Adaptive wavelet Galerkin methods: Extension to unbounded domains and fast evaluation of system matrices

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Overview

Motivation: Wavelet methods in numerical finance

Adaptive wavelet (Galerkin) methods on unbounded domains

Fast evaluation of system matrices
**Challenges in numerical solutions of PIDEs arising in numerical finance**

Computation of European option prices via numer. solution of PIDE problem ([RSW10])

\[
\begin{cases}
\frac{\partial u(\tau, x)}{\partial \tau} - B^X[u](\tau, x) = 0, & x \in \mathbb{R}^n, \ \tau \in (0, T], \\
u(0, x) = h(x), & x \in \mathbb{R}^n,
\end{cases}
\]

where \( X = (X^1_t, \ldots, X^n_t)_{t \geq 0} \) is Lévy process with **infinitesimal generator**

\[
B^X[w] := \frac{1}{2} \sum_{i,j=1}^n Q_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^n \gamma_i \frac{\partial w}{\partial x_i} + \int_{\mathbb{R}^n} (w(\cdot + y) - w + 1_{\{|y| \leq 1\}} \sum_{i=1}^n y_i \frac{\partial w}{\partial x_i}) \nu(\,d y).
\]

- **Unbounded** domain:
  \( \rightsquigarrow \) A priori versus adaptive truncation.

- **Non-local** operator \( B^X \):
  \( \rightsquigarrow \) Wavelet compression.

- **Spatial dimension**:
  \( \rightsquigarrow \) Avoid curse of dimension by sparse grids or adaptive methods.

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For the computation of European option prices in multi-dimensional Lévy models

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\begin{aligned}
\frac{\partial}{\partial \tau} u_R(\tau, x) - B^X[u_R](\tau, x) &= 0, \quad x \in (-R, R)^n, \ \tau \in (0, T], \\
 u_R(0, x) &= h(x), \quad x \in (-R, R)^n, \\
 u_R(\tau, x) &= 0, \quad x \in \mathbb{R}^n \setminus (-R, R)^n, \ \tau \in (0, T],
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  \item \textbf{Unbounded} domain:
    \implies A priori versus adaptive truncation.
  \item \textbf{Non-local} operator \( B^X \):
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  \item \textbf{Spatial dimension}:
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\end{itemize}
Adaptive solution of the option pricing problem

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u(0, x) = h(x), & x \in \mathbb{R}^n.
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\]

For the discretization of \( B^X \), a common assumption is (compare, e.g., [Hep11]):

\[
\langle v_1 \otimes \cdots \otimes v_n, B^X[w_1 \otimes \cdots \otimes w_n] \rangle = \left( \sum_{m=1}^{M} \alpha_m \cdot a_m^{(1)}(v_1, w_1) \otimes \cdots \otimes a_m^{(n)}(v_n, w_n) \right),
\]

where for \( i \in \{1, \ldots, n\} \) at most one univariate bilinear form \( a_m^{(i)} \) is non-local.

Where wavelets find applications...

- Fast approximate evaluation of non-local bilinear forms (\( \rightsquigarrow \) wavelet compression).
- Non-smooth payoff \( h \) requires adaptive refinement (see also [BHPS12]).
- Treatment of unbounded domain \( \mathbb{R}^n \): Towards optimal balancing of truncation and discretization error.
- Fast exact evaluation of local bilinear forms.

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Overview

Motivation: Wavelet methods in numerical finance

Adaptive wavelet (Galerkin) methods on unbounded domains

Fast evaluation of system matrices
Given $f \in \mathcal{X}'$, consider a linear, well-posed operator equation:

$$\mathcal{A}[u] = f \text{ in } \mathcal{X'},$$

- $\mathcal{X}$ is a Sobolev space over an unbounded domain $\Omega$ (e.g., $\mathcal{X} = H^1(\mathbb{R}^n)$),
- $\mathcal{A} : \mathcal{X} \rightarrow \mathcal{X}'$ is boundedly invertible, self-adjoint.

