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# THE ZAK TRANSFORM: A SIGNAL TRANSFORM FOR SAMPLED TIME-CONTINUOUS SIGNALS

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**Abstract**

In this paper we study the Zak transform which is a signal transform relevant to time-continuous signals sampled at a uniform rate and an arbitrary clock phase. Besides being a time-frequency representation for time-continuous signals, the Zak transform is a particularly useful tool in signal theory. It can be used to provide a framework within which celebrated themes like the Fourier inversion theorem, Parseval's theorem, the Nyquist criterion, the Shannon sampling theorem, and the Gabor representation problem fit. Moreover, the Zak transform is of particular use for solving linear integral equations, the kernel of which is the autocorrelation function of a cyclostationary random process. This paper lists and interprets the numerous properties of the Zak transform; to one of these properties, the occurrence of zeros of Zak transforms, special attention is paid. Furthermore, the relation with other time-frequency representations such as the Wigner distribution and the radar ambiguity function is given, and the Gabor representation problem is tackled. Finally, an example of how to use the Zak transform is given by solving data transmission problems where a robust filter, with or without decision feedback, has to be designed with optimal performance properties with respect to intersymbol interference and noise surpression.

Keywords: Cyclostationary processes, equalization, time-frequency transforms, Zak transform.

## 1. Introduction

To introduce the notion of Zak transform, we give an example where it arises quite naturally. There are practical situations where time-continuous signals are sampled at a certain uniform rate and at an unknown or uncontrolled sampling phase. The fact that this sampling phase is unspecified usually presents no problems when the sampling rate is well above the Nyquist rate. However, for undersampled signals, the particular sampling phase may play a significant role. To be more specific, the expression

$$(Zf)_T(t,\nu) = T^{\frac{1}{2}} \sum_{k=-\infty}^{\infty} f(t+kT) \exp(-2\pi i k \nu T),$$
$$0 \leqslant t \leqslant T,\ 0 \leqslant \nu \leqslant T^{-1}, \quad (1.1)$$

which we call the Zak transform of $f$, may depend quite heavily on $t$. In (1.1), $f$ is the time-continuous signal, and $T$ is the sampling time. We recognize in (1.1) the discrete Fourier transform of the sequence $f(t+kT)$, $k \in \mathbb{Z}$. Problem areas in signal processing where signals sampled well below the Nyquist rate occur, are video and data transmission.

There are quite a few instances in the mathematical, physical and signal theoretic literature where the Zak transform has been used explicitly or implicitly. Without being exhaustive we mention refs 1 to 7 (mathematical literature), refs 8 to 13 (physical literature), and refs 14 to 23 (signal theory literature). The Zak transform was studied more or less systematically for the first time by Zak [8])*), but some of its properties were already known much longer (according to Schempp [6]), the Zak transform, termed Weil-Brezin map by him, was already known to Gauss). Despite the fact that the Zak transform shows up at several places in the applied and theoretical literature, it appears that no systematic study of it has been made in the context of signal theory as an analysis or description tool. It is this gap which this paper tries to fill in.

The Zak transform of a signal can be considered as a mixed time-frequency representation of $f$, as one clearly sees from (1.1). Indeed, there exist intriguing relations with bilinear signal representations, such as the Wigner distribution, the Rihaczek distribution, and the radar ambiguity function. To mention such a relation, we have

$$\sum_{n,m} R_f(t+nT, \nu+\frac{m}{T}) = |(Zf)_T(t,\nu)|^2, \quad 0 \leqslant t \leqslant T,\ 0 \leqslant \nu \leqslant T^{-1}, \quad (1.2)$$

where $R_f$ is the Rihaczek distribution of $f$. As with the usual Fourier transform, however, the significance of the Zak transform does not only derive from its interpretation in terms of distribution of signal energy, but also from its capability as a tool to bring certain linear equations into a more manageable form. The first author*) to recognize this point was Zak [8]). In this paper the Schrödinger equation for an electron subjected to a periodic potential in a constant magnetic field (as encountered in solid-state physics applications) is derived and brought into a form more apt for physical applications by means of the Zak transform. (It is remarkable that, as with the Wigner distribution in quantum thermodynamics and the coherent states representation in quantum radiation-field theory for nonparametric and parametric (spectrograms) time-frequency representations, respectively, this is another occasion where a concept originally introduced in quantum mechanics turns out to be useful in signal analysis.) A recent example, with potential applications in digital data transmission, is discussed in ref. 23 by Bergmans and Janssen. In this application one has to solve equations of the type

$$\int_{-\infty}^{\infty} R(t,s) f(s)\, ds + N_0 f(t) = g(t), \quad -\infty < t < \infty, \quad (1.3)$$

where $R$ is the autocorrelation function of a cyclostationary process (so that $R(t+T, s+T) = R(t,s)$ for all $t$, s), $N_0 > 0$, $g$ is given, and $f$ is an unknown impulse response function. With the aid of the Zak transform, this equation is brought into the form

$$\int_0^T \Phi(t,s;\nu)(Zf)_T(s,\nu)\, ds + N_0 (Zf)_T(t,\nu) = (Zg)(t,\nu),$$
$$0 \leqslant t \leqslant T,\ 0 \leqslant \nu \leqslant T^{-1}, \quad (1.4)$$

which is easier to solve and lends itself more easily for incorporating causality conditions on $g$ (when feedback is used). Here

*) Zak calls his transform the $k$-$q$ representation. In view of the references given above, it is a somewhat delicate matter to attach the name of a single author to the transform. Since Zak was, to our knowledge, the first one to recognize the merits of the transform in a wider context, we think it is fair to name the transform after him.

*) After completion of the paper it was brought to the autor's attention that I.M. Gel'fand in "Eigenfunction expansions for an equation with periodic coefficients", Dokl. Akad. Nauk SSSR 76 (1950), pp 1117-1120 (Russian) has used the transform in (1.1) to prove a theorem on eigenfunction expansions for Schrödinger operators with periodic potentials. Zak was apparently not aware of this work, and it would therefore be fair to call the transform the Gel'fand-Zak transform. In "Geometric and Arithmetic Methods in the Spectral Theory of Multidimensional Periodic Operators", Proc. Steklov Institute of Mathematics, Vol 171 issue 2, AMS. Providence, Rhode Island, 1987 by M.M. Skriganov the transform is called the Gel'fand mapping. The transform and Gel'fand's theorem also occur in M. Reed and B. Simon, "Methods of Modern Mathematical Physics IV", Academic Press, New York, 1978, Ch XIII. 16.

$$\Phi(t,s;\nu) = T^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} R(t+nT,s)\exp(-2\pi i n \nu T),$$

$$0 \leq t,\ s \leq T,\ 0 \leq \nu \leq T^{-1}, \quad (1.5)$$

is to be regarded as the natural generalization to cyclostationary processes of the notion of spectral density function for stationary time-discrete stochastic processes.

We now give a summary of the contents of this paper. Since the Zak transform, just as the Wigner distribution, provides a means to represent signals in time and frequency, there are quite a few properties of the Wigner distribution that have counterparts in the Zak transform. The plan of the paper[24]), to which much of the present popularity of the Wigner distribution in the signal analytic community is owed, is therefore taken as a guideline to some extent for the present paper. Hence, in sec. 2 we develop notations, give the definition of the Zak transform and its basic properties (such as shift properties, marginal properties, support properties), and show how time or frequency limitations on the signal are reflected by corresponding properties of the Zak transforms. While establishing these properties, we shall, with great ease, rediscover various important theorems in signal analysis, such as the Fourier inversion theorem, Parseval's theorem, Nyquist's first criterion, Shannon's sampling theorem. In sec. 3 we shall calculate the Zak transforms of some test signals like periodic signals, chirps, Gaussians, exponentials, impulse responses and so on. It is apparent from this section that the Zak transform can be of some help in displaying the time-frequency characteristics of signals. As an unexpected by-product of considering Zak transforms of chirps, we obtain generalizations of identities involving quadratic Gauss sums which require several pages of proof when derived by more conventional methods. In sec. 4 we consider the effect of linear transformations, such as modulations, convolutions, and integral transformations as in the left-hand side of (1.3). At this point we consider autocorrelation functions of cyclostationary processes in some more detail. We give a representation theorem of these autocorrelation functions, show how the Zak transform comes up in this representation, and indicate in what sense the $\Phi(t, s; \nu)$ in (1.5) is to be interpreted as a spectral density. In sec. 5 we shall study the zeros of the Zak transform (from now on we deviate from the set-up followed in ref. 24). We have the following striking property: any Zak transform that is continuous everywhere must vanish somewhere in the fundamental rectangle $[0, T] \times [0, T^{-1}]$. Hence, when $f(kT)$, $k = -\infty, \cdots, \infty$, is the impulse response of a filter with specific properties (such as band-pass, equalizing), it happens for certain sampling phases that the specific character is destroyed by the occurrence of spectral zeros. This phenomenon has been observed in disguised form[17,22]), but a rigorous proof of this fact has not been published in the signal analysis literature as far as we know.





The contents of this paper, except perhaps the digression on cyclostationary processes in sec.4, should be accessible to most readers up to this point. This may not be the case with the remainder of the paper which is of a more specific nature. In sec. 6 we present relations between the Zak transform and other time-frequency transformations, such as the Wigner distribution, the (generalized) Rihaczek distribution, and the radar ambiguity function. In sec. 7 we devote attention to problems associated with the determination of the coefficients $c_{nm}(f)$ in the Gabor series expansion[25])

$$f(t) = \sum_{n,m} c_{nm}(f)\, g_{nT,mT^{-1}}(t) \quad -\infty < t < \infty, \quad (1.6)$$

where $g_{a,b}$ is a time-frequency translate (see sec. 2) of a fixed function $g$. A related problem also studied in sec. 6, is the completeness problem of the set of functions $\{g_{a,b} \mid a = nT,\ b = mT^{-1},\ n,\ m \text{ integer}\}$. It will turn out that the sets where $(Zg)_T$ and $(Zf)_T$ vanish play a crucial role. In sec. 8 the Zak transform is used to solve certain equalization problems from digital communication theory. These problems have been included to show the merits of the Zak transform for problems with a more practical background.

The present paper is intrinsically more 'formula-oriented' than the paper[24]) on the Wigner distribution. This is so since, contrary to the Wigner distribution, the Zak transform derives its usefulness as a convenient mathematical tool for signal analysis applications, rather than as a clear display tool. Many of the results in this paper can be found somewhere in the literature (especially in refs 4, 5, 8 to 13, 19 and 20), and do not need to be proved. However, some of the proofs shed further light on the nature of the Zak transform, and will for this reason be included. All the author can say is that he has not seen some of the results in subsec. 2.2.7, subsec. 3.4, sec. 4, the end of sec. 5, sec. 6 and sec. 8 anywhere else in the literature.

## 2. Definition of the Zak transform and basic properties

### 2.1. *Definitions and notations*

In this subsection we give the definition of the Zak transform as well as various other linear operators acting on time-continuous signals. Let $f(t)$ be a complex-valued, time-continuous signal. We define

$$(Zf)(\tau,\Omega) = \sum_{k=-\infty}^{\infty} f(\tau+k) \exp(-2\pi i k \Omega), \quad -\infty<\tau,\Omega<\infty. \quad (2.1)$$

Notice that, in comparison with (1.1), we have taken the sampling time $T$ equal to unity, and that we use normalized time and frequency variables $\tau$ and $\Omega$. The formulas and properties to follow remain, however, valid for the more general case $T \neq 1$ when they are properly modified. In this paper we shall be rather careless about the set of time functions $f$ for which we would like to consider $Zf$. Mostly, they will be rather smooth and fast decaying, but we will also allow delta functions, periodic functions, chirps, and so on. It is possible to cast almost everything that is presented in this paper in an entirely rigorous form by using suitable spaces of generalized functions, pretty much as this was done in ref. 12. This would, however, seriously affect the tutorial value of the paper, and we prefer to state and prove our results without too much attention to rigour.

The knowledgeable reader will have noticed the formal resemblance of the Zak transform and the modified $z$-transform[15]). In the latter transform the summation is only over non-negative $k$, and the complex exponential $\exp(2\pi i \Omega)$ is replaced by a complex $z$, usually outside the unit circle. One can find in ref. 15 quite a few properties which have a counterpart for the Zak transform in the present paper. However, in our approach the time variable $\tau$ and the frequency variable $\Omega$ occur symmetrically, as follows from formulas (2.11 and 2.14).

