Admissible strategies in semimartingale portfolio selection

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Abstract

The choice of admissible trading strategies in mathematical modelling of financial markets is a delicate issue, going back to Harrison and Kreps [HK79]. In the context of optimal portfolio selection with expected utility preferences this question has been a focus of considerable attention over the last twenty years.

We propose a novel notion of admissibility that has many pleasant features – admissibility is characterized purely under the objective measure P; the wealth of any admissible strategy is a supermartingale under all pricing measures; local boundedness of the price process is not required; neither strict monotonicity, strict concavity nor differentiability of the utility function are necessary; the definition encompasses both the classical mean-variance preferences and the monotone expected utility.

For utility functions finite on $\mathbb{R}$, our class represents a minimal set containing simple strategies which also contains the optimizer, under conditions that are milder than the celebrated *reasonable asymptotic elasticity* condition on the utility function.

Key words: utility maximization – non locally bounded semimartingale – incomplete market – σ-localization and $\mathcal{I}$-localization – σ-martingale measure – Orlicz space – convex duality

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1 Introduction

A central concept of financial theory is the notion of a self-financing strategy H, whose wealth is expressed mathematically by the stochastic integral

$$x + H \cdot S_t := x + \int_0^t H_s dS_s,$$

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where S is a semimartingale process representing discounted prices of d traded assets and x is the initial wealth. According to stochastic integration theory, there are some minimal requirements for the integral above to exist (see [Pr05]). The class of predictable processes H for which the integral exists is denoted by $L(S;P)$ or simply $L(S)$. It turns out, however, that for general S the class $L(S)$ is *not appropriate* for financial applications. Specifically, Harrison and Kreps [HK79] noted that when all trading strategies in $L(S)$ are allowed, arbitrage opportunities arise even in the standard Black-Scholes model. This is not a problem of the model – the reason is that the theory of stochastic integration operates with a set of integrands that is far too rich for such applications.

In the context of expected utility maximization a good set of trading strategies is a subset of $L(S)$ on which the utility maximization is well posed and which contains the optimizer. The search for a good definition of admissibility has proved to be a difficult task and it has evolved in two streams. For utility functions finite on a half-line, for example a logarithmic utility, there is a natural definition: admissible strategies are those with wealth bounded below by a constant, see [KS99, KS03]. For utility functions finite on the whole $\mathbb{R}$, the situation is more complicated. The definition via uniformly bounded below wealth works only to a certain extent. Although the utility maximization over these strategies is well defined, some restrictions have to be imposed on S. In fact, S should be locally bounded (or σ-bounded) if one wants to have non trivial strategies which are admissible both in the long and short position, i.e. both H and $-H$ admissible. Moreover, the optimal wealth process itself may not be bounded from below, as it happens even in classical models such as the Merton problem.

A natural choice in this situation is to consider all strategies whose wealth is a martingale under all (suitably defined) pricing measures. This works well for exponential utility, see [DGRSSS02, KabStr02]. The seminal work of Schachermayer [Sch03] shows that in general the martingale class is too narrow and the optimal strategy only exists among strategies whose wealth is a *supermartingale* under all pricing measures. Thus, for utility functions finite on $\mathbb{R}$ the *supermartingale class* is now considered the appropriate notion of admissibility as it contains the optimizer.

As transpires from the above discussion, admissibility is currently defined in a primal way for utility functions finite on $\mathbb{R}_+$ but for utilities finite on $\mathbb{R}$ the definition is dual, via pricing measures. A connection of sorts between the two approaches can be found in Bouchard et al. [BTZ04] who postulate that a strategy is admissible if the utility of its terminal wealth can be approximated in $L^1(P)$ by strategies whose wealth is bounded below. In this definition of admissibility not all strategies belong to the supermartingale class, but, crucially, the optimizer does. We note for completeness that the idea of $L^1(P)$ approximation of terminal utility was used first by Schachermayer [Sch01] and Owen [O02].

All of the papers cited above deal with *locally bounded price processes*. Biagini and Frittelli [BF07] show that Schachermayer's supermartingale class of strategies contains the optimizer also when S is not locally bounded. In a subsequent paper [BF08], they provide a unified treatment for utility functions finite on a half-line as well as those finite on the whole $\mathbb{R}$ in the unbounded case. The unified framework encompasses a host of features and is very flexible. The wealth of the admissible strategies here is controlled from below by an exogenously given random variable.

The optimal strategy, however, may not be in this class and it is not clear whether it can be approximated by strategies with controlled losses. Also, the solution may in principle depend on the choice of the loss control.

The key point of the present paper is that we do not ask for approximation of terminal utility *only*, but we also require an approximation by simple integrands at intermediate times, in the spirit of Kallsen, cf. [ČK07, Definition 2.2]. The numerous advantages, mathematical and economic, of our definition have been anticipated in the abstract and they are thoroughly discussed in the main body of the paper. Here we mention only that our definition implies all admissible strategies are in the supermartingale class, that the optimizer belongs to this class under very mild conditions and, as a byproduct, we obtain an extremely compact proof of the supermartingale property of the optimal solution.

The paper is organized as follows. In Sections 2.1-2.3 there are basic definitions from convex analysis, theory of Orlicz spaces and stochastic integration. Section 2.4 contains a new result on σ-localization. In Section 3.1 we discuss conditions imposed on the price process S and the corresponding definitions of simple strategies. In Section 3.2 we prove the martingale property of simple strategies. In Section 3.3 we define the admissible strategies and prove their supermartingale property. In Sections 4.1 and 4.2 we discuss the customary conditions of *reasonable asymptotic elasticity* and other related conditions used in the literature and we contrast them with a weaker Inada condition at $+\infty$ employed in this paper. The main result (Theorem 4.8) is stated in Section 4.3. Section 5 contains a detailed discussion on the advantages of our framework compared to the existing literature.

2 Mathematical preliminaries

2.1 Utility functions

A utility function U is a proper, concave, non-decreasing, upper semi-continuous function. Its effective domain is

$$\operatorname{dom} U := \{x \mid U(x) > -\infty\}. \tag{1}$$

We assume that U is strictly increasing in a neighborhood of 0. Without loss of generality, suppose also $U(0) = 0$. Let $U(+\infty) := \lim_{x\to+\infty} U(x)$ and define

$$\overline{x} := \inf\{x \mid U(x) = U(+\infty)\}. \tag{2}$$

For strictly increasing utility functions $\overline{x} = +\infty$, but for truncated utility functions (which feature for example in shortfall risk minimization) one has $\overline{x} < +\infty$ and then $\overline{x}$ acts as a satiation point (bliss point) of the utility function.

The convex conjugate of U,

$$V(y) := \sup_{x\in\mathbb{R}}\{U(x) - xy\},$$

is a proper, convex, lower semi-continuous function, equal to $+\infty$ on $(-\infty, 0)$, and verifying $V(0) = U(+\infty)$. In the sequel we will often exploit the following form of the Fenchel inequality, obtained as a simple consequence of the definition of V:

$$U(x) \leq xy + V(y). \tag{3}$$

2.2 Orlicz spaces and the Orlicz space induced by U

Let Ψ be a *Young function*, that is an even, convex, lower semi-continuous, $[0,\infty]$-valued function with $\Psi(0) = 0$. Consider the corresponding Orlicz space

$$L^{\Psi}(P) = \{X \in L^0(P) \mid E[\Psi(c|X|)] < \infty \text{ for some } c > 0\}.$$

Orlicz spaces are generalizations of L^p spaces, since when $\Psi(x) = |x|^p, p \geq 1$, then $L^{\Psi} = L^p$ and if $\Psi(x) = +\infty I_{\{|x|>1\}}$ then $L^{\Psi} = L^{\infty}$.

The Morse subspace of L^{Ψ}, also called "Orlicz heart", is given by

$$M^{\Psi}(P) = \{X \in L^0(P) \mid E[\Psi(c|X|)] < \infty \text{ for all } c > 0\}.$$

In the context of this paper the Young function will be, from Section 3 onwards,

$$\hat{U}(x) := -U(-|x|),$$

meaning that the Orlicz space in consideration is generated by the lower tail of the utility function. For utility functions with lower tail which is asymptotically a power, say p, one has $L^{\hat{U}} = M^{\hat{U}} = L^p$. When U is exponential, say $U(x) = 1 - e^{-x}$, $\hat{U}(x) = e^{|x|} - 1$, and it is easy to check that $L^{\hat{U}} \supsetneq M^{\hat{U}} \supseteq L^{\infty}$. For utility functions with half-line as their effective domain, such as $U(x) = \ln(1+x)$, one has $L^{\hat{U}} = L^{\infty}$ and $M^{\hat{U}} = \{0\}$.

The link between U and $\hat{U}$ gives

$$X \in L^{\hat{U}} \text{ iff } E[U(-\alpha|X|)] > -\infty \text{ for some } \alpha > 0. \tag{4}$$

The reader interested in the general theory of Orlicz spaces is referred to the book by Rao and Ren [RR91].

2.3 Semimartingale norms

There are two standard norms in stochastic calculus. Let S be an $\mathbb{R}^d$-valued semimartingale on the filtered space $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ and let $S_t^* = \sum_{i=1}^d \sup_{0\leq s\leq t} |S_s^i|$ be the corresponding maximal process. For $p \in [1,\infty]$ let

$$\|S\|_{\mathscr{S}^p} := \|S_T^*\|_{L^p},$$

and denote the class of semimartingales with finite $\mathscr{S}^p$-norm also by $\mathscr{S}^p$. This definition is due to Meyer [M78]. We extend the definition slightly to allow for an arbitrary Orlicz space $L^{\Psi}(P)$ or its Morse subspace $M^{\Psi}(P)$,

$$\mathscr{S}^{\Psi} := \{\text{semimartingale } S \mid S_T^* \in L^{\Psi}\}, \tag{5}$$

$$\widetilde{\mathscr{S}}^{\Psi} := \{\text{semimartingale } S \mid S_T^* \in M^{\Psi}\}. \tag{6}$$

Note for future use that $\mathscr{S}^\Psi$ and $\widetilde{\mathscr{S}}^\Psi$ are stable under stopping, that is if S belongs to $\mathscr{S}^\Psi$ or $\widetilde{\mathscr{S}}^\Psi$ and if τ is a stopping time, then the stopped process $S^\tau := (S_{\tau\wedge t})_t$ is in $\mathscr{S}^\Psi$ or $\widetilde{\mathscr{S}}^\Psi$, respectively.

Following Protter [Pr05], for any special semimartingale S with canonical decomposition into local martingale part M and predictable finite variation part A, $S = S_0 + M + A$, we define the following semimartingale norm,

$$\|S\|_{\mathscr{H}^p} = \|S_0\|_{L^p} + \|[M,M]_T^{1/2}\|_{L^p} + \|\mathrm{var}(A)_T\|_{L^p},$$

where $\mathrm{var}(A)$ denotes the absolute variation of process A. The class of processes with finite $\mathscr{H}^p$-norm is denoted by $\mathscr{H}^p$. As usual we let

$$\mathscr{M}^p := \mathscr{H}^p \cap \mathscr{M},$$

where $\mathscr{M}$ is the set of uniformly integrable P-martingales.

