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Gabor Representation of Generalized Functions*

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0. INTRODUCTION

This paper deals with Gabor representation of (generalized) functions. By this we mean that a given function is to be expanded in a series involving Gabor functions that are located in the points of a given lattice in the time–frequency plane. We shall mainly consider lattices in which the area of the elementary cells of the lattice equals 1.

In 1946, Gabor used these expansions for the simultaneous analysis of signals in time and frequency (cf. [8]). Gabor stated that the above mentioned expansions exist for every reasonable signal, and that the coefficients in the expansion are uniquely determined by the signal. This statement is true when interpreted carefully, and the aim of this paper is to find out what kind of (generalized) functions are sufficiently well-behaved to allow a development in a Gabor series. We shall also consider the question of the uniqueness of the coefficients in the expansions, as well as questions concerning convergence. It turns out that the uniqueness questions can be handled with the aid of the main results of [10] (in particular 2.12 and 2.13); the questions on the existence of expansions are harder to answer, and require special care depending on the particular function. We consider L^p-functions ($1 \leqslant p \leqslant 2$) in detail, and show existence of Gabor representation for tempered distribution in general.

1. NOTATIONS AND PRELIMINARIES

We use exactly the same notations as in [10]; also the preliminaries are the same. Any notion, definition or notational convention not explained explicitly in this paper can be found in the introduction or notations and preliminaries section of [10]. In particular, $\mathscr{S}(\mathscr{S}')$ is the space of test functions of rapid decrease (tempered distributions).

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2. Preparation

2.1. In this section we introduce the notion of Gabor series, and we give conditions on the coefficients in the series that ensure convergence. Also, we give some results on the behavior of the inner products of (generalized) functions with Gabor functions.

2.2. If $a \in \mathbb{R}$, $b \in \mathbb{R}$, $\gamma > 0$, then the Gabor function $G_\gamma(a, b)$ is defined as

$$(G_\gamma(a, b))(t) = \left(\frac{2}{\gamma}\right)^{1/4} \exp\left(-\frac{\pi}{\gamma}(t-a)^2 + 2\pi ibt - \pi iab\right) \qquad (t \in \mathbb{R}).$$

We can regard $G_\gamma(a, b)$ as a function located at the point (a, b) of the time–frequency plane. We note that $G_\gamma(a, b) = e^{-\pi iab} R_{-b} T_{-a} g_\gamma$, where $g_\gamma = \curlyvee_t (2/\gamma)^{1/4} \exp(-\pi\gamma^{-1}t^2)$ (cf. [10, Introduction for the notations]); we also note that $G_\gamma(a, b)$ is an eigenfunction of the operator $(\gamma/2\pi)(d/dt) + t$, with eigenvalue $a + ib\gamma$. It is easy to see that $G_\gamma(a, b) \in S$, the space of smooth functions. The Wigner distribution $W(G_\gamma(a, b))$ of $G_\gamma(a, b)$ is given by

$$W(G_\gamma(a, b)) = \curlyvee_{(x,y)}\, 2\exp(-2\pi\gamma^{-1}(x-a)^2 - 2\pi\gamma(y-b)^2)$$

(cf. [1, 27.12.1.2]).

2.3. A Gabor series is a series of the form $\sum_{n,m} c_{nm} G_\gamma(n\alpha, m\beta)$, where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and where $(c_{nm})_{n\in\mathbb{Z}, m\in\mathbb{Z}}$ is a double sequence in $\mathbb{C}$ (we assert nothing about the convergence of the series).

Note that we consider $G_\gamma(n\alpha, m\beta)$ instead of $G_\gamma((n+\frac{1}{2})\alpha, (m+\frac{1}{2})\beta)$ as was done in [10, Sect. 2]. By means of [10, 2.2] we can easily carry over the results of [10], and we shall freely use these in this paper, of course with the proper modifications if necessary. As in [10, Sect. 2] we shall assume $\gamma = 1$, and we write $G(a, b)$ instead of $G_\gamma(a, b)$.

2.4. The following lemmas are used to settle questions about convergence of the series $\sum_{n,m} c_{nm} G(n\alpha, m\beta)$. Note that $(F, G(a, b)) = e^{\pi iab}(T_a R_b F, g_1)$ if $F \in \mathscr{S}'$, $a \in \mathbb{R}$, $b \in \mathbb{R}$ (cf. 2.2).

Lemma. (i) *Let $f \in \mathscr{S}$. Then $\curlyvee_{(a,b)}(G(a, b), f) \in \mathscr{S}^2$.*

(ii) *Let $F \in \mathscr{S}'$. There exists an $M > 0$, $N \in \mathbb{N}$ such that*

$$|(F, G(a, b))| \leqslant M(1 + a^2)^N (1 + b^2)^N \qquad (a \in \mathbb{R}, b \in \mathbb{R}).$$

Proof. (i) We have $\curlyvee_{(t,a)}\, 2^{1/4} \exp(-\pi(t-a)^2)\overline{f(t)} \in \mathscr{S}^2$. The Fourier transform of this function with respect to the first variable is again an element of $\mathscr{S}^2$ and equals $\curlyvee_{(a,b)}\, e^{\pi iab}(G(a, b), f)$.

(ii) We have, for $a \in \mathbb{R}$, $b \in \mathbb{R}$,

$$(F, G(a, b)) = e^{\pi iab - \pi a^2}(F, R_{b+ia}\, g).$$

Now apply [10, 2.10] to derive the desired inequality for $(F, G(a, b))$. ∎

Remark. Since $\curlyvee_{(a,b)}(F, G(a, b))$ is continuous if $F \in \mathscr{S}'$, it is a regular tempered distribution (after embedding in $\mathscr{S}'$, of course).

2.5. Lemma. *Let $(c_{nm})_{n,m}$ be a double sequence in $\mathbb{C}$ with*

$$c_{nm} = O((1 + n^2)^N (1 + m^2)^N) \qquad (n \in \mathbb{Z}, m \in \mathbb{Z})$$

for some $N \in \mathbb{N}$, and let $\alpha > 0$, $\beta > 0$. The series $\sum_{n,m} c_{nm} G(n\alpha, m\beta)$ is convergent in $\mathscr{S}'$-sense, and the convergence is unconditional.

Proof. Let $f \in \mathscr{S}$. It follows easily from 2.4 that the series $\sum_{nm} c_{nm}(G(n\alpha, m\beta), f)$ is absolutely convergent. We conclude that $\sum_{n,m} c_{nm} G(n\alpha, m\beta)$ is convergent in $\mathscr{S}'$-sense, and that the order of the terms in the series is immaterial. ∎

2.6. Definition. Let $\mathscr{K}$ be the class of all double sequences $(c_{nm})_{n,m}$ in $\mathbb{C}$ for which there exists an $N \in \mathbb{N}$ with

$$c_{nm} = O((1 + n^2)^N (1 + m^2)^N) \qquad (n \in \mathbb{Z}, m \in \mathbb{Z}).$$

Note that $\sum_{n,m} c_{nm} G(n\alpha, m\beta)$ converges in $\mathscr{S}'$-sense if $(c_{nm})_{n,m} \in \mathscr{K}$, and that $(F, G(n\alpha, m\beta))_{n,m} \in \mathscr{K}$ if $F \in \mathscr{S}'$.

