Localization properties of the weighted frames

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Abstract

The frames localized with respect to the Riesz basis and the frames self-localized are considered with semi-normalized weights put on atoms. The preservation of the localization properties of the frame and its dual frame under the applied weights is verified.

Introduction

Frames $\mathcal{G} = (g_i)_{i\in I}$ generalize the idea of a basis in a Hilbert space $\mathcal{H}$ and consist of the indexed families such that the so-called **frame operator** S

$$Sf = \sum_{i\in I} \langle f, g_i\rangle g_i \tag{1}$$

is invertible. Hence every element $f \in \mathcal{H}$ has an expansion of the form ([4])

$$f = SS^{-1}f = \sum_{i\in I} \langle S^{-1}f, g_i\rangle g_i = \sum_{i\in I} \langle f, S^{-1}g_i\rangle g_i. \tag{2}$$

The family $(\tilde{g}_i)_{i\in I} = (S^{-1}g_i)_{i\in I}$ is again a frame and is called the **canonical dual frame**. It is not the only dual frame unless the frame is a basis. In general, any frame $(\gamma_i)_{i\in I}$ that allows the expansion of any $f \in \mathcal{H}$ as follow:

$$f = \sum_{i\in I} \langle f, g_i\rangle \gamma_i = \sum_{i\in I} \langle f, \gamma_i\rangle g_i \tag{3}$$

is called a **dual frame**. However, the canonical dual $(\tilde{g}_i)_{i\in I}$ provides the coefficients $(\langle f, \tilde{g}_i\rangle)_{i\in I}$ of minimal l^2-norm ([4]).

Localization with respect to a Riesz basis

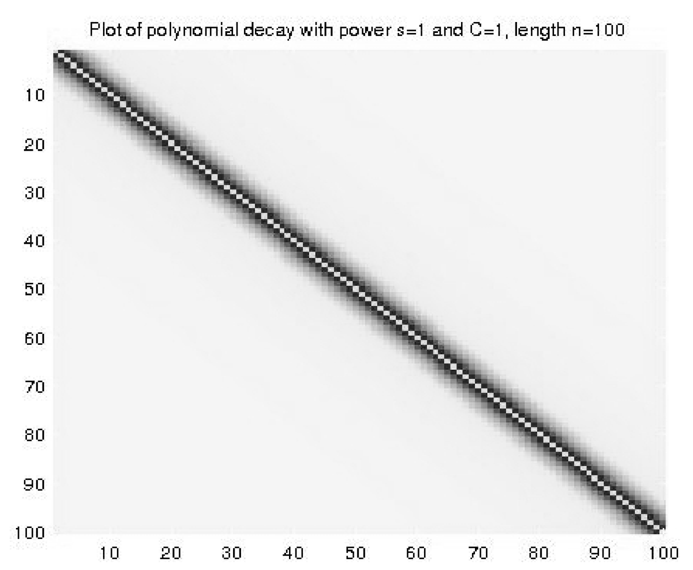
Original definition of frame localization requires an auxiliary Riesz basis ([1,6]). Let $\mathcal{N}$ and $\mathcal{I}$ denote countable index sets in $\mathbb{R}^d$ that are both separated, i.e. $\inf_{i,j\in\mathcal{I}, i\neq j} |i-j| = \delta > 0$. Let $(h_n)_{n\in\mathcal{N}}$ be a Riesz basis of a separable Hilbert space $\mathcal{H}$ with dual basis $(\tilde{h}_n)_{n\in\mathcal{N}}$.

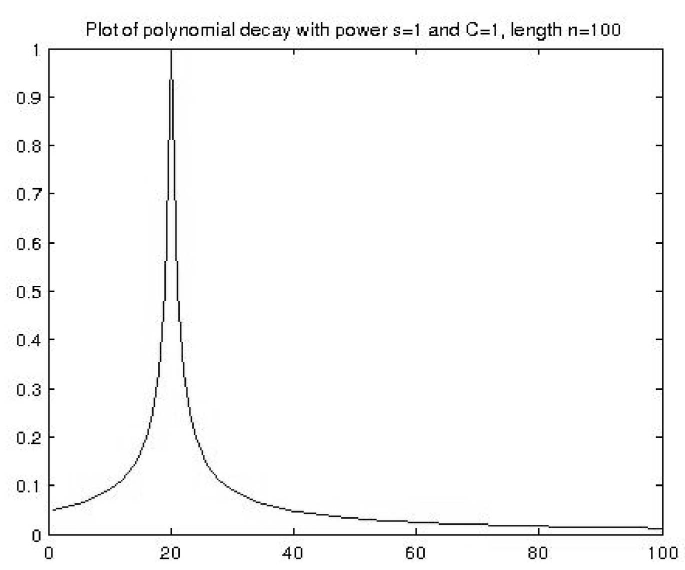
Definition 1. *A countable set $\mathcal{G} = (g_i)_{i\in I} \subseteq \mathcal{H}$ is **localized** with respect to the Riesz basis $(h_n)_{n\in\mathcal{N}}$, if the matrix with entries $(\langle g_i, h_n\rangle)_{i,n}$ possesses a prescribed decay in $|n-i|$, and similarly for the matrix $(\langle g_i, \tilde{h}_n\rangle)$ with respect to the biorthogonal basis $(\tilde{h}_n)_{n\in\mathcal{N}}$. For the polynomial decay the condition takes the following form for some $s > d$*

$$max\{|\langle g_i, h_n\rangle|, |\langle g_i, \tilde{h}_n\rangle|\} \leq C(1+|i-n|)^{-s} \tag{4}$$

for all $n \in \mathcal{N}$ and $i \in \mathcal{I}$.