2.4 Localization and beyond: σ-localization and $\mathcal{I}$-localization

Recall that for a given semimartingale S on $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, P)$, $L(S)$ indicates the class of predictable and $\mathbb{R}^d$-valued, S-integrable processes H *under* P, while $H\cdot S$ indicates the integral process. When H is a scalar predictable process belonging to $\cap_{i=1}^d L(S^i)$ we write, with a slight abuse of notation, $H\cdot S$ for the process $(H\cdot S^1, \dots, H\cdot S^d)$.

Now, let $\mathscr{C}$ be some fixed class of semimartingales. The following methods of extending $\mathscr{C}$ appear in the literature:

i) $S \in \mathscr{C}_{\mathrm{loc}}$, i.e. S is *locally* in $\mathscr{C}$, if there is a sequence of stopping times τ_n increasing to $+\infty$ such that each of the stopped processes S^{τ_n} is in $\mathscr{C}$.

ii) $S \in \mathscr{C}_\sigma$, i.e. S is σ-*locally* in $\mathscr{C}$, if there is a sequence of predictable sets D_n increasing to $\Omega\times\mathbb{R}_+$ such that each of the processes $I_{D_n}\cdot S$ is in $\mathscr{C}$.

iii) $S \in \mathscr{C}_\mathcal{I}$, i.e. S is $\mathcal{I}$-*locally* in $\mathscr{C}$, if there is some scalar process $\varphi \in \cap_{i=1}^d L(S^i)$, $\varphi > 0$ such that $\varphi\cdot S$ is in $\mathscr{C}$.

The first two items are standard (cf. [JS03, I.1.33], [Ka04]) while the third item is an ad hoc definition. By construction, for an arbitrary semimartingale class $\mathscr{C}$ one has $\mathscr{C}_\sigma \supseteq \mathscr{C}_{\mathrm{loc}} \supseteq \mathscr{C}$. However it is not *a priori* clear what inclusions hold for $\mathscr{C}_\mathcal{I}$, apart from the obvious $\mathscr{C}_\mathcal{I} \supseteq \mathscr{C}$. Émery [E80, Proposition 2] has shown that when $\mathscr{C} = \mathscr{M}^p$ or $\mathscr{H}^p$, the following equalities hold

$$\mathscr{M}^p_\sigma = \mathscr{M}^p_\mathcal{I}, \qquad \mathscr{H}^p_\sigma = \mathscr{H}^p_\mathcal{I}, \qquad \text{for } p\in[1,+\infty). \tag{7}$$

The name $\mathcal{I}$-localization ($\mathcal{I}$ standing for integral) is probably a misnomer, since no localization procedure is involved. But we choose it since in Emery's result, $\mathcal{I}$-localization coincides with σ-localization. In general, however, $\mathscr{C}_\mathcal{I} \neq \mathscr{C}_\sigma$. Intuition suggests that the two localizations coincide whenever the primary class $\mathscr{C}$ is defined via some sort of integrability properties, as in the case above: martingale property and its generalizations, boundedness or more generally Orlicz integrability conditions on the maximal process. The next result in this direction appears to be new.

Proposition 2.1. *For any Orlicz space L^Ψ, its Morse subspace M^Ψ and the corresponding semimartingale normed spaces $\mathscr{S}^\Psi, \widetilde{\mathscr{S}}^\Psi$, the following identities hold: $\mathscr{S}_\sigma^\Psi = \mathscr{S}_\mathcal{I}^\Psi$ and $\widetilde{\mathscr{S}_\sigma}^\Psi = \widetilde{\mathscr{S}_\mathcal{I}}^\Psi$.*

Proof. i) Assume $S \in \mathscr{S}_\sigma^\Psi$. There are predictable sets D_n increasing to $\Omega\times\mathbb{R}_+$ such that $(I_{D_n}\cdot S)_T^* \in L^\Psi$ for all n. Thus there is $\alpha_n \in (0,1]$ such that $0 \le b_n := E[\Psi(\alpha_n(I_{D_n}\cdot S)_T^*)] < +\infty$. Let

$$\varphi := \sum_n \eta_n I_{D_n}, \qquad \eta_n := \frac{c}{2^n}\frac{\alpha_n}{1+b_n}, \qquad c := 1/(\sum_n 2^{-n}(1+b_n)^{-1}).$$

Then,

$$E[\Psi((\varphi\cdot S)_T^*)] \le E[\Psi(\sum_n \eta_n (I_{D_n}\cdot S)_T^*)] \le \sum_n \frac{\eta_n}{\alpha_n} E[\Psi(\alpha_n(I_{D_n}\cdot S)_T^*)]$$
$$= \sum_n \frac{\eta_n b_n}{\alpha_n} \le c < 2(1+b_1), \quad (8)$$

where the first inequality follows from monotonicity of Ψ and $(\varphi\cdot S)_T^* \le \sum_n \eta_n (I_{D_n}\cdot S)_T^*$. The second inequality follows from $\sum_n \eta_n/\alpha_n = 1$ and from the following simple pointwise argument, which ensures that, when $\sum_{n\ge1}\alpha_n x_n$ is summable, $\Psi(\sum_{n\ge1}\alpha_n x_n) \le \sum_{n\ge1}\alpha_n\Psi(x_n)$ whenever the weights α_n are nonnegative and the series $\sum_{n\ge1}\alpha_n = 1$. The argument is: from convexity and $\Psi(0)=0$, $\Psi(\sum_{n=1}^N \alpha_n x_n) \le \sum_{n=1}^N \alpha_n \Psi(x_n)$. As $\sum_{n=1}^N \alpha_n x_n \to \sum_{n\ge1}\alpha_n x_n$, from lower semicontinuity of Ψ we get $\Psi(\sum_{n\ge1}\alpha_n x_n) \le \liminf_n \Psi(\sum_{n=1}^N \alpha_n x_n)$ and the $\liminf$ is clearly dominated by $\sum_{n\ge1}\alpha_n\Psi(x_n)$. By construction, $\varphi > 0$ and inequality (8) implies $S \in \mathscr{S}_\mathcal{I}^\Psi$.

ii) To prove the opposite inclusion, $\mathscr{S}_\sigma^\Psi \supseteq \mathscr{S}_\mathcal{I}^\Psi$, consider $\varphi > 0$ such that $\varphi\cdot S \in \mathscr{S}^\Psi$. If $D_n = \{\frac{1}{n} < \varphi < n\}$, then $(D_n)_n$ is a sequence of predictable sets increasing to $\Omega\times\mathbb{R}_+$. In addition, $I_{D_n}\cdot S \in \mathscr{S}_{\text{loc}}^\Psi$ for all n. In fact, let $\tau_k^n = \inf\{t \mid (I_{D_n}\cdot S)^* > k\}$. Then

$$\begin{aligned}(I_{D_n}\cdot S^{\tau_k^n})_T^* &\le (I_{D_n}\cdot S^{\tau_k^n})_{T-}^* + |(I_{D_n}\cdot S^{\tau_k^n})_T| \\ &\le k + |(I_{D_n}\cdot S^{\tau_k^n})_{T-}| + |\Delta(I_{D_n}\cdot S^{\tau_k^n})_T| \le 2k + |\Delta(I_{D_n}\cdot S^{\tau_k^n})_T|\end{aligned}$$

and the latter jump verifies

$$|\Delta(I_{D_n}\cdot S^{\tau_k^n})_T| = |\Delta\left((I_{D_n}\frac{1}{\varphi})\cdot(\varphi\cdot S^{\tau_k^n})\right)_T| = (I_{D_n}\frac{1}{\varphi})_T|\Delta(\varphi\cdot S^{\tau_k^n})_T| \le n2(\varphi\cdot S)_T^*$$

so that

$$(I_{D_n}\cdot S^{\tau_k^n})_T^* \le 2k + 2n(\varphi\cdot S)_T^* \in L^\Psi.$$

This shows that $S \in (\mathscr{S}_{\text{loc}}^\Psi)_\sigma$. As $\mathscr{S}^\Psi$ is stable under stopping, a result by Kallsen [Ka04, Lemma 2.1] ensures $(\mathscr{S}_{\text{loc}}^\Psi)_\sigma = \mathscr{S}_\sigma^\Psi$, which completes the proof for $\mathscr{S}^\Psi$. The proof for $\widetilde{\mathscr{S}}^\Psi$ is analogous. □

3 The strategies

3.1 Conditions on S and simple strategies

Let S be a d-dimensional semimartingale which models the discounted evolution of d underlyings. As hinted in the introduction, to accommodate popular models for S, including exponential Lévy, we do not assume that S is locally bounded. However, to make sure that there is a sufficient number of well-behaved simple strategies we impose the following condition on S:

Assumption 3.1. *$S \in \mathscr{S}_\sigma^{\hat{U}}$. In other words, there exists a $\mathcal{I}$-localizing integrand φ, namely $\varphi \in \cap_{i=1}^d L(S^i; P)$, $\varphi > 0$ and $(\varphi \cdot S)_T^* \in L^{\hat{U}}$.*

The σ-localization in Assumption 3.1 provides a substantial amount of flexibility. For example, in the Black-Scholes model the risky asset $S \notin \mathscr{S}^{\hat{U}}$ when U stands for the exponential utility. On the other hand S is continuous and therefore locally bounded which means $S \in \mathscr{S}_{\text{loc}}^\infty \subseteq \mathscr{S}_{\text{loc}}^{\hat{U}} \subseteq \mathscr{S}_\sigma^{\hat{U}}$ for any utility function satisfying our assumptions, including the exponential. The price process class $\mathscr{S}_\sigma^{\hat{U}}$ introduced in this paper appears to be the most general ever studied in the context of utility maximization.

Definition 3.2. *i) For $S \in \mathscr{S}_{\text{loc}}^{\hat{U}}$ we say H is a* simple integrand *if it is of the form $H = \sum_{k=1}^N H_k I_{]T_{k-1},T_k]}$ where each H_k is $\mathcal{F}_{T_{k-1}}$-measurable and bounded, and $T_1 \leq \cdots \leq T_N$ are stopping times dominated by some localizing time τ_n for S from the definition of $\mathscr{S}_{\text{loc}}^{\hat{U}}$.*

ii) For $S \in \mathscr{S}_\sigma^{\hat{U}} \setminus \mathscr{S}_{\text{loc}}^{\hat{U}}$ fix $\varphi \in \cap_{i=1}^d L(S^i; P)$, $\varphi > 0$ such that $(\varphi \cdot S)_T^* \in L^{\hat{U}}$. We say that H is a simple integrand *if it is of the form $H = (\sum_{k=1}^N H_k I_{]T_{k-1},T_k]}) \cdot \varphi$ where each H_k is $\mathcal{F}_{T_{k-1}}$-measurable and bounded, and $T_1 \leq \cdots \leq T_N$ are stopping times.*

The vector space of all simple integrands is denoted by $\mathcal{H}$.

For $S \in \mathscr{S}_{\text{loc}}^{\hat{U}}$ every simple integral represents a buy-and-hold strategy over finitely many trading dates and it is thus a very natural financial object. To motivate the definition of simple strategies for $S \in \mathscr{S}_\sigma^{\hat{U}} \setminus \mathscr{S}_{\text{loc}}^{\hat{U}}$ we now define dual asset pricing measures.

