NuHAG Numerical Harmonic Analysis Group

Function Spaces and Invariance Properties

Hans G. Feichtinger, Univ. Vienna & TUM
hans.feichtinger@univie.ac.at
www.nuhag.eu

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SUBMITTED ABSTRACT

It is the purpose of this talk to discuss a variety of situations where invariance properties of function spaces under a certain group of operators, specifically time-frequency shifts or dilations, help to derive atomic characterizations, find minimal or maximal spaces, or prove boundedness properties of certain operators.
Aside from the well-known characterization of real Hardy spaces via atomic decompositions ([2]) we can mention the work on the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ in the context of Gabor analysis (see [4]), but also the proof of Wiener's Third Tauberian Theorem (see [6]) for functions of bounded p-means on $\mathbb{R}^d$ (Wiener did the case $d = 1, p = 2$ in his book [16]).

SUBMITTED ABSTRACT II I

We will also present some known results concerning the *Fofana spaces* $(\boldsymbol{L}^q, \ell^p)^\alpha$ (see [10, 13, 11, 9]). These spaces are defined as subspaces of Wiener Amalgam spaces $\boldsymbol{W}(\boldsymbol{L}^q, \ell^p)(\mathbb{R}^d)$, for $1 \leq p < \alpha < q \leq \infty$ (otherwise they are trivial). In particular we are able to describe them as dual Banach spaces and provide atomic characterizations of the predual.

This talk will be about construction of function spaces based on particular invariance properties.
Given various families of popular Banach spaces of functions resp. distributions with additional invariance properties one can always ask whether there is a smallest resp. a largest in the family. As we shall show those minimal spaces have typically atomic characterizations.
We will discuss a couple of established examples, including the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and its dual, or functions of bounded p-means arising in the theory of Tauberian theorems, but also indicate some new results involving a family of spaces discussed in the work of I. Fofana and J. Feuto. The approach provides a characterization of the predual of these spaces.

Generalities and View-points

The talk will also shed some light on the strategies behind the various constructions (Wiener amalgams, modulation spaces, coorbit spaces, double module spaces, Banach Gelfand triples). Many results have not been published explicitly because they arise as special cases of results of a more general nature.
But I admit that one needs a guidance and detailed explanations to understand the situation. So, for example, duality and pointwise multiplier results on Wiener amalgam spaces (as introduced in [5]) have been only given in the framework of decomposition spaces ([7]).

The personal view on modulation spaces

The theory of modulation spaces has been developed in the early 1980, culminating in the well-known technical 1983 report on **Modulation spaces on locally compact Abelian groups**, and the first "public appearence" of modulation spaces at the conference in Kiew: *A new family of functional spaces on the Euclidean n-space*, in the same year.
They have been first designed as Wiener amalgam spaces on the Fourier transform side, using BUPUs, but soon the connection to the STFT and the Heisenberg group began to play a role.
Around 1986-1989 the appearance of wavelets suggested to look for a unified theory of wavelet analysis and time-frequency analysis, based on the common group-theoretical basis. The results have been published under the name of coorbit theory with K. Gröchenig in 1988/89.

Basic Facts about Wiener Amalgams

Wiener amalgam spaces, as the name says, had their origin in the work of Norbert Wiener, mostly in connection with his investigations around the Tauberian theorem (see [16]).

The so-called *Wiener algebra* $\boldsymbol{W}(\mathbb{R}^d)$, according to current systematic conventions $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$, was given as an interesting example of a so-called *Segal algebra* in Hans Reiter's book [14], see [3].

At that time J. Fournier and J. Stewart (see [12]) gave a nice survey on the role of the spaces they called $\ell^q(\boldsymbol{L}^p)$, while Busby and Smith observed the convolution properties of the classical amalgam space ([1]).

Advantages of the family $\ell^q(L^p)$

One of the draw-backs of the classical Banach spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, with $1 \leq p \leq \infty$ is the fact that there are no inclusion relations between any two of these spaces. However, the obstacles are of a different nature.