We define some more operators now. The Fourier transform $\mathcal{F}f$ of $f$ and the inverse transform $\mathcal{F}^{-1}f$ are defined by

$$(\mathcal{F}f)(\nu) = \int_{-\infty}^{\infty} \exp(-2\pi i t \nu) f(t)\, dt,\ (\mathcal{F}^{-1}f)(\nu) = (\mathcal{F}f)(-\nu),$$
$$-\infty<\nu<\infty. \quad (2.2)$$

We shall often write capitals for the Fourier transform of the functions denoted by the corresponding lower case symbol. We have $\mathcal{F}\mathcal{F}^{-1}f = \mathcal{F}^{-1}\mathcal{F}f = f$. The translations $T_af$, $R_bf$ for real $a$, $b$ are defined by

$$(T_af)(t) = f(t+a),\ (R_bf)(t) = \exp(-2\pi i b t) f(t), \quad -\infty<t<\infty. \quad (2.3)$$

We shall sometimes write $f_{a,b}$ for the function $R_b T_a f$. We have





$$(Zf)(\tau,\Omega) = \sum_{k=-\infty}^{\infty} (R_\Omega T_\tau f)(k) = \sum_{k=-\infty}^{\infty} f_{\tau,\Omega}(k), \quad -\infty<\tau,\Omega<\infty. \quad (2.4)$$

The dilations $D_\gamma f$ with $\gamma > 0$ are defined by

$$(D_\gamma f)(t) = \gamma^{\frac{1}{2}} f(\gamma t), \quad -\infty<t<\infty. \quad (2.5)$$

The inner product $(f,g)$ of the functions $f$ and $g$ is defined by

$$(f,g) = \int_{-\infty}^{\infty} f(t)\,(g(t))^*\, dt, \quad (2.6)$$

where the asterisk refers to taking complex conjugates.
The norm $\|f\|$ of $f$ is defined by

$$\|f\| = \Big(\int_{-\infty}^{\infty} |f(t)|^2\, dt\Big)^{\frac{1}{2}}.$$

The operators $\mathcal{F}$, $T_a$, $R_b$, $D_\gamma$ are norm-preserving, that is,

$$\|f\| = \|\mathcal{F}f\| = \|T_af\| = \|R_bf\| = \|D_\gamma f\|, \quad (2.8)$$

and we have for the functions $f$ and $g$

$$(T_af,g) = (f,T_{-a}g), \quad (R_bf,g) = (f,R_{-b}g) \text{ and } (D_\gamma f,g) = (f,D_{1/\gamma}g). \quad (2.9)$$

Finally, we denote by $f_-$ and $f^*$ the time-reversed and complex conjugated functions, defined respectively by

$$f_-(t) = f(-t),\ f^*(t) = (f(t))^*, \quad -\infty<t<\infty. \quad (2.10)$$

### 2.2. *Basic properties of the Zak transform*

In this subsection we give a list of basic properties of the Zak transform.

#### 2.2.1. *The Zak transform and the Fourier transform*

The Zak transform $Zf$ of $f$ can also be defined by using the Fourier transform $F = \mathcal{F}f$ of $f$ instead of $f$ itself. This is so since we have

$$(Zf)(\tau,\Omega) = \exp(2\pi i\tau\Omega)\,(Z\mathcal{F}f)(\Omega,-\tau), \quad -\infty<\tau,\Omega<\infty. \qquad (2.11)$$

This formula is so important that we give its proof. When $g(t)$ is a time-continuous function with Fourier transform $G(\nu)=(\mathcal{F}g)(\nu)$, see (2.2), then the sample values of $g$ and $G$ at integer points $k$ are related by the Poisson summation formula*) according to

$$\sum_{k=-\infty}^{\infty} g(k) = \sum_{k=-\infty}^{\infty} G(k). \qquad (2.12)$$

When we take $g=R_\Omega T_\tau f$ (see 2.4), and note that

$$G(\nu) = (\mathcal{F}R_\Omega T_\tau f)(\nu)$$
$$= \exp(2\pi i\tau\Omega)\,(R_{-\tau}T_\Omega\mathcal{F}f)(\nu), \quad -\infty<\nu<\infty, \qquad (2.13)$$

formula (2.11) follows.
Furthermore,

$$(Z\mathcal{F}f)(\Omega,\tau) = \exp(2\pi i\Omega\tau)\,(Zf)(-\tau,\Omega), \quad -\infty<\tau,\Omega<\infty. \qquad (2.14)$$

and

$$(Z\mathcal{F}^{-1}f)(\Omega,\tau) = \exp(2\pi i\tau\Omega)\,(Zf)(\tau,-\Omega), \quad -\infty<\tau,\Omega<\infty. \qquad (2.15)$$

We note at this point a very simple proof of the Fourier inversion formula $\mathcal{F}^{-1}\mathcal{F}f=\mathcal{F}\mathcal{F}^{-1}f=f$: just apply (2.14) and (2.15) to obtain $Z(\mathcal{F}^{-1}\mathcal{F}f)=Z(\mathcal{F}\mathcal{F}^{-1}f)=Zf$ (we shall see later that $Zf=Zg$ implies that $f=g$).

The formula (2.14) can also be used to derive Nyquist's first criterion. When $f$ is a time-continuous signal with Fourier transform $F$, one is interested in a condition for $F$ that ensures that $f(k)=0$, integer $k\neq 0$, $f(0)=1$. By (2.14) we see that $(Zf)(0,\Omega)=(ZF)(\Omega,0)$ for $-\infty<\Omega<\infty$.
Hence

$$(Zf)(0,\Omega) = \sum_{k=-\infty}^{\infty} f(k)\exp(-2\pi ik\Omega) = 1, \quad -\infty<\Omega<\infty, \qquad (2.16)$$

if and only if

*) One can say that, mathematically, Zak transform theory is nothing more than cleverly employing the Poisson summation formula (2.12).





$$(ZF)(\Omega,0) = \sum_{k=-\infty}^{\infty} F(\Omega+k) = 1, \quad -\infty<\Omega<\infty. \qquad (2.17)$$

Therefore, $f(k)=0$, $k\neq 0$, $f(0)=1$ if and only if (2.17) holds. This is Nyquist's first criterion. Compare ref. 26.

### 2.2.2. *The Zak transform and time and frequency shifts*

We have for $-\infty<a,b<\infty$ the formulas

$$(ZT_af)(\tau,\Omega) = (Zf)(\tau+a,\Omega), \quad -\infty<\tau,\Omega<\infty, \qquad (2.18)$$

and

$$(ZR_bf)(\tau\Omega) = \exp(-2\pi ib\tau)\,(Zf)(\tau,\Omega+b), \quad -\infty<\tau,\Omega<\infty. \qquad (2.19)$$

The formula (2.18) expresses that a time translation of the signal over distance $a$ is reflected by a corresponding translation of $Zf$ in the time variable. The second formula has a similar interpretation, except for the factor $\exp(-2\pi ib\tau)$.

The Zak transform satisfies the following periodicity relations. We have

$$(Zf)(\tau+1,\Omega) = \exp(2\pi i\Omega)\,(Zf)(\tau,\Omega), \quad -\infty<\tau,\Omega<\infty, \qquad (2.20)$$

and

$$(Zf)(\tau,\Omega+1) = (Zf)(\tau,\Omega), \quad -\infty<\tau,\Omega<\infty. \qquad (2.21)$$

Hence, $Zf$ is periodic in the frequency variable and is what we call quasi-periodic in the time variable. As a consequence, we have for integer $n$ and $m$

$$Z(R_mT_nf)(\tau,\Omega) = \exp(-2\pi im\tau+2\pi in\Omega)\,(Zf)(\tau,\Omega),$$
$$-\infty<\tau,\Omega<\infty. \qquad (2.22)$$

### 2.2.3. *The Zak transforms and dilations*

Formulas for $ZD_\gamma f$ in terms of $Zf$ are usually rather involved. This is so since sample values $f(\gamma\tau+\gamma k)$ are to be related to sample values $f(\tau+\ell)$, integer $k$, $\ell$. In terms of the unnormalized transform $(Zf)_T$ in (1.1) with $T>0$, we have

$$(Z D_\gamma f)(\tau,\Omega) = (Zf)_\gamma(\gamma\tau, \frac{\Omega}{\gamma}), \quad -\infty < \tau, \Omega < \infty. \qquad (2.23)$$

Formula (2.23) can be used to translate the results for the normalized transform (2.1) to the unnormalized one (1.1).

When $\gamma$ is rational there is a formula that expresses $Z D_\gamma f$ in terms of $Zf$. We have, with $p$, $q$ positive integers, the formulas [13])

$$(Z D_p f)(\tau,\Omega) = p^{-\frac{1}{2}} \sum_{r=0}^{p-1} (Zf)(p\,\tau, \frac{\Omega + r}{p}), \quad -\infty < \tau, \Omega < \infty, \qquad (2.24)$$

$$(Z D_{1/q} f)(\tau,\Omega) = q^{-\frac{1}{2}} \sum_{\ell=0}^{q-1} \exp(-2\pi i\,\ell\,\Omega)(Zf)(\frac{\tau+\ell}{q}, \Omega q), \quad -\infty < \tau, \Omega, < \infty. \qquad (2.25)$$

These formulas indicate how $Zf$ must be folded to obtain the Zak transforms of $D_p f$ and $D_{q^{-1}} f$. If we combine —(2.14) and (2.15), we get

$$(Z D_{p/q} f)(\tau,\Omega) = (p\,q)^{-\frac{1}{2}} \sum_{r=0}^{p-1} \sum_{\ell=0}^{q-1} \exp\left(-2\,\pi i\,\ell\frac{\Omega+r}{p}\right)(Zf)\left(\frac{p\tau+\ell}{q}, \frac{\Omega+r}{p}q\right) =$$
$$= (p\,q)^{-\frac{1}{2}} \sum_{\ell=0}^{q-1} \sum_{r=0}^{p-1} \exp(-2\,\pi i\,\ell\,\Omega)(Zf)\left(\frac{\tau+\ell}{q}p, \frac{q\,\Omega+r}{p}\right), \quad -\infty < \tau, \Omega < \infty, \qquad (2.26)$$

Together with formula (2.23) this formula is useful when dealing with rational sample rate conversions.

### 2.2.4. *The Zak transform and time-reversal, complex conjugation*

We have the formulas

$$(Zf_-)(\tau,\Omega) = (Zf)(-\tau, -\Omega), (Zf^*)(\tau,\Omega) = (Zf)^*(\tau, -\Omega), \quad -\infty < \tau, \Omega < \infty, \qquad (2.27)$$

In particular, when $f$ is real, we see that $Zf$ is a Hermitean function in the frequency variable $\Omega$. When $f$ is real and even, we have

$$(Zf)(\tau,\Omega) = (Zf)(-\tau,-\Omega) = (Zf)^*(\tau, -\Omega) = (Zf)(-\tau,\Omega), \quad -\infty < \tau, \Omega < \infty. \qquad (2.28)$$





### 2.2.5. *Marginal properties of the Zak transform*

We have the formulas

$$f(t) = \int_0^1 (Zf)(t,\Omega)\,d\Omega, \quad -\infty < t < \infty, \qquad (2.29)$$

$$F(\nu) = \int_0^1 \exp(-2\pi i\,\nu\,\tau)(Zf)(\tau,\nu)\,d\tau, \quad -\infty < \nu < \infty. \qquad (2.30)$$

These formulas parallel the correct marginal formulas for the Wigner distribution in ref. 24, eq. 2.3. Among other things we see that the mapping $f \to Zf$ is invertible, and that $f$ and $F$ are continuous when $Zf$ is continuous (the converse of the latter result does not hold).

### 2.2.6. *Restriction of Zak transforms to unit squares*

Let $S$ denote a square of the form $[a, a+1] \times [b, b+1]$. We have

$$\iint_S (Zf)(\tau,\Omega)(Zg)^*(\tau,\Omega)\,d\tau\,d\Omega = (f,g), \qquad (2.31)$$

where the right-hand side is the usual inner product for square integrable functions. In particular, we have

$$\iint_S |(Zf)(\tau,\Omega)|^2\,d\tau\,d\Omega = \|f\|^2, \qquad (2.32)$$

which shows that $f \to Zf$ is an energy preserving transformation.

