q, p, t) - \nabla_x S(\hat{x}(q, p, t), t) \]

evolves from its initial value \( a(0) = p - \nabla S_0(q) \), according to the ODE

\[ \frac{d}{dt} a(t) = -B(t)a(t), \quad B(t) = (b_{ij}(t) = S_{x_ix_j}(\hat{x}(q, p, t), t)) \]
If \( \Phi(t) = \{\phi_{ij}(t)\}_{i,j=1,...,N} \) \( \Phi(0) = I_N \), is the fundamental solution, then

\[
a(t) = \Phi(t)a(0).
\]

Similarly, the matrix \( C_k(t) = (c_{ijk}(t) = S_{xixjxk}(\hat{x}(q,p,t),t)) \) with initial data \( C_k(0) = (c_{ijk}(0) = S_{0,qiqjqk}(q)) \) evolves according to the ODE

\[
\frac{d}{dt} C_k(t) = -B(t)C_k(t) - C_k(t)B(t) - b_{kl}(t)C_l(t) - V_k(t) + O(t|a(0)|)
\]
and it is given by

$$C_k(t) = \Phi(t)(C_l(0) + U_l(t) + O(t|a(0)|))\Phi^T(t)\phi_{kl}(t)$$

where $V_k(t) = \left(V_{x_i x_j x_k}(x(q, p, t, t), t)\right)$, and

$$\phi_{kl} \frac{d}{dt} U_l = -\Phi^{-1} V_k \Phi^{-T}, \quad U_k(0) = 0, \quad k, l = 1, 2, \ldots, N$$
Key observation: The semiclassical Wigner

\[ \tilde{W}^\epsilon(x, k, t) = \left(2\epsilon^{-\frac{2}{3}}\right)^N A^2(x, t) P_N \left(-2\epsilon^{-\frac{2}{3}}\hat{a}(t), \hat{C}_k(t)\right) \]

where \( \hat{C}_k(t) := C_k(\hat{q}, \hat{p}, t) = C_k(\hat{q}(x, k, t), \hat{p}(x, k, t), t) \) obeys the evolution "law"

\[ \tilde{W}^\epsilon(x, k, t) = \tilde{W}_0^\epsilon(\hat{q}, p) *_p P_N \left(-2\epsilon^{-\frac{2}{3}}p, \hat{U}_k(t) + O(t|\hat{a}(0)|)\right) \bigg|_{p=\hat{p}} \]

\[ \times (1 + O(t^2|\hat{a}(0)|)) \left(2\epsilon^{-\frac{2}{3}}\right)^N \]
"Evolution law": at the exception of error terms $O(t^2|\hat{a}(0)|)$, the semiclassical Wigner $\tilde{W}^\epsilon(x, k, t)$ is equal to the convolution of $\tilde{W}_0^\epsilon(\hat{q}, p)$ with a suitable phase integral, the convolution being evaluated at the point $p = \hat{p}(x, k, t)$.
We set

\[ \tilde{G}^\epsilon(p, \hat{U}_k(t)) := \left(2\epsilon^{-\frac{2}{3}}\right)^N P_N(-2\epsilon^{-\frac{2}{3}}p, \hat{U}_k(t)) \]

and we then define \( \tilde{f}^\epsilon(x, k, t) \) as

\[ \tilde{f}^\epsilon(x, k, t) := \tilde{G}^\epsilon(p, \hat{U}_k(t)) \ast_p \tilde{W}_0^\epsilon(\hat{q}, p) \bigg|_{p=\hat{p}} \]

\[ = \tilde{G}^\epsilon(p, \hat{U}_k(t)) \ast_p \tilde{f}_0^\epsilon(\hat{q}, p) \bigg|_{p=\hat{p}}. \]
Remarks on the structure of $\tilde{f}^\epsilon$

1. $\tilde{f}^\epsilon(x, k, t)$ is "near" to $\tilde{W}^\epsilon(x, k, t)$, for $|\hat{a}(0)|$ small (i.e., when phase space dynamics remains locally near the Lagrangian manifold)

2. The evolved semiclassical Wigner is in fact a $P_N$–phase integral, given by

$$\tilde{f}^\epsilon(x, k, t) = \left(2\epsilon^{-\frac{2}{3}}\right)^N A_0^2(\hat{q}) P_N \left(-2\epsilon^{-\frac{2}{3}} \hat{a}(0), \hat{C}_k(0) + \hat{U}_k(t)\right),$$

and it follows easily that

$$\tilde{f}^\epsilon(x, k, t) \to f^0(x, k, t) = A^2(x, t) \delta(k - \nabla_x S(x, t)), \quad \text{as} \quad \epsilon \to 0.$$
3. \( \tilde{f}^\epsilon(x, k, t) \) is a smooth function iff the point \((x, \nabla S(x, t))\) on \( \Lambda_t \) is a non-degenerate point. In particular, for any \( 0 < t < t_c \), the approximation \( \tilde{f}^\epsilon \) provides a \( P_N \)-regularization of \( f^0 \) away from degenerate points.