### Requirements for numerical scheme

- Adaptive domain truncation and local refinement.
- Optimal convergence rate under weak smoothness assumptions.
- Linear complexity for a large class of operators $\mathcal{A}$.

### New approach

Existing methods (Infinite Elements, BEM, . . .) do either not cover all of these requirements or are only applicable for special classes of $\mathcal{A}$.

- *Equivalent formulation* in an infinite-dimensional sequence space $\ell_2$ of *wavelet coefficients*,
  $$\mathcal{A}[u] = f \text{ in } \mathcal{X'} \iff \mathcal{A}u = f \text{ in } \ell_2.$$  

- Approximation of $u$ by means of *adaptive wavelet methods*.
Univariate Riesz wavelet bases

Let $\Omega \subseteq \mathbb{R}$ be a domain (possibly unbounded).

We consider a Riesz wavelet basis $\Psi = \{\psi_\lambda : \lambda \in \mathcal{J}\}$ that characterizes univariate Sobolev spaces $\mathcal{H}^s(\Omega)$ (possibly incorporating essential homogeneous bc’s)

$$
\overline{C}_{\mathcal{H}^s} \|v\|^2_{\ell_2(\mathcal{J})} \leq \sum_{\lambda \in \mathcal{J}} v_\lambda \psi_\lambda / \|\psi_\lambda\|_{\mathcal{H}^s(\Omega)}^2 \leq \overline{C}_{\mathcal{H}^s} \|v\|^2_{\ell_2(\mathcal{J})}, \quad \forall v = v^\top D^s \Psi \in \mathcal{H}^s(\Omega),
$$

where $-\tilde{\gamma} < s < \gamma$. Here, $\Psi$ is a column vector and $D^s = \text{diag} \left[ (\|\psi_\lambda\|_{\mathcal{H}^s}^{-1})_{\lambda \in \mathcal{J}} \right]$ a bi-infinite diagonal matrix.

Standard wavelet assumptions / notations

- $\psi_\lambda := 2^{i/2} \psi^{(i)}(2^j \cdot -k)$, $\lambda = (i, j, k)$.
- $\psi_\lambda$ are piecewise polynomials of order $d$.
- Local support: diam supp $\psi_\lambda \approx 2^{-j}$.
- Example:

$$
\Psi_{L_2(\mathbb{R})} := \{2^{j_0/2} \phi(2^{j_0} \cdot -k) : k \in \mathbb{Z}\} \cup \{2^{i/2} \psi(2^j \cdot -k) : j \geq j_0, k \in \mathbb{Z}\}
$$
Tensor product Riesz wavelet bases

Let now $\Omega := \Omega_1 \times \cdots \times \Omega_n$ be a **product domain** (possibly unbounded).

With $n$ univariate Riesz wavelet bases $\psi^{(i)}$ for $L_2(\Omega_i)$ ($i \in \{1, \ldots, n\}$),

$$
\Psi := \psi^{(1)} \otimes \cdots \otimes \psi^{(n)} = \{\psi_\lambda := \psi_{\lambda_1} \otimes \cdots \otimes \psi_{\lambda_n} : \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{J}\} 
$$

is a Riesz wavelet basis for $L_2(\Omega)$ where $\mathcal{J} := \mathcal{J}^{(1)} \times \cdots \times \mathcal{J}^{(n)}$.

For Sobolev spaces $\mathcal{X}$ over $\Omega$ that can be characterized by (intersections of) tensor products of univariate Sobolev spaces,

$$
\Psi^\mathcal{X} := D^\mathcal{X} \Psi := \{\psi_\lambda / \|\psi_\lambda\|_\mathcal{X} : \lambda \in \mathcal{J}\}, \quad D^\mathcal{X} := \text{diag} \left[(\|\psi_\lambda\|_\mathcal{X}^{-1})_{\lambda \in \mathcal{J}}\right],
$$

is a tensor product Riesz wavelet basis for $\mathcal{X}$.