The proof of (2.31) is straightforward: just plug in the formula (2.1) for $Zf$ and $Zg$, perform the integration with respect to $\Omega$ and note that

$$\sum_n \int_a^{a+1} h(\tau+n)\,d\tau = \int_{-\infty}^{\infty} h(t)\,dt.$$

At this point we note the following simple proof of Parseval's theorem: if we take for $S$ in (2.31) the sets $[0,1] \times [0,1]$ and $[-1,0] \times [0,1]$, respectively, and use (2.11), we get

$$(\mathcal{F}f,\mathcal{F}g) = (f,g). \tag{2.33}$$

As an aside we note that, more generally, for $1 \leqslant p \leqslant 2$

$$\iint_S |(Zf)(\tau,\Omega)|^p \, d\tau d\Omega \leqslant \int_{-\infty}^{\infty} |f(t)|^p \, dt; \tag{2.34}$$

this result cannot be generalized to the case $p > 2$.

As a consequence of the periodicity relations (2.20) and (2.21), it is sufficient to consider Zak transforms on unit squares only. Conversely, we have the following result. If we have a function $Z(\tau,\Omega)$ satisfying

$$Z(\tau+1,\Omega) = \exp(2\pi i \Omega) Z(\tau,\Omega) \text{ and } Z(\tau,\Omega+1) = Z(\tau,\Omega), \quad -\infty < \tau,\Omega < \infty, \tag{2.35}$$

which is square integrable over $S$, then there is a function $f(t)$, square integrable over $-\infty < t < \infty$, such that $Z = Zf$. This $f$ is unique and given by

$$f(t) = \int_0^1 Z(t,\Omega) \, d\Omega, \quad -\infty < t < \infty. \tag{2.36}$$

A very important formula, which generalizes (2.31), is obtained when we consider the Fourier expansion of $Zf \cdot (Zg)^*$ over $[0,1] \times [0,1]$. Due to the periodic relations (2.20) and (2.21), the latter function is indeed periodic in its two variables. The formula reads

$$(Zf)(\tau,\Omega)(Zg)^*(\tau,\Omega) = \sum_{n,m=-\infty}^{\infty} (f, R_{-m} T_n g) \exp(2\pi i n \Omega + 2\pi i m \tau), \quad -\infty < \tau,\Omega < \infty. \tag{2.37}$$

Formula (2.31) follows upon integration of (2.37) over a unit square; on the other hand, formula (2.37) can be derived from formula (2.31) by calculating the Fourier coefficients of $Zf \cdot (Zg)^*$ using (2.22).

The importance of formula (2.37) lies in the facts that it can be used to determine the effect of convolutions and modulations, that one can use it to present relations between the Zak transform and Wigner distribution, Rihaczek distribution and ambiguity function, and that one can solve problems





related to Gabor's expansion with it. These topics will be addressed in secs 4, 6 and 7, respectively.

*2.2.7. Zak transforms of time- or band-limited signals*

Let $0 \leqslant a,b \leqslant \frac{1}{2}$. We have the formulas

$$(Zf)(\tau,\Omega) = f(\tau), \quad |\tau| \leqslant \tfrac{1}{2}, \quad -\infty < \Omega < \infty, \tag{2.38}$$

$$(Zf)(\tau,\Omega) = \exp(2\pi i \tau \Omega) F(\Omega), \quad -\infty < \tau < \infty, \quad |\Omega| \leqslant \tfrac{1}{2}, \tag{2.39}$$

when $f$ is time-limited to $[-a, a]$ and band-limited to $[-b,b]$, respectively. Hence, when $f$ is time-limited to $[-a,a]$, then $(Zf)(\tau,\Omega) = 0$ for $a \leqslant |\tau| \leqslant \frac{1}{2}$, and when $f$ is band-limited to $[-b,b]$, then $(Zf)(\tau,\Omega) = 0$ for $b \leqslant |\Omega| \leqslant \frac{1}{2}$.

We note at this point that formula (2.39) can be used to give a quick proof of the Shannon sampling theorem. Indeed, take $\tau = 0$, multiply (2.39) by $\exp(2\pi i t \Omega)$, and integrate the identity over $\Omega \in [-b,b]$. At the right we obtain $f(t)$ by the Fourier inversion theorem, and the left-hand side is found by using the definition of the Zak transform and integrating the exponentials. The result is

$$\sum_{k=-\infty}^{\infty} f(k) \frac{\sin 2\pi b (k-t)}{\pi (k-t)} = f(t), \quad -\infty < t < \infty, \tag{2.40}$$

which is Shannon's formula.

In a similar fashion it follows from Parseval's theorem for Fourier series and (2.39) that

$$\sum_{k=-\infty}^{\infty} |f(\tau+k)|^2 = \int_{-b}^{b} |F(\Omega)|^2 \, d\Omega, \quad -\infty < \tau < \infty, \tag{2.41}$$

is independent of $\tau$ whenever $f$ is band-limited to $[-b,b]$, $b \leqslant \frac{1}{2}$ and also that

$$\sum_{k=-\infty}^{\infty} |F(\Omega+k)|^2 = \int_{-a}^{a} |f(\tau)|^2 \, d\tau, \quad -\infty < \Omega < \infty, \tag{2.42}$$

is independent of $\Omega$ whenever $f$ is time-limited to $[-a,a]$, $a \leqslant \frac{1}{2}$.

There are some interesting formulas for functions $f$ that are either time-limited to $[-\frac{1}{2}L, \frac{1}{2}L]$ or band-limited to $[-\frac{1}{2}M, \frac{1}{2}M]$, where $L$ and $M$ are positive integers. Such a formula reads

$$f(\tau) = \frac{1}{L} \sum_{\ell=0}^{L-1} (Zf)(\tau, \frac{\ell}{L}), \quad -\frac{L}{2} \leqslant \tau \leqslant \frac{L}{2}, \tag{2.43}$$

and holds for signals $f$ time-limited to $[-\frac{1}{2}L, \frac{1}{2}L]$. Correspondingly,

$$F(\Omega) = \frac{1}{M} \sum_{m=0}^{M-1} \exp\left(-2\pi i \frac{m}{M} \Omega\right) (Zf)\left(\frac{m}{M}, \Omega\right), \quad -\frac{M}{2} \leqslant \Omega \leqslant \frac{M}{2}, \tag{2.44}$$

holds for signals $f$ band-limited to $[-\frac{1}{2}M, \frac{1}{2}M]$. The formulas (2.43) and (2.44) show that the marginal integrals (2.18) and (2.19) can be 'discretized' when $f$ is time- or band-limited. It goes without saying that this yields considerable computational savings when one wants to recover $f$ or $F$ from $Zf$. Remarkably, formulas like (2.43) and (2.44) also hold for $|Zf|^2$: we have

$$\int_0^1 |(Zf)(\tau,\Omega)|^2 \, d\Omega = \frac{1}{L} \sum_{\ell=0}^{L-1} |(Zf)(\tau, \frac{\ell}{L})|^2, \quad -\frac{L}{2} \leqslant \tau \leqslant \frac{L}{2}, \tag{2.45}$$

and

$$\int_0^1 |(Zf)(\tau,\Omega)|^2 \, d\tau = \frac{1}{M} \sum_{m=0}^{M-1} |(Zf)(\frac{m}{M}, \Omega)|^2, \quad -\frac{M}{2} \leqslant \Omega \leqslant \frac{M}{2}, \tag{2.46}$$

when $f$ is time-limited to $[-\frac{1}{2}L, \frac{1}{2}L]$ and band-limited to $[-\frac{1}{2}M, \frac{1}{2}M]$, respectively.

## 3. The Zak transform of certain test signals

In this section we calculate the Zak transform for certain test signals such as periodic signals, Gaussians, chirps, exponentials, and we display $|Zf|$ for some signals $f$.





*3.1.*

We start with the signal

$$f(t) = \begin{cases} 1, & 0 \leqslant t < 1, \\ 0, & t < 0, \ t \geqslant 1, \end{cases} \tag{3.1}$$

which is obviously discontinuous. We obtain

$$(Zf)(\tau,\Omega) = \exp(2\pi i [\tau] \Omega), \quad -\infty < \tau, \Omega < \infty, \tag{3.2}$$

where $[\tau]$ denotes the largest integer $\leqslant \tau$. Note that this $Zf$ has unit modulus everywhere (and hence no zeros), and that it is discontinuous at all integer $\tau$.

*3.2*

Consider now a periodic signal

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \exp(2\pi i k t), \quad -\infty < t < \infty. \tag{3.3}$$

We obtain

$$(Zf)(\tau,\Omega) = f(\tau) \sum_{\ell=-\infty}^{\infty} \delta(\Omega - \ell), \quad -\infty < \tau, \Omega < \infty, \tag{3.4}$$

where $\delta$ is the usual delta function at zero. Note that $Zf$ is supported completely by the lines $\Omega = \ell$, $\ell$ integer. It is essential here that the period of $f$ and the time extent of the fundamental square are equal. When, for instance, $f$ is periodic with a rational period $q/p$ (where the greatest common divisor of $q$ and $p$ equals 1), the Zak transform $Zf$ is supported by the lines $\Omega = m/q$, $m$ integer; a factorization as in (3.4) does not hold. When $f$ is periodic with an irrational period, the support of $Zf$ is usually a dense set in the $(\tau,\Omega)$-plane.

There are similar results for functions $f$ of the form

$$f(t) = \sum_{k=-\infty}^{\infty} d_k \, \delta(t - k\alpha), \quad -\infty < t < \infty. \tag{3.5}$$

*3.3*

The next signal we consider is the Gaussian

$$f(t) = (2\gamma)^{\frac{1}{4}} \exp(-\pi \gamma t^2), \quad -\infty < t < \infty. \tag{3.6}$$

We now obtain

$$(Zf)(\tau,\Omega) = (2\gamma)^{\frac{1}{4}} \exp(-\pi\gamma\tau^2)\, \theta_3(\pi(\Omega - i\gamma\tau); \exp(-\pi\gamma)), \quad -\infty < \tau, \Omega < \infty, \tag{3.7}$$

where $\theta_3$ is the third theta function, given by

$$\theta_3(z;q) = 1 + 2 \sum_{n=-\infty}^{\infty} q^{n^2} \cos(2\pi n z), \quad z \text{ complex}. \tag{3.8}$$

See for instance ref. 26, ch. 21.

The formula (2.11) can be used to derive Jacobi's formula (3.10) below by nothing that

$$(\mathcal{F} f)(\nu) = \left(\frac{2}{\gamma}\right)^{\frac{1}{4}} \exp(-\pi\gamma^{-1}\nu^2), \quad -\infty < \nu < \infty, \tag{3.9}$$

so that

$$\theta_3\left(-i\gamma^{-1}z; \exp\left(-\frac{\pi}{\gamma}\right)\right) = \gamma^{\frac{1}{2}} \exp\left(\frac{z^2}{\pi\gamma}\right) \theta_3(z; \exp(-\pi\gamma)), \quad z \text{ complex}. \tag{3.10}$$

It is known that the zeros of the function at the right-hand side of (3.7) are $(\tau,\Omega) = (k+\frac{1}{2}, \ell+\frac{1}{2})$, $k$, $\ell$ integer. Hence, e.g. $(Zf)(\frac{1}{2},\frac{1}{2}) = 0$.

In fig. 1 we display $|(Zf)(\tau,\Omega)|$ as a function of $(\tau,\Omega)$ with $\gamma = 1$. Note that now $\mathcal{F} f = f$, so that $|(Zf)(\tau,\Omega)| = |(Zf)(\Omega,\tau)|$ by formula (2.11).

*3.4.*

We now consider the Zak transform of the chirp

$$f(t) = \exp(\pi i t^2), \quad -\infty < t < \infty. \tag{3.11}$$





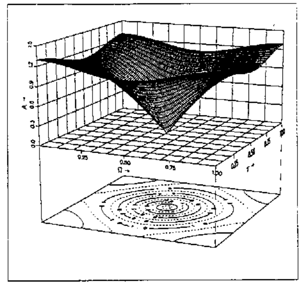

Fig. 1. The modulus of the Zak transform of $f$ in (3.6) with $\gamma = 1$.

It turns out that

$$(Zf)(\tau,\Omega) = \exp(\pi i \tau^2) \sum_{k=-\infty}^{\infty} \delta(\tau - \Omega - \tfrac{1}{2} - k), \quad -\infty < \tau, \Omega < \infty, \tag{3.12}$$

which shows that $Zf$ is concentrated on the lines $\Omega = \tau - \frac{1}{2} - k$, $k$ integer. This result is satisfactory, since the sweep rate can be nicely read off from the slope of the lines on which $Zf$ is concentrated.