2.7. *Remark.* If F is (the embedding of an) $L^2(\mathbb{R})$-function, then we have $\sum_{n,m} |(F, G(n\alpha, m\beta))|^2 < \infty$. Also, if $(c_{nm})_{n,m}$ satisfies $\sum_{n,m} |c_{nm}|^2 < \infty$, then $\sum_{n,m} c_{nm} G(n\alpha, m\beta)$ converges in L^2-sense. The proofs of these facts are not hard, although not completely trivial. If F is (the embedding of an) $L^1(\mathbb{R})$-function, then $(F, G(n\alpha, m\beta))$ is bounded in n and m (even $(F, G(n\alpha, m\beta)) \to 0$ if $n^2 + m^2 \to \infty$). We conclude by the Riesz–Thorin theorem that $\sum_{n,m} |(F, G(n\alpha, m\beta))|^q < \infty$ if $F \in L^p(\mathbb{R})$ and $1 < p < 2$ (q conjugate exponent of p).

2.8. The inner products of Gabor functions play an important role in this paper. We can evaluate them explicitly.

Lemma. *For $a \in \mathbb{R}$, $b \in \mathbb{R}$, $x \in \mathbb{R}$, $y \in \mathbb{R}$ we have*

$$(G(a, b), G(x, y)) = \exp\left(-\frac{\pi}{2}(a-x)^2 - \frac{\pi}{2}(b-y)^2 + \pi ibx - \pi iay\right).$$

Proof. This follows from a straightforward calculation. ∎

Observe that the Gabor functions are not orthogonal.

3. Tempered Distributions and Gabor Coefficients

3.1. Let $F \in \mathscr{S}'$, and let $(c_{nm})_{n,m} \in \mathscr{H}$. In this section we shall investigate what we can say about the relation between the c_{nm}'s and F if it is known that $F = \sum_{n,m} c_{nm} G(n\alpha, m\beta)$. We shall mainly consider the case that $\alpha\beta = 1$ (this is the case Gabor considered in his investigations).

3.2. Let $F \in \mathscr{S}'$, and let $(c_{nm})_{n,m} \in \mathscr{H}$. If $F = \sum_{n,m} c_{nm} G(n\alpha, m\beta)$, then we have

$$\sum_{n,m} c_{nm}(G(n\alpha, m\beta), G(k\alpha, l\beta)) = (F, G(k\alpha, l\beta)) \tag{1}$$

for all $k \in \mathbb{Z}$, $l \in \mathbb{Z}$. If $\alpha\beta < 1$, and (1) holds for all k and l, then we know from [10, Theorem 2.8] that $F = \sum_{n,m} c_{nm} G(n\alpha, m\beta)$. If $\alpha\beta > 1$, and (1) holds for all k and l, then it is not always true that F has much to do with $\sum_{n,m} c_{nm} G(n\alpha, m\beta)$ (cf. [10, 2.9]). The case that $\alpha\beta = 1$ is more interesting, and in the remainder of this paper we shall only deal with that case.

3.3. With the aid of Lemma 2.8 we can write (1) as

$$\sum_{n,m} c_{nm} \exp\left(-\frac{\pi\alpha}{2}(n-k)^2 - \frac{\pi\beta^2}{2}(m-l)^2 + \pi imk - \pi inl\right) = (F, G(k\alpha, l\beta)) \tag{2}$$

for $k \in \mathbb{Z}$, $l \in \mathbb{Z}$. Now $\exp(\pi imk - \pi inl) = \exp(\pi i(n-k)(m-l) - \pi inm + \pi ikl)$, so if we define $c'_{nm} := (-1)^{nm} c_{nm}$, $d'_{kl} := (-1)^{kl}(F, G(k\alpha, l\beta))$, then we get

$$\sum_{n,m} c'_{nm} \exp\left(\frac{-\pi\alpha^2}{2}(n-k)^2 - \frac{\pi\beta^2}{2}(m-l)^2 + \pi i(n-k)(m-l)\right) = d'_{kl} \tag{3}$$

for $k \in \mathbb{Z}$, $l \in \mathbb{Z}$.

3.4. With every $a = (a_{nm})_{n,m} \in \mathscr{H}$ we associate the periodic distribution F_a (of two variables), given formally by

$$F_a = \sum_{n,m} a_{nm} e^{2\pi inz + 2\pi imw}$$

(cf. [14, Chap. VII, Sect. 1 and further]; the condition $a \in \mathscr{H}$ guarantees convergence of the series). If we put $c' := (c'_{nm})_{n,m}$, $d' := (d_{k,l})_{k,l}$, $a := (\exp(-(\pi\alpha^2/2)\,n^2 - (\pi\beta^2/2)\,m^2 + \pi inm))_{n,m}$, then the system of equations in 3.3 (3) can be written (at least formally) as $F_a \cdot F_{c'} = F_{d'}$. This can be justified as follows. We note that F_a is analytic and periodic (period 1) in both of its variables. Hence F_a is a multiplier in the space of all periodic distributions, and the Fourier coefficients of $F_a \cdot F_{c'}$ can be calculated by simply convoluting the Fourier coefficients of F_a and $F_{c'}$.

3.5. We are going to study the following problem in detail: given the periodic distribution B, find a periodic distribution K such that $F_a \cdot K = B$. To solve this problem we have to analyze F_a. In what follows we write

$$\Theta = \Upsilon_{(z,w)} \sum_{n,m} \exp\left(-\frac{\pi\alpha}{2}n^2 - \frac{\pi\beta^2}{2}m^2 + 2\pi inz + 2\pi imw + \pi inm\right)$$

instead of F_a.