**Tensor product wavelets**

- $\psi_\lambda$ are piecewise polynomials.
- Local **anisotropic** support:

$$
|\text{supp } \psi_\lambda| \sim 2^{-|\lambda_1| + \cdots + |\lambda_n|}.
$$
Wavelet discretization of operator equations

Unique expansion of $u$ in $D^x \Psi$, $u = u^\top (D^x \Psi) := \sum_{\lambda \in \mathcal{J}} u_\lambda D^x_\lambda \psi_\lambda$, yields ([CDD01])

\[ \langle v, A[u] \rangle = \langle v, f \rangle, \ \forall v \in X \iff \langle D^x \Psi, A[D^x \Psi] \rangle u = \langle D^x \Psi, f \rangle \iff Au = f \text{ in } \ell_2(\mathcal{J}). \]

Infinite load vector $f = D^x [(\langle \psi_\lambda, f \rangle)_{\lambda \in \mathcal{J}}] \in \ell_2(\mathcal{J})$.

Boundedly invertible bi-infinite system matrix $A = D^x [(\langle \psi_\lambda, A[\psi_\mu] \rangle)_{\lambda, \mu \in \mathcal{J}}] D^x$.

$\implies$: This discretization only requires a Riesz basis (independent of the domain).

Error estimate: Riesz basis property guarantees for both bounded and unbounded domains $\Omega$ that

\[ C \|u - u_\Lambda\|\ell_2 \leq \varepsilon \implies \|u - u_\Lambda^\top (D^x \Psi)\|X \leq \varepsilon \]

$\implies$: Local refinement and adaptive domain truncation via selecting significant wavelet indices from $u$.

Adaptive approximation of $Au = f$

Solve finite Galerkin systems on nested index sets $\Lambda_k \subset \Lambda_{k+1} \subset \cdots \subset J$ ([GHS07]).

\textbf{AWGM}[$\varepsilon$]

\textbf{for} $k = 0; \|r_k\|_2 \leq \varepsilon ; \, k = k + 1$ \textbf{do}

Compute approx. solution $w_{\Lambda_k}$ of

$$A_{\Lambda_k} u_{\Lambda_k} = f_{\Lambda_k}.$$ 

Compute approximation $r_k$ of the \textit{infinite} residual

$$f - A w_{\Lambda_k}.$$ 

Compute smallest $\Lambda_{k+1} \supset \Lambda_k$ s.t.

$$\|P_{\Lambda_{k+1}} r_k\|_2 \geq \mu \|r_k\|_2 \quad \text{(1)}$$

for $\mu \in (0, 1)$ where $P_{\Lambda} v := v|_{\Lambda}$.

\textbf{end for}

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&\quad A_{\Lambda_k} u_{\Lambda_k} = f_{\Lambda_k}. \\
\quad &\text{Compute approximation } r_k \text{ of the infinite residual} \\
&\quad f - Aw_{\Lambda_k}. \\
\quad &\text{Compute smallest } \Lambda_{k+1} \supset \Lambda_k \text{ s.t.} \\
&\quad \|P_{\Lambda_{k+1}} r_k\|_2 \geq \mu \|r_k\|_2 \quad \text{(1)} \\
&\quad \text{for } \mu \in (0, 1) \text{ where } P_{\Lambda} v := v|_{\Lambda}. \\
\text{end for}
\end{align*}

Idea from [KU12]:

In (1), significant wavelet indices $\lambda \in \mathcal{J}$ are added for

- local refinement of singularities, and
- domain extension.

Approximation of the infinite residual

\begin{align*}
\Rightarrow &\quad \|\text{APPLY}[w_{\Lambda_k}, \eta] - Aw_{\Lambda_k}\|_2 \leq \eta, \\
\Rightarrow &\quad \|\text{RHS}[\eta] - f\|_2 \leq \eta.
\end{align*}

For suff. small $\eta$, define $r_k$ as

\[ r_k := \text{RHS}[\eta] - \text{APPLY}[w_{\Lambda_k}, \eta]. \]

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\textbf{end for}

Best \( N \)-term approximation:

\[
\| u - u_N \|_2 \leq CN^{-s}.
\]

\( \Rightarrow \) (Tensor-)Besov regularity of \( u \) ([SU09]).