4. Combined evolution mechanism: the semiclassical Wigner \( \tilde{f}^\epsilon \) is not only transported by the Hamiltonian flow but also dispersed in the \( k \)-direction, through its convolution with the kernel \( \tilde{G}^\epsilon \). Such a behavior is in complete agreement with the transport–dispersive character of the Wigner equation.
Remarks on the structure of $\tilde{f}^\epsilon$

5. In the $\epsilon = 0$ limit

$$\tilde{W}_0^\epsilon(\hat{q}, \hat{p}) \rightarrow f_0^0(\hat{q}, \hat{p})$$

and

$$\tilde{G}^\epsilon(p, \hat{U}_k(t)) \rightarrow \delta(p), \ \epsilon \rightarrow 0$$

Consequently, as $\epsilon \rightarrow 0$, we recover Liouvillian dynamics

$$\tilde{f}^\epsilon(x, k, t) = \tilde{G}^\epsilon(p, \hat{U}_k(t)) \ast_p \tilde{W}_0^\epsilon(\hat{q}, p) \Big|_{p=\hat{p}}$$

$$\rightarrow \delta(p) \ast_p f_0^0(\hat{q}, p) \Big|_{p=\hat{p}} = W_0^0(\hat{q}, \hat{p})$$

where

$$W_0^0(\hat{q}, \hat{p}) = A_0(\hat{q})\delta(\hat{p} - \nabla S_0(\hat{q}))$$
In order to arrive at the definition of $\tilde{f}^\epsilon$ we required the evolved semiclassical Wigner to be in agreement with $\tilde{W}^\epsilon$. In particular all the underlying analysis was restricted in the time interval $(0, T)$, $T < t_c$.

Once however we define $\tilde{f}^\epsilon$, it follows that $\tilde{f}^\epsilon$ is well defined even at a caustic point $(x_c, t_c)$, if this point is a non–degenerate point.

At such a point, the integration $\int_{\mathbb{R}_k^N} \tilde{f}^\epsilon(x_c, k, t_c) dk$ is now meaningful.
Example with caustic: elliptic umbilic-1

The initial WKB function

\[ A_0(q) \equiv 1 \quad S_0(q_1, q_2) = \frac{1}{3} q_1^3 - q_1 q_2^2 - a(q_1^2 + q_2^2), \quad a > 0 \]

Assuming zero potential, the bicharacteristics are given by

\[ q = \hat{q}(x, k, t) = x - kt, \quad p = \hat{p}(x, k, t) = k \]

and the caustic in \((x_1, x_2, t)\)–space is given by

\[
\frac{x_1 - q_1}{t} - \frac{\partial}{\partial q_1} S_0(q_1, q_2) = 0, \quad \frac{x_2 - q_2}{t} - \frac{\partial}{\partial q_2} S_0(q_1, q_2) = 0,
\]

\[
(1 + t \frac{\partial^2 S_0}{\partial^2 q_1})(1 + t \frac{\partial^2 S_0}{\partial^2 q_2}) - t^2 \left( \frac{\partial^2 S_0}{\partial q_1 \partial q_2} \right)^2 = 0
\]
Geometry of the caustic: The $t = \text{const.}$ sections are hypocycloids with cusps $A$, $B$, $C$
For

\[ u_1 := \frac{x_1}{t}, \quad u_2 := \frac{x_2}{t}, \quad v := \frac{1}{2t} - a, \]

the equations of the caustic are

\[ u_1 = q_1^2 - q_2^2 + 2q_1v \]
\[ u_2 = -2q_1q_2 + 2q_2v \]
\[ v^2 = q_1^2 + q_2^2. \]

For \( v > 0 \), we set \( q_1 = v \cos \theta, \ q_2 = v \sin \theta \), and

\[ u_1 = v^2(\cos 2\theta + 2 \cos \theta) \]
\[ u_2 = -v^2(\sin 2\theta - 2 \sin \theta) \]
For fixed $v$ (that is, fixed $t$) this describes a hypocycloid with three cusps. A similar analysis shows that the picture remains the same for $v < 0$.

All three points move to infinity as $t$ tends to either 0 or infinity. Also, at $t = 1/2a$, the hypocycloid reduces to a point (the origin) which is called the focus of the caustic.
The initial Wigner function $f_0^c$

The initial scaled Wigner function is

$$f_0^c(q, p) = \frac{1}{(\epsilon \pi)^2} \int_{\mathbb{R}^2} e^{\frac{i}{\epsilon} F(q, p; \sigma)} d\sigma$$

where

$$F(q, \frac{x - q}{t}; \sigma) = -2\hat{z} \cdot \sigma + \frac{1}{3} (2\sigma_1^3 - 6\sigma_1\sigma_2^2),$$

with

$$\hat{z} = (\hat{z}_1, \hat{z}_2)$$
$$\hat{z}_1(u, v; q) := u_1 - 2vq_1 + q_2^2 - q_1^2,$$
$$\hat{z}_2(u, v; q) := u_2 - 2vq_2 + 2q_1q_2,$$
The integral $P_2$