A similar result is true for the moral general chirp

$$f(t) = \exp(\pi i \frac{p}{q} t^2), \quad -\infty < t < \infty, \tag{3.13}$$

with integer $p$, $q$. We have now

$$(Zf)(\tau,\Omega) = f(\tau) \sum_{k=-\infty}^{\infty} \delta(q\Omega - p\tau - \tfrac{1}{2}pq - k) \times \sum_{\ell=0}^{q-1} (-1)^{\ell p} \exp(\pi i \frac{p}{q}\ell^2 - 2\pi i \frac{\ell k}{q}), \quad -\infty < \tau, \Omega < \infty, \tag{3.14}$$

and from this formula we see that $Zf$ is concentrated on the lines $\{(\Omega,\tau) \mid q\Omega - p\tau = \frac{1}{2}pq + k\}$, $k$ integer.

Let us employ formula (2.11)*) to obtain the identity (3.19) below on Gaussian sums. We have

*) This part of the text is merely meant to show once more the power of formula (2.11), and may be skipped by readers not interested in formula (3.19).

$$(\mathcal{F}f)(\nu) = \sqrt{\frac{iq}{p}} \exp(-\pi i \frac{q}{p} \nu^2), \quad -\infty < \nu < \infty, \tag{3.15}$$

and, as in (3.14),

$$(Z\mathcal{F}f)(\tau,\Omega) = (\mathcal{F}f)(\tau) \sum_{k=-\infty}^{\infty} \delta(p\Omega + q\tau - \tfrac{1}{2}pq - k) \times \sum_{\ell=0}^{p-1} (-1)^{\ell q} \exp\Big(-\pi i \frac{q}{p}\ell^2 - 2\pi i \frac{\ell k}{p}\Big), \quad -\infty < \tau, \Omega < \infty. \tag{3.16}$$

When we use formula (2.11), we get an identity of the form

$$\exp(2\pi i \tau\Omega) f(-\Omega) \sum_{k=-\infty}^{\infty} c_k \delta(p\Omega + q\tau - \tfrac{1}{2}pq - k) = (\mathcal{F}f)(\tau) \sum_{k=-\infty}^{\infty} d_k \delta(p\Omega + q\tau - \tfrac{1}{2}pq - k), \quad -\infty < \tau, \Omega < \infty. \tag{3.17}$$

Hence

$$\exp(2\pi i \tau\Omega) f(-\Omega) c_k = (\mathcal{F}f)(\tau) d_k, \tag{3.18}$$

whenever $p\Omega + q\tau - \frac{1}{2}pq - k = 0$, $k$ integer. This yields for integer $k$

$$\sum_{\ell=0}^{q-1} (-1)^{\ell p} \exp(\pi i \frac{p}{q}\ell^2 - 2\pi i \frac{\ell k}{q}) = (\mathcal{F}f)\Big(\frac{k}{q} + \frac{p}{2}\Big) \sum_{\ell=0}^{p-1} (-1)^{\ell q} \exp\Big(-\pi i \frac{q}{p}\ell^2 - 2\pi i \frac{\ell k}{p}\Big), \tag{3.19}$$

a result a special case of which requires several pages of proof in ref. 27. The fact that the Zak transform can yield considerable shortcuts in proofs of certain identities was also observed in ref. 13.

As we see from formula (3.14), the expression for $Zf$ gets more complicated with increasing value of $qp$, although the sweep rate can still be read off from the slopes of the lines on which $Zf$ is concentrated. The case of irrational sweep rate is undoubtedly much more complicated.

In fig. 2 we display the supporting set of $Zf$ in (3.14), as far as contained in the unit square $[0,1] \times [0,1]$ for $p = 7$, $q = 5$ and $p = 5$, $q = 8$, respectively.





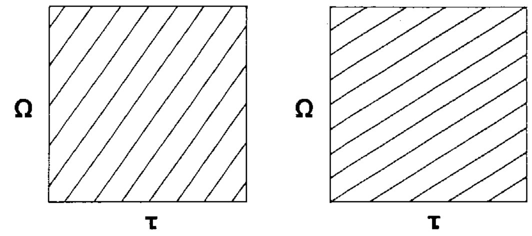


Fig. 2. The supporting set of the Zak transforms in (3.14) with $p = 7$, $q = 5$ and $p = 5$, $q = 8$, respectively.

*3.5.*

We now consider the case that

$$f(t) = \exp(-2\pi\alpha|t|), \quad -\infty < t < \infty, \tag{3.20}$$

with $\alpha > 0$. It can be checked that

$$(Zf)(\tau,\Omega) = \frac{zr^\tau}{z-r} + \frac{zr^{1-\tau}}{1-rz}, \quad 0 \leqslant \tau, \Omega \leqslant 1, \tag{3.21}$$

where

$$z = \exp(2\pi i\Omega),\ r = \exp(-2\pi\alpha), \quad 0 \leqslant \Omega \leqslant 1. \tag{3.22}$$

It follows that, within the unit square $[0,1] \times [0,1]$, $Zf$ has exactly one zero, viz. at $(\tau,\Omega) = (\frac{1}{2}, \frac{1}{2})$.

When we consider

$$g(t) = \begin{cases} \exp(-2\pi\alpha t), & t \geqslant 0, \\ 0 & , t < 0, \end{cases} \tag{3.23}$$

we get

$$(Zg)(\tau,\Omega) = \frac{r^\tau z}{z-r}, \quad 0 \leqslant \tau < 1,\ 0 \leqslant \Omega \leqslant 1, \tag{3.24}$$

with $z$ and $r$ as in (3.22). This $Zg$ has no zeros in the unit square, but is discontinuous at all integer $\tau$.

Formula (2.11) can be used to find the Zak transform of the function

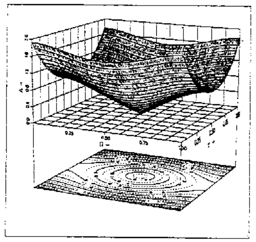

Fig. 3. The modulus of the Zak transform of $f$ in (3.20) with $\alpha = \frac{1}{2}$.

$$F(\nu) = \frac{1}{\pi\alpha} \frac{1}{1+(\nu/\alpha)^2}, \quad -\infty < \nu < \infty, \tag{3.25}$$

which is the Fourier transform of $f$ in (3.20). In figs 3 and 4 we display $|(Zf)(\tau,\Omega)|$ and $|(ZF)(\tau,\Omega)|$ as a function of $(\tau,\Omega)$ with $\alpha=\frac{1}{2}$. Observe that $|(Zf)(\tau,\Omega)| = |(ZF)(\Omega,\tau)|$.

## 4. The effect of certain linear transformations on the Zak transform

In this section we shall find out how certain linear transformations on signals are reflected by their Zak transforms. The transformations we consider here are modulations, convolutions, and integral transformations as in (1.3) with a kernel $R$ satisfying $R(t+1, s+1) = R(t,s)$ for all real $t$, $s$. Except for the last case, the resulting formulas parallel the corresponding formulas for the Wigner distribution.

### *4.1. The effect of linear filtering on the Zak transform*

Let $f$ and $h$ be two signals, and denote by $f*h$ the convolution

$$(f*h)(t) = \int_{-\infty}^{\infty} f(s)\, h(t-s)\, ds, \quad -\infty < t < \infty, \tag{4.1}$$

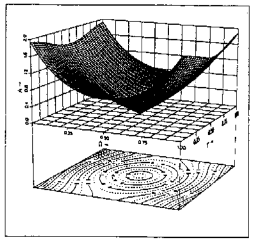

Fig. 4. The modulus of the Zak transform of $F$ in (3.25) with $\alpha = \frac{1}{2}$.





of $f$ and $h$. We then have

$$Z(f*h)(\eta,\Omega) = \int_0^1 (Zf)(\tau,\Omega)\,(Zh)(\eta-\tau,\Omega)\, d\tau, \quad -\infty < \eta,\Omega < \infty, \tag{4.2}$$

or, in short-hand notation, $Z(f*h) = Zf *_\tau Zh$. This formula results from formula (2.37) by choosing $g = T_{-\eta} h_-^*$ and integration over $\tau$.

An important special case arises when $h$ has the form $h(t) = \sum_k a_k \delta(t-k)$.

Then $Z(f*h)(\tau,\Omega) = A(\Omega)\,(Zf)(\tau,\Omega)$, where $A(\Omega) = \sum_k a_k \exp(-2\pi i k \Omega)$.

### *4.2. The effect of modulation on the Zak transform*

Let $f$ and $m$ be two signals, and denote by $f \cdot m$ the product function

$$(f\cdot m)(t) = f(t)\, m(t), \quad -\infty < t < \infty. \tag{4.3}$$

We then have

$$Z(f\cdot m)(\tau,\Lambda) = \int_0^1 (Zf)(\tau,\Omega)\,(Zm)(\tau,\Lambda-\Omega)\, d\Omega, \quad -\infty < \tau,\Lambda < \infty, \tag{4.4}$$

or, in short-hand notation, $Z(f\cdot m) = Zf *_\Omega Zm$. This formula results from formula (2.37) by choosing $g = R_{-\Lambda} m^*$ and integration over $\Omega$.

An important special case arises when $m(t)$ has the form

$$m(t) \sum_k b_k \exp(-2\pi i k t).$$

Then $Z(f\cdot m)(\tau,\Omega) = m(\tau)\,(Zf)(\tau,\Omega)$.

Hence we see that the effect of convolutions and multiplications in time-domain are reflected in the Zak transforms by convolutions in the time- and frequency-domain, respectively.

### *4.3. The effect of a special linear transformation on the Zak transform*

We consider in this subsection integral operators

$$(T_R f)(t) = \int_{-\infty}^{\infty} R(t,s) f(s)\,\mathrm{d}s, \quad -\infty < t < \infty, \tag{4.5}$$

where the kernel $R$ satisfies

$$R(t+1,\, s+1) = R(t,\, s), \quad -\infty < t,s < \infty. \tag{4.6}$$

An example of such an $R$ is the autocorrelation function of a cyclostationary process [28]), i.e. a stochastic process $\underline{x}$ with

$$E[\underline{x}(t+1)] = E[\underline{x}(t)],\ E[\underline{x}(t+1)[\underline{x}^*(s+1)] = E[(\underline{x}(t)\underline{x}^*(s)], \quad -\infty < t,s < \infty. \tag{4.7}$$

A somewhat degenerate example of a $T_R$ as in (4.5) is the multiplication by a periodic function $m$. One can then take

$$R(t,s) = m\left(\frac{t+s}{2}\right)\delta(t-s), \quad -\infty < t,s < \infty, \tag{4.8}$$

which yields $T_R f = f \cdot m$. This example is of some historical interest since in Zak's problem [8]) periodic multipliers (= potentials) occur.

We have the following result: for all signals $f$

$$(Z\,T_R f)(\tau,\Omega) = \int_0^1 \Phi(\tau,\mu;\Omega)\,(Zf)(\mu,\Omega)\,\mathrm{d}\mu, \quad 0 \leqslant \tau,\Omega \leqslant 1, \tag{4.9}$$

where

$$\Phi(\tau,\mu;\Omega) = \sum_{n=-\infty}^{\infty} R(\tau+n,\mu)\exp(-2\pi i n\Omega), \quad 0 \leqslant \tau,\nu,\mu \leqslant 1. \tag{4.10}$$

Formula (4.9) contains formula (4.2) as a special case, with $\Phi(\tau,\mu;\Omega) = (Zh)(\tau-\mu,\Omega)$.

The formula (4.9) should be interpreted as the Zak transform version of the convolution theorem. If $R(t+a,\, s+a) = R(t,s)$ for all real $t$, $s$ and all real $a$, i.e. if $R(t,\,s)$ is a function of the form $\varphi(t-s)$, then $T_R f$ in (4.5) equals $\varphi * f$, and $(\mathcal{F}T_R f)(\Omega) = \Phi(\Omega)\,F(\Omega)$. Formula (4.9) exhibits an additional integration over $\mu \in [0,1]$ since $R(t+a,\, s+a) = R(t,s)$ for all real $t$, $s$ is valid for integer $a$ only.