If $p_1 < p_2$ there are functions (locally like $x^{-\alpha}$, for a suitable value of $\alpha > 0$) which are locally in $\boldsymbol{L}^{p_1}$ but not in $\boldsymbol{L}^{p_2}$.
In constract, for $p_1 < p_2$ there are (step) functions in $\boldsymbol{L}^{p_1} \setminus \boldsymbol{L}^{p_2}$.
For Wiener amalgams the situation is quite easy:

$$\boldsymbol{W}(\boldsymbol{L}^{p_1}, \ell^{q_1}) \subset \boldsymbol{W}(\boldsymbol{L}^{p_2}, \ell^{q_2}) \quad \Leftrightarrow \quad p_2 \leq p_1 \text{ and } q_1 \leq q_1.$$

Hence $\boldsymbol{W}(L^\infty, \ell^1)$ is the smallest space in this family (with $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$ as the closure of the test functions), while $\boldsymbol{W}(\boldsymbol{L}^1, \ell^\infty)$ is the largest, closed in the dual of $\boldsymbol{W}(\mathbb{R}^d)$.

The Magic Square for Wiener Amalgams

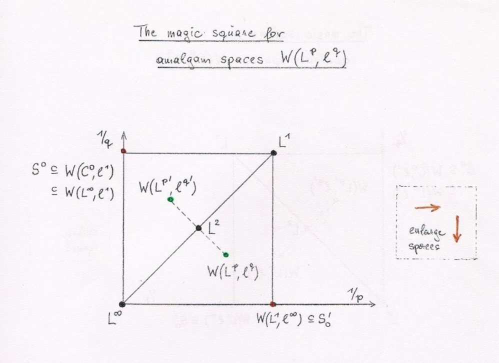

Figure: The inclusion relations: magic square

BUT overall classical Wiener Amalgams do not behave well under the Fourier transform!

The Hausdorff-Young Result for Amalgams

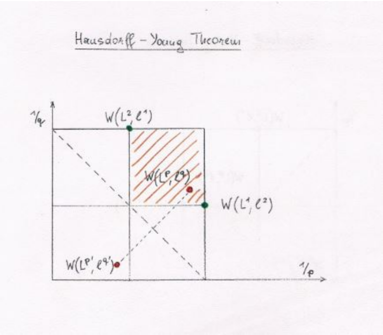

Figure: Hausdorff-Young theorem for Wiener amalgams

Wiener's Third Tauberian Theorem

Theorem

Let $f \in \mathbf{L}^p_{loc}(\mathbb{R}^n)$ *be given for some* p *with* $1 < p < \infty$, *such that*

$$\|f\|_{[p]} := \sup_{T \geq 1} \left(T^{-n} \int_{|x| \leq T} |f(x)|^p dx \right)^{1/p} < \infty \qquad (1)$$

and (kernels) $K_1, K_2 \in L^1_{n/p}$ *(Beurling algebra) be given. If* K_1 *satisfies the Tauberian condition* $\widehat{K_1}(t) \neq 0 \quad$ *for all* $t \in \mathbb{R}^n$, *then one may conclude from the relation*

$$\lim_{T \to \infty} \left(T^{-n} \int_{|x| \leq T} |K_1 * f(x)|^p dx \right)^{1/p} \qquad (2)$$

that the same relation is valid for K_2.

We write $\mathbf{M}^p(\mathbb{R}^n)$ for the Banach space of all f with $\|f\|_{[p]} < \infty$.