Theorem. (i) *$\Theta(z, w) \geqslant 0$ ($z \in \mathbb{R}$, $w \in \mathbb{R}$), and $\Theta(z, w) = 0$ if and only if $z = \frac{1}{2} (\mathrm{mod}\ 1)$, $w = \frac{1}{2} (\mathrm{mod}\ 1)$.*

(ii) $$\frac{\partial^2\Theta}{\partial z^2}\left(\frac{1}{2}, \frac{1}{2}\right) > 0, \qquad \frac{\partial^2\Theta}{\partial w^2}\left(\frac{1}{2}, \frac{1}{2}\right) > 0.$$

(iii) $$\frac{\partial^{k+l}\Theta}{\partial z^k\,\partial w^l}\left(\frac{1}{2}, \frac{1}{2}\right) = 0 \qquad \textit{if}\quad k \textit{ or } l \textit{ is odd.}$$

Proof. We have, for $z \in \mathbb{C}$, $w \in \mathbb{C}$,

$$\Theta(z, w) = \sum_m \exp\left(\frac{-\pi\beta^2}{2} m^2 + 2\pi imw\right) \times \sum_n \exp\left(\frac{-\pi\alpha^2}{2} n^2 + 2\pi in\left(z + \frac{1}{2}m\right)\right).$$

Now

$$\sum_n \exp\left(-\frac{\pi\alpha^2}{2} n^2 + 2\pi in\left(z + \frac{1}{2}m\right)\right) = \theta_3\left(\pi\left(z + \frac{1}{2}m\right), e^{-\pi\alpha^2/2}\right),$$

where θ_3 denotes the third theta function (notation as in [18, Chap. XXI]). Since θ_3 is periodic with period π we have

$$\Theta(z, w) = \sum_{m\ \mathrm{even}} \exp\left(-\frac{\pi\beta^2}{2} m^2 + 2\pi imw\right) \theta_3(\pi z, e^{-\pi\alpha^2/2}) + \sum_{m\ \mathrm{odd}} \exp\left(-\frac{\pi\beta^2}{2} m^2 + 2\pi imw\right) \theta_3\left(\pi\left(z + \frac{1}{2}\right), e^{-\pi\alpha^2/2}\right).$$

The two sums can be expressed as theta functions too, and we get

$$\Theta(z, w) = \theta_3(\pi z, e^{-\pi\alpha^2/2})\,\theta_3(2\pi w, e^{-2\pi\beta^2}) + \theta_4(\pi z, e^{-\pi\alpha^2/2})\,\theta_2(2\pi w, e^{-2\pi\beta^2}).$$

Using the relations between the theta functions and their translates (cf. [18, 21.11]) we get

$$\Theta(\tfrac{1}{2}, \tfrac{1}{2}) = \theta_4(0, e^{-\pi\alpha^2/2})\,\theta_3(0, e^{-2\pi\beta^2}) - \theta_3(0, e^{-\pi\alpha^2/2})\,\theta_2(0, e^{-2\pi\beta^2}).$$

It follows from [18, 21.51, Example 1] that $\Theta(\frac{1}{2}, \frac{1}{2}) = 0$ ($\alpha\beta = 1$).

We next show that $(\frac{1}{2}, \frac{1}{2})$ is the only zero of Θ in the square $[0, 1] \times [0, 1]$. Note first that $\theta_4(\pi z, e^{-\pi\alpha^2/2}) > 0$, $\theta_3(2\pi w, e^{-2\pi\beta^2}) > 0$ for all z and w in $\mathbb{R}$. Hence, if $z \in [0, 1]$, $w \in [0, 1]$, then $\Theta(z, w) = 0$ if and only if

$$T_1(z) := \frac{\theta_3(\pi z, e^{-\pi\alpha^2/2})}{\theta_4(\pi z, e^{-\pi\alpha^2/2})} = -\frac{\theta_2(2\pi w, e^{-2\pi\beta^2})}{\theta_3(2\pi w, e^{-2\pi\beta^2})} =: T_2(w).$$

We shall show that T_1 is minimal at $z = \frac{1}{2}$ (strictly), and that T_2 is maximal at $w = \frac{1}{2}$ (strictly). From this (i) follows at once.

We have, by [18, 2.16, Example 2],

$$\frac{dT_1}{dz} = -\pi\theta_2^2\,\frac{\theta_1(\pi z, e^{-\pi\alpha^2/2})\,\theta_2(\pi z, e^{-\pi\alpha^2/2})}{\theta_4^2(\pi z, e^{-\pi\alpha^2/2})} = 0$$

if and only if $z = m$ or $z = m + \frac{1}{2}$ with $m \in \mathbb{Z}$ (cf. [18, 21.12]). It is clear that T_1 is maximal at $z = 0$ and minimal at $z = \frac{1}{2}$, for

$$\theta_3(0, e^{-\pi\alpha^2/2}) = \theta_4(\tfrac{1}{2}\pi, e^{-\pi\alpha^2/2}) = \sum_n e^{(-\pi\alpha^2/2)n^2},$$

$$\theta_3(\tfrac{1}{2}\pi, e^{-\pi\alpha^2/2}) = \theta_4(0, e^{-\pi\alpha^2/2}) = \sum_n (-1)^n\, e^{(-\pi\alpha^2/2)n^2}.$$

As to T_2 we note that

$$\frac{\theta_2(2\pi w, e^{-2\pi\beta^2})}{\theta_3(2\pi w, e^{-2\pi\beta^2})} = \frac{\theta_1(2\pi w + (1/2)\pi, e^{-2\pi\beta^2})}{\theta_4(2\pi w + (1/2)\pi, e^{-2\pi\beta^2})},$$

and that by [18, 21.6],

$$\frac{d}{dv}\left(\frac{\theta_1(v, e^{-2\pi\beta^2})}{\theta_4(v, e^{-2\pi\beta^2})}\right) = \theta_4^2\,\frac{\theta_2(v, e^{-2\pi\beta^2})\,\theta_3(v, e^{-2\pi\beta^2})}{\theta_4^2(v, e^{-2\pi\beta^2})} = 0$$

if and only if $v = (m + \frac{1}{2})\pi$ with $m \in \mathbb{Z}$ (cf. [18, 21.12]). It is not hard to see now that T_2 is maximal at $w = \frac{1}{2}$.

As to (ii) we note that it is enough to show that $(\partial^2\Theta/\partial z^2)(\frac{1}{2}, \frac{1}{2}) > 0$. We have

$$\frac{\partial^2\Theta}{\partial z^2} = \pi^2\{\theta_3''(\pi z, e^{-\pi\alpha^2/2})\,\theta_3(2\pi w, e^{-2\pi\beta^2}) + \theta_4''(\pi z, e^{-\pi\alpha^2/2})\,\theta_2(2\pi w, e^{-2\pi\beta^2})\}.$$

Noting that $\theta_3''(\frac{1}{2}\pi, e^{-\pi\alpha^2/2}) = \theta_4''(0, e^{-\pi\alpha^2/2}) > 0$, $\theta_4''(\frac{1}{2}\pi, e^{-\pi\alpha^2/2}) = \theta_3''(0, e^{-\pi\alpha^2/2}) < 0$ (cf. [18, 21.41]) and that $\theta_3(\pi, e^{-2\pi\beta^2}) > 0$, $\theta_2(\pi, e^{-2\pi\beta^2}) < 0$, we see that $(\partial^2\Theta/\partial z^2)(\frac{1}{2}, \frac{1}{2}) > 0$.