In the case where $R$ is the autocorrelation function of a cyclostationary process $\underline{x}$, $\Phi$ can be considered as a kind of spectral density function of $\underline{x}$. This is the generalization of the notion of spectral density function for stationary times series $(\underline{x}_k)_{-\infty<k<\infty}$ with discrete time $k$. Our $\Phi$ is, however, not positive in the ordinary sense, but in the sense of positive definite functions: we have

$$\int_0^1\int_0^1 \Phi(\tau,\mu;\Omega)\,\varphi(\tau)\,\varphi^*(\mu)\,\mathrm{d}\tau\mathrm{d}\mu \geqslant 0 \tag{4.11}$$

for all functions $\varphi$ defined on [0,1] and all $\Omega$. In particular

$$\Phi(\tau,\tau;\Omega) \geqslant 0, \quad 0 \leqslant \tau,\Omega \leqslant 1. \tag{4.12}$$

The result (4.11) can be shown to hold by observing that for any function $\varphi$ defined on [0,1] the time-discrete stochastic process

$$\left(\int_0^1 \underline{x}(\tau+k)\,\varphi(\tau)\,\mathrm{d}\tau\right)_{-\infty<k<\infty} \tag{4.13}$$

is wide-sense stationary, with spectral density function given by the left-hand side of (4.11).

As an example of an $R$ as above, we have

$$R(t,s) = E\left[\sum_{i=-\infty}^{\infty} k^{\Delta}(t-i)\,k^{\Delta *}(s-i)\right], \quad -\infty < t,s < \infty, \tag{4.14}$$

where $k^{\Delta}(t)$ is a stochastic time-continuous function and the expectation is taken over $\Delta$. In this case we get

$$\Phi(\tau,\mu;\Omega) = E[(Zk^{\Delta})(\tau,\Omega)\,(Zk^{\Delta})^*(\mu,\Omega)], \quad 0 \leqslant \tau,\mu,\Omega \leqslant 1, \tag{4.15}$$

so that the left-hand side of (4.11) equals

$$E\left|\int_0^1 (Zk^{\Delta})(\tau,\Omega)\,\varphi(\tau)\,\mathrm{d}\tau\right|^2. \tag{4.16}$$

There is a striking converse result: any autocorrelation function $R$ that satisfies $R(t+1, s+1) = R(t,s)$, $-\infty < t,s < \infty$, can be cast into the form (4.14) with a suitable choice for the stochastic function $k^{\Delta}(t)$.

We give some further examples.

1. Consider the process

$$\underline{x}(t) = \sum_{k=-\infty}^{\infty} a_k h(t-k-\theta_k), \quad -\infty < t < \infty, \tag{4.17}$$

with $a_k = \pm 1$, independent and identically distributed, $\theta_k$ independent and identically distributed, $Ea_k = 0$. Then

$$R_{\underline{x}}(t,s) = E\left[\sum_{k=-\infty}^{\infty} h(t-k-\theta)\,h^*(s-k-\theta)\right], \quad -\infty < t,s < \infty, \tag{4.18}$$

where the expectation $E$ is taken over $\theta$, and

$$\Phi_{\underline{x}}(\tau,\mu;\Omega) = E[(Zh)(\tau-\theta,\Omega)\,(Zh)^*(\mu-\theta,\Omega)], \quad 0 \leqslant \tau,\mu,\Omega \leqslant 1. \tag{4.19}$$

2. Let $R_{\underline{x}}$ be the autocorrelation function of a wide-sense stationary process $\underline{x}$, so that

$$R_{\underline{x}}(t,s) = K(t-s), \quad -\infty < t,s < \infty. \tag{4.20}$$

Then

$$\Phi_{\underline{x}}(\tau,\mu;\Omega] = (ZK)(\tau-\mu,\Omega), \quad 0 \leqslant \tau,\mu,\Omega \leqslant 1. \tag{4.21}$$

In particular, for a white noise process $\underline{x}$ with $K(t) = \delta(t)$, we get

$$\Phi_{\underline{x}}(\tau,\mu;\Omega) = \delta(\tau-\mu), \quad 0 \leqslant \tau,\mu,\Omega \leqslant 1. \tag{4.22}$$

3. Let $m$ be a function, periodic with period 1, and let $R$ be given by (4.8). Then





$$\Phi(\tau,\mu;\Omega) = m\left(\frac{\tau+\mu}{2}\right)\delta(\tau-\mu), \quad 0 \leqslant \tau,\mu,\Omega \leqslant 1. \tag{4.23}$$

More generally, when $R$ has the form

$$R(t,s) = m\left(\frac{t+s}{2}\right)K(t-s), \quad -\infty < t,s < \infty, \tag{4.24}$$

then we have

$$\Phi(\tau,\mu;\Omega) = \tfrac{1}{2}m\left(\frac{\tau+\mu}{2}\right)[(ZK)(\tau-\mu,\Omega) + (ZK)(\tau-\mu,\Omega+\tfrac{1}{2}] + {} + \tfrac{1}{2}m\left(\frac{\tau+\mu+1}{2}\right)[(ZK)(\tau-\mu,\Omega) - (ZK)(\tau-\mu,\Omega+\tfrac{1}{2})] \quad 0 \leqslant \tau,\mu,\Omega \leqslant 1. \tag{4.25}$$

Here we mention the representation ref. 28, (1 to 6)

$$R(t,s) = \sum_{n=-\infty}^{\infty} K_n(t-s)\exp(\pi i n(t+s)), \quad -\infty < t,s < \infty, \tag{4.26}$$

with

$$K_n(s) = \int_0^1 R(t+\tfrac{1}{2}s, t-\tfrac{1}{2}s)\exp(-2\pi i n t)\,\mathrm{d}t, \quad -\infty < s < \infty, \tag{4.27}$$

which holds for any $R$ with $R(t+1, s+1) = R(t,s)$, $-\infty < t,s < \infty$.

## 5. The zeros of the Zak transform

We shall regard in this section the Zak transform $(Zf)(\tau,\Omega)$ as a Fourier spectrum of the impulse response function $f(\tau+k)$, $-\infty < k < \infty$, of a time-discrete, linear, time-invariant system, where $\tau$ is to be interpreted as a timing error. We sketch here a situation for which the results of this section are relevant.

In digital data transmission one is faced with problems of the following type. A data sequence $a_k = \pm 1$, uncorrelated and with zero expectation, is transmitted over a noisy, dispersive channel, and its output is sampled at the symbol rate. This gives a time-discrete signal $r_k$ of the form

$$r_k = \sum_{i=-\infty}^{\infty} a_{k-i} f(i+\tau) + n_k, \quad -\infty < k < \infty, \tag{5.1}$$

where $f$ is the impulse response function of the channel, $\tau$ is the sampling phase, and $n_k$ is a white noise sequence with variance $N_0$. To estimate the data $a_k$ from $r_k$, a time-discrete filter with impulse response function $c_k$, $-\infty < k < \infty$, is applied to $r_k$ and designed in such a way that

$$E[|(c * r)(k) - a_k|^2] = \bar{\varepsilon}(\tau) \tag{5.2}$$

is minimal. Here $*$ indicates the discrete convolution, and $E$ refers to taking the expectation over data $a$ and noise $n$. Normally, one will design $c$ such that (5.2) is minimal for the nominal case $\tau = 0$, or one will incorporate knowledge of the distribution of the sampling phase $\tau$ by minimizing $E_\tau[\bar{\varepsilon}(\tau)]$, where $E_\tau$ represents the expectation over $\tau$. At any rate, when the sampling phase equals $\tau$, the average error with the $c$ thus obtained is at least equal to $\bar{\varepsilon}(\tau)$. It now turns out that the $c(\tau)$ that minimizes (5.2), is determined by

$$C(\tau;\Omega) = \sum_{j=-\infty}^{\infty} c_j(\tau) \exp(-2\pi i j \Omega) = \frac{(Zf)^*(\tau,\Omega)}{N_0 + |(Zf)(\tau,\Omega)|^2}, \quad 0 \leq \Omega \leq 1, \tag{5.3}$$

and that

$$\bar{\varepsilon}(\tau) = \int_0^1 \frac{N_0}{N_0 + |(Zf)(\tau,\Omega)|^2} \mathrm{d}\Omega. \tag{5.4}$$

It follows that the $\bar{\varepsilon}(\tau)$ depends quite heavily on the extent to which $(Zf)(\tau,\Omega)$ vanishes as a function of $\Omega$.

The following result shows that the performance can be bad when the sampling phase $\tau$ is allowed to range through an interval of length unity.

Assume that $(Zf)(\tau,\Omega)$ is a continuous*) function of the two variables $\tau$ and $\Omega$. Then $(Zf)(\tau_0,\Omega_0) = 0$ for some $\tau_0$, $\Omega_0$ with $0 \leq \tau_0, \Omega_0 \leq 1$.

*) A sufficient condition on a signal $f$ for continuity of $Zf$ is that $f$ is continuous and that $f(t) = 0(|t|^{-\alpha})$, $-\infty < t < \infty$, for some $\alpha > 1$. There are several other sufficient conditions of similar type.





The proof of this result is instructive and runs as follows. Suppose that $Zf$ is continuous and has no zeros. Then we can write

$$(Zf)(\tau,\Omega) = |(Zf)(\tau,\Omega)| \exp(2\pi i \varphi(\tau,\Omega)), \quad -\infty < \tau, \Omega < \infty, \tag{5.5}$$

where $\varphi$ is a continuous function of its two variables. We have by the periodicity relations (2.20) and (2.21)

$$(Zf)(1,\Omega) = (Zf)(0,\Omega) \exp(2\pi i \Omega), \; (Zf)(\tau,1) = (Zf)(\tau,0), \quad -\infty < \tau, \Omega < \infty. \tag{5.6}$$

Hence

$$\varphi(1,\Omega) = \varphi(0,\Omega) + \Omega + k, \; \varphi(\tau,1) = \varphi(\tau,0) + \ell, \quad -\infty < \tau, \Omega < \infty, \tag{5.7}$$

where $k$ and $\ell$ are integers independent of $\Omega$ and $\tau$, respectively, by continuity of $\varphi$. Now we calculate $\varphi(1,1)$ in two different ways:

$$\varphi(1,1) = \varphi(0,1) + 1 + k = \varphi(0,0) + \ell + 1 + k \tag{5.8}$$

and

$$\varphi(1,1) = \varphi(1,0) + \ell = \varphi(0,0) + 0 + k + \ell. \tag{5.9}$$

This is a contradiction. Hence $Zf$ must have a zero.

There is an interesting comment to make on the proof just given. Assume that $(Zf)(0,\Omega)$ (and hence $(Zf)(1,\Omega)$) is a smooth, nonvanishing function of $\Omega$. Then we can write $(Zf)(0,\Omega)$ and $(Zf)(1,\Omega)$ in the form (5.5) with $\tau = 0$ and $\tau = 1$ and $\varphi(\tau,\Omega)$ replaced by smooth functions $\varphi_0(\Omega)$ and $\varphi_1(\Omega)$, respectively. By the first periodicity relation in (5.6) we get the following relation between the group delays $t(1,\Omega) = -\varphi_1'(\Omega)$, $t(0,\Omega) = -\varphi_0'(\Omega)$:

$$t(1,\Omega) = t(0,\Omega) - 1, \quad -\infty < \Omega < \infty, \tag{5.10}$$

which is what we should expect.

The fundamental result given above has been proved in refs 11 and 12; in the data transmission literature, the performance breakdown of the estimation procedure given in the introduction of this section has been noted[17,22]), but an explanation, based on the zeros of Zak transforms, has not been presented thus far.

The proof of the fundamental result does not give a clue as to where to find the zeros of the Zak transform. We shall now give some results in this direction. We assume that $Zf$ is continuous.