Theorem

For $f \in \mathbf{L}^p_{loc}(\mathbb{R}^n)$ and $1 \leq p \leq \infty$ we define the sequence $(d_k)_{k\geq 1}$ of p-norms over dyadic annuli, i.e.

$$d_1 = \|f \cdot \mathbf{1}_{|x|\leq 2}\|_p, \quad d_k = \|f \cdot \mathbf{1}_{2^{k-1}\leq |x| \leq 2^k}\|_p \text{ for } k \geq 2 \tag{3}$$

and write

$$\|f\|_{(p)} := \sup_{k\geq 1} 2^{-nk/p} d_k. \tag{4}$$

Then $f \in \mathbf{M}^p(\mathbb{R}^n)$ if and only if $\|f\|_{(p)} < \infty$. The closure of the compactly supported functions in $\mathbf{M}^p$, denoted by $\mathbf{I}^p(\mathbb{R}^n)$, is characterized by the condition

$$\lim_{k\to\infty} 2^{-kn/p} d_k = 0. \tag{5}$$

Theorem

For $1 < p \leq \infty$ the space $\boldsymbol{M}^p(\mathbb{R}^n)$ is the dual space of $\boldsymbol{E}^q(\mathbb{R}^n)$, with the conjugate index $1/q + 1/p = 1$. The space $\boldsymbol{E}^q(\mathbb{R}^n)$ can be defined for $1 \leq q \leq \infty$ by a weighted ℓ^1-norm over the dyadic rings:

$$\|h\|_{\boldsymbol{E}^q} := \sum_{k=1}^{\infty} 2^{kn/p} \|h \cdot \mathbf{1}_{2^{k-1} \leq |x| \leq 2^k}\|_q < \infty. \tag{6}$$

$\boldsymbol{E}^q(\mathbb{R}^n)$ can be identified naturally with the dual of $\boldsymbol{I}^p(\mathbb{R}^n)$. All three spaces are Banach modules with respect to convolution over the Beurling algebra $\boldsymbol{L}^1_{n/p}(\mathbb{R}^n)$.

Various Operators

Definition (Operators defined on functions)

1. translation by x is the operator T_x given by

$$T_x f(z) = [T_x f](z) := f(z-x) \quad x,z \in \mathbb{R}^d; \tag{7}$$

2.

$$\text{involution} \quad f \mapsto f^{\checkmark} \text{ with } f^{\checkmark}(z) := f(-z) \tag{8}$$

3. modulation M_s: Multiplication with the character χ_s:

$$[M_s f](z) := e^{2\pi i s\cdot z} f(z) = \chi_s(t) f(z), \quad x,s \in \mathbb{R}^d; \tag{9}$$

4. Fourier transform $\mathcal{F}, \mathcal{F}^{-1}$:

$$\mathcal{F}: f \mapsto \hat{f}: \quad \hat{f}(s) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i s\cdot t} dt \tag{10}$$

Dilation operators

Operators		
T_z	$T_z f(x) = f(x - z)$	translation by z
M_s	$M_s f(x) = e^{2\pi i s \cdot x} f(x)$	modulation operator
St_ρ	$\mathrm{St}_\rho f(x) = \rho^{-d} f(x/\rho)$	stretching operator
D_ρ	$\mathrm{D}_\rho f(x) = f(\rho x)$	dilation operator
$f^\checkmark$	$f^\checkmark(x) = f(-x)$	flip operator
f^*	$f^*(x) = \overline{f(-x)}$	$\boldsymbol{L}^1$-involution
$\overline{f}$	$\overline{f}(x) = \overline{f(x)}$	conjugation operator

Translation and modulation are isometric on *all the* $\boldsymbol{L}^p$-spaces, $1 \leq p \leq \infty$. The stretching operator is isometric on $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, while D_ρ is isometric on $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ hence $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ (or $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$).