Finally (iii). Let k and l be integers. Now

$$\frac{\partial^{k+l}\Theta}{\partial z^k\,\partial w^l}\left(\frac{1}{2}, \frac{1}{2}\right) = (2\pi i)^{k+l} \sum_{n,m} n^k m^l \times \exp\left(-\frac{\pi\alpha^2}{2}n^2 - \frac{\pi\beta^2 m^2}{2}\right)(-1)^{n+m+nm},$$

and it is easy to see that the series equals 0 if k or l is odd. ∎

3.6. The function Θ can be expanded as

$$\sum_{k\,\text{even},\,l\,\text{even}} r_{kl}(z - \tfrac{1}{2})^k (w - \tfrac{1}{2})^l$$

around the point $(\frac{1}{2}, \frac{1}{2})$. Here $r_{00} = 0$, $r_{20} > 0$, $r_{02} > 0$.

The division problem $\Theta \cdot K = B$ of 3.5 is not of the type discussed in [13, Chap. V]; nevertheless, one can carry out the division (compare with [14, Chap. VII, Sect. 1, Exemples et applications 2, Équations aux différences finies, case $k = \pm 1$]). We find the following result. There exists an infinite number of periodic distributions K with $\Theta \cdot K = B$. Each two solutions differ by a finite linear combination of $\sum_{n,m} \delta_{n+1/2}^{(p)} \otimes \delta_{m+1/2}^{(q)}$ ($p \geqslant 0$, $q \geqslant 0$). The proof goes roughly as follows (cf. [15] for the proof of a more general theorem).[1] As B is a periodic distribution there exists a continuous periodic function C of two variables and $k \in \mathbb{N}$, $l \in \mathbb{N}$ such that $B = \partial^{k+l}C/\partial z^k\,\partial w^l$. Let $m_1 \in \mathbb{N}$, $m_2 \in \mathbb{N}$ be so large that $(\partial^{k+l}/\partial z^k\,\partial w^l)(\phi/\Theta)$ is continuous if ϕ is a test function whose derivatives of order $\leqslant(m_1, m_2)$ vanish at $(\frac{1}{2}, \frac{1}{2})$, and let Φ be the class of all these ϕ's. Now we can define $(\phi, B/\Theta)$ for all $\phi \in \Phi$ in such a way that $(\Theta \cdot \psi, B/\Theta) = (\psi, B)$ if ψ satisfies $\Theta \cdot \psi \in \Phi$. It must then be shown that the mapping $\phi \to (\phi, B/\Theta)$ can be defined for all test functions ϕ in a linear, continuous way such that $(\Theta \cdot \phi, B/\Theta) = (\phi, B)$. It is seen at once that the solutions of the homogeneous problem ($B = 0$) are finite linear

[1] I thank W. A. J. Luxemburg for calling my attention to this reference.

combinations of the functions $\sum_{n,m} \delta^{(p)}_{n+1/2} \otimes \delta^{(q)}_{m+1/2}$ with $p \geqslant 0$, $q \geqslant 0$ (cf. [16, Chap. 24, Theorem 24.6]).

3.7. To express the non-uniqueness in the coefficients c'_{nm} (cf. 3.3, 3.4): if $(c'_{nm})_{n,m}$ and $(c''_{nm})_{n,m}$ give rise to $F_{c'}$ and $F_{c''}$ with $\Theta \cdot F_{c'} = \Theta \cdot F_{c''} = F_{d'}$, then there exists a polynomial p in two variables such that

$$c'_{nm} - c''_{nm} = (-1)^{n+m}\, p(n, m) \qquad (n \in \mathbb{Z}, m \in \mathbb{Z}).$$

This is easily seen by observing that $\sum_{n,m}(-1)^{n+m}\, n^p m^q (-2\pi i)^{p+q} e^{2\pi i n z + 2\pi i m w}$ is the Fourier series of $\sum_{n,m} \delta^{(p)}_{n+1/2} \otimes \delta^{(q)}_{m+1/2}$, where p and q are non-negative integers.

3.8. For later references we calculate $\sum_{n,m} c_{nm}\, G(n\alpha, m\beta) = \sum_{n,m}(-1)^{nm} c'_{nm}\, G(n\alpha, m\beta)$ (cf. 3.3) with $c'_{nm} = (-1)^{n+m}(a_1 + a_2 n + a_3 m + a_4 nm)$. For notational convenience we consider the case with $\alpha = \beta = 1$ only (the general case presents no particular problems).

Introduce the function ψ by

$$\psi(t) := \sum_{n=-\infty}^{\infty} (-1)^n \exp(-\pi(t-n)^2) \qquad (t \in \mathbb{C}).$$

This ψ is analytic, periodic with period 2 and satisfies

$$\psi(t+1) = -\psi(t), \psi(-t) = \psi(t) \qquad (t \in \mathbb{C}).$$

We further have $\psi(\frac{1}{2}) = \psi''(\frac{1}{2}) = \cdots = 0$, $\psi'(\frac{1}{2}) \neq 0$. With this ψ we find

$$\sum_{n,m} (-1)^{n+m+nm}(a_1 + a_2 n + a_3 m + a_4 nm)\, G(n, m)$$

$$= \frac{\psi'(1/2)}{2\pi} \sum_m (-1)^m \left\{ \left(a_2 - i a_3 - \frac{i a_4}{2\pi} \right) \delta'_{m+1/2} - i a_4 (m + 1/2)\, \delta_{m+1/2} \right\}.$$

The above function is a tempered distribution and has the form $\sum_m F_m$, where F_m is concentrated in $m + \frac{1}{2}$ for $m \in \mathbb{Z}$. This is of course no surprise: we have (c_{nm} as above)

$$\left(\sum_{n,m} c_{nm}\, G(n, m), G(k, l) \right) = 0 \qquad (k \in \mathbb{Z}, l \in \mathbb{Z}),$$

so that we can apply [10, 2.12] (also cf. 2.2). Note that we have to shift over a distance $\frac{1}{2}$ as we consider $G(n, m)$ instead of $G(n + \frac{1}{2}, m + \frac{1}{2})$ with integer values of n and m.

We also see that there exist $(c_{nm})_{n,m} \in \mathscr{K}$ with $c_{nm} \not\equiv 0$ such that $\sum_{n,m} c_{nm}\, G(n, m) = 0$.