Definition 3.3. *$Q \ll P$ is a σ-martingale measure for S iff S is a σ-martingale under Q. The set of all σ-martingales measures for S is denoted by $\mathcal{M}$ and the subset of equivalent measures by $\mathcal{M}^e$.*

The concept of σ-martingale measure was introduced to mathematical finance by Delbaen and Schachermayer [DS98]. When S is (locally) bounded, it can be shown that $\mathcal{M}$ coincides with the absolutely continuous (local) martingale measures for S (see e.g. Protter [Pr05, Theorem 91]). Therefore, σ-martingales are a natural generalization of local martingales in the case when S is not locally bounded and the elements of $\mathcal{M}$ which are equivalent to P can be used as arbitrage-free pricing measures for the derivative securities whose payoff depends on S. The recent book [DS06] contains an extensive treatment of the financial applications of this mathematical concept.

Thanks to Émery's equality (7) the set of absolutely continuous σ-martingale measures for S is the same as the set of σ-martingale measures for $S' := \varphi \cdot S$. Specifically, $Q \ll P$ is a σ-martingale measure for S by (7) if and only if there exists a Q-positive, predictable process $\psi_Q \in \cap_{i=1}^d L(S^i; Q)$ such that $\psi_Q \cdot S$ is a Q-martingale. And this happens if and only if $\psi_Q' \cdot (\varphi \cdot S)$ is a Q-martingale, where $\psi_Q' = \frac{\psi_Q}{\varphi}$.

Since the sets of σ-martingale measures for S and S' are the same the *dual* problem to the utility maximization is unchanged. Under suitable conditions (see the statement of the main Theorem

4.8), we end up with the same optimizer, irrespective of a specific choice of the $\mathcal{I}$-localizing strategy φ. The price we pay for the extra flexibility in allowing $S \in \mathscr{S}_\sigma^{\hat{U}} \setminus \mathscr{S}_{\text{loc}}^{\hat{U}}$ is that simple strategies can no longer be interpreted as buy-and-hold vis-à-vis the original price process S.

3.2 Properties of simple strategies

Simple integrals have good mathematical properties with respect to σ-martingale measures with *finite relative entropy.*

Definition 3.4. *A probability Q has finite relative entropy, notation: $Q \in P_V$, if there is $y_Q > 0$ such that*

$$v_Q(y_Q) := E[V(y_Q \frac{dQ}{dP})] < \infty. \tag{9}$$

This definition differs from the classic formulation of finite relative entropy, also called finite V-divergence, which requires $y_Q = 1$ (see Liese and Vajda [LV87], Kramkov and Schachermayer [KS99], Bellini and Frittelli [BeF02], Goll and Rüschendorf [GR01] and basically all the contemporary literature on utility maximization).

Lemma 3.5. *The wealth process $X = H \cdot S$ of every $H \in \mathcal{H}$ is a uniformly integrable martingale under all $Q \in \mathcal{M} \cap P_V$.*

Proof. i) $S \in \mathscr{S}_\sigma^{\hat{U}} \setminus \mathscr{S}_{\text{loc}}^{\hat{U}}$. Since $H \in \mathcal{H}$, the maximal functional X^* verifies $X_T^* \leq c(\varphi \cdot S)_T^*$ for some constant $c > 0$ and some $\mathcal{I}$-localizing integrand φ from Assumption 3.1. Then $E[U(-\alpha(\varphi \cdot S)_T^*)] \in \mathbb{R}$ for some $\alpha > 0$ and, as a consequence,

$$0 \geq E[U\left(-\frac{\alpha}{c} X_T^*\right)] > -\infty.$$

For any fixed $Q \in \mathcal{M} \cap P_V$, the Fenchel inequality $U(x) - xy \leq V(y)$ applied with $x = -\frac{\alpha}{c} X_T^*, y = y_Q \frac{dQ}{dP}$ gives

$$U(-\frac{\alpha}{c} X_T^*) + \frac{\alpha}{c} X_T^* y_Q \frac{dQ}{dP} \leq V(y_Q \frac{dQ}{dP}),$$

whence $0 \leq \frac{\alpha}{c} y_Q X_T^* \frac{dQ}{dP} \leq V(y_Q \frac{dQ}{dP}) - U(-\frac{\alpha}{c} X_T^*)$, and therefore X_T^* is in $L^1(Q)$. As Q is a σ-martingale probability for S, X is also a Q-σ-martingale. Since its maximal process is integrable, X is in fact a Q-uniformly integrable martingale (see Protter [Pr05, Chapter IV-9]).

ii) $S \in \mathscr{S}_{\text{loc}}^{\hat{U}}$. Proceed as in i), replacing φ with $I_{[0,\tau_n]}$. □

In financial terms, the message of the above Lemma is that each $Q \in \mathcal{M} \cap P_V$ represents a pricing rule that assigns a correct price to every simple self-financing strategy.

3.3 Admissible integrands and integrals

Since simple integrands are very basic tools, it is clear that their class may not contain the solution of the utility maximization problem. The appropriate class is an extension given in terms of suitable limits of strategies in $\mathcal{H}$.

Definition 3.6. *$H \in L(S)$ is an* admissible integrand *if $U(H \cdot S_T) \in L^1(P)$ and if there exists an approximating sequence $(H^n)_n$ in $\mathcal{H}$ such that:*

i) $H^n \cdot S_t \to H \cdot S_t$ in probability for all $t \in [0,T]$;

ii) $U(H^n \cdot S_T) \to U(H \cdot S_T)$ in $L^1(P)$.

The set of all admissible integrands is denoted by $\overline{\mathcal{H}}$.

The two requirements above are quite natural assumptions if considered *separately.* In fact, fix any integral $H \cdot S$. By the very definition of stochastic integral, one can always find a sequence of simple integrands that approximate $H \cdot S$ as required by item i): $H^n = \sum_{i=1}^n H_{T_i^n} I_{]T_i^n, T_{i+1}^n]}$ whenever $(T_i^n)_i$ is a finite random partition of $[0,T]$ via stopping times, with mesh going to zero in probability when $n \to +\infty$. Item ii), on the other hand ensures that utility of an admissible strategy can be approximated by the utility from simple strategies. Definition 3.6 combines these *two desirable approximation features* together.

While for $H \in \mathcal{H}$ the wealth process $H \cdot S$ is always a martingale under $Q \in \mathcal{M} \cap P_V$, the following result shows that $\overline{\mathcal{H}}$ is a subset of the supermartingale class of strategies $\mathcal{H}^{\text{sup}}$ introduced by [Sch03],

$$\begin{aligned}\mathcal{H}^{\text{sup}} := \{H \in L(S) \mid H \cdot S &\text{ is a local martingale} \\ &\text{and a supermartingale under any } Q \in \mathcal{M} \cap P_V\}.\end{aligned} \tag{10}$$

Proposition 3.7. $\overline{\mathcal{H}} \subseteq \mathcal{H}^{\text{sup}}$.

Proof. Let $X = H \cdot S$ for some $H \in \overline{\mathcal{H}}$ and let $(X^n := H^n \cdot S)_n$ with $H^n \in \mathcal{H}$ be an approximating sequence. Fix a $Q \in \mathcal{M} \cap P_V$ and a corresponding scaling y_Q as in Definition 3.4. Item *i*) applied at time T implies $(X_T^n)^-$ converge in P and therefore Q-probability to X_T^-. Moreover, Fenchel inequality gives

$$U(X_T^n) - V(y_Q \frac{dQ}{dP}) \leq X_T^n y_Q \frac{dQ}{dP}.$$

From item *ii*), the left hand side above converges in $L^1(P)$, whence the family $(Y^n)_n, Y^n := (X_T^n)^- \frac{dQ}{dP}$ is P-uniformly integrable, so $((X_T^n)^-)_n$ is Q-uniformly integrable. Uniform integrability plus convergence in probability ensures $(X_T^n)^- \to X_T^-$ in $L^1(Q)$. By passing to a subsequence, we can construct

$$W^Q := \sum_n |(X_T^{n+1})^- - (X_T^n)^-| \in L^1(Q),$$

and the associated uniformly integrable Q-martingale Z^Q with $Z_T^Q = W^Q$. Note that when $\operatorname{dom} U$ is a half-line we could also have chosen trivially $W^Q := -\inf \operatorname{dom} U$.

Since $X_T^n \geq -W^Q$ and process X^n is a Q-martingale for all n by Lemma 3.5, we obtain

$$X_t^n = E_Q[X_T^n \mid \mathcal{F}_t] \geq -E_Q[W^Q \mid \mathcal{F}_t] = Z_t^Q, \tag{11}$$

so that the sequence X^n is controlled from below by the Q-martingale Z^Q. Therefore by Delbaen and Schachermayer compactness result [DS99, Theorem D] (in the version stated in Section 5,

[DS98]) there exists a limit càdlàg supermartingale $\widetilde{V}$ to which a sequence $K^n \cdot S$, where K^n is a suitable convex combinations of tails $K^n \in \mathrm{conv}(H^n, H^{n+1}, \ldots)$, converges Q-almost surely for every rational time $0 \leq q \leq T$. By item *i*), $((X_t^n))_n$ converges in P-probability to X_t for every t, thus $K^n \cdot S_t$ converges to X_t for every t as well. Therefore $\widetilde{V}$ coincides Q-a.s. with X on rational times, and since X is also càdlàg as it is an integral, X and $\widetilde{V}$ are indistinguishable, so that X is a Q-supermartingale. By assumption Q is a σ-martingale measure, so $X = H \cdot S = (\frac{1}{\varphi} H) \cdot (\varphi \cdot S)$ where $\varphi > 0$ and $(\varphi \cdot S)$ is a Q-martingale. As $X \geq Z^Q$, Ansel and Stricker lemma [AS94, Corollaire 3.5] implies that X is a local Q-martingale. □

The importance of the supermartingale property will become clear in Section 4.

Remark 3.8. Proposition 3.7 would go through if one replaced $\mathcal{H}$ in Definition 3.6 with the set

$$\mathcal{H}' = \{H \in L(S) \mid H \cdot S \geq c \text{ for some } c \in \mathbb{R}\}, \tag{12}$$

as in Schachermayer [Sch01], when S is locally bounded, or more generally the set of those $H \in L(S)$ whose losses are in some sense well controlled as in Biagini and Frittelli [BF05, BF08] or Biagini and Sîrbu [BS09]. An application of the Ansel and Stricker lemma [AS94, Corollaire 3.5] shows that wealth processes for strategies in $\mathcal{H}'$ are local martingales and supermartingales under any $Q \in \mathcal{M} \cap P_V$ – but not martingales in general. However, $\mathcal{H}' \subseteq \mathcal{H}^{\sup}$ is exactly what is needed from a *mathematical* point of view in the utility maximization problem.