\( \Rightarrow \) Asymptotic dimension-independent convergence rates are possible.

\[\text{[GHS07, Theorem 2.7]}\]

\textbf{Given suitable} routines \textbf{APPLY} and \textbf{RHS}:

\[
\| u - u_{\Lambda_k} \|_2 \sim \| u - u_{N_k} \|_2 \leq N_k^{-s}
\]

\text{where} \( N_k := \text{supp} w_{\Lambda_k} \).

\( \Rightarrow \) Linear complexity.

\( \Rightarrow \) Optimal scheme for bounded and unbounded domains
## Adaptive wavelet algorithms: From bounded to unbounded domains

### Bounded domain: $\Omega = (a, b)$

- $\mathcal{J}^\Omega := \{(j, k) : j \geq j_0, k \in \mathbb{I}_j\}, j_0 \geq 0$.
- ✓ Fixed minimal level $j_0$.
- ✓ Realization of RHS (e.g. [GHS07]):

### Unbounded domain: $\mathbb{R}$ (cf. [KU12])

- $\mathcal{J}^\mathbb{R} := \{(j, k) : j \geq j_0, k \in \mathbb{Z}\}, j_0 \in \mathbb{Z}$.
- ✓ Good choice of $j_0$:
  - $\bowtie$ Diameter initial domain: $\approx 2^{-j_0}$.
- ✓ Construct finite $\nabla_\eta \subset \mathcal{J}^\mathbb{R}$ with
  $$\|f - f|_{\nabla_\eta}\|_{\ell^2} \leq \eta.$$  
  - $\bowtie$ Bound for translation indices.
- ✓ Special treatment of negative levels:
  $$A = \begin{pmatrix} A_{+-} & A_{++} \\ A_{-+} & A_{--} \end{pmatrix}.$$  
- ✓ Adaptive domain truncation, local refinement, convergence.

---


Adaptive approximation of \( \mathbf{A} \mathbf{u} = \mathbf{f} \): Numerical experiment

Solve finite Galerkin systems on nested index sets \( \Lambda_k \subset \Lambda_{k+1} \subset \cdots \subset \mathcal{J} \) ([GHS07]).

\[
\text{AWGM}[\varepsilon]
\]
\[
\text{for } k = 0; \| \mathbf{r}_k \|_2 \leq \varepsilon; k = k + 1 \text{ do}
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Compute approx. solution \( \mathbf{w}_{\Lambda_k} \) of
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\mathbf{f} - \mathbf{A} \mathbf{w}_{\Lambda_k}.
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\| \mathbf{P}_{\Lambda_{k+1}} \mathbf{r}_k \|_2 \geq \mu \| \mathbf{r}_k \|_2
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for \( \mu \in (0, 1) \) where \( \mathbf{P}_\Lambda \mathbf{v} := \mathbf{v}|_{\Lambda} \).
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\[+: \text{ barycenter of supp } \psi_{\lambda} \text{ for } \lambda \in \Lambda_k\]
AWGM: Extension to unbounded domains and fast evaluation of system matrices | Adaptive wavelet (Galerkin) methods on unbounded domains

Adaptive approximation of $Au = f$: Numerical experiment

Solve **finite Galerkin systems** on nested index sets $\Lambda_k \subset \Lambda_{k+1} \subset \cdots \subset \mathcal{J}$ ([GHS07]).

**AWGM**[$\varepsilon$]

**for** $k = 0$; $\|r_k\|_2 \leq \varepsilon$; $k = k + 1$ **do**

Compute approx. solution $w_{\Lambda_k}$ of

$$A_{\Lambda_k} u_{\Lambda_k} = f_{\Lambda_k}.$$  

Compute approximation $r_k$ of the **infinite** residual

$$f - Aw_{\Lambda_k}.$$  

Compute smallest $\Lambda_{k+1} \supset \Lambda_k$ s.t.

$$\|P_{\Lambda_{k+1}} r_k\|_2 \geq \mu \|r_k\|_2$$

for $\mu \in (0, 1)$ where $P_{\Lambda} v := v|_\Lambda$.