- When $f$ is even, we have $(Zf)(\frac{1}{2},\frac{1}{2})=0$
- When $f$ is real, we have $(Zf)(\tau,\frac{1}{2})=0$ for some $\tau\in[0,1]$.
- When $f$ is odd, we have $(Zf)(0,0)=0$, $(Zf)(0,\frac{1}{2})=0$, $(Zf)(\frac{1}{2},0)=0$.
- When $f$ is real and even and $f''(t)>0$, $t>0$, we have

$$\mathrm{Re}\,(Zf)(0,\Omega)>0, \quad -\infty<\Omega<\infty, \tag{5.11}$$

$$(Zf)(\tau,\tfrac{1}{2})>0,\ 0\leqslant\tau<\tfrac{1}{2};\ (Zf)(\tau,\tfrac{1}{2})<0,\ \tfrac{1}{2}<\tau\leqslant 1. \tag{5.12}$$

- When $f$ is real and even and $f'''(t)<0$, $t>0$, we have

$$\mathrm{Im}(Zf)(\tau,\Omega)>0, \quad 0<\tau\leqslant 1,\ 0<\Omega<\tfrac{1}{2}, \tag{5.13}$$

$$\mathrm{Im}\,(Zf)(\tau,\Omega)<0, \quad 0<\tau\leqslant 1,\ \tfrac{1}{2}<\Omega<1. \tag{5.14}$$

Hence, we conclude from this list that the Zak transform of a real, even $f$ with $f'(t)>0$, $f''(t)<0$, $t>0$, has exactly one zero in the unit square, viz. at $(\tau,\Omega)=(\frac{1}{2},\frac{1}{2})$. This is a rather stringent condition, satisfied e.g. by $f(t)=\exp(-2\pi\alpha|t|)$, but not by $f(t)=\exp(-\pi t^2)$, see subsec. 3.5 and 3.3.

A better idea of the zero set of $Zf$ or the extent to which $|Zf|$ is small (as is necessary, for example, to assess the performance in the data transmission example in the beginning of this section, see (5.4)) is obtained by visual inspection of a plot of $|Zf|$. We have already given some examples in sec. 3, and we present now some more examples which serve as illustrations of the peculiar properties of the zero sets of Zak transforms. In all examples the normalization constant $c$ has been chosen such that the resulting function has unit energy.

In fig. 5 we display the modulus of the Zak transform of $c(F(\nu)-F(\nu-1))$, where $F$ is given by (3.25). This example is meant to illustrate the theorem that the zero set of $ZG$, with

$$G(\nu)=\sum_{k=-\infty}^{\infty}a_k F(\nu-k), \quad -\infty<\nu<\infty, \tag{5.15}$$





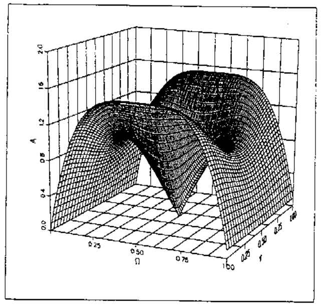

Fig. 5. The modulus of the Zak transform of $c(F(\nu)-F(\nu-1))$ with $F$ given by (3.25) and $\alpha=\frac{1}{2}$.

is contained in the zero set of $ZF$ itself. This follows from the formula

$$(ZG)(\tau,\Omega)=A(\Omega)(ZF)(\tau,\Omega), \quad -\infty<\tau,\Omega<\infty, \tag{5.16}$$

with

$$A(\Omega)=\sum_{k=-\infty}^{\infty}a_k\exp(2\pi i k\Omega), \quad -\infty<\Omega<\infty. \tag{5.17}$$

In the particular example we get, in addition to the zero at $(\frac{1}{2},\frac{1}{2})$, zero lines $\Omega=0$, $\Omega=1$.

In fig. 6 we display the modulus of the Zak transform of $cg'(t)$ with $g$ given by (3.6) with $\gamma=1$. Note that $cg'(t)$ is an odd function so that $c(Zg')(\tau,\Omega)$ exhibits zeros at $(0,0)$, $(0,\frac{1}{2})$, $(\frac{1}{2},0)$ and by the periodicity relations (2.20), (2.21) also at $(0,1)$, $(1,0)$, $(\frac{1}{2},1)$, $(1,\frac{1}{2})$, $(1,1)$.

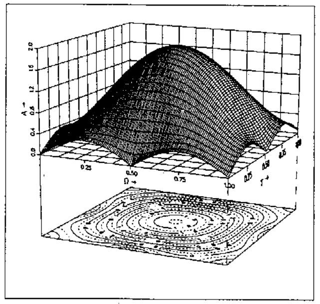

Fig. 6. The modulus of the Zak transform of $cg'(t)$ with $g$ given by (3.6) with $\gamma=1$.

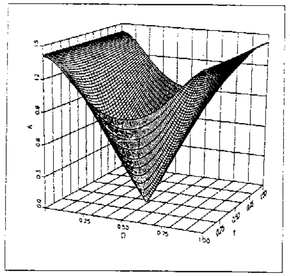

Fig. 7. The modulus of the Zak transform of $f = cf_1 * f_2$ with $f_1, f_2$ given in (5.18).

In fig. 7 we show the modulus of the Zak transform of $f = cf_1 * f_2$, with

$$f_1(t) = \begin{cases} 1, & 0 \leqslant t < 1 \\ 0, & t < 0,\ t \geqslant 0 \end{cases}, \quad f_2(t) = \begin{cases} \exp(-\frac{1}{2}\pi t), & t \geqslant 0 \\ 0, & t < 0 \end{cases} \tag{5.18}$$

The signal $f$ can be viewed as the response of a cable with impulse response $f_2$ to a unit pulse $f_1$. Note that both $Zf_1$ and $Zf_2$ are discontinuous and have no zeros in the unit square. However, $Zf$ is continuous and has hence a zero in the unit square, viz. on the line $\Omega = \frac{1}{2}$ (and not at $\tau = \Omega = \frac{1}{2}$, since this $f$ is not even), as can be expected from one of the properties given above.

In figs 8 and 9 we display the modulus of the Zak transforms of the functions

$$w_\beta(t) = \frac{\sin \pi t}{\pi t} \frac{\cos \pi \beta t}{1 - (2\beta t)^2}, \quad -\infty < t < \infty, \tag{5.19}$$

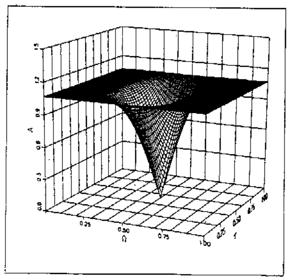

Fig. 8. The modulus of the Zak transform of $w_\beta$ in (5.19) with $\beta = 0.5$.





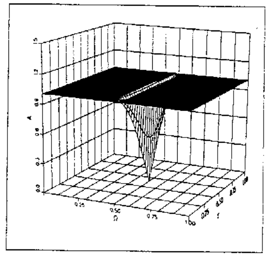

Fig. 9. The modulus of the Zak transform of $w_\beta$ in (5.19) with $\beta = 0.05$.

whose Fourier transforms $W_\beta$ (raised cosines) are given by

$$W_\beta(\nu) = \begin{cases} 1 & 0 \leqslant |\nu| \leqslant \frac{1-\beta}{2}, \\ \frac{1}{2}(1 - \sin \frac{\pi}{\beta}(|\nu| - \frac{1}{2})), & \frac{1-\beta}{2} \leqslant |\nu| \leqslant \frac{1+\beta}{2}, \\ 0 & , |\nu| \geqslant \frac{1+\beta}{2}, \end{cases} \tag{5.20}$$

with the roll-off parameter $\beta$ equal to 0.5 and 0.05, respectively. The choice $\beta = 0.5$ gives rise to a quite commonly used time-continuous filter, whereas $\beta = 0.05$ is an extreme case. Note that the Zak transform in fig. 9 is barely continuous, but has, indeed, a zero at $\tau = \Omega = \frac{1}{2}$.

## 6. Relations with other time-frequency representations

We shall present in this section some relations with time-frequency representations like the ambiguity function, the Wigner distribution and the (generalized) Rihaczek distribution. These are defined for finite-energy signals $f$, $g$ by

$$A_{f,g}(\nu,\tau) = \int_{-\infty}^{\infty} \exp(-2\pi i \nu s) f(s + \tfrac{1}{2}\tau)\ g^*(s - \tfrac{1}{2}\tau)\, ds, \quad -\infty < \nu, \tau < \infty, \tag{6.1}$$

and

$$C_{f,g}^{(\alpha)}(t,\omega) = \int_{-\infty}^{\infty} \exp(-2\pi i s \omega) f(t + (\tfrac{1}{2} - \alpha)s)\, g^*(t - (\tfrac{1}{2} + \alpha)s\, ds, \quad -\infty < t, \omega < \infty, \tag{6.2}$$

where $-\infty<\alpha<\infty$. The expression (6.1) is called the (cross) ambiguity function of $f$ and $g$, and in (6.2) the (cross) Wigner distribution $W_{f,g}$ results upon taking $\alpha = 0$, while the (cross) Rihaczek distribution $R_{f,g} = \exp(-2\pi i t\omega) f(t) G^*(\omega)$ results upon taking $\alpha=\frac{1}{2}$.

It is useful to note that Wigner distribution and ambiguity function are related according to the formulas

$$W_{f,g}(t,\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(2\pi i\nu t-2\pi i\tau\omega)A_{f,g}(\nu,\tau)\,d\nu d\tau, \qquad -\infty<t,\omega<\infty, \tag{6.3}$$

and

$$A_{f,g}(\nu,\tau)=2W_{f,g^-}(\tfrac{1}{2}\tau,\tfrac{1}{2}\nu), \qquad -\infty<\tau,\nu<\infty. \tag{6.4}$$

We refer to refs 24, 29 and 30 for background and basic properties of these time-frequency functions.

The results of this section are largely based on (2.37). Using that

$$(f,R_{-m}T_n g)=(-1)^{nm}A_{f,g}(m,-n), \qquad \text{integer } m,\, n, \tag{6.5}$$

we can write this formula as

$$(Zf)(\tau,\Omega)(Zg)^*(\tau,\Omega)=\sum_{n,m=-\infty}^{\infty}(-1)^{nm}A_{f,g}(m,-n)\exp(2\pi i n\Omega+2\pi i m\tau), \qquad -\infty<\tau,\Omega<\infty. \tag{6.6}$$

An interesting formula is obtained when formulas (6.6) and (6.3) are combined and the Poisson summation formula is applied. The result is (we limit ourselves to the case $f=g$)

$$|(Zf)(\tau,\Omega)|^2=\tfrac{1}{2}\sum_{n,m=-\infty}^{\infty}(-1)^{nm}W_{f,f}(\tau+\tfrac{1}{2}n,\Omega+\tfrac{1}{2}m), \qquad -\infty<\tau,\Omega<\infty. \tag{6.7}$$

Hence $|(Zf)(\tau,\Omega)|^2$ can be obtained as an infinite double series of shifted versions of $W_{f,f}$ with proper choises of the sign.

A similar formula holds for $R_{f,f}$ (and, in fact, for $^{(\alpha)}$ with $\alpha=k+\frac{1}{2}$, $k$ integer):





$$|(Zf)(\tau,\Omega)|^2=\sum_{n,m=-\infty}^{\infty}R_{f,f}(\tau+n,\Omega+m), \qquad -\infty<\tau,\Omega<\infty. \tag{6.8}$$

A consequence of the formulas (6.7) and (6.8), obtained by integrating these formulas over the unit square, is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W_{f,f}(t,\omega)\,dt\,d\omega=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R_{f,f}(t,\omega)\,dt\,d\omega=\|f\|^2. \tag{6.9}$$

The formula (6.7) can be used to reestablish the result in subsec. 3.4 on the supporting set of $Zf$, where $f(t)$ is the chirp $\exp(\pi i\alpha t^2)$ whose Wigner distribution is proportional to $\delta(\omega-\alpha t)$. Thus one sees that $Zf$ is supported by a discrete set of straight lines when $\alpha$ is rational, and that the support of $Zf$ is a dense subset of $\mathbb{R}^2$ when $\alpha$ is irrational. When we examine the case $\alpha=p/q$, where $p$ and $q$ are integers with greatest common divisor equal to 1, it can be shown with the aid of (6.7) that the support of $Zf$ consists exactly of the lines $\{(\tau,\Omega)\,|\,\Omega q-\tau p=\frac{1}{2}pq+k\}$, $k$ integer.

There are more interesting formulas that follow from (6.6), such as

$$\sum_{n,m=-\infty}^{\infty}|A_{f,g}(n,m)|^2=\sum_{n,m=-\infty}^{\infty}A_{f,f}(n,m)A^*_{g,g}(n,m)=\int_0^1\int_0^1|(Zf)(\tau,\Omega)(Zg)(\tau,\Omega)|^2\,d\tau d\Omega \tag{6.10}$$

(see ref. 6), but these fall somewhat out of the scope of this paper.