We summarize the advantages of $\overline{\mathcal{H}}$ over current definitions of admissibility in the following list:

a) Definition 3.6 is *primal*. No pricing measures come into play, and admissibility can thus be checked under P.

b) The present definition is *dynamic*, that is the whole wealth process, rather than just its terminal value, is involved in the definition of $\overline{\mathcal{H}}$. As a result all admissible strategies are in the supermartingale class. In contrast, in [BTZ04, Section 4] only *the optimal strategy* is guaranteed to be in the supermartingale class.

c) The loss controls required in the proof of the supermartingale property are generated endogenously, via approximating sequences. This provides a great deal of flexibility and ensures that for U finite on $\mathbb{R}$ the optimizer is in $\overline{\mathcal{H}}$ under very mild conditions, milder than the conditions assumed to obtain the supermartingale property of the optimizer in [Sch03, BF07].

d) Approximation by strategies in $\mathcal{H}$ is built into the definition of admissibility, it does not have to be deduced separately (cf. [Str03]).

e) The desirable properties above hold without any technical assumptions on U. It can be finite on $\mathbb{R}$ or only on a half-line; bounded above or not, or even truncated; neither strict monotonicity, strict convexity nor differentiability are required.

f) Our definition is compatible with the existing definition of admissibility for non-monotone quadratic preferences, see Remark 3.9 below. We have therefore found a good notion of admissibility which encompasses both the classical mean-variance preferences *and* monotone expected utility.

Remark 3.9. For the purpose of this remark only, we admit non-monotone U. Specifically, let $U(x) := x - x^2/2$, which represents a normalized quadratic utility. In such case, $H \in \overline{\mathcal{H}}$ if and only if there is a sequence of $H^n \in \mathcal{H}$ such that: 1) $H^n \cdot S_t \to H \cdot S_t$ in probability for all $t \in [0,T]$ and 2) $H^n \cdot S_T \to H \cdot S_T$ in $L^2(P)$. In other words, when U is quadratic the admissibility criterion in Definition 3.6 coincides with the notion of admissibility pioneered by Kallsen in [ČK07, Definition 2.2], which inspired our work. Since 1) above and i) in Definition 3.6 coincide, the only thing to prove is that ii) in our definition is equivalent to 2) above:

- $\Rightarrow$ Suppose first $H \in \overline{\mathcal{H}}$. The $L^1(P)$ convergence of utilities implies $E[U(X_T^n)] \to E[U(X_T)]$ so that X_T^n are uniformly bounded in $L^2(P)$. Since $L^2(P)$ is a reflexive space there is a sequence of convex combinations of tails of $(X_T^n, X_T^{n+1}, \ldots)$, say $\widetilde{X}_T^n$, which converges in $L^2(P)$ to a square integrable random variable which necessarily is $X_T = H \cdot S_T$ thanks to Definition 3.6-i). By considering the corresponding convex combinations of strategies, which are again simple, we obtain the existence of an approximating sequence à la Kallsen for H.
- $\Leftarrow$ Conversely, let $X = H \cdot S$ be an integral approximated à la Kallsen by simple integrals $(X^n)_n$. $L^1(P)$ convergence of the utilities $U(X_T^n)$ to $U(X_T)$ is then a consequence of the Cauchy-Schwartz inequality.

4 Optimal trading strategy is in $\overline{\mathcal{H}}$

The optimal investment problem can be formulated over $\mathcal{H}$ or over $\overline{\mathcal{H}}$, respectively,

$$u_{\mathcal{H}}(x) := \sup_{H \in \mathcal{H}} E[U(x + H \cdot S_T)], \tag{13}$$

$$u_{\overline{\mathcal{H}}}(x) := \sup_{H \in \overline{\mathcal{H}}} E[U(x + H \cdot S_T)]. \tag{14}$$

Alongside, we consider an auxiliary complete market utility maximization problem, fixing an arbitrary $Q \in \mathcal{M} \cap P_V$,

$$u_Q(x) := \sup_{X \in L^1(Q), E_Q[X] \le x} E[U(X)]. \tag{15}$$

The value functions $u_{\mathcal{H}}(x), u_{\overline{\mathcal{H}}}(x), u_Q(x)$ are also known as indirect utilities (from the respective domains of maximization). The next lemma is an easy consequence of the definition of $\overline{\mathcal{H}}$, of the Fenchel inequality (3) and of the supermartingale property of the strategies in $\overline{\mathcal{H}}$. The proof is omitted.

Lemma 4.1. *For any $x > \inf \operatorname{dom} U$ and for any $Q \in \mathcal{M} \cap P_V$*

$$u_{\mathcal{H}}(x) = u_{\overline{\mathcal{H}}}(x) \le u_Q(x). \tag{16}$$

4.1 Reasonable Asymptotic Elasticity and Inada conditions

It is well known in the literature that the existence of an optimizer is not guaranteed yet, neither in the set $\overline{\mathcal{H}} \supseteq \mathcal{H}$ nor in the larger supermartingale class $\mathcal{H}^{\sup} \supseteq \overline{\mathcal{H}}$. An additional condition has to be imposed, essentially to ensure that the expected utility functional $X \to E[U(X)]$ is upper semicontinuous with respect to some weak topology on terminal wealths.

Kramkov and Schachermayer were the first to address this issue in [KS99, Sch01] for regular U, that is utilities that are strictly increasing, strictly concave and differentiable in the interior of their effective domain. To the end of recovering an optimizer they introduced the celebrated reasonable asymptotic elasticity condition on U (RAE(U)),

$$\limsup_{x\to+\infty} \frac{xU'(x)}{U(x)} < 1, \tag{17}$$

$$\text{and also} \quad \liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1, \text{ when } U \text{ is finite on } \mathbb{R}, \tag{18}$$

as a necessary and sufficient condition to be imposed on the utility U *only*, regardless of the probabilistic model. This condition is now very popular, see [OŽ09, RS05, Sch03, B02] just to mention a few contributions.

In subsequent work, in the context of utilities finite on $\mathbb{R}_+$, Kramkov and Schachermayer [KS03] put forward less restrictive conditions*, *imposed jointly on the model and on the preferences*, in order to recover the optimal terminal wealth. Here they work under assumptions which are equivalent to the existence of $Q \in \mathcal{M}^e \cap P_V$ and the following Inada condition on the indirect utility $u_{\mathcal{H}'}$, where the class $\mathcal{H}'$ is defined in (12):

$$\lim_{x\to+\infty} u_{\mathcal{H}'}(x)/x = 0. \tag{19}$$

It is important to note that for utility functions finite on a half-line the modulus of the conjugate function $V(y)$ grows only linearly for large y and therefore the following implication holds automatically:

$$Q \in \mathcal{M} \cap P_V \Rightarrow v_Q(y) < \infty \text{ for all } y \text{ sufficiently high.} \tag{20}$$

On the other hand, for utilities finite on $\mathbb{R}$ condition (20) has to be imposed explicitly, together with an appropriate generalization of condition (19).

Assumption 4.2. *Condition* (20) *is satisfied and*

$$\textit{there exists } Q \in \mathcal{M} \cap P_V \textit{ such that } \lim_{x\to+\infty} u_Q(x)/x = 0. \tag{21}$$

We provide a detailed discussion of Assumption 4.2 and its relation to RAE(U) and the Inada condition (19) in Section 5.1. The results of the next Section already go in that direction.

*The interested reader is referred also to the recent Biagini and Guasoni [BG09] for counterexamples and a different, *relaxed* framework that allows optimal terminal wealth to be a measure and not only a random variable.

4.2 Complete market duality

Here we study a complete market $Q \in P_V$ and hence no model for S is required. We provide, among other results, an alternative characterization of the Inada condition (21) in terms of the relative entropy of Q.

Lemma 4.3. *Consider the function v_Q defined in (9). For any fixed $Q \in P_V$ and $x > \inf \operatorname{dom} U$,*

$$u_Q(x) = \min_{y \geq 0} \{xy + v_Q(y)\} < +\infty. \tag{22}$$

The proof of Lemma 4.3 follows standard arguments and it is given in Section 4.4 for completeness.

Corollary 4.4. *Fix $Q \in P_V$. The following statements are equivalent:*

i) u_Q verifies the Inada condition at $+\infty$: $\lim_{x \to +\infty} u_Q(x)/x = 0$;

ii) there is $y_Q > 0$ such that

$$v_Q(y) = E[V(y\frac{dQ}{dP})] < +\infty \text{ for all } y \in (0, y_Q]. \tag{23}$$

Proof. ii) $\Rightarrow$ *i)* Suppose that $v_Q(y)$ is finite in a right neighborhood of 0. By Fenchel inequality, $E[U(X)] - E[y\frac{dQ}{dP}X] \leq E[V(y\frac{dQ}{dP})]$ for all $X \in L^1(Q)$ so that $u_Q(x) \leq xy + v_Q(y)$ for all $y > 0$. Fixing y one obtains $\lim_{x \to +\infty} u_Q(x)/x \leq y$ and on letting $y \to 0$ equation (21) follows.
i) $\Rightarrow$ *ii)* For a given $x > \inf \operatorname{dom} U$, select one dual minimizer in (22) and denote it by y_x. Now, $u_Q(x) = xy_x + v_Q(y_x)$, $v_Q(y_x)$ is finite, and the chain of inequalities

$$u_Q(x) = xy_x + v_Q(y_x) \overset{Jensen}{\geq} xy_x + V(y_x) \geq xy_x \geq 0$$

holds for any x as V is nonnegative. Dividing by $x > 0$ and sending x to $+\infty$, (21) implies $\lim_{x \to +\infty} y_x = 0$. Finiteness of v_Q over the set $\{y_x\}_x$, whose closure contains 0, and convexity of v_Q finally imply v_Q is finite in the interval $(0, y_Q]$, with y_Q from (9). □

The next proposition contains a novel characterization of the condition

$$u_Q(x) < U(+\infty),$$

which is a kind of "no utility-based arbitrage" condition, when Q has finite entropy. Bliss utility $U(+\infty)$ cannot be reached if the initial capital x is smaller than the satiation point $\bar{x}$ and vice versa.

Proposition 4.5. *For $Q \in P_V$ and $x > \inf \operatorname{dom} U$ the following statements are equivalent:*

i) $x < \bar{x}$;

ii)

$$u_Q(x) = \min_{y > 0} \{xy + v_Q(y)\} < U(+\infty). \tag{24}$$

Proof. *ii)* $\Rightarrow$ *i)* $U(x) \le u_Q(x) < U(+\infty)$ implies $x < \bar{x}$.
i) $\Rightarrow$ *ii)* Let $Z := y_Q dQ/dP$, with y_Q from (9). When $U(+\infty) = V(0) = +\infty$ there is nothing to prove in view of (22). Consider therefore the remaining case $0 < U(+\infty) = V(0) < +\infty$. Function $f(y) := V(y) + xy$ is convex and by Rockafellar [R70, Theorem 23.5] it attains its minimum at $\hat{y} := U'_-(x) > 0$ with $f(\hat{y}) = V(\hat{y}) + x\hat{y} = U(x)$. Convexity then gives

$$f(y) \le f(0) - y\frac{f(0) - f(\hat{y})}{\hat{y}} = U(+\infty) - y\frac{U(+\infty) - U(x)}{\hat{y}} \qquad \text{for } y \in [0, \hat{y}],$$
$$f(y/k) \le f(0) + \frac{f(y) - f(0)}{k} \le U(+\infty) + \frac{f(y)}{k} \qquad \text{for } k \ge 1, y \ge 0.$$

For $k \ge 1$ these estimates imply

$$\begin{aligned} E[f(Z/k)] &= E[f(Z/k)1_{\{Z \le k\hat{y}\}}] + E[f(Z/k)1_{\{Z > k\hat{y}\}}] \\ &\le U(+\infty) - \frac{1}{k}\left(\frac{U(+\infty) - U(x)}{\hat{y}} E[Z1_{\{Z \le k\hat{y}\}}] - E[f(Z)1_{\{Z > k\hat{y}\}}]\right), \end{aligned}$$

and, as $x < \bar{x}$ implies $U(x) < U(+\infty)$, for sufficiently large k $E[f(Z/k)] < U(+\infty) = V(0)$, which completes the proof. $\square$

Remark 4.6. Proposition 4.5 should be contrasted with an example by Schachermayer [Sch01, Lemma 3.8], where the author constructs an arbitrage-free complete market with unique pricing measure Q for which $u_Q(x) \equiv U(+\infty)$, while U is strictly increasing and bounded above (and therefore it satisfies the Inada condition at $+\infty$). This is possible because the measure Q in question does not belong to P_V.