**end for**

**: barycenter of supp $\psi_\lambda$ for $\lambda \in \Lambda_k$$
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AWGM: Extension to unbounded domains and fast evaluation of system matrices

Adaptive wavelet (Galerkin) methods on unbounded domains

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Solve **finite Galerkin systems** on nested index sets $\Lambda_k \subset \Lambda_{k+1} \subset \cdots \subset \mathcal{J}$ ([GHS07]).

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for $k = 0; \|r_k\|_{\ell^2} \leq \varepsilon; k = k + 1$ do

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for $\mu \in (0, 1)$ where $P_{\Lambda} v := v|_{\Lambda}$.

end for

$+$: barycenter of supp $\psi_{\lambda}$ for $\lambda \in \Lambda_k$
Adaptive wavelet methods on unbounded domains: Remarks

Product domains are **not** mandatory.

Works also within heuristic wavelet schemes (e.g. [BK06]).

Same proceeding for **non-linear** problems on unbounded domains (when isotropic wavelet bases are used).

Adaption of special multiwavelet bases for constant coefficient PDE operators (e.g. [DS10]) is possible (cf. [K.12]). On the right, applied to the problem

\[-\Delta u + u = f \text{ in } H^{-1}(\mathbb{R}^2).\]

Work in progress: **Non-local** operators.

---


Overview

Motivation: Wavelet methods in numerical finance

Adaptive wavelet (Galerkin) methods on unbounded domains

Fast evaluation of system matrices
Tensor structure of the system matrix $A$

Consider $\Psi = \psi^{(1)} \otimes \cdots \otimes \psi^{(n)}$ and let $\mathcal{B}(v, w) := \langle v, A[w] \rangle$ be such that

$$
\mathcal{B}(\otimes_{i=1}^{n} v_i, \otimes_{i=1}^{n} w_i) := \sum_{m=1}^{M} \prod_{i=1}^{n} a_{m}^{(i)}(v_i, w_i),
$$

where $a_{m}^{(i)}$ are local, univariate bilinear forms related to coordinate direction $e_i$. Now,

$$
A = D \mathcal{B}(\Psi, \Psi) D = D \left[ \sum_{m=1}^{M} \bigotimes_{i=1}^{n} \tilde{S}_{m}^{(i)} \right] D, \quad \tilde{S}_{m}^{(i)} := a_{m}^{(i)}(\psi^{(i)}, \psi^{(i)}).
$$

Poisson’s equation ($n = 2$, $\Omega = (0, 1)^2$)

$$
\mathcal{B}(\Psi, \Psi) = \tilde{A} \otimes \tilde{M} + \tilde{M} \otimes \tilde{A},
$$

where $\psi^{(1)} = \psi^{(2)}$ and

$$
\tilde{A} := [\int_{0}^{1} \partial \psi_{\lambda} \partial \psi_{\mu}]_{\lambda, \mu \in J}, \quad \tilde{M} := [\int_{0}^{1} \psi_{\lambda} \psi_{\mu}]_{\lambda, \mu \in J}.
$$

$\implies \mathcal{B}(\Psi, \Psi)$ is not sparse!
Three fundamental principles

Splitting into unidirectional operations

\[ \tilde{S} \otimes \tilde{S} = (\tilde{S} \otimes \text{Id}) \circ (\text{Id} \otimes \tilde{S}) = (\text{Id} \otimes \tilde{S}) \circ (\tilde{S} \otimes \text{Id}). \]

Sequential application of up- and down operations

\[ \tilde{S} = [a(\psi_\lambda, \psi_\mu)]_{\lambda, \mu \in \mathcal{J}} = \tilde{L} + \tilde{U}, \quad \tilde{L} = [a(\psi_\lambda, \psi_\mu)]_{|\lambda| > |\mu|}, \quad \tilde{U} = [a(\psi_\lambda, \psi_\mu)]_{|\lambda| \leq |\mu|}. \]

Multi-level structure of univariate wavelet bases

\[ \psi = \bigcup_{\ell \in \mathbb{N}_0} \psi_j, \quad \psi_\ell := \{\psi_\lambda : \lambda \in \mathcal{J}, |\lambda| = \ell\} \quad \text{Bi-directional transformations in linear complexity: FWT, IFWT.} \]

These principles have been introduced in \textbf{sparse grid} settings (using in particular \textit{hierarchical} bases), see, e.g., [BG04, Bun92 BZ96, Zei11, Zen91].