The main result about the localized frames with respect to a Riesz basis, presented in ([6]) says that the canonical dual frame has the same localization property.





Relations

Relations between the localization properties and duality are presented on the diagram below. The horizontal arrows bear the fact that the frame localized with respect to some Riesz basis is self-localized ([5] and Thm 1). We do not know however if the converse is true ([5]).

The vertical arrows show the fact that the canonical dual frame possesses the same localization property ([6] for localization with respect to some Riesz basis and [3] for self-localization).

Frame localized with respect to Riesz basis	$\Longrightarrow$	Frame self-localized
$\Downarrow$		$\Downarrow$
Canonical dual frame localized with respect to Riesz basis	$\Longrightarrow$	Canonical dual frame self-localized

Self-localization

An alternative and more general definition for the localization of frame was proposed in ([5]). It does not require an underlying Riesz basis anymore, therefore this new notion of localization was called *self-localization* of a frame or *intrinsic localization* of frame ([3]).

Definition 2. *A countable set $\mathcal{G} = (g_i)_{i\in I} \subseteq \mathcal{H}$ is **self-localized** if its Gramian matrix with entries $(\langle g_i, g_j\rangle)_{i,j\in\mathcal{I}}$ has a prescribed off-diagonal decay in $|i-j|$, which in the case of polynomial decay gives the following condition for some $s > d$*

$$|\langle g_i, g_j\rangle| \leq C(1+|i-j|)^{-s} \qquad \forall i,j \in \mathcal{I}$$

Theorem 1 ([5])**.** *If $\mathcal{G}$ is localized with respect to some Riesz basis, then $\mathcal{G}$ is self-localized.*

Weighted frames

Let $\mathcal{G} = (g_i)_{i\in I}$ be a frame in $\mathcal{H}$ and $W = (w_i)_{i\in I}$ be a sequence of real scalars. When $W\mathcal{G} = (w_i \cdot g_i)_{i\in I}$ is the frame?

Theorem 2 ([2])**.** *Let $\mathcal{G} = (g_i)_{i\in I}$ be a frame in $\mathcal{H}$ with the frame bounds m and M and let W be a semi-normalized real sequence with bounds a, b i.e. such that $0 < a \leq b < \infty$ and $\forall i\ a \leq |w_i| \leq b$. Then the family $W\mathcal{G} = (w_i \cdot g_i)_{i\in I}$ is also a frame with the bounds a^2m and b^2M.*

Theorem 3 ([2])**.** *Let $\mathcal{G} = (g_i)_{i\in I}$ be a frame in $\mathcal{H}$, $D\mathcal{G} = (\tilde{g}_i)_{i\in I}$ be a canonical dual frame to $\mathcal{G}$ and W be a semi-normalized real sequence with bounds a, b as in Thm 2. Then the sequence $x = (x_i)_{i\in I}$ such that $x_i = 1/w_i$ forms a semi-normalized real sequence with bounds $1/b$, $1/a$ and the family $iWD\mathcal{G} = (x_i \cdot \tilde{g}_i)_{i\in I}$ constitutes a dual frame to the weighted frame $W\mathcal{G} = (w_i \cdot g_i)_{i\in I}$ (but is not necessarily its canonical dual one).*

Localization properties of weighted frames

The frame weighted with semi-normalized weights inherits the same localization properties: localization with respect to some Riesz basis (Proposition 1) and self-localization (Proposition 2).

Proposition 1. *Let $\mathcal{G} = (g_i)_{i\in I}$ be a frame for $\mathcal{H}$ localized with respect to the Riesz basis $(h_n)_{n\in\mathcal{N}}$ and $W = (w_i)_{i\in I}$ be a real sequence of weights with bounds $0 < a, b < \infty$. Then the weighted frame $W\mathcal{G} = (w_i \cdot g_i)_{i\in I}$ is localized with respect to the same Riesz basis $(h_n)_{n\in\mathcal{N}}$.*

Proof. The proof consists on the verification of Definition 1:

$$max\{|\langle w_i \cdot g_i, h_n\rangle|, |\langle w_i \cdot g_i, \tilde{h}_n\rangle|\} = |w_i| max\{|\langle g_i, h_n\rangle|, |\langle g_i, \tilde{h}_n\rangle|\} \leq b \cdot C(1+|i-n|)^{-s} \tag{5}$$

□

Proposition 2. *Let $\mathcal{G} = (g_i)_{i\in I}$ be the self-localized frame $\mathcal{H}$ and $W = (w_i)_{i\in I}$ be a real sequence of weights with bounds $0 < a, b < \infty$. Then the weighted frame $W\mathcal{G} = (w_i \cdot g_i)_{i\in I}$ is self-localized frame for $\mathcal{H}$.*

Proof. According to the Definition 2 we make the following calculations

$$|\langle w_i \cdot g_i, w_j \cdot g_j\rangle| = |w_i \cdot \bar{w}_j||\langle g_i, g_j\rangle| \leq$$

$$\leq |w_i||w_j| C(1+|i-j|)^{-s} \leq b^2 C(1+|i-j|)^{-s}$$

□

Conclusions

We recall the fact that the canonical dual frame keeps the same localization properties of the localized frame and we show that the localized frame weighted with semi-normalized weights remains localized, hence both duality and weightening maintain the localization properties of the localized frame.

Therefore all considered frames: $\mathcal{G}$, weighted $W\mathcal{G}$, canonical dual $D\mathcal{G}$ and the inversely weighted dual frame $iWD\mathcal{G}$ (see Thm 3) possess the same localization properties (either localization with respect to some Riesz basis or self-localization).

References

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