3.9. If we combine 3.6 and [10, Theorem 2.12 and Corollary 2.13] then we find the following result.

THEOREM. *Let $F \in \mathscr{S}'$ and $\alpha > 0$, $\beta > 0$, $\alpha\beta = 1$. There exists an infinite number of solutions $(c_{nm})_{n,m}$ in $\mathscr{K}$ of the system of equations*

$$\sum_{n,m} c_{nm}(G(n\alpha, m\beta), G(k\alpha, l\beta)) = (F, G(k\alpha, l\beta)) \qquad (k \in \mathbb{Z}, l \in \mathbb{Z}). \qquad (*)$$

If $(c_{nm})_{n,m}$ and $(d_{nm})_{n,m}$ are solutions of $(*)$ then there exists a polynomial p of two variables such that $c_{nm} - d_{nm} = (-1)^{n+m+nm}\, p(n, m)$. Also, if $(c_{nm})_{n,m}$ is a solution of $(*)$ then there exists a $Q \in \mathscr{S}'$ concentrated in the point $\frac{1}{2}$ such that

$$\left(F - \sum_{n,m} c_{nm}\, G(n\alpha, m\beta) \right) \cdot T_{1/2}\, g_1 = \sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi n^2} T_n Q.$$

Moreover, if both F and $\sum_{n,m} c_{nm}\, G(n\alpha, m\beta)$ are regular tempered distributions, then $F = \sum_{n,m} c_{nm}\, G(n\alpha, m\beta)$, where the series converges in $\mathscr{S}'$-sense.

4. GABOR REPRESENTATION FOR TEMPERED DISTRIBUTIONS

4.1. In this section we prove theorems about the existence of Gabor representation for tempered distributions. The criterion in Theorem 3.9 for the existence of such a representation is not very useful, since it is not easy to see whether or not a particular $(c_{nm})_{n,m}$ gives rise to a regular tempered distribution. The following theorem gives a condition that is much easier to check. For the sake of notational convenience we take $\alpha = \beta = 1$ in this section, although the general case presents no real problems.

THEOREM. *Assume that $F \in \mathscr{S}'$ is regular, and let $(c_{nm})_{n,m} \in \mathscr{K}$ be a solution of the system of equations $(*)$ in 3.9 such that $c_{nm} \to 0$ if $n^2 + m^2 \to \infty$. Then $F = \sum_{n,m} c_{nm}\, G(n, m)$ with convergence in $\mathscr{S}'$-sense.*

Proof. It follows from [10, 2.11] that there exists a polynomial Q such that

$$\left(R_z \left(F - \sum_{n,m} c_{nm}\, G(n, m) \right), g_1 \right) = Q(z)\, H_{1,1}(z)$$

for $z \in \mathbb{C}$ (cf. [10, 2.3]; $g_1 = G(0,0)$). If $z \in \mathbb{R}$, then

$$\begin{aligned}\Big(R_z\Big(\sum_{n,m} c_{nm} G(n,m)\Big), g_1\Big) &= \sum_{n,m} c_{nm}(G(n,m), G(0,z)) \\ &= \sum_{n,m} c_{nm} \exp\Big(-\frac{\pi}{2} n^2 - \frac{\pi}{2}(m-z)^2 - \pi i n z\Big)\end{aligned}$$

by 2.8. As $c_{nm} \to 0$ $(n^2+m^2 \to \infty)$ we easily conclude that

$$\Big(R_z\Big(\sum_{n,m} c_{nm} G(n,m)\Big), g_1\Big) \to 0$$

if $z \to \infty$. Also, F is a regular tempered distribution, so $F \cdot g_1 \in L^1(\mathbb{R})$. Hence, by the Riemann–Lebesgue lemma,

$$(R_z F, g_1) = \int_{-\infty}^{\infty} e^{-2\pi i z t} F(t)\, g_1(t)\, dt \to 0$$

if $z \to \infty$. Now $H_{1,1}$ is periodic, and Q is a polynomial, whence $H_{1,1}(z)$. $Q(z) \to 0$ $(z \to \infty)$ if and only if $Q \equiv 0$. ∎

Remarks. (1) The condition on $(c_{nm})_{n,m}$ can be weakened somewhat as the proof of the above theorem shows. For example, the proof works as well if we replace $c_{nm} \to 0$ $(n^2+m^2 \to \infty)$ by $\lim_{m \to \infty} c_{nm} = 0$ for all n.

(2) If $F \in \mathscr{S}'$ and $F = \sum_{n,m} c_{nm} G(n,m)$, where $(c_{nm})_{n,m} \in \mathscr{K}$, then $\mathscr{F}F = \sum_{n,m} c_{-mn} G(n,m)$ as $\mathscr{F}G(a,b) = G(b,-a)$ for $a \in \mathbb{R}$, $b \in \mathbb{R}$.

4.2. Let $F: \mathbb{R} \to \mathbb{C}$ be sufficiently well-behaved, so that the manipulations below can be justified (e.g., $F \in \mathscr{S}$). In the division problem of 3.5 the right hand side of the equation $F_a \cdot K = B$ is given by

$$B(z,w) := \sum_{k,l} (F, G(k,l))(-1)^{kl} e^{2\pi i k z + 2\pi i l w}$$

for $z \in [0,1]$, $w \in [0,1]$. We can rewrite B as

$$\begin{aligned}B(z,w) &= \sum_{k,l} \int_{-\infty}^{\infty} F(t) \exp(-\pi(t-k)^2 - 2\pi i l t + 2\pi i k z + 2\pi i l w)\, dt \\ &= \sum_k e^{2\pi i k z} \sum_m F(w-m)\, e^{-\pi(w-m-k)^2}\end{aligned}$$

(here we applied the Poisson sum formula), and we get

$$B(z,w) = \psi_1(z,w)\, \psi_2(z,w)$$

for $z \in [0,1]$, $w \in [0,1]$, where

$$\psi_1(z,w) = \sum_m F(w-m)\, e^{-2\pi i m z},$$

$$\psi_2(z,w) = \sum_k e^{-\pi(w-k)^2 + 2\pi i k z},$$

for $z \in [0,1]$, $w \in [0,1]$. Observe that $\psi_2(\frac{1}{2},\frac{1}{2}) = 0$ (cf. 3.9), and that ψ_2 is entire. We see that we can carry out the division B/Θ directly (that is, without using the result of 3.6). Assuming, e.g., that $\psi_1 \in L^\infty([0,1] \times [0,1])$ we get $B/\Theta \in L^p([0,1] \times [0,1])$ for every $p < 2$. By the Hausdorff–Young theorem we conclude that the Fourier coefficients of B/Θ are in l^q for every $q > 2$. Hence Theorem 4.1 applies, and we conclude that F has a Gabor representation.

If we have an F such that $\psi_1 \in L^1([0,1] \times [0,1])$, and such that $|\psi_1|^p$ is integrable in a neighborhood of $(\frac{1}{2},\frac{1}{2})$ for some $p > 2$, then $B/\Theta \in L^1([0,1] \times [0,1])$ and $c_{nm} \to 0$ if $n^2+m^2 \to \infty$ (this also holds if $\psi_1 \in L^1([0,1] \times [0,1])$ and ψ_1 has a Lebesgue point at $(\frac{1}{2},\frac{1}{2})$). Theorem 4.1 applies again.