Three fundamental principles

Splitting into unidirectional operations

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Multi-level structure of univariate wavelet bases

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Three fundamental principles

Splitting into unidirectional operations

\[ \vec{S} \otimes \vec{S} = (\vec{S} \otimes \vec{1d}) \circ (\vec{1d} \otimes \vec{S}) = (\vec{1d} \otimes \vec{S}) \circ (\vec{S} \otimes \vec{1d}). \]

Sequential application of up- and down operations

\[ \vec{S} = [a(\psi_\lambda, \psi_\mu)]_{\lambda, \mu \in \mathcal{J}} = \vec{L} + \vec{U}, \quad \vec{L} = [a(\psi_\lambda, \psi_\mu)]_{|\lambda| > |\mu|}, \quad \vec{U} = [a(\psi_\lambda, \psi_\mu)]_{|\lambda| \leq |\mu|}. \]

Multi-level structure of univariate wavelet bases

\[ \psi = \bigcup_{\ell \in \mathbb{N}_0} \psi_j, \quad \psi_\ell := \{\psi_\lambda : \lambda \in \mathcal{J}, |\lambda| = \ell\} \]  
\[ \text{clos}_{L^2} \left( \text{span} \bigcup_{0 \leq \ell \leq j} \psi_\ell \right) = \text{clos}_{L^2} \left( \text{span} \Phi_j \right) \]  
Bi-directional transformations in linear complexity: **FWT, IFWT**.
Example: Matrix-vector multiplication on sparse grids

Given a refinement level $j \in \mathbb{N}_0$, consider the (two-dimensional) sparse grid space

$$\Lambda_j := \bigcup_{\ell_1 + \ell_2 \leq j} \Lambda_{(\ell_1, \ell_2)}, \quad \Lambda_{(\ell_1, \ell_2)} := \{ (\lambda_1, \lambda_2) : |\lambda_1| = \ell_1, |\lambda_2| = \ell_2 \}, \quad \#\Lambda_j \approx (j + 1) 2^j.$$
Example: Matrix-vector multiplication on sparse grids

Given a refinement level $j \in \mathbb{N}_0$, consider the (two-dimensional) sparse grid space

$$\Lambda_j := \bigcup_{\ell_1 + \ell_2 \leq j} \Lambda(\ell_1, \ell_2), \quad \Lambda(\ell_1, \ell_2) := \{(\lambda_1, \lambda_2) : |\lambda_1| = \ell_1, |\lambda_2| = \ell_2\}, \quad \#\Lambda_j \approx (j + 1)2^j.$$

Within the matrix-vector multiplication on a sparse grid, by the splitting $\tilde{S} = \tilde{L} + \tilde{U}$,

$$P_{\Lambda_j}(\tilde{S} \otimes \tilde{S})E_{\Lambda_j} = P_{\Lambda_j}(\tilde{L} \otimes \text{Id})E_{\Lambda_j} P_{\Lambda_j}(\text{Id} \otimes \tilde{S})E_{\Lambda_j} + P_{\Lambda_j}(\text{Id} \otimes \tilde{S})E_{\Lambda_j} P_{\Lambda_j}(\tilde{U} \otimes \text{Id})E_{\Lambda_j}$$

we do not leave the sparse grid index set $\Lambda_j$!
Towards adaptivity: multi-tree structured index sets

A (univariate) index set $\Lambda \subseteq J$ is a (univariate) tree when for all $\lambda \in \Lambda$

$$\text{supp } \psi_\lambda \subseteq \bigcup_{\mu \in \Lambda, |\mu| = |\lambda| - 1} \text{supp } \psi_\mu.$$ 

Theorem ([KS12a]): When $\tilde{\Lambda}, \Lambda \subseteq J$ are trees, then $P_{\tilde{\Lambda}} \tilde{X} E_{\Lambda} = [a(\psi_\lambda, \psi_\mu)]_{\lambda \in \tilde{\Lambda}, \mu \in \Lambda}$ for $\tilde{X} \in \{\tilde{S}, \tilde{L}, \tilde{U}\}$ can be applied within $O(\#\tilde{\Lambda} + \#\Lambda)$ operations.