4.3 The main result

The general theory of [BF08] shows that the dual problem may contain singular parts whenever $S \in \mathscr{S}_\sigma^{\hat{U}} \setminus \widetilde{\mathscr{S}_\sigma}^{\hat{U}}$ but for $S \in \widetilde{\mathscr{S}_\sigma}^{\hat{U}}$ the singular parts in the dual problem disappear. Our main result hinges on the absence of singularities in the dual problem, which is what we now assume. Connections with general Orlicz space duality are spelled out in Section 5.1.

Assumption 4.7. *For any $x < \bar{x}$, the following dual relation holds:*

$$u_{\mathcal{H}}(x) = \min_{Q \in \mathcal{M} \cap P_V} u_Q(x) = \min_{y \ge 0, Q \in \mathcal{M} \cap P_V} \{xy + v_Q(y)\}. \tag{25}$$

When U is finite on $\mathbb{R}$, the Orlicz heart $M^{\hat{U}} \supseteq L^\infty$ is nontrivial, and consequently the class $\widetilde{\mathscr{S}_\sigma}^{\hat{U}}$ is also nontrivial. By virtue of [BF08, Proposition 44] and Proposition 4.5 $S \in \widetilde{\mathscr{S}_\sigma}^{\hat{U}}$ is sufficient for Assumption 4.7 to hold. Note that (25) represents no loss of generality for utility functions whose left tail behaves asymptotically like a power, x^p, with $p > 1$, since then $\widetilde{\mathscr{S}_\sigma}^{\hat{U}} = \mathscr{S}_\sigma^{\hat{U}} = \mathscr{S}_\sigma^{p}$.

However, Assumption 4.7 is weaker than $S \in \widetilde{\mathscr{S}_\sigma}^{\hat{U}}$ — it may be applied also when $S \in \mathscr{S}_\sigma^{\hat{U}} \setminus \widetilde{\mathscr{S}_\sigma}^{\hat{U}}$. This includes utility functions finite on a half-line for which both the Orlicz heart and consequently also $\widetilde{\mathscr{S}_\sigma}^{\hat{U}}$ are trivial. Thus, within the confines of Assumption 4.7, we provide a unified treatment for utility function defined both on a half-line and on the whole of $\mathbb{R}$.

Theorem 4.8. *Under Assumptions 3.1, 4.2 and 4.7, for any initial wealth x such that $x < \bar{x}$:*

a) $u_{\mathcal{H}}(x) = u_{\overline{\mathcal{H}}}(x) = \min_{y>0, Q\in\mathcal{M}\cap P_V}\{xy + E[V(y\frac{dQ}{dP})]\}$.

Any optimal dual couple is indicated by $(\hat{y}, \hat{Q})$, dependence on x is understood. The lack of uniqueness of the optimal dual couple is due to the lack of strict convexity of V, which in turn is due to the lack of strict concavity of U.

b) There exists a $(-\infty, +\infty]$-valued claim $\hat{f}$, not unique in general, with the following properties

i) $\hat{f} < +\infty$ *whenever* $\mathcal{M}^e \cap P_V \neq \emptyset$;

ii) $\hat{f}$ *realizes the optimal expected utility, in the sense that*

$$E[U(\hat{f})] = u_{\mathcal{H}}(x);$$

iii) $E_{\hat{Q}}[\hat{f}] = x$, *and the following equalities hold P-a.s. for any dual optimizer $\hat{Q}$:*

$$V(\hat{y}\frac{d\hat{Q}}{dP}) = U(\hat{f}) - \hat{f}\hat{y}\frac{d\hat{Q}}{dP},$$
$$\{\hat{f} \geq \bar{x}\} = \{\frac{d\hat{Q}}{dP} = 0\},$$

where $\bar{x} \in (0, +\infty]$ is the satiation point of U.

iv) $\hat{f} \in L^1(Q)$ *and* $E_Q[\hat{f}] \leq x$ *for all* $Q \in \mathcal{M} \cap P_V$;

v) In case U is strictly concave, V is strictly convex and the solutions of primal and dual problem $\hat{f}, \hat{y}, \hat{Q}$ are unique. If in addition U is differentiable, the unique solutions satisfy $\hat{y}\frac{d\hat{Q}}{dP} = U'(\hat{f})$;

c) There is an approximating sequence of simple strategies $H^n \in \mathcal{H}$ with terminal values $f^n := x + H^n \cdot S_T$ such that:

i)

$$f_n \overset{P\text{-a.s.}}{\rightarrow} \hat{f}, \tag{26}$$

provided $\mathcal{M}^e \cap P_V \neq \emptyset$ *or* $\bar{x} = +\infty$;

ii)

$$U(f_n) \overset{L^1(P)}{\rightarrow} U(\hat{f}); \tag{27}$$

iii)

$$f_n \overset{L^1(\hat{Q})}{\rightarrow} \hat{f}, \tag{28}$$

and, provided (26) holds, for any $Q \in \mathcal{M} \cap P_V$ *such that* $E_Q[\hat{f}] = x$

$$f_n \overset{L^1(Q)}{\rightarrow} \hat{f}; \tag{29}$$

iv) There exists an integral representation $\hat{f} = x + \hat{H} \cdot S_T$ with $\hat{H} \in L(S; \hat{Q})$, and $\hat{H} \cdot S$ is a $\hat{Q}$-martingale. When there is $Q \in \mathcal{M}^e \cap P_V$ such that $E_Q[\hat{f}] = x$, this representation holds also under P, $\hat{H} \in \overline{\mathcal{H}}$ and thus the utility maximization problem admits a maximizer

$$u_{\mathcal{H}}(x) = \max_{H \in \overline{\mathcal{H}}} E[U(x + H \cdot S_T)].$$

Proof. a) This follows directly from Assumption 4.7, from the straightforward identity $u_{\mathcal{H}}(x) = u_{\hat{Q}}(x)$, from $x < \bar{x}$ and from Proposition 4.5.

b) Let us fix a pair $\hat{y}, \hat{Q}$ of dual minimizers. For ease of notation and without loss of generality we let $x = 0$ throughout.

i.1) Select a maximizing sequence $(k_n)_n, k_n = K^n \cdot S_T, K^n \in \mathcal{H}$ so that $E[U(k_n)] \uparrow u_{\mathcal{H}}(0)$.

i.2) Fix $Q \in \mathcal{M} \cap P_V$. We claim the sequence $(k_n)_n$ is bounded in $L^1(Q)$. In a general case this follows from the auxiliary Proposition 4.9, which in turn is a consequence of the Inada condition (21). In a special case when $\operatorname{dom} U$ is a half-line, $L^1(Q)$-boundedness also follows trivially from $k_n \geq \inf \operatorname{dom} U$ and $E_Q[k_n] = 0$, which is a consequence of Lemma 3.5. In a second special case where U is bounded above the claim can be alternatively deduced from the boundedness of $U^-(f_n)$ and the Fenchel inequality (3).

i.3) $L^1(Q)$ boundedness of $(k_n)_n$ enables the application of the Komlós theorem, so that there exists a sequence of convex combinations $(f_n)_n$ with $f_n \in \operatorname{conv}(k_n, k_{n+1}, \ldots)$, that converges Q-a.s. to a certain random variable $f \in L^1(Q)$. As $\mathcal{H}$ is a vector space, these f_n are terminal values of simple integrals $f_n = H^n \cdot S_T$, $H^n \in \mathcal{H}$. By concavity, the f_n are still maximizers, i.e. $E[U(f_n)] \uparrow u_{\mathcal{H}}(0)$.

i.4) When $\mathcal{M}^e \cap P_V$ is non-empty we choose $Q \sim P$ in b.i.2) and b.i.3) and then $\hat{f} := f$ is a well defined element of $L^0(\Omega, \mathcal{F}_T, P)$ with $f_n \overset{P\text{-}a.s.}{\rightarrow} \hat{f}$.

ii.1) When $\mathcal{M}^e \cap P_V$ is empty we perform the above construction with $Q = \hat{Q}$ and define $\hat{f}$ in the following way

$$\hat{f} := f I_{\{\frac{d\hat{Q}}{dP} > 0\}} + \bar{x} I_{\{\frac{d\hat{Q}}{dP} = 0\}}.$$

In both cases it is easily seen that for $y > 0$

$$\limsup_n (U(f_n) - f_n y \frac{d\hat{Q}}{dP}) \leq U(\hat{f}) - \hat{f} y \frac{d\hat{Q}}{dP}, \tag{30}$$

using the convention $+\infty \cdot 0 = 0$. Additionally, Fenchel inequality $U(\hat{f}) - \hat{f}\hat{y}\frac{d\hat{Q}}{dP} \leq V(\hat{y}\frac{d\hat{Q}}{dP})$ implies $U(\hat{f})$ is integrable.

ii.2) We now show that $\hat{f}$ is an optimizer in the sense that $u_{\mathcal{H}}(0) = E[U(\hat{f})]$. Thanks to condition (20) $v_{\hat{Q}}(y) < +\infty$ for all $y \geq \hat{y}$. For fixed n, Fenchel pointwise inequality gives

$$U(f_n) - f_n y \frac{d\hat{Q}}{dP} \leq V(y \frac{d\hat{Q}}{dP}).$$

Given the integrability of $V(y\frac{d\hat{Q}}{dP})$, the Fatou lemma applies and in view of (30)

$$\begin{aligned} u_{\mathcal{H}}(0) &= \limsup_n E[U(f_n) - f_n y \frac{d\hat{Q}}{dP}] \\ &\le E[\limsup_n (U(f_n) - f_n y \frac{d\hat{Q}}{dP})] \le E[U(\hat{f}) - \hat{f} y \frac{d\hat{Q}}{dP}]. \end{aligned} \tag{31}$$

Given the integrability of $U(\hat{f})$, this holds for all $y \geq \hat{y}$ iff $E_{\hat{Q}}[\hat{f}] \leq 0$.