In general it does not seem to be easy to prove more precise theorems about the convergence of the series $\sum_{n,m} c_{nm} G(n,m)$. But if F is even, then $\psi_1(\frac{1}{2},\frac{1}{2}) = 0$. Hence $B/\Theta \in L^2([0,1] \times [0,1])$ under some continuity condition on ψ_1 in $(\frac{1}{2},\frac{1}{2})$. Now, $\sum_{n,m} |c_{nm}|^2 < \infty$, and it follows from 2.7, Remark that $\sum_{n,m} (-1)^{nm} c_{nm} G(n,m)$ converges in $L^2(\mathbb{R})$-sense to F.

4.3. We are now going to prove theorems on the Gabor representation of $f \in L^p(\mathbb{R})$ with $1 \leqslant p \leqslant 2$. We use the following theorem.

THEOREM. (i) *If $f \in L^2(\mathbb{R})$, then $Tf := \curlyvee_{(z,w)} \sum_{n=-\infty}^{\infty} f(w-n)\, e^{2\pi i n z} \in L^2([0,1] \times [0,1])$. Also, T is bijective and norm preserving.*

(ii) *If $f \in L^1(\mathbb{R})$, then $Tf := \curlyvee_{(z,w)} \sum_{n=-\infty}^{\infty} f(w-n)\, e^{2\pi i n z} \in L^1([0,1] \times [0,1])$, and $\|Tf\|_{1,[0,1]\times[0,1]} \leqslant \|f\|_{1,\mathbb{R}}$.*

Proof. (i) Take $f \in L^2(\mathbb{R})$. The functions $\curlyvee_{(z,w)} f(w-n)\, e^{-2\pi i n z}$ are orthogonal, so

$$\begin{aligned}\int_0^1 \int_0^1 \Big| \sum_{n=-\infty}^{\infty} f(w-n)\, e^{-2\pi i n z} \Big|^2 dz\, dw &= \sum_{n=-\infty}^{\infty} \int_0^1 \int_0^1 |f(w-n)\, e^{-2\pi i n z}|^2\, dz\, dw \\ &= \int_{-\infty}^{\infty} |f(w)|^2\, dw.\end{aligned}$$

Hence T is well-defined, maps $L^2(\mathbb{R})$ into $L^2([0,1]\times[0,1])$ and is norm preserving.

We now show that T is bijective. So let $g\in L^2([0,1]\times[0,1])$. Let $(c_{nm})_{n,m}$ the sequence of Fourier coefficients of g, and put $g_n:=\sum_{m=-\infty}^{\infty} c_{nm}e^{-2\pi imw}$ for $n\in\mathbb{Z}$. Then $g_n\in L^2([0,1])$, and $\int_0^1|g_n(w)|^2\,dw=\sum_{m=-\infty}^{\infty}|c_{nm}|^2$. If we define $f(w-n):=g_n(w)$ $(w\in[0,1],\ n\in\mathbb{Z})$, then $f\in L^2(\mathbb{R})$, and the Fourier coefficients of $\forall_{(z,w)}\sum_{n=-\infty}^{\infty} f(w-n)\,e^{-2\pi inz}$ equal those of g.

(ii) Let $f\in L^1(\mathbb{R})$. Then

$$\int_0^1\int_0^1\left|\sum_{n=-\infty}^{\infty} f(w-n)\,e^{-2\pi inz}\right|dz\,dw\leqslant\int_0^1\int_0^1\sum_{n=-\infty}^{\infty}|f(w-n)|\,dz\,dw$$
$$=\int_{-\infty}^{\infty}|f(w)|\,dw.$$

Hence $Tf\in L^1([0,1]\times[0,1])$, and $\|Tf\|_{1,[0,1]\times[0,1]}\leqslant\|\ \|_{1,\mathbb{R}}$. ∎

Remarks. (1) By the Riesz–Thorin theorem the mapping T extends to $L^p(\mathbb{R})$ and maps $L^p(\mathbb{R})$ into $L^p([0,1]\times[0,1])$ such that $\|Tf\|_{p,[0,1]\times[0,1]}\leqslant\|f\|_{p,\mathbb{R}}$ for $f\in L^p(\mathbb{R})$ (we assume $1\leqslant p\leqslant 2$).

(2) T cannot be defined on $L^p(\mathbb{R})$ with $p>2$: there is an $f\in L^p(\mathbb{R})$ such that $\forall_{(z,w)}\sum_{n=-\infty}^{\infty} f(z-n)\,e^{2\pi inw}$ belongs to none of the spaces $L^q([0,1]\times[0,1])$ with $q\geqslant 1$ (cf. also [19, Chap. XII, 2, p. 102]).

4.4. We know from 4.2 that for a well-behaved function f the Gabor coefficients are found by taking $(-1)^{nm}c_{nm}$, where c_{nm} is the (nm)th Fourier coefficient of $\psi_2\cdot Tf/\Theta$ (ψ_2 and Θ as in 4.2; Tf as in 4.3). For an $f\in L^p(\mathbb{R})$ with $1\leqslant p\leqslant 2$ this is not possible in general. However, if $f\in L^p(\mathbb{R})$, then

$$c'_{nm}:=\int_0^1\int_0^1\frac{(Tf)(z,w)\,\psi_2(z,w)}{\Theta(z,w)}\,(e^{-2\pi inz-2\pi imw}-(-1)^{n+m})\,dz\,dw$$

makes sense for integers n and m, and we show in 4.6 that $f=\sum_{n,m}c'_{nm}G(n,m)$.

4.5. Let $f\in L^p(\mathbb{R})$ with $1\leqslant p\leqslant 2$, and let c'_{nm} be as in 4.4.

LEMMA. (i) *If $1\leqslant p<2$, then there is a $C>0$ such that*

$$|c'_{nm}|\leqslant C\,\|f\|_p(|n|^{2/p-1}+|m|^{2/p-1})\qquad(n\in\mathbb{Z},m\in\mathbb{Z}).$$

(ii) *If $p=2$, then there is a $C>0$ such that*

$$|c'_{nm}|\leqslant C\,\|f\|_2((\log|n|)^{1/2}+(\log|m|^{1/2})\qquad(n\in\mathbb{Z},m\in\mathbb{Z}).$$

Proof. Assume $1<p\leqslant 2$. By Hölder's inequality and 4.3, Remark 1 we get for $n\in\mathbb{Z}$, $m\in\mathbb{Z}$ (q denotes the conjugate exponent)

$$|c'_{nm}|\leqslant\|f\|_p\left(\int_0^1\int_0^1\frac{|e^{-2\pi inz-2\pi imw}-(-1)^{n+m}|^q}{\Theta(z,w)}\times|\psi_2(z,w)|^q\,dz\,dw\right)^{1/q}.$$

We have by 4.2 and 3.5 for some $C>0$

$$\left|\frac{\psi_2(z,w)}{\Theta(z,w)}\right|\leqslant C\left(\left(z-\frac{1}{2}\right)^2+\left(w-\frac{1}{2}\right)^2\right)^{-1/2}.$$