Compatibility of Operators	
$T_z \circ M_s = e^{-2\pi i s\cdot z} M_s \circ T_z$	translation with modulation
$\mathcal{F} \circ M_s = T_s \circ \mathcal{F}$	translation and Fourier
$\mathcal{F} \circ \mathrm{St}_\rho = \mathrm{D}_\rho \circ \mathcal{F}$	dilation and Fourier
$\mathcal{F} \circ \mathrm{D}_\rho = \mathrm{St}_\rho \circ \mathcal{F}$	dilation and Fourier
$M_s(g * f) = M_s f * M_s g$	modulation and convolution
$T_x(h \cdot f) = T_x h \cdot T_x f$	translation and multiplication
$\mathrm{D}_\rho(h \cdot f) = \mathrm{D}_\rho h \cdot \mathrm{D}_\rho f$	dilation and multiplication
$\mathrm{St}_\rho(g * f) = \mathrm{St}_\rho f * \mathrm{St}_\rho g$	stretching and convolution
$(f * g)^* = g^* * f^*$	convolution and involution
$\overline{h \cdot f} = \overline{h} \cdot \overline{f}$	multiplication and conjugation

Minimality of $\boldsymbol{E}^q(\mathbb{R}^n)$

Theorem

For any given $q \in [1, \infty]$ the Banach space $\boldsymbol{E}^q(\mathbb{R}^n)$ is the smallest among all Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ within $\boldsymbol{L}^1(\mathbb{R}^n)$ containing the following set of q-atoms:

1. *it contains all the $\boldsymbol{L}^q$-functions f with $\|f\|_q = 1$ and $\mathrm{supp}(f) \subset B_1(0)$;*
2. *the semigroup of dilation operators $(\mathrm{St}_\rho)_{\rho \geq 1}$ acts uniformly bounded on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.*

Hence we have the following atomic characterization for $\boldsymbol{E}^q(\mathbb{R}^n)$:

Corollary

$$\boldsymbol{E}^q(\mathbb{R}^n) = \{f \mid f = \sum_{k=1}^{\infty} \mathrm{St}_{\rho_k} f_k, f_k \text{ is } q - atom, \rho_k \geq 1\}$$

Banach convolution property

The fact that $f \mapsto \mathrm{St}_\rho(f)$ is an automorphism for the Banach convolution algebra $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, i.e. the property

$$\mathrm{St}_\rho(f * g) = \mathrm{St}_\rho f * \mathrm{St}_\rho g, \quad f, g \in \boldsymbol{L}^1(\mathbb{R}^d), \rho > 0$$

easily implies that

Theorem

$(\boldsymbol{E}^q(\mathbb{R}^n), \|\cdot\|_{\boldsymbol{E}^q(\mathbb{R}^n)})$ is a Banach algebra with respect to convolution and of course isometrically invariant under the dilation semigroup $(\mathrm{St}_\rho)_{\rho \geq 1}$.

Note that these characterizations have for $q = 1$ the space $\boldsymbol{L}^1(\mathbb{R}^n)$ as a special case. Thus the family $\boldsymbol{E}^q(\mathbb{R}^n)$ interpolates between the case $q = 2$ relevant for Wiener's original theorem and the $\boldsymbol{L}^1$-case.

Wiener's inversion theorem

With the last theorem we have essentially all the tools which are required to give Wiener's proof of his inversion theorem, essentially in the same way as found in Hans Reiter's book [14] reps. [15].

Theorem

Assume that $g \in \boldsymbol{E}^q(\mathbb{R}^n)$ satisfies for some compact set $S \subset \widehat{\mathbb{R}^d}^n$ the non-vanishing condition $\hat{g}(s) \neq 0$ for all $s \in S$. Then there exists $f \in \boldsymbol{E}^q(\mathbb{R}^n)$ such that $1/\hat{g}(s) = \widehat{f}(s)$ for all $s \in S$.
In particular, if $\widehat{g}(s) \neq 0$ for all $s \in \widehat{\mathbb{R}^d}^n$ then the set of translates of g is norm total in $\boldsymbol{E}^q(\mathbb{R}^n)$.