ii.3) The Fenchel inequality and (31) yield

$$u_{\mathcal{H}}(0) \le E[U(\hat{f})] - \hat{y} E_{\hat{Q}}[\hat{f}] \le E[V(\hat{y}\frac{d\hat{Q}}{dP})] = u_{\mathcal{H}}(0), \tag{32}$$

whereby necessarily $E_{\hat{Q}}[\hat{f}] = 0$ and $E[U(\hat{f})] = u_{\mathcal{H}}(0)$.

iii) The Fenchel optimal relation $U(\hat{f}) - \hat{f}\hat{y}\frac{d\hat{Q}}{dP} \overset{P\text{-a.s.}}{=} V(\hat{y}\frac{d\hat{Q}}{dP})$ follows from (32). From here we conclude

$$\frac{d\hat{Q}}{dP} = 0 \Leftrightarrow U(\hat{f}) = U(\bar{x}) = U(+\infty).$$

The forward implication follows from $V(0) = U(+\infty)$ and the converse from $\hat{y} > 0$. The equality $E_{\hat{Q}}[\hat{f}] = 0$ was shown in b.ii.3).

iv.1) Since $\limsup_n (U(f_n) - f_n y \frac{d\hat{Q}}{dP}) \leq (U(\hat{f}) - \hat{f} y \frac{d\hat{Q}}{dP})$ and the inequalities in (31) are also equalities for $y = \hat{y}$, one has $\limsup_n U(f_n) = U(\hat{f}) = U(+\infty)$ on $A := \{\frac{d\hat{Q}}{dP} > 0\}$. Therefore, modulo the passage to a subsequence converging to the limsup, still denoted by $U(f_n)$, $U(f_n)I_A \to U(\hat{f})I_A$, whence globally

$$U(f_n) \overset{P\text{-a.s.}}{\to} U(\hat{f}). \tag{33}$$

iv.2) Given (33) necessarily $\liminf_n f_n I_A \geq \bar{x} I_A$ and therefore $\liminf_n |f_n| \geq |\hat{f}|$. Additionally, $(f_n)_n$ is $L^1(Q)$ bounded: $E_Q[f_n] = 0$ and $(E[U(f_n)])_n$ is bounded below, so Proposition 4.9 applies again. The Fatou lemma yields

$$E_Q[|\hat{f}|] \le E_Q[\liminf_n |f_n|] \le \liminf_n E_Q[|f_n|] \le \text{const},$$

which proves $\hat{f} \in L^1(Q)$.

iv.3) Convergence of utility (33) yields $\liminf_n f_n \geq \hat{f}$. By the Fenchel inequality,

$$U(f_n) - f_n y \frac{dQ}{dP} \le V(y\frac{dQ}{dP})$$

and similarly as in b.ii.2) an application of the Fatou lemma for y sufficiently large yields $E_Q[\hat{f}] \leq 0$.

v) Finally, the results when U is strictly concave and differentiable follow now from the pointwise identity $U(x) - xU'(x) = V(U'(x))$.

c) i) This follows by construction when $\mathcal{M}^e \cap P_V \neq \emptyset$, cf. item b.i.4) above, and otherwise from $U(f_n) \to U(\hat{f})$ when $\bar{x} = +\infty$, cf. item b.ii.3).

ii) Since $U(f_n) \overset{P\text{-a.s.}}{\to} U(\hat{f})$, the L^1 convergence of the utilities is equivalent to showing uniform integrability of $(U(f_n))_n$. Given the convergence of the expected utility, $E[U(f_n)] \uparrow E[U(\hat{f})]$, an argument "à la Scheffé" shows* that the uniform integrability of $(U(f_n))_n$ is equivalent to uniform integrability of any of the two families $(U^-(f_n))_n$, $(U^+(f_n))_n$. $U(0) = 0$ and strict monotonicity of U in a neighborhood of 0 imply $U^-(f_n) = -U(-f_n^-)$ and $U^+(f_n) = U(f_n^+)$. Suppose by contradiction that the family $(U^+(f_n))_n \equiv (U(f_n^+))_n$ is *not* uniformly integrable, and proceed as in [KS03, Lemma 1]. Given the supposed lack of uniform integrability, there exist disjoint measurable sets $(A_n)_n$ and a constant $\alpha > 0$ such that

$$E[U(f_n^+)I_{A_n}] \geq \alpha.$$

Set $g_n = \sum_{i=1}^n f_i^+ I_{A_i}$ and fix a $Q \in \mathcal{M} \cap P_V$ satisfying the Inada condition (21). $(f_n)_n$ is $L^1(Q)$ bounded by Proposition 4.9 and clearly $E_Q[g_n] \leq nC$ where C is a positive bound on the $L^1(Q)$ norms of the sequence $(f_n)_n$. In addition, $E[U(g_n)] \geq n\alpha$ because the $(A_n)_n$ are disjoint. Therefore,

$$\frac{u_Q(nC)}{nC} \geq \frac{E[U(g_n)]}{nC} \geq \frac{\alpha}{C} > 0$$

and passing to the limit when $n \uparrow \infty$ the conclusion contradicts (21). So the family $(U^+(f_n))_n$ is uniformly integrable, and $(U(f_n))_n$ as well, which means $U(f_n)$ tends in $L^1(P)$ to $U(\hat{f})$.

iii) To see that $f_n \to \hat{f}$ in $L^1(\hat{Q})$, from $U(\hat{f}) - \hat{f}\hat{y}\frac{d\hat{Q}}{dP} = V(\hat{y}\frac{d\hat{Q}}{dP}) \geq U(f_n) - f_n\hat{y}\frac{d\hat{Q}}{dP}$ the difference $U(\hat{f}) - U(f_n) - (\hat{f} - f_n)\hat{y}\frac{d\hat{Q}}{dP}$ is nonnegative and has P-expectation which tends to zero. Henceforth such difference is $L^1(P)$ convergent to 0, which, thanks to $L^1(P)$ convergence of $U(f) - U(f_n)$, yields $L^1(P)$ convergence to 0 of $(\hat{f} - f_n)\frac{d\hat{Q}}{dP}$.

From Fenchel inequality,

$$f_n^- y_Q \frac{dQ}{dP} \leq V(y_Q\frac{dQ}{dP}) - U(-f_n^-) \leq V(y_Q\frac{dQ}{dP}) + |U(f_n)|$$

and given the P-uniform integrability of $(U(f_n))_n$, proved in c.ii), the Q-uniform integrability of $(f_n^-)_n$ follows. Admitting $f_n \overset{P\text{-a.s.}}{\to} \hat{f}$ and $E_Q[\hat{f}] = 0$, and in view of $0 = \lim_n E_Q[f_n]$, an application of the Scheffé lemma again yields $f_n \overset{L^1(Q)}{\to} \hat{f}$.

iv) Recall that $X^n := H^n \cdot S$ are all $\hat{Q}$ *uniformly integrable* martingales by Lemma 3.5. Moreover, $\hat{Q}$ is a σ-martingale measure for S, so $X^n = (H^n \frac{1}{\varphi_{\hat{Q}}}) \cdot (\varphi_{\hat{Q}} \cdot S)$, where $M = \varphi_{\hat{Q}} \cdot S$

*The a.s. convergence and convergence of expected values do not necessarily imply convergence in L^1 if the variables are not positive (or uniformly bounded below by a r.v. in L^1): take e.g. $g_n, n \geq 2$, defined on $[0,1]$ as n on $[0, 1/n[$, $-n$ on $]1 - 1/n, 1]$ and 0 in $[1/n, 1 - 1/n]$. $g_n \to 0$ dx a.s. and $\int g_n(x)dx = 0$, but $(g_n)_n$ do not converge in $L^1(dx)$ to 0. So, some extra work is needed here.

is a $\hat{Q}$ martingale and $\varphi_{\hat{Q}} > 0$ holds $\hat{Q}$-a.s. The convergence (29) permits a straightforward application of a celebrated result by Yor [Yor78] on the closure of stochastic integrals, which gives an integral representation with respect to M of the limit $\hat{f}$ under $\hat{Q}$, $\hat{f} = H^* \cdot M_T = \widehat{H} \cdot S_T$, with $\widehat{H} = H^* \varphi_{\hat{Q}}$, and the optimal process $\widehat{X} := \widehat{H} \cdot S$ is also a $\hat{Q}$-uniformly integrable martingale.

When there is $Q \in \mathcal{M}^e \cap P_V$ with $E_Q[\hat{f}] = 0$ convergence (26) applies and by virtue of c.iii) the construction of $\widehat{H}$ can be performed under Q instead of $\hat{Q}$ and therefore $\widehat{H} \in L(S,P)$. To show $\widehat{H} \in \overline{\mathcal{H}}$, note we have already proved (27) so we only need convergence in P-probability of the wealth process at intermediate times. The convergence in (29) and the martingale property of the X^n and of $\widehat{H} \cdot S$ under Q imply

$$E_Q[|X^n_t - \widehat{H} \cdot S_t|] = E_Q\left[|E_Q[X^n_T - \widehat{H} \cdot S_T \mid \mathcal{F}_t]|\right] \overset{Jensen}{\leq} E_Q[|X^n_T - \widehat{H} \cdot S_T|].$$

Therefore, for any t, $X^n_t \to \widehat{H} \cdot S_t$ in $L^1(Q)$ and henceforth in Q-probability, which is equivalent to convergence in P-probability. Thus, $\widehat{H} \in \overline{\mathcal{H}}$ follows.

□

4.4 Auxiliary results

Proof of Lemma 4.3. $u_Q(x) < +\infty$ follows from Fenchel inequality and from finite entropy of Q: if X satisfies $E_Q[X] \leq x$, $E[U(X)] \leq xy_Q + v_Q(y_Q)$, with y_Q from Definition 3.4.