---

Towards adaptivity: multi-tree structured index sets

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$\Lambda \subset \mathcal{J}$ is a multi-tree (cf. [KS12a]) when for any $\lambda = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)$

$$\Lambda_{e_i, \lambda} := \{\mu \in \mathcal{J} : (\lambda_1, \ldots, \lambda_{i-1}, \mu, \lambda_{i+1}, \ldots, \lambda_n) \in \Lambda\}, \quad i \in \{1, \ldots, n\},$$

is a tree (or the empty set). “A multi-tree $\Lambda$, when frozen in $n - 1$ coordinate directions, is a tree in the remaining coordinate.”

Towards adaptivity: multi-tree structured index sets

\[ \Lambda = J \otimes J \] is a multi-tree when for any \( \lambda \in J \)

\[ \Lambda_{e_1,\lambda} := \{ \mu \in J : (\mu, \lambda) \in \Lambda \}, \quad \Lambda_{e_2,\lambda} := \{ \mu \in J : (\lambda, \mu) \in \Lambda \}, \]

is a tree or the empty set.
Towards adaptivity: multi-tree structured index sets

\[
\Lambda \subset \mathcal{J} = \mathcal{J} \otimes \mathcal{J} \text{ is a multi-tree when for any } \lambda \in \mathcal{J}
\]

\[
\Lambda_{e_1,\lambda} := \{ \mu \in \mathcal{J} : (\mu, \lambda) \in \Lambda \}, \quad \Lambda_{e_2,\lambda} := \{ \mu \in \mathcal{J} : (\lambda, \mu) \in \Lambda \},
\]

is a tree or the empty set.
Theorem ([KS12a]): Consider two (poss. different) tensor product Riesz wavelet bases
\[ \tilde{\Psi} := \{ \tilde{\psi}_\lambda : \lambda \in \tilde{J} \} := \tilde{\psi}^{(1)} \otimes \cdots \otimes \tilde{\psi}^{(n)}, \quad \hat{\Psi} := \{ \hat{\psi}_\lambda : \lambda \in \hat{J} \} := \hat{\psi}^{(1)} \otimes \cdots \otimes \hat{\psi}^{(n)} \]
for \( L_2(\Omega) \). If \( \tilde{\Lambda} \subset \tilde{J} \), \( \hat{\Lambda} \subset \hat{J} \) are multi-trees, then the matrix-vector multiplication w.r.t.
\[ P_{\tilde{\Lambda}} B(\tilde{\Psi}, \hat{\Psi}) E_{\hat{\Lambda}} := P_{\tilde{\Lambda}} \left[ \sum_{m=1}^{M} \prod_{i=1}^{n} a_{m}^{(i)}(\tilde{\psi}^{(i)}, \hat{\psi}^{(i)}) \right] E_{\hat{\Lambda}} \]
can be computed within \( O(\#\tilde{\Lambda} + \#\hat{\Lambda}) \) operations.

This result generalizes results from adaptive sparse grids (see, e.g., [Pfl10]):

- Different trial- and test bases (\( \leadsto \) Petrov-Galerkin methods).
- Very general tree concept.
- Different input and output sets.
- New approximate residual approximation (cf. [KS12b]).

The resulting algorithm uses the decomposition \( \tilde{S} = \tilde{L} + \tilde{U} \) analogously to sparse grid schemes and is recursive in the dimension \( n \).

Numerical experiment 1: Poisson’s equation with variable coefficients

\[
\begin{align*}
\nabla \cdot (p \nabla u) &= f \quad \text{on } \Box := (0, 1)^2 \\
u|_{\partial \Box} &= 0
\end{align*}
\]

- Constant right-hand side \( f \equiv 20 \).
- Biorthogonal wavelets \((d = 3)\).
- AWGM with multi-tree constraint and new approximate residual.