So we must estimate (replace $z-\frac{1}{2}$ by z and $w-\frac{1}{2}$ by w)

$$A:=\left(\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\frac{|e^{-2\pi inz-2\pi imw}-1|^q}{(z^2+w^2)^{q/2}}\,dz\,dw\right)^{1/q}.$$

Write $e^{-2\pi inz-2\pi imw}-1=f_1(z,w)+f_2(z,w)+f_3(z,w)$, where $f_2(z,w)=e^{2\pi inz}-1$, $f_3(z,w)=f_2(w,z)$, $f_1=f_2\cdot f_3$. By Minkowsky's inequality we get

$$A\leqslant\sum_{i=1}^{3}\left(\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\frac{|f_i(z,w)|^q}{(z^2+w^2)^{q/2}}\,dz\,dw\right)^{1/q}.$$

To estimate the second term in the above sum we note that $|f_2(z,w)|\leqslant 2\pi\min(|nz|,2)$. Hence the qth power of the second term can be estimated by

$$\int_{|z|\leqslant 1/n}\int_{-1/2}^{1/2}\frac{|2\pi nz|^q}{(z^2+w^2)^{q/2}}\,dz\,dw+\int_{|z|\geqslant 1/n}\int_{-1/2}^{1/2}\frac{(4\pi)^q\,dz\,dw}{(z^2+w^2)^{q/2}}.$$

Now

$$\int_{|z|\leqslant 1/n}\int_{-1/2}^{1/2}\frac{|z|^q}{(z^2+w^2)^{q/2}}\,dz\,dw$$
$$=\int_{|z|\leqslant 1/n}|z|\left(\int_{-1/2|z|}^{1/2|z|}\frac{dv}{(1+v^2)^{q/2}}\right)dz\leqslant\pi n^{-2}$$

as $q\geqslant 2$. And

$$\int_{|z|\geqslant 1/n}\frac{dz}{(z^2+w^2)^{q/2}}=|w|^{1-q}\int_{|v|\geqslant 1/n|w|}\frac{dv}{(1+v^2)^{q/2}}$$
$$\leqslant\pi\,|w|^{1-q}\min(1,(n\,|w|)^{q-1}).$$

Hence, if $p < 2$,

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{|f_2(z, w)|^q}{(z^2 + w^2)^{q/2}}\, dz\, dw$$
$$\leqslant 2^q \pi^{q+1} n^{q-2} + \pi \left(\int_{-1/2}^{1/2} |w|^{1-q} \min(1, (n\,|w|))^{q-1}\right) dw$$
$$\leqslant \frac{\pi(2\pi n)^q}{n^2} + \pi \left(\int_{-1/n}^{1/n} n^{q-1}\, dw + \int_{|w| \geqslant 1/n} |w|^{1-q}\, dw\right) \leqslant a n^{q-2},$$

where $a = 2^q \pi^{q+1} + 2\pi + 2/(2-q)$. And if $p = 2$, we get

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{|f_2(z, w)|^q}{(z^2 + w^2)^{q/2}}\, dz\, dw \leqslant 2^q \pi^{q+1} + 2\pi + 2\pi \log n.$$

The first and the third terms in the above sum can be treated in a similar way. We find

$$A = O(|n|^{q-2} + |m|^{q-2})^{1/q}) \qquad \text{or}$$
$$= O((\log |n| + \log |m|)^{1/2})$$

according as $1 < p < 2$ or $p = 2$. Now note that $(x + y)^{1/q} \leqslant x^{1/q} + y^{1/q}$ for all $x \geqslant 0$, $y \geqslant 0$, $q \geqslant 1$, and that $(q - 2)/q = 2/p - 1$.

The case $p = 1$ can be handled in the same way (there is only notational difference). ∎

4.6. Now let $f \in L^p(\mathbb{R})$ with $1 \leqslant p \leqslant 2$. Let $(f_k)_k$ be a sequence in $\mathscr{S}$ such that $f_k \to f$ in $L^p(\mathbb{R})$-sense. Now Tf_k (cf. 4.2 and 4.3) is continuous. Let $c_{nm}^{(k)}$ be $(-1)^{nm}$ times the Fourier coefficient of $Tf_k \cdot \psi_2/\Theta$ (that is, the $c_{nm}^{(k)}$'s are the coefficients in the Gabor representation of f_k). Let c'_{nm} be as in 4.4, and let $c_{nm} := (-1)^{nm}\, c'_{nm}$.

We have

$$c_{nm}^{(k)} = (-1)^{nm} \int_0^1 \int_0^1 \frac{(Tf_k)(z, w)\, \psi_2(z, w)}{\Theta(z, w)}$$
$$\times (e^{-2\pi i n z - 2\pi i m w} - (-1)^{n+m})\, dz\, dw + (-1)^{n+m+nm}\, c^{(k)},$$

where

$$c^{(k)} := \int_0^1 \int_0^1 \frac{(Tf_k)(z, w)\, \psi_2(z, w)}{\Theta(z, w)}\, dz\, dw.$$

Put $d_{nm}^{(k)} = c_{nm}^{(k)} - (-1)^{n+m+nm}\, c^{(k)}$. Now $d_{nm}^{(k)} \to c_{nm}$ $(k \to \infty)$ for all n and m by 4.3, Remark 1 as $f_k \to f$ in L^p-sense. And

$$f_k = \sum_{n,m} c_{nm}^{(k)} G(n, m)$$
$$= \sum_{n,m} d_{nm}^{(k)} G(n, m) + c^{(k)} \sum_{n,m} (-1)^{n+m+nm}\, G(n, m)$$
$$= \sum_{n,m} d_{nm}^{(k)} G(n, m)$$

by 3.8. It is easy to see from Lemma 4.5 that $\sum_{n,m} d_{nm}^{(k)} G(n, m) \to \sum_{n,m} c_{nm} G(n, m)$ in $\mathscr{S}'$-sense. Hence $f = \sum_{n,m} c_{nm} G(n, m)$.

We thus proved the following theorem.

THEOREM. *Let $f \in L^p(\mathbb{R})$ with $1 \leqslant p \leqslant 2$. If $c_{nm} = (-1)^{nm}\, c'_{nm}$, where c'_{nm} is as in 4.4 for all n and m, then*

$$c_{nm} = O(|n|^{2/p-1} + |m|^{2/p-1}) \qquad \text{or}$$
$$= O((\log |n|)^{1/2} + (\log |m|)^{1/2})$$

according as $1 \leqslant p < 2$ or $p = 2$, and $f = \sum_{n,m} c_{nm} G(n, m)$, where the convergence is in $\mathscr{S}'$-sense.

Remark. From 4.3 and the proof of 4.5 it follows that there exists an $f \in L^2(\mathbb{R})$ whose Gabor coefficients are unbounded. This is shown by using the Banach–Steinhaus theorem.