The utility maximization problem $\sup_{E_Q[X] \leq x} E[U(X)]$ can be rewritten over the utility-induced Orlicz space $L^{\hat{U}}(P)$ defined in (2.2). This can be done because: i) the supremum will be reached over those X such that $E[U(X)]$ is finite, so that $-X^- \in L^{\hat{U}}(P)$; ii) if X^- satisfies $E[U(-X^-)]$ is finite, then the truncated sequence $X_n = X \wedge n$ is also in the Orlicz space and by Fatou Lemma in the limit it delivers the same expected utility from X; iii) $L^{\hat{U}}(P) \subseteq L^1(Q)$, which follows from $Q \in P_V$, from (4) and Fenchel inequality (this also implies Q is in the topological dual of $L^{\hat{U}}$). Therefore, $u_Q(x) = \sup_{X \in L^{\hat{U}}, E_Q[X] \leq x} E[U(X)]$. On $L^{\hat{U}}$, the concave functional $I_U(X) := E[U(X)]$ is proper:

$$X \in L^{\hat{U}} \Rightarrow X \in L^1(P) \text{ so that } E[U(X)] \overset{\text{Jensen}}{\leq} U(E[X]) < +\infty.$$

Moreover, I_U has a continuity point which belongs to the maximization domain $D = \{X \in L^{\hat{U}} \mid E_Q[X] \leq x\}$. This is more subtle to check, but it can be proved that the set

$$\mathcal{B} := \{X \in L^{\hat{U}} \mid E[U(-(1+\epsilon)X^-)] > -\infty \text{ for some } \epsilon > 0\},$$

coincides with the interior of the proper domain of I_U (see [BFG08, Lemma 4.1] modulo a sign change), where I_U is automatically continuous by the Extended Namioka Theorem (see e.g. [BF09]). Then, as $x > \inf \operatorname{dom} U$, the constant x is in $\mathcal{B} \cap D$.
The dual formula (22) is thus a consequence of Fenchel Duality Theorem [Bre83, Chapter 1], of the fact that the polar set of the constraint $C := \{X \mid E_Q[X] \leq x\} \supseteq -L^\infty_+$, i.e. the set $\{\mu \in (L^{\hat{U}})^* \mid$

$\mu(X) \leq x \,\forall X \in C\}$, by the Bipolar Theorem is the positive ray $\{yQ \mid y \geq 0\}$, and of the expression of the convex conjugate $(I_U)^*$ of I_U over the variables $y\frac{dQ}{dP}$: $(I_U)^*(y\frac{dQ}{dP}) = E[V(y\frac{dQ}{dP})] = v_Q(y)$. □

Proposition 4.9. *Suppose $(k_n)_n$ is a sequence of random variables such that $(E[U(k_n)])_n$ is bounded below and assume $(E_{\tilde{Q}}[k_n])_n$ is bounded above for some $\tilde{Q} \in P_V$ satisfying the Inada condition* (21)*. Then the following statements hold*

i) $U(k_n)$ is $L^1(P)$-bounded;

ii) k_n is $L^1(Q)$-bounded for any *$Q \in P_V$ for which $E_Q[k_n]$ is bounded above. The indirect utility u_Q need not satisfy the Inada condition* (21)*.*

Proof. In this proof c refers to a constant, not necessarily the same on each line.

i) By hypothesis there is $0 < y_1 < y_2$ such that $v_{\tilde{Q}}(y_i) < +\infty$ for $i = 1, 2$. The Fenchel inequality implies

$$E[U(-k_n^-)] \leq v_{\tilde{Q}}(y_2) - y_2 E_{\tilde{Q}}[k_n^-], \tag{34}$$

$$E[U(k_n^+)] \leq v_{\tilde{Q}}(y_1) + y_1 E_{\tilde{Q}}[k_n^+], \tag{35}$$

which yields

$$E[U(k_n)] \leq c + y_1 E_{\tilde{Q}}[k_n] - (y_2 - y_1) E_{\tilde{Q}}[k_n^-]. \tag{36}$$

By assumption, $(E_{\tilde{Q}}[k_n])_n$ is bounded above and $(E[U(k_n)])_n$ is bounded below, whereby one concludes from (36) and from $y_2 - y_1 > 0$ that $(E_{\tilde{Q}}[k_n^-])_n$ is bounded and consequently $(E_{\tilde{Q}}[k_n^+])_n$ is also bounded. Finally, by (35) the sequence $(E[U(k_n^+)])_n$ is bounded. Since $U(k_n^+) \geq 0, U(-k_n^-) \leq 0$ and $(E[U(k_n)])_n$ is bounded below the $L^1(P)$-boundedness of $U(k_n)$ follows.

ii) By construction the inequality (34) applies for any $Q \in P_V$, i.e. there is $y_Q > 0$ such that

$$E[U(-k_n^-)] \leq c - y_Q E_Q[k_n^-]. \tag{37}$$

By i) the sequence $(E[U(-k_n^-)])_n$ is bounded below whereby $(E_Q[k_n^-])_n$ must be bounded. As in i), this and boundedness from above of the expectations $(E_Q[k_n])_n$ ensure $(E_Q[k_n^+])_n$ is also bounded, which completes the proof.

□

Lemma 4.10. *Let $Q \in P_V$ verify $E_Q[X_T] = 0$ for all $X = H \cdot S, H \in \mathcal{H}$. Then $Q \in \mathcal{M} \cap P_V$.*

Proof. We just need to show $Q \in \mathcal{M}$. Consider $S \in \mathscr{S}_\sigma^{\hat{U}} \setminus \mathscr{S}_{\text{loc}}^{\hat{U}}$, fix any $\mathcal{I}$-localizing φ from Assumption 3.1 and let $S' = \varphi \cdot S$. For any $A \in \mathcal{F}_s, s \in [0, T[, t > s$ let $H = I_A \varphi I_{]s,t]}$, which is in $\mathcal{H}$. Since $H \cdot S = (I_A I_{]s,t]}) \cdot S'$ and $E_Q[H \cdot S_T] = E_Q[I_A(S'_t - S'_s)] = 0$, for all $A \in \mathcal{F}_s, s < t$, S' is a then Q-martingale, and hence $Q \in \mathcal{M}$. For $S \in \mathscr{S}_{\text{loc}}^{\hat{U}}$ we proceed as above, replacing φ with $I_{[0,\tau_n]}$. □

5 Connections to literature

5.1 Sufficient conditions

5.1.1 On Assumption 3.1

The class $\mathscr{S}_\sigma^{\hat{U}}$ introduced in this paper appears to be the most comprehensive class of price processes to have been studied in the context of utility maximization to date. Most papers in the literature consider $S \in \mathscr{S}_{\mathrm{loc}}^{\infty}$. Sigma-bounded processes, that is processes in $\mathscr{S}_\sigma^{\infty}$, appear in Kramkov and Sîrbu [KSi06]. It can be shown, cf. [ČK07, Lemma A.2] that for $p > 1$ the class of processes which are *locally in* L^p coincides with $\mathscr{S}_{\mathrm{loc}}^{p}$. These processes feature in Delbaen and Schachermayer [DS96]. There seem to be no papers that explicitly use $S \in \mathscr{S}_\sigma^{p}$.

The $\mathcal{I}$-localizing strategy φ from Assumption 3.1 has already appeared in the literature on utility maximization. It plays an important role in the work of Biagini [Bia04] where the maximal process $(\varphi \cdot S)^*$ is taken as a dynamic loss control for the strategies in the utility maximization problem. Within setups of increasing generality in Biagini and Frittelli [BF05, BF08] φ gives rise to so-called *suitable* and *(weakly) compatible* loss control variables $W := (\varphi \cdot S)_T^*$.

5.1.2 On Assumption 4.2

Condition (20) is automatically satisfied for utilities finite on a half-line. For utilities finite on $\mathbb{R}$ it makes sure that the claim $\hat{f}$ constructed via the Komlós theorem satisfies the budget constraint $E_Q[\hat{f}] \leq x$ for every $Q \in \mathcal{M} \cap P_V$.

Since for any $Q \in \mathcal{M} \cap P_V$ one has $u_Q(x) \geq u_{\mathcal{H}}(x) \geq U(x)$, and U is monotone, condition (21) implies an identical Inada condition both on the indirect utility $u_{\mathcal{H}}$ and also on the original utility U at $+\infty$. For this reason condition (21) is slightly stronger than the truly necessary and sufficient condition $\lim_{x\to+\infty} u_{\mathcal{H}'}(x)/x = 0$ in [KS03] when U is finite on a half-line. It is an open question whether condition (21) can be weakened to

$$\text{there exists } Q \in \mathcal{M} \cap P_V \text{ and } \lim_{x\to+\infty} u_{\mathcal{H}}(x)/x = 0. \tag{38}$$

At the same time, to the best of our knowledge Assumption 4.2 is strictly weaker than any other assumption used in the current literature for U finite on $\mathbb{R}$. In current references, the typical assumption is RAE(U), which implies $v_Q(y) < +\infty$ for all $y > 0$ and for all $Q \in P_V$ by [Sch01, Corollary 4.2], whence Assumption 4.2 necessarily holds. In the non-smooth utility case studied by Bouchard et al. [BTZ04], equivalent asymptotic elasticity conditions are imposed on the Fenchel conjugate V,

$$\lim_{y\to 0_+} \frac{|V'_-(y)|y}{V(y)} < +\infty, \qquad \lim_{y\to+\infty} \frac{|V'_+(y)|y}{V(y)} < +\infty. \tag{39}$$

These again imply $v_Q(y) < +\infty$ for all $y > 0$ and for all $Q \in P_V$, see [BTZ04, Lemma 2.3].

On the other hand, Biagini and Frittelli [BF05, BF08] do not require RAE(U), but instead assume that $v_Q(y)$ is finite for all $Q \in \mathcal{M} \cap P_V$ and all $y > 0$, which is weaker than RAE(U) but clearly stronger than Assumption 4.2 by virtue of Corollary 4.4. Therefore Assumption 4.2 seems the best for a unified treatment of utility maximization problems, regardless of the domain of U.

5.1.3 On Assumption 4.7

General duality theory and Orlicz spaces theory show that the dual problem associated with the utility maximization over a general Orlicz space may contain singular parts, see [BF08]. The dual variables have, in general, a two-way decomposition $z = z_r + z_s$, where z_r only can be identified with a measure absolutely continuous wrt P. The normalized dual variables are given by

$$D := \{z \in (L^{\hat{U}})^*_+ \mid z(\Omega) = 1, E[V(y\frac{dz_r}{dP})] = 0 \text{ for some } y > 0, \text{ and } \langle z, H \cdot S_T \rangle \leq 0 \, \forall H \in \mathcal{H}\},$$

where $\langle \cdot, \cdot \rangle$ indicates the bilinear form for the dual system $(L^{\hat{U}}, (L^{\hat{U}})^*)$. Clearly, as $\mathcal{H}$ is a vector space, the inequality $\langle z, H \cdot S_T \rangle \leq 0$ may be replaced by an equality. [BF08, Theorem 21] yields the following characterization:

Theorem 5.1. *Under Assumptions 3.1 and $u_{\mathcal{H}}(x) < U(+\infty)$, the following dual relation holds:*

$$u_{\mathcal{H}}(x) = \min_{y \geq 0, z \in D} \{y(x + \|z_s\|) + v_{z_r}(y)\}. \tag{40}$$

Now, Lemma 4.10 guarantees that whenever $z \in D$ has zero singular part, it is an element of $\mathcal{M} \cap P_V$. This shows that Assumption 4.7 is equivalent to requiring $z_s = 0$ in the general duality (40). Note that (40) holds automatically under Assumptions 3.1, 4.2 and $x < \bar{x}$. In fact, Assumption 4.2 implies in particular $\mathcal{M} \cap P_V \neq \emptyset$ and when $x < \bar{x}$ Proposition 4.5 implies $u_{\mathcal{H}}(x) \leq u_Q(x) < U(+\infty)$ for any $Q \in \mathcal{M} \cap P_V$. *Therefore the assumptions in Theorem 4.8 already imply the general duality* (40).

The appropriate modification of Theorem 4.8 which would work without Assumption 4.7 remains an interesting area for future research.

5.2 Characterization of the optimal solution

As a general comment, in items b) and c) of Theorem 4.8 we provide a desirable approximation result both for the value function $u_{\mathcal{H}}(x) = u_{\overline{\mathcal{H}}}(x)$ and for the optimizer $\hat{f}$, in a unified framework irrespective of the effective domain of U. The approximation holds under very mild conditions on U which may lack strict monotonicity and strict concavity. These results are novel for utility functions finite on a half-line.

For U finite on $\mathbb{R}$ we extend results of Bouchard et al. [BTZ04] to non-locally bounded S under a weaker condition from Assumption 4.2 instead of RAE (39), while considerably simplifying the required proofs thanks to the Orlicz duality approach. In the remainder of this section we restrict our attention to U finite on $\mathbb{R}$.