Numerical experiments realized with Library for Adaptive Wavelet Applications (http://lawa.sourceforge.net)
Numerical experiment 1: Poisson’s equation with variable coefficients

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Computation times for matrix-vector multiplication (Galerkin system) w.r.t.

\[
P_{\Lambda_k} (\tilde{A} \otimes \tilde{M} + \tilde{M} \otimes \tilde{A}) E_{\Lambda_k},
\]

where \( N_k := \# \Lambda_k + \# \Lambda_k \).
Numerical experiment 1: Poisson’s equation with variable coefficients

\[
\begin{cases}
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\]

- Constant right-hand side \( f \equiv 20 \).
- Biorthogonal wavelets \((d = 3)\).
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Computation times for matrix-vector multiplication (residual) w.r.t.

\[ P_{\tilde{\Lambda}_k}(\tilde{A} \otimes \tilde{M} + \tilde{M} \otimes \tilde{A})E_{\Lambda_k}, \]

where, for residual computation (cf. [KS12b]), \( \tilde{\Lambda}_k \supset \Lambda_k \) is the “multi-tree extension of \( \Lambda_k \) by one level for each coordinate direction”, \( N_k := \# \tilde{\Lambda}_k + \# \Lambda_k \).
Numerical experiment 2: Poisson’s equation with constant coefficients

\[
\begin{aligned}
\nabla \cdot (\nabla u) &= f \text{ on } \Box := (0, 1)^3 \\
u|_{\partial \Box} &= 0
\end{aligned}
\]

- Constant right-hand side \( f \equiv 100 \).
- \( L_2 \)-orthonormal multiwavelets.
- AWGM with multi-tree constraint and new approximate residual.

Multitree in 3d obtained by AWGM with refinements in the corners and along the edges.

Numerical experiments realized with Library for Adaptive Wavelet Applications (http://lawa.sourceforge.net)
Numerical experiment 2: Poisson’s equation with constant coefficients

\[
\begin{cases}
\nabla \cdot (\nabla u) = f \text{ on } \square := (0, 1)^3 \\
u|_{\partial \square} = 0
\end{cases}
\]

- Constant right-hand side \( f \equiv 100 \).
- \( L_2 \)-orthonormal multiwavelets.
- AWGM with multi-tree constraint and new approximate residual.

Computation times for matrix-vector multiplication (Galerkin system) w.r.t.

\[
P_{\Lambda_k}(A \otimes I \otimes I + I \otimes A \otimes I + I \otimes I \otimes A)E_{\Lambda_k},
\]

where \( N_k := \#\Lambda_k + \#\Lambda_k \).

![Graph showing computation times for matrix-vector multiplication](image)

![Graph showing CPU time as a function of N_k](image)
Numerical experiment 2: Poisson’s equation with constant coefficients

\[
\begin{align*}
\nabla \cdot (\nabla u) &= f \text{ on } \square := (0, 1)^3 \\
u|_{\partial \square} &= 0
\end{align*}
\]

- Constant right-hand side \( f \equiv 100 \).
- \( L_2 \)-orthonormal multiwavelets.
- AWGM with multi-tree constraint and new approximate residual.

Computation times for matrix-vector multiplication (residual) w.r.t.

\[
P_{\tilde{\Lambda}_k}(\tilde{A} \otimes \mathbf{1d} \otimes \mathbf{1d} + \mathbf{1d} \otimes \tilde{A} \otimes \mathbf{1d} + \mathbf{1d} \otimes \mathbf{1d} \otimes \tilde{A})E_{\Lambda_k},
\]

where, for residual computation (cf. [KS12b]), \( \tilde{\Lambda}_k \supset \Lambda_k \) is the “multi-tree extension of \( \Lambda_k \) by one level for each coordinate direction”, \( N_k := \# \tilde{\Lambda}_k + \# \Lambda_k \).

![CPU time graph](image)
Contact

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Thank you for your attention!
Questions / Remarks . . .