4.7. Theorem 4.6 enables us to prove existence of Gabor representation for arbitrary tempered distributions.

THEOREM. *Let $F \in \mathscr{S}'$. There exists a sequence $(d_{nm})_{n,m}$ in $\mathscr{K}$ such that $F = \sum_{n,m} d_{nm} G(n, m)$, where the series converges in $\mathscr{S}'$-sense.*

Proof. Define the operators A_1 and A_2 of $\mathscr{S}'$ by $A_1 G := ((1/2\pi)(d/dt) + t)G$, $A_2 G := ((-1/2\pi)(d/dt) + t)G$ $(G \in \mathscr{S}')$, respectively. Let ψ_n denote the nth Hermite function for $n = 0, 1, \ldots$ (normalization as in [1, 27.6.3]). Then

$$A_1 \psi_n = (n/\pi)^{1/2}\, \psi_{n-1}\ (n = 1, 2, \ldots),$$
$$A_2 \psi_n = ((n+1)/\pi)^{1/2}\, \psi_{n+1}\ (n = 0, 1, \ldots)$$

by [1, 27.6.3]. We know from [12, Appendix to V.3] that there is an integer $k > 0$ such that $((F, \psi_n)\, n^{-k})_n \in l^2$.

Now solve the equation $A_1^{2k}G = F$ $(G \in \mathscr{S}')$ in terms of Hermite coefficients. We get

$$(F, \psi_n) = (A_1^{2k}G, \psi_n) = (G, A_2^{2k}\psi_n) = c_{n,k}(G, \psi_{n+2k}),$$

where

$$c_{n,k} := ((n+1)(n+2) \cdots (n+2k))^{1/2}\, \pi^{-k}$$

for $n = 0, 1, \ldots$. So $G := \sum_n c_{n,k}^{-1}(F, \psi_n)\, \psi_{n+2k}$ satisfies $F = A_1^{2k}G$, and $G \in L^2(\mathbb{R})$.

Let $(e_{nm})_{n,m}$ be such that $G = \sum_{n,m} e_{nm} G(n, m)$, where the series converges in $\mathscr{S}'$-sense (cf. Theorem 4.6). Then

$$F = \sum_{n,m} e_{nm}(n+im)^{2k}\, G(n, m)$$

according to 2.2. It is obvious that $(e_{nm}(n+im)^k)_{n,m} \in \mathscr{K}$, and that the series converges in $\mathscr{S}'$-sense.

5. Final Remarks

We conclude this paper by some remarks and comments. In 1946, Gabor claimed that every signal F can be developed $(\alpha\beta = 1)$ as $\sum_{n,m} c_{nm} G(n\alpha, m\beta)$, and that the coefficients c_{nm} are uniquely determined by F.

The statement about the uniqueness of the coefficients does not hold if F is a tempered distribution (cf. Theorem 3.9), and even the case that F is an element of $L^2(\mathbb{R})$ is not easy to handle (cf. 4.6). Nevertheless, $\alpha\beta = 1$ seems to be the only reasonable choice. For if $\alpha\beta < 1$, it will be hard to find a canonical solution $(c_{nm})_{n,m}$ to the coefficients problem (it is likely that we have many solutions in this case; if we have a solution, then the corresponding series represents F by 3.2). And if $\alpha\beta > 1$, then the series corresponding with a solution of the coefficients problem has in general not very much to do with F. It is likely that in the latter case the coefficients problem is easy to solve (if, e.g., $\alpha\beta = 2$, then we can introduce a function Θ as in 3.5, and this Θ turns out to be strictly positive; also, the mapping

$$(c_{kl})_{k,l} \to \left(\sum_{k,l} \exp\left(\frac{-\pi\alpha^2}{2}(k-n)^2 - \frac{\pi\alpha^2}{2}(l-m)^2\right) c_{kl}\right)_{n,m}$$

is invertible as a mapping of l^p with $1 \leqslant p \leqslant \infty$ if $\alpha^2 \geqslant 1.005$).

Among the functions that can be developed in a Gabor series are the polynomials and the distributions with period 1. If F is a periodic tempered distribution (with period 1), then the Gabor series for F has the form $\sum_{n,m}(-1)^{nm} c_m G(n, m)$, where the c_m's are the Fourier coefficients of the periodic distribution F/g with $g = \Psi_t \sum_n \exp(-\pi(t-n)^2)$. This is one of the reasons that we did not restrict ourselves to Gabor coefficients that are in l^1, l^2 or l^∞. Another reason is that certain physical interesting signals F have the property that $(F, G(a, b))$ is unbounded as a function of a and b. It may be proved, e.g., that this is the case for almost every realization F of a Gaussian white noise process.

We finally make some comments on related results on the subject in literature. There is a connection with certain classical interpolation theorems stated in [17, Chap. V, Sect. 12]: if f is an entire function with $\limsup_{\xi\to\infty} M(\xi)/\xi^2 < \pi/2$, where $M(\xi) = \max_{|z|=\xi} |f(z)|$, then we have

$$f(z) = \sigma(z) \sum_{n,m} \exp\left(-\frac{\pi}{2}(n^2+m^2)\right) f(n+im)/(z-n-im),$$

where σ is the Weierstrasz σ-function. We could use this result with $f = \Psi_z \exp(\pi z^2/2)(R_z F, g)$, where $F \in \mathscr{S}'$ and $g = \Psi_z \exp(-\pi z^2)$, but unfortunately such an f will satisfy $\limsup_{\xi\to\infty} M(\xi)/\xi^2 \geqslant \pi/2$ in general.

Some of the sums arising in the proof of Theorem 3.5 can also be found in [5, 6, 11], where completeness properties of the functions $G(n, m)$ $(n \in \mathbb{Z}, m \in \mathbb{Z})$ are considered (in the references just given, and in [4], completeness means: if $f \in L^2(\mathbb{R})$ satisfies $(f, G(n, m)) = 0$ for all n and m, then $f \equiv 0$; compare also [10]). In [11, (38)] an example of coefficients $(c_{nm})_{n,m}$ with $\sum_{n,m} c_{nm} G(n, m) = 0$ is given (cf. also 3.8); here the expansion problem is considered briefly.

The operator T of 4.3 also occurs in [5] and [4], and can be used to give a quite simple proof of the completeness (in the sense of the previous paragraph) of the $G(n, m)$'s.

We refer to [1, 27.12.1.5; 9; and 2, Theorem 2.6] for continuous versions of Gabor representation of generalized functions.

In a recent paper [3], the expansion problem is also considered, i.e., expressions for the coefficients are given; the methods used there are somewhat different from ours, and are probably more closely related to the approach suggested above in connection with the classical interpolation theory.

Some numerical results are given in [7],[2] where it was found that the Gabor expansions have poor convergence properties in general. This is no surprise, of course, in view of the behavior (and non-uniqueness) of the coefficients (also cf. 4.2 and 4.5).

[2] I thank Alan Weinstein for calling my attention to this reference.

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