## 7. The Zak transform and the Gabor representation problem

In this section we study some problems related to the Gabor representation problem with the aid of the Zak transform. Such problems were also treated in refs 5,10 to 12,14 and 19 to 21. In 1946, Gabor[25]) suggested a signal representation in which to each finite energy signal $f$ there is assigned a double array $c_{nm}(f)$, integer $n$, $m$, where $|c_{nm}(f)|^2$ should be a measure of the energy of the signal $f$ at the $(n,m)$-th logon, i.e. the square of area one with center $(n,m)$. More precisely, Gabor intended to obtain a series representation

$$f(t) = \sum_{n,m=-\infty}^{\infty} c_{nm}(f) g_{n,m}(t), \quad -\infty < t < \infty, \tag{7.1}$$

with $g$ a fixed function (for which Gabor took the Gaussian $\exp(-\pi t^2)$) and $g_{n,m} = R_m T_n g$, as usual. It is clear that the coefficients $c_{nm}(f)$ in (7.1) also depend on $g$. Apart from being a means to describe the energy distribution of the signal $f$ over time and frequency, the representation (7.1) can also be useful when one has efficient transmission of the signal $f$ in mind.

We shall obtain formulas for $c_{nm}(f)$ in terms of $Zf$ and $Zg$, and indicate the problems associated with the representation (7.1). Using formula (2.22), we can write (7.1) as

$$(Zf)(\tau,\Omega) = (Zg)(\tau,\Omega) \sum_{n,m=-\infty}^{\infty} c_{nm}(f) \exp(-2\pi i m \tau + 2\pi i n \Omega),$$
$$-\infty < \tau, \Omega < \infty. \tag{7.2}$$

Hence, at least formally,

$$c_{nm}(f) = \int_0^1 \int_0^1 \frac{(Zf)(\tau,\Omega)}{(Zg)(\tau,\Omega)} \exp(2\pi i m \tau - 2\pi i n \Omega) \, d\tau d\Omega,$$
$$\text{integer } n,m, \tag{7.3}$$

i.e. $c_{nm}(f)$ are Fourier coefficients of the function $Zf/Zg$ (which is indeed periodic in its two variables). Formula (7.3) shows quite clearly that there are problems with the $c_{nm}(f)$'s when $Zg$ is continuous (so that $Zg$ has at least one zero in the unit square).

In general it will be the case that

$$\sum_{n,m=-\infty}^{\infty} |c_{nm}(f)|^2 = \infty, \tag{7.4}$$

even when $f$ and $g$ are well-behaved, rapidly decaying, finite-energy signals. For instance, when $g(t) = \exp(-\pi t^2)$, then $Zg$ has a zero to the extent that

$$\int_0^1 \int_0^1 \frac{d\tau d\Omega}{|(Zg)(\tau,\Omega)|} < \infty = \int_0^1 \int_0^1 \frac{d\tau d\Omega}{|(Zg)(\tau,\Omega)|^2}. \tag{7.5}$$





Hence, if $Zf$ is continuous and bounded on the unit square and $(Zf)(\frac{1}{2},\frac{1}{2}) \neq 0$, we see that $c_{nm}(f) \to 0$ as $n^2 + m^2 \to \infty$, but that the double series in (7.4) diverges.

An example of a (discontinuous) $g$ such that $Zg$ has no zeros is obtained when we take $g$ as in subsec. 3.1. Then it turns out that

$$c_{nm}(f) = \int_0^1 f(\tau + n) \exp(2\pi i m \tau) \, d\tau, \quad \text{integer } n, m \tag{7.6}$$

so that the Gabor representation of $f$ amounts to a Fourier series representation of the functions $f(\tau + n)$, $0 \leq \tau \leq 1$, $n$ integer.

The problems with the zeros of $Zg$ are much less severe when $Zf$ has its zeros where $Zg$ has them. An example where this happens is when $g(t) = \exp(-\pi t^2)$ and $f$ is either time-limited to $[-a,a]$ with $0 < a < \frac{1}{2}$, or band-limited to $[-b,b]$ with $0 < b < \frac{1}{2}$. Then $Zf/Zg$ is square integrable over the unit square (assuming that $f$ has finite energy), and the left-hand side of (7.4) is finite.

We consider the case

$$f(t) = \exp\left(\pi i \frac{p}{q} t^2\right), \quad g(t) = \exp(-\pi t^2), \quad -\infty < t < \infty, \tag{7.7}$$

where $p$, $q$ are integers with greatest common divisor equal to 1, in some more detail. We have seen that $Zf$ is concentrated on the lines $\{(\tau,\Omega) \mid \Omega q - \tau p = \frac{1}{2} p q + k\}$, $k$ integer, and the point $(\frac{1}{2},\frac{1}{2})$ does not lie on any of these lines. It can therefore be expected that the $c_{nm}(f)$'s behave reasonably in this case. Indeed, when $p = q = 1$, it turns out that[21])

$$|c_{nm}(f)|^2 = |a_{n-m}|^2, \quad \text{integer } n, m, \tag{7.8}$$

where $a_n$ are complex numbers, decaying roughly as as $\exp(-\pi |n|/2)$ as $|n| \to \infty$. This result is not unexpected, as the Wigner distribution of $f$ equals $\delta(t - \omega)$. Similar, but more complicated formulas hold for the general case $p \neq 1$ or $q \neq 1$. The $c_{nm}(f)$'s are then roughly a function of $m - np/q$, whose decay rate is determined by the minimum distance of $(\frac{1}{2},\frac{1}{2})$ to the set of supporting lines of $Zf$. This decay rate becomes quite low when $p$ and $q$ get large.

A problem closely related to the Gabor representation problem is about the completeness of the set $g_{n,m}$, $n$, $m$ integer. Here one wants to find out whether there are finite-energy signals $f$ such that

$$(f,g_{n,m}) = 0, \quad \text{integer } n, m \tag{7.9}$$

This problem is solved as follows. We have by formula (2.37) that (7.9) holds if and only if $Zf{\cdot}(Zg)^*$ vanishes everywhere. In the case that $g(t) = \exp(-\pi t^2)$ (with only one zero of $Zg$), it is seen that (7.9) fails to hold (unless $f=0$) since $Zf$ cannot be concentrated at a single point as a square integrable function. In this case it is even so that a square integrable $f$, for which $(f,g_{n,m}) = 0$ for all except one integer pair $(n,m)$, must be the null function. However, there are square integrable functions $f \neq 0$ such that $(f,g_{n,m}) = 0$ for all except two integer pairs $(n,m)$. In the case of a general $g$, we see that the solution to the completeness problem is entirely determined by the number and nature of the zeros of $Zg$ in the unit square.

## 8. The Zak transform and some equalization problems

We describe in this section problems in digital communication theory about optimal estimation of time-discrete binary data, transmitted through a stochastic time-continuous channel and contaminated by additive stationary noise. To solve these problems we require, besides the theory of subsec. 4.3, some knowledge of the theory of spectral factorization for which one may consult ref. 32.

To describe the problems more specifically, we assume that we have zero-mean, uncorrelated data $a_k = \pm 1$, $k$ integer, that pass through a time-continuous channel $h^{\Delta}(t)$, $-\infty < t < \infty$, where $\Delta$ represents the stochastic nature of the channel, such as random variation of channel parameters and timing errors. To the transmitted signal there is added stationary noise $n(t)$, $-\infty < t < \infty$, with zero-mean and known autocorrelation function $E_n(n(t)\,n^*(s)) = \psi(t-s)$, $-\infty < t, s < \infty$. The noise is uncorrelated with the data and the channel stochastics. Hence, we receive at the filter end the signal

$$r(t) = \sum_{k=-\infty}^{\infty} a_k h^{\Delta}(t-k) + n(t), \quad -\infty < t < \infty, \tag{8.1}$$

and the objective is to reconstruct $a_k$ from $r(t)$. We consider two ways to do this, viz. one without and one with decision-feedback. In the first case one performs a time-continuous filter operation on $r(t)$ (impulse response $w(t)$, $-\infty < t < \infty$) to obtain the estimate





$$\tilde{a}_k = (r * w)(k) = \sum_{i=-\infty}^{\infty} a_{k-i} (h^{\Delta} * w)(i) + (n * w)(k). \tag{8.2}$$

On this $\tilde{a}_k$ a decision operation is performed so that we get $\hat{a}_k = +1$ or $-1$ according as $\tilde{a}_k \geqslant 0$ or $\tilde{a}_k < 0$. In the second method one uses in addition previous estimates $\hat{a}_{k-i}$, $i = 1,2,\ldots$, to get the estimate

$$\tilde{a}_k = (r * w)(k) - \sum_{i=-\infty}^{\infty} \hat{a}_{k-i} p_i, \tag{8.3}$$

where the discrete-time impulse response $p_i$ equals 0 for $i \leqslant 0$. Schematically we have for this method the configuration shown in fig. 10.

In the second case we shall operate under the assumption that $\hat{a}_k = a_k$ for all integer $k$ which is a common assumption in this field of communication theory, see ref. 33, p. 1343. We thus obtain a theoretical upper bound for the performance of the above filter structure, or, alternatively, indicate how it performs as long as no errors have occurred. Hence we get in this case

$$\tilde{a}_k = \sum_{i=-\infty}^{\infty} a_{k-i} [(h^{\Delta} * w)(i) - p_i] + (n * w)(k). \tag{8.4}$$

The problem we set ourselves now is to determine $w$ (and $p$) such that

$$\bar{\varepsilon} = E_{\Delta,a,n} |\tilde{a}_k - a_k|^2 \tag{8.5}$$

is minimal, where the expectation is over the channel stochastics, the data and the noise. Such problems have received some attention in the existing literature, see e.g. refs 33 and 34. However, as far as we know, the case with stationary, non-white noise and the case with decision-feedback and a stochastic channel have not been treated. We present here a framework, based on the Zak transform, that properly takes into account the mixed discrete/continuous-time aspects of the problems, and within which the solutions of both the previously considered and the new problems fit while also

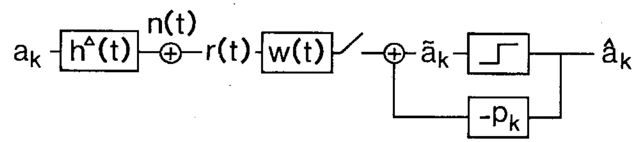


Fig. 10. Filter structure of decision-feedback filter to produce estimate $\hat{a}_k$ of $a_k$ from $r(t)$.

bandlimitedness assumptions on channel and noise can be incorporated properly.

Our approach to the above stated problems is as follows. We insert (8.2) or (8.4) into (8.5), use the assumptions on $h^{\Delta}$, $a$, $n$ and set the variation of the expression thus obtained with respect to $w$ or $w$ and $p$ to zero. We get in case of (8.2) the condition

$$\int_{-\infty}^{\infty} R(t,s)\,\hat{w}(s)\,\mathrm{d}s + \int_{-\infty}^{\infty} \psi(t-s)\,\hat{w}(s)\,\mathrm{d}s = k(t), \qquad -\infty<t<\infty, \tag{8.6}$$

for the optimal $\hat{w}$, and in case of (8.4) the two conditions

$$\int_{-\infty}^{\infty} R(t,s)\,\hat{w}(s)\,\mathrm{d}s + \int_{-\infty}^{\infty} \psi(t-s)\,\hat{w}(s)\,\mathrm{d}s = \sum_{i=-\infty}^{\infty} (\delta_i+\hat{p}_i)\,k(t-i), \ -\infty<t\leq\infty, \tag{8.7}$$

$$\hat{p}_i = (h*w)(i) = (k_-^{*}*\hat{w})(i), \quad i=1,2,\ldots; \quad \hat{p}_i=0,\ i=0,-1,\ldots . \tag{8.8}$$

Here $\delta_i$ is Kronecker's delta, and

$$R(t,s) = E_{\Delta}\Big[\sum_{i=-\infty}^{\infty} k^{\Delta}(t-i)\,k^{\Delta *}(s-i)\Big], \qquad -\infty<t,s<\infty, \tag{8.9}$$

$$k(t) = E\,k^{\Delta}(t), \quad k^{\Delta}(t) = h^{\Delta *}(-t), \qquad -\infty<t<\infty. \tag{8.10}$$

Observe that $R$ has the form (4.14) so that the theory of subsec. 4.3 applies.