Modulation Spaces

Modulation spaces *can be defined* as subspaces of the Schwartz space $\mathcal{S}'(\mathbb{R}^d)$ by the behaviour of the short-time Fourier transform of their elements (with respect to some Gaussian window, or any non-zero, non-negative Schwartz window $g \in \mathcal{S}(\mathbb{R}^d)$)

$$V_g(\sigma)(\lambda) = \sigma(\pi(\lambda)g) \text{ with } [\pi(t,s)g](x) = e^{2\pi i s \cdot x} g(x-t). \quad (11)$$

The (unweighted) *modulation spaces* $\boldsymbol{M}^{p,q}(\mathbb{R}^d)$ and also the spaces $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$ are defined by mixed norms on the continuous short-time Fourier transform. They can be characterized by Gabor expansions with coefficients in the corresponding mixed norm sequence spaces (see the course of K. Gröchenig).

Minimality and Maximility

Within this family the case $p = q$ are of particular interest. $\sigma \in \boldsymbol{M}^p(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p)(\mathbb{R}^d)(\mathbb{R}^d)$ if and only if $V_g(\sigma) \in \boldsymbol{L}^p(\mathbb{R}^{2d})$, with the corresponding norm. All if these spaces are Fourier invariant, with Gabor coefficients in $\ell^p(\mathbb{Z}^d)$. The space $\boldsymbol{M}^1(\mathbb{R}^d)$ is also known as Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$, while the dual $\boldsymbol{S}_0'(\mathbb{R}^d)$ is just $\boldsymbol{M}^\infty(\mathbb{R}^d)$.
These spaces can be characterized as the *smallest* respectively *largest* Banach space of tempered distributions for which time-frequency shifts act uniformly bounded.
Hence it is not surprising that one has an atomic characterization of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, starting from (any) single, non-zero element $g \in \boldsymbol{S}_0(\mathbb{R}^d)$:

Atomic characterization

Definition

$$\mathcal{S}_{atom} := \{f \in \boldsymbol{L}^2(\mathbb{R}^d) \,|\, f = \sum_{k=1}^{\infty} c_k \pi(\lambda_k) g_0 \, , \, \sum_{k=1}^{\infty} |c_k| < \infty\}$$

It is easy to verify that this space is a Banach space with the natural quotient norm

$$\|f\|_{atom} := \inf\{\sum_k |c_k|, \text{over all admiss. representations of } f\}.$$

weak-star convergence in $S_0'(\mathbb{R}^d)$

This description provides an alternative approach to $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ and implies among others: For any $0 \neq g \in S_0(\mathbb{R}^d)$ one has the following characterization of w^*-convergence within $S_0'(\mathbb{R}^d)$:

$$\sigma_0 = w^*\text{-}\lim \sigma_n \Leftrightarrow V_g(\sigma_n)(\lambda) \to V_g(\sigma_0)(\lambda), \forall \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

(or equivalently: uniformly over compact subsets of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$).

Fofana Spaces

Let us recall that for any $\alpha \in [1, \infty]$ there is an isometric version of the (commutative) group of dilation operators:

$$[\mathrm{St}_\rho^{(\alpha)} f](x) = \rho^{-d/\alpha} f(x/\rho). \qquad (12)$$

Clearly one has

$$\|\mathrm{St}_\rho^{(\alpha)} f\|_\alpha = \|f\|_\alpha \quad \rho > 0$$

and also

$$\mathrm{supp}(\mathrm{St}_\rho^{(\alpha)} f) = \rho \cdot \mathrm{supp}(f).$$

Lemma

Clearly for any fixed α the operators $(\mathrm{St}_\rho^{(\alpha)})_{\rho>0}$ form a group of operators isomorphic to $(\mathbb{R}_+^*, \cdot)$, i.e. for $\rho_1, \rho_2 > 0$ one has:

$$\mathrm{St}_{\rho_1}^{(\alpha)} \circ \mathrm{St}_{\rho_2}^{(\alpha)} = \mathrm{St}_{\rho_1 \cdot \rho_2}^{(\alpha)} = \mathrm{St}_{\rho_2}^{(\alpha)} \circ \mathrm{St}_{\rho_1}^{(\alpha)}$$