Let

$$P_2(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i[z \cdot \sigma + \frac{1}{3}(2\sigma_1^3 - 6\sigma_1\sigma_2^2)]} d\sigma$$

and define the new phase integral

$$\tilde{P}_2(z, w) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i[z \cdot \sigma + w|\sigma|^2 + \frac{1}{3}(2\sigma_1^3 - 6\sigma_1\sigma_2^2)]} d\sigma$$

so that

$$\tilde{P}_2(z, 0) = P_2(z).$$

Then, the following projection identity holds

$$\int_{\mathbb{R}^2} P_2(-\gamma \hat{z}(u, v; q)) dq = \frac{2^{1/3} 4\pi^2}{\gamma} |\tilde{P}_2(-2^{2/3} \gamma u, 2^{1/6} \gamma^{1/2} v)|^2$$
The evolved Wigner function $\tilde{f}^\epsilon-1$

\[ f^\epsilon(x, k, t) = \frac{1}{(\epsilon \pi)^2} \int_{\mathbb{R}^2} e^{\frac{i}{\epsilon} (-2\hat{z}_1 \sigma_1 - 2\hat{z}_2 \sigma_2 + \frac{1}{3} (2\sigma_1^3 - 6\sigma_1 \sigma_2^2))} d\sigma \]

\[ = (2\epsilon^{-\frac{2}{3}})^2 P_2 (-2\epsilon^{-\frac{2}{3}} \hat{z}(u, v; q)) \]

and integrating wrt. $k$

\[ |\psi^\epsilon(x, t)|^2 = \frac{1}{(2\pi \epsilon t)^2} \left| \int_{\mathbb{R}^2} e^{\frac{i}{\epsilon} (\frac{1}{3} \xi_1^3 - \xi_1 \xi_2^2 + v(\xi_1^2 + \xi_2^2) - u_1 \xi_1 - u_2 \xi_2)} d\xi_1 d\xi_2 \right|^2 \]

where

\[ u = \frac{x}{t}, \quad v = \frac{1}{2t} - a. \]
The evolved Wigner function $\tilde{f}^\epsilon-2$

This formula is valid at any point including the points on the caustic.
For instance, at the focus of the caustic $(x_1, x_2, t) = (0, 0, \frac{1}{2a})$, that is $(u_1 = u_2 = v = 0)$, we get

$$|\psi^\epsilon(0, 0, \frac{1}{2a})| = O(\epsilon^{-1/3}) .$$
The projection identity (Berry & Wright-1980)-1

Let

\[ \psi_E(C_1, C_2, C_3) = \frac{1}{2\pi} \int e^{i\phi_E(S_1, S_2; C_1, C_2, C_3)} dS_1 dS_2 \]

with

\[ \phi_E(S_1, S_2; C_1, C_2, C_3) := -C_1 S_1 - C_2 S_2 - C_3 (S_1^2 + S_2^2) + S_1^3 - 3S_1 S_2^2 \]

Put

\[ \tilde{C}_1 := 2^{2/3} (C_1 + 2C_3 U_1 + 3(U_2^2 - U_1^2)) \]

\[ \tilde{C}_2 := 2^{2/3} (C_2 + 2C_3 U_2 + 6U_1 U_2) \]

Then

\[ |\psi_E(C_1, C_2, C_3)|^2 = \frac{2^{1/3}}{\pi} \int_{\mathbb{R}^2} \psi_E(\tilde{C}_1, \tilde{C}_2, 0) dU_1 dU_2 \]
The projection identity -2

The important fact here is that the zero section of $\psi_E$ is expressed in terms of the Airy functions $Ai$&$Bi$, and in this sense the wave function is expressed in terms of lower order singularities.
We have introduced the semiclassical Wigner function $\tilde{f}^\epsilon$ as a formal approximation of the scaled Wigner transform of the WKB solution to the problem.

- This approximation is valid near the manifold $k = \nabla_x S(x, t)$.
- $\tilde{f}^\epsilon$ is an object that is “richer” than the limit Wigner function $f^0$ which is a Dirac mass concentrated on $k = \nabla_x S(x, t)$.
- In particular, $\tilde{f}^\epsilon$ is an $\epsilon$–dependent oscillatory integral that tends to $f^0$ as $\epsilon$ tends to zero. Moreover, it obeys an evolution law which is in agreement with the transport–dispersive character of the Wigner equation.
B. If \((x_c, t_c)\) is a caustic point,

- the restriction of \(f^0\) at this point, that is \(f^0(x_c, k, t_c)\), is not a well defined distribution in \(\mathbb{R}^N_k\), and as a consequence the amplitude at \((x_c, t_c)\) cannot be computed via the projection identity.

- On the contrary, \(\tilde{f}^\epsilon(x_c, k, t_c)\) is a well defined function in \(\mathbb{R}^N_k\), and therefore the integral of \(\tilde{f}^\epsilon\) with respect to \(k\) is meaningful and is expected to approximate the \((\epsilon\text{-dependent})\) exact amplitude, at least on simple caustics. However, an approximation result in this direction is still missing.
More generally, the asymptotic nearness of $\tilde{f}^\epsilon$ and $f^\epsilon$ is an open question. The difficulty in settling these questions stems from the fact that one deals with complicated multidimensional oscillatory integrals.