5.2.1 $\hat{Q}$ equivalent to P

In the mathematically pleasant case $\hat{Q} \sim P$ (see Remark 5.2 for a list of sufficient conditions for $\hat{Q} \sim P$) our framework is an improvement over the current literature: [Sch01], [KabStr02], [Str03], [OŽ09], [BTZ04] all assume S locally bounded. Moreover, when $\hat{Q} \sim P$ we provide a sequence of simple strategies which approximate the optimal strategy. Approximation by simple strategies has so far been shown only for exponential utility [Str03, Theorem 5] for locally bounded S and for expected utility only – not in the stronger sense of $L^1(P)$ convergence given here.

For comparison, Schachermayer [Sch01] proves an approximation similar to (27) for the optimal solution via integrals bounded from below. He considers locally bounded S and regular U, satisfying RAE(U) and this is the very first article to show in such generality an integral representation for $\hat{f} = \widehat{H} \cdot S_T$ when $\hat{Q} \sim P$. This work is extended further by Bouchard et al. [BTZ04] who allow for non-differentiable and non-monotone utility functions. Moreover, in [Sch03] $\widehat{H}$ is shown to be in the supermartingale class of strategies through a (hard) contradiction argument, which is later extended by [BF07] to non locally bounded S along the same line of proofs. Here we show the supermartingale property of $\widehat{H}$ in a general setup and in a very natural way, as $\widehat{H} \in \overline{\mathcal{H}}$, and moreover provide its approximation via simple integrands.

Remark 5.2 (On sufficient conditions for $\hat{Q} \sim P$)*.* The following two conditions are well-established in the literature. First, utility function unbounded above implies $\hat{Q} \sim P$. This is straightforward, as $V(0) = U(+\infty) = +\infty$ while $E[V(\hat{y}\frac{d\hat{Q}}{dP})]$ must be finite. Second, when U is strictly monotone but bounded, a sufficient condition for $\hat{Q} \sim P$ is the existence of an equivalent σ-martingale measure with finite entropy. This follows from b.i) and b.iii) in Theorem 4.8, on observing that $\bar{x} = +\infty$.

5.2.2 $\hat{Q}$ not equivalent to P

In the less appealing case when $\hat{Q}$ is not equivalent to P, which may happen only when U is bounded above, we also improve the known literature. Note that all results in Theorem 4.8 still hold true, apart from the last part of item c.iv). If U is strictly monotone (a typical example is the exponential utility) an approximation result for $\hat{f}$ via terminal values of integrals under P was first shown by Acciaio [A05], under the following technical conditions,

i) U is differentiable, monotone, strictly concave and it satisfies RAE (17, 18);

ii) S is locally bounded;

iii) the stopping times of the filtration are predictable.

Acciaio builds a sequence of integrals $\widetilde{H}_n \cdot S_T$, whose expected utility tends to the optimum, and which satisfies $(x + \widetilde{H}_n \cdot S_T) \to \hat{f}$ P-a.s.

Our setup allows us to remove the technical conditions above and to prove the stronger convergence in (27), $U(x + f_n) \overset{L^1(P)}{\to} U(\hat{f})$, with $f_n = H^n \cdot S_T, H^n \in \mathcal{H}$, which clearly implies convergence of expectations. And when U is invertible, as in [A05], by passing to a subsequence if necessary, $x + f_n \to \hat{f}$ P-a.s.

When U is not strictly monotone, that is $\overline{x} < +\infty$, the sufficient conditions for $\hat{Q} \sim P$ known in the monotone case do not work; here typically $\hat{Q}$ is *not* equivalent to P even when there are equivalent probabilities in $\mathcal{M} \cap P_V$. We nonetheless recover existence of the optimal solution provided that $E_Q[\hat{f}] = x$ for *some* $Q \in \mathcal{M}^e \cap P_V$. This mild sufficient condition appears to be new in the literature.

References

[A05] B. Acciaio. *Absolutely continuous optimal martingale measures.* Statistics and Decisions 23, 81-100, 2005.

[AS94] J.-P. Ansel and C. Stricker. *Couverture des actifs contingents et prix maximum.* Annales de l'Institut Henri Poincaré. Probabilités et Statistiques 30(2), 303-315,1994.

[BeF02] F. Bellini and M. Frittelli. *On the existence of minimax martingale measures.* Mathematical Finance 12(1), 1-21, 2002.

[Bia04] S. Biagini. *A new class of strategies and application to utility maximization for unbounded processes.* Technical report of the University of Perugia, 2004 available at http://sites.google.com/site/sarabiagini/.

[BF05] S. Biagini and M. Frittelli. *Utility Maximization in Incomplete Markets for Unbounded Processes.* Finance and Stochastics 9, 493-517, 2005.

[BF07] S. Biagini and M. Frittelli. *The Supermartingale Property of the Optimal Wealth Process for General Semimartingales.* Finance and Stochastics, 11(2) 253-266, 2007.

[BF08] S. Biagini and M. Frittelli, *A Unified Framework for Utility Maximization Problems: an Orlicz Space Approach.* Annals of Applied Probability 18(3), 929-966, 2008.

[BF09] S. Biagini and M. Frittelli, *On the extension of the Namioka-Klee Theorem and on the Fatou property for risk measures.* In F. Delbaen, M. Rasonyi, C. Stricker eds. Optimality and Risk – Modern Trends in Mathematical Finance: Youri Kabanov's Festschrift, Springer.

[BFG08] S. Biagini, M. Frittelli and M. Grasselli. *Indifference price with general semimartingales.* To appear in Mathematical Finance, preprint version available at http://sites.google.com/site/sarabiagini/.

[BG09] S. Biagini and P. Guasoni. *Relaxed Utility Maximization in Complete Markets.* Submitted 2009, available at http://sites.google.com/site/sarabiagini/.

[BS09] S. Biagini and M. Sîrbu. *A note on investment opportunities when the credit line is infinite.* Submitted 2009, available at http://sites.google.com/site/sarabiagini/.

[B02] B. Bouchard. *Utility maximization on the real line under proportional transaction costs.* Finance and Stochastics 6(4), 495-516, 2002.

[Bre83] H. Brezis. *Analyse fonctionnelle.* Masson, 1983.

[BTZ04] B. Bouchard, N. Touzi and A. Zeghal. *Dual Formulation of the Utility Maximization Problem: The Case of Nonsmooth Utility.* Annals of Applied Probability, 14(2), 678-717, 2004.

[ČK07] A. Černý and J. Kallsen. *On the structure of general mean–variance hedging strategies.* The Annals of Probability, 35(4), 1479-1531, 2007.

[DGRSSS02] F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer and Ch. Stricker. *Exponential hedging and entropic penalties.* Mathematical Finance, 12(2), 99-123, 2002.

[DS96] F. Delbaen and W. Schachermayer. *Attainable Claims with p'th Moments.* Annales de l'Institut Henri Poincaré 32, 743-763, 1996.

[DS98] F. Delbaen and W. Schachermayer. *The Fundamental Theorem of Asset Pricing for Unbounded Stochastic Processes.* Mathematische Annalen 312, 215-250, 1998.

[DS99] F. Delbaen and W. Schachermayer. *A Compactness Principle for Bounded Sequences of Martingales with Applications.* Proceedings of the Seminar on Stoch. Analysis, Random Fields and Applications, Progress in Probability 45, 137-173, 1999.

[DS06] F. Delbaen and W. Schachermayer. *The Mathematics of Arbitrage.* Springer Finance, Springer 2006.

[CSW01] J. Cvitanić, W. Schachermayer, and H. Wang. *Utility Maximization in Incomplete Markets with Random Endowment.* Finance and Stochastics 5, 237-259, 2001.

[E80] M. Émery. *Compensation de Processus à Variation Finie Non Localement Intégrables.* Séminaire de Probabilités XIV, 1978/79, Lecture Notes in Mathematics, 784, 152-160, Springer Berlin 1980.

[GR01] T. Goll and L. Rüschendorf. *Minimax and minimal distance martingale measures and their relationship to portfolio optimization.* Finance and Stochastics 5, 557-581, 2001.

[HK79] J. M. Harrison and D. M. Kreps. *Martingales and Arbitrage in Multiperiod Securities Market.* Journal of Economic Theory 20, 381-408, 1979.

[JS03] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, second edition, 2003.

[KabStr02] Y. Kabanov and C. Stricker. *On the optimal portfolio for the exponential utility maximization: Remarks to the six-authors paper.* Mathematical Finance 12(2), 125-134, 2002.

[Ka04] J. Kallsen. *σ-Localization and σ-Martingales.* Theory of Probability and its Applications 48(1), 152-163, 2004.

[KS99] D. Kramkov and W. Schachermayer. *The asymptotic elasticity of utility functions and optimal investment in incomplete markets.* Annals of Applied Probability 9(3), 904-950.

[KS03] D. Kramkov and W. Schachermayer. *Necessary and sufficient conditions in the problem of optimal investment in incomplete markets.* Annals of Applied Probability 13(4), 1504-1516, 2003.

[KSi06] D. Kramkov amd M. Sîrbu. *On the two-times differentiability of the value functions in the problem of optimal investment in incomplete markets.* Annals of Applied Probability, 16(3), 1352-1384, 2006.

[LV87] F. Liese and I. Vajda. *Convex Statistical Distances.* Leipzig: Teubner, 1987.

[M78] P.-A. Meyer. *Inégalités de normes pour les intégrales stochastiques.* Séminaire de Probabilités XII, 1976/77, Lecture Notes in Mathematics, 649, 757-762, Springer Berlin 1978.

[O02] M. P. Owen. *Utility based optimal hedging in incomplete markets.* The Annals of Applied Probability, 12(2), 691709, 2002.

[OŽ09] M. P. Owen and G. Žitković. *Optimal Investment with an Unbounded Random Endowment and Utility-Based Pricing.* Mathematical Finance 19(1), 129-159, 2009.

[RR91] M. M. Rao and Z. D. Ren. *Theory of Orlicz Spaces.* Marcel Dekker Inc., NY, 1991.

[Pr05] P. Protter. *Stochastic Integration and Differential Equations.* Springer, 2nd ed, 2005.

[RS05] M. Ràsonyi and L. Stettner. *On utility maximization in discrete-time financial market models.* The Annals of Applied Probability 15(2), 1367-1395, 2005.

[R70] R. T. Rockafellar. *Convex Analysis.* Princeton University Press, Princeton 1970.

[Sch03] W. Schachermayer. *A super-martingale property of the optimal portfolio process.* Finance and Stochastics 4, 433-457, 2003.

[Sch01] W. Schachermayer. *Optimal investment in incomplete markets when wealth may become negative.* Annals of Applied Probability 11(3), 694-734, 2001.

[Str03] C. Stricker. *Simple strategies in exponential utility maximization.* Séminaire de Probabilités, XXXVI 1801, 415-418, 2003.

[Yor78] M. Yor. *Sous-espaces denses dans L^1 ou H^1.* Séminaire de Probabilités XII, Lect. Notes Mathematics 649, 265-309, 1978.