Fofana Spaces II

An elegant way to characterize the Fofana spaces $(\boldsymbol{L}^q, \ell^p)^\alpha(\mathbb{R}^d)$ is to make use of the dilation family for the parameter $\alpha \in (p, q)$:

Theorem

The Banach space $(\boldsymbol{L}^q, \ell^p)^\alpha$ with the norm $f \mapsto \|f\|_{q,p,\alpha}$ coincides with the following subspace of $(\boldsymbol{L}^q, \ell^p) = \boldsymbol{W}(\boldsymbol{L}^q, \ell^p)(\mathbb{R}^d)$:

$$\{f \in \boldsymbol{W}(\boldsymbol{L}^q, \ell^p)(\mathbb{R}^d) \mid \sup_{\rho>0} \|\mathrm{St}_\rho^{(\alpha)}(f)\|_{\boldsymbol{W}(\boldsymbol{L}^q, \ell^p)} < \infty\}, \tag{13}$$

and associated norms are equivalent. It contains $\boldsymbol{L}^\alpha(\mathbb{R}^d)$. The dilation group $(\mathrm{St}_\rho^{(\alpha)})_{\rho>0}$ acts isometrically

$$\|\mathrm{St}_\rho^{(\alpha)} f\|_{q,p,\alpha} = \|f\|_{q,p,\alpha}, \quad \forall f \in (\boldsymbol{L}^q, \ell^p)^\alpha(\mathbb{R}^d) \quad (***).$$

and is the largest subspace of $\boldsymbol{W}(\boldsymbol{L}^q, \ell^p)(\mathbb{R}^d)$ with this property.

Predual space

This characteriziation of $(\boldsymbol{L}^q, \ell^p)^\alpha(\mathbb{R}^d)$ suggest an atomic characterization of its predual space:

Theorem

The predual must be a Banach space between $\boldsymbol{L}^{\alpha'}$ and $\boldsymbol{W}(\boldsymbol{L}^{q'}, \ell^{p'})$. It is the smallest subspace $\mathrm{St}_\rho^{(\beta)}$-invariant subspace of $\boldsymbol{L}^\beta$, with $1/\beta + 1/\alpha = 1$, containing $\boldsymbol{W}(\boldsymbol{L}^{q'}, \ell^{p'})$. and can be characterized as follows:

$$\mathrm{St}_\rho^{(\beta)} - hull(\boldsymbol{W}(\boldsymbol{L}^{q'}, \ell^{p'})) = \{f \in \boldsymbol{L}^\beta(\mathbb{R}^d) \mid f = \sum_{n=1}^{\infty} c_n \mathrm{St}_{\rho_n}^{(\beta)} f_n\} \quad (14)$$

with the side condition that $\sum_{n=1}^\infty |c_n| < \infty$, $f \in \boldsymbol{W}(\boldsymbol{L}^{q'}, \ell^{p'})$ with $\|f_n\|_{\boldsymbol{W}(\boldsymbol{L}^{q'}, \ell^{p'})} \leq 1$ and $(\rho_n)_{n=1}^\infty$ being an arbitrary sequence of positive dilation parameters, with norm equivalence.

Real Hardy Spaces

Finally we would like to mention that the atomic spaces have of course a great similarity with the atomic characterization of the *real Hardy space* using atoms (compactly supported and bounded functions with vanishing integral), and $\boldsymbol{L}^1$-normalized dilation! See [2].

There are also various connections to *Morrey spaces* and so on. The *exotic Banach space* in [8] arose by considerations to add $\boldsymbol{L}^2$-normalized dilation to the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, in order to obtain a Fourier and TF-shift invariant and dilation invariant Banach space. Some questions about atomic decompositions are still open in this case.

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