Before we proceed further to solving eqs (8.6), (8.7) and (8.8) it is instructive to see how we would deal with them if they were replaced by the stationary, time-continuous versions

$$\int_{-\infty}^{\infty} \varphi(t-s)\,\hat{w}(s)\,\mathrm{d}s + \int_{-\infty}^{\infty} \psi(t-s)\,\hat{w}(s)\,\mathrm{d}s = k(t), \qquad -\infty<t<\infty, \tag{8.6'}$$

and





$$\int_{-\infty}^{\infty} \varphi(t-s)\,\hat{w}(s)\,\mathrm{d}s + \int_{-\infty}^{\infty} \psi(t-s)\,\hat{w}(s)\,\mathrm{d}s =$$
$$= \int_{-\infty}^{\infty} (\delta(s)+\hat{p}(s))\,k(t-s)\,\mathrm{d}s, \qquad -\infty<t<\infty, \tag{8.7'}$$

$$\hat{p}(t) = (h*w)(t) = (k_-^{*}*\hat{w})(t), \quad t\geq 0,\ \hat{p}(t)=0 \text{ and } t<0. \tag{8.8'}$$

Here $\delta(s)$ is Dirac's delta function, and

$$\varphi(t-s) = E_{\Delta}\Big[\int_{-\infty}^{\infty} k^{\Delta}(t-x)\,k^{\Delta *}(s-x)\,\mathrm{d}x\Big], \qquad -\infty<t,s<\infty. \tag{8.11}$$

It is natural to apply the Fourier transform to both equations (8.6′) and (8.7′). Then (8.6′) becomes

$$M(\nu)\,\hat{W}(\nu) = K(\nu), \qquad -\infty<\nu<\infty, \tag{8.12}$$

where $M$ is the Fourier transform of $m = \varphi+\psi$, so that

$$M(\nu) = \Phi(\nu) + \Psi(\nu), \qquad -\infty<\nu<\infty, \tag{8.13}$$

and (8.7′) becomes

$$M(\nu)\,\hat{W}(\nu) = (1+\hat{P}(\nu))\,K(\nu), \qquad -\infty<\nu<\infty. \tag{8.14}$$

Equation (8.12) can be solved at once, while for eq. (8.14) a Wiener-Hopf approach to incorporate (8.8′) is required. To that end we set

$$q = k_-^{*}*w, \quad Q(\nu) = K^{*}(\nu)\,W(\nu), \quad T(\nu) = 1+\hat{P}(\nu)-Q(\nu), \qquad -\infty<\nu<\infty, \tag{8.15}$$

so that $T(\nu)$ is anti-causal, while $P(\nu)$ is causal. Then (8.14) becomes

$$(M(\nu)-K(\nu)\,K^{*}(\nu))\,W(\nu) = T(\nu)\,K(\nu), \qquad -\infty<\nu<\infty \tag{8.16}$$

(it has some didactical advantage to write $|K(\nu)|^2$ as $K(\nu)\,K^{*}(\nu)$). By solving

$\hat{W}(\nu)$ from both (8.14) and (8.16), equating the two expressions thus obtained and dividing by $K(\nu)$ we obtain (formally only)

$$1+\hat{p}(\nu)=\frac{T(\nu)}{1-M(\nu)^{-1}K(\nu)\,K^*(\nu)},\qquad -\infty<\nu<\infty. \tag{8.17}$$

Let

$$\chi(\nu)=1-M(\nu)^{-1}K(\nu)\,K^*(\nu),\qquad -\infty<\nu<\infty. \tag{8.18}$$

We want to factorize

$$\frac{1}{\chi(\nu)}=\Gamma(\nu)\,\Gamma^*(\nu)=|\Gamma(\nu)|^2,\qquad -\infty<\nu<\infty, \tag{8.19}$$

with $\Gamma$, $1/\Gamma$ causal and minimum phase and $\Gamma^*$, $1/\Gamma^*$ anti-causal and maximum phase. To that end we note that

$$\Psi(\nu)\geqslant 0,\qquad \Phi(\nu)-K(\nu)\,K^*(\nu)=E_\Delta|K^\Delta(\nu)-E\,K^\Delta(\nu)|^2\geqslant 0,\qquad -\infty<\nu<\infty, \tag{8.20}$$

so that $0\leqslant\chi(\nu)\leqslant 1$. Hence the above factorization exists and is unique whenever the Paley-Wiener condition

$$\int_{-\infty}^{\infty}\frac{-\ln\chi(\nu)}{1+\nu^2}\,\mathrm{d}\nu<\infty \tag{8.21}$$

is satisfied. Then (8.17) becomes

$$\frac{1+\hat{P}(\nu)}{\Gamma(\nu)}=T(\nu)\,\Gamma^*(\nu),\qquad -\infty<\nu<\infty, \tag{8.22}$$

which is an identity between a causal function at the left and an anti-causal one at the right. Hence, both sides of (8.22) are equal to a constant, $\gamma_\infty^{-1}$, say, determined by $\Gamma$, and we obtain as our solution

$$\hat{P}(\nu)=\gamma_\infty^{-1}\,\Gamma(\nu)-1,\qquad \hat{W}(\nu)=\gamma_\infty^{-1}\Gamma(\nu)\,M(\nu)^{-1}K(\nu),\qquad -\infty<\nu<\infty. \tag{8.23}$$

We now return to the equations (8.6) and (8.7), (8.8), and we follow the line of reasoning just given with the Zak transform instead of the Fourier





transform. Just as in the Zak transform version (4.9) of the convolution theorem, multiplication operators must be replaced by integral operators and divisions must be replaced by inversions of these integral operators. In addition, we need to factorize a Fourier series rather than a Fourier integral. To present our solution it is necessary to develop some notation. When $N(\tau,\mu;\Omega)$ and $F(\tau,\Omega)$, $G(\tau,\Omega)$ are functions on $0\leqslant\tau,\mu,\Omega\leqslant 1$ and $0\leqslant\tau,\Omega\leqslant 1$, respectively, we set

$$NF\equiv(NF)(\tau,\Omega)=\int_0^1 N(\tau,\mu;\Omega)\,F(\mu,\Omega)\,\mathrm{d}\mu,\qquad 0\leqslant\tau,\Omega\leqslant 1, \tag{8.24}$$

$$(F,G)_\tau\equiv(F,G)_\tau(\Omega)=\int_0^1 F(\tau,\Omega)\,G^*(\tau,\Omega)\,\mathrm{d}\tau,\qquad 0\leqslant\Omega\leqslant 1, \tag{8.25}$$

$$F\otimes G\equiv(F\otimes G)(\tau,\mu;\Omega)=F(\tau,\Omega)\,G(\mu,\Omega),\qquad 0\leqslant\tau,\mu,\Omega\leqslant 1. \tag{8.26}$$

Hence, just as $\Phi$ in (4.9), $N$ operates on the time variable only. Now (8.6) takes the form (compare (8.12))

$$M\,Z\hat{w}=Zk, \tag{8.27}$$

and (8.7) becomes (compare (8.14))

$$M\,Z\hat{w}=(1+\hat{P})\cdot Zk, \tag{8.28}$$

while incorporating (8.7) into (8.27) leads to (compare (8.16))

$$(M-Zk\otimes(Zk)^*)\cdot Z\hat{w}=\mathrm{T}\cdot Zk. \tag{8.29}$$

Here we have set (compare (8.13))

$$M(\tau,\mu;\Omega)=\Phi(\tau,\mu;\Omega)+(Z\,\psi)(\tau-\mu;\Omega),\qquad 0\leqslant\tau,\mu,\Omega\leqslant 1, \tag{8.30}$$

with (as in subsec. 4.3)

$$\Phi(\tau,\mu;\Omega)=\sum_{n=-\infty}^{\infty}R(\tau+n,\mu)\exp(-2\,\pi i n\,\Omega),\qquad 0\leqslant\tau,\mu,\Omega\leqslant 1, \tag{8.31}$$

and $Zk \otimes (Zk)^*$ given according to (8.26), while

$$q = k_-^* * w, \quad Q(\Omega) = \sum_{j=-\infty}^{\infty} q(j) \exp(-2\pi i j \Omega), \quad 0 \leqslant \Omega \leqslant 1, \quad (8.32)$$

$$\hat{P}(\Omega) = \sum_{j=1}^{\infty} q(j) \exp(-2\pi i j \Omega), \quad T(\Omega) = 1 + \hat{P}(\Omega) - Q(\Omega), \quad 0 \leqslant \Omega \leqslant 1. \quad (8.33)$$

We then get the following results. Set (compare (8.18))

$$\chi(\Omega) = 1 - (M^{-1} Zk, Zk)_\tau(\Omega), \quad 0 \leqslant \Omega \leqslant 1. \quad (8.34)$$

Then $0 \leqslant \chi(\Omega) \leqslant 1$, and

A. The optimal $\hat{w}$ in case of (8.2) is given by

$$Z\hat{w} = M^{-1} Zk, \quad (8.35)$$

the optimal mean-square $\bar{\varepsilon}$ is given by

$$\bar{\varepsilon} = \int_0^1 \chi(\Omega)\, d\Omega, \quad (8.36)$$

and for the overall impulse response function $q = h * \hat{w} = k^* * w$ we have

$$Q(\Omega) = \sum_{j=-\infty}^{\infty} q(j) \exp(-2\pi i j \Omega) = 1 - \chi(\Omega), \ 0 \leqslant \Omega \leqslant 1. \quad (8.37)$$

B. The optimal $\hat{w}$ and $\hat{p}$ in case of (8.4) are given by

$$\hat{P} = \gamma_0^{-1} \Gamma - 1, \quad Z\hat{w} = \gamma_0^{-1} \Gamma \cdot M^{-1} Zk. \quad (8.38)$$

Here $\Gamma$ is the unique causal minimum phase factor in the factorization

$$\frac{1}{\chi} = \Gamma \cdot \Gamma^*, \quad (8.39)$$





and $\gamma_0$ is the zeroth coefficient in the Fourier expansion $\Sigma_{k=0}^{\infty} \gamma_k \exp(-2\pi i j \Omega)$ of $\Gamma$. It is assumed here that the Paley-Wiener condition

$$\int_0^1 -\log \chi(\Omega)\, d\Omega < \infty \quad (8.40)$$

is satisfied. For the optimal mean-square error $\bar{\varepsilon}$ we have

$$\bar{\varepsilon} = \frac{1}{\gamma_0^2} = \exp\left(\int_0^1 \log \chi(\Omega)\, d\Omega\right), \quad (8.41)$$

and for the overall impulse response $q = h * \hat{w} = k^* * \hat{w}$ we have

$$Q(\Omega) = \sum_{j=-\infty}^{\infty} q(j) \exp(-2\pi i j \Omega) = \gamma_0^{-1}\left(\Gamma(\Omega) - \frac{1}{\Gamma^*(\Omega)}\right), \quad 0 \leqslant \Omega \leqslant 1. \quad (8.42)$$

We would like to make some comments on this solution

a. Both in case $A$ and $B$ we obtain $Z\hat{w}$ instead of $\hat{w}$ itself. To obtain $\hat{w}$ one should use (2.29).

b. The derivation is only formal, and it should be checked that $M^{-1}$ indeed exists, since both solution $A$ and $B$ involve $M^{-1} Zk$. While it can be shown that, as a linear integral operator, $M$ is positive semi-definite, so that

$$(MF, F)_\tau(\Omega) \geqslant 0, \quad 0 \leqslant \Omega \leqslant 1, \quad (8.43)$$

for all $F$ defined on $0 \leqslant \tau, \Omega \leqslant 1$, it may happen that $M^{-1}$ does not exist. However, when the noise spectrum $\Phi$ satisfies $\Phi(\nu) \geqslant m > 0$, $-\infty < \nu < \infty$, we have

$$(MF, F)_\tau(\Omega) \geqslant m(F, F)_\tau(\Omega), \quad 0 \leqslant \Omega \leqslant 1, \quad (8.44)$$

so that $M$ is positive definite and hence invertible. When $\Phi$ is allowed to vanish, there are cases where the equation $MZv = Zk$ has many solutions and others where there are no solutions. However, any solutions $v$ to the equation $MZv = Zk$ satisfies $0 \leqslant (Zv, Zk)_\tau \leqslant 1$.

c. Solving the equation $MZv = Zk$, written out in full as

$$\int_0^1 M(\tau,\mu;\Omega)\,(Zv)\,(\mu,\Omega)\,\mathrm{d}\mu = (Zk)\,(\tau,\Omega), \quad 0 \leqslant \tau,\Omega \leqslant 1, \quad (8.45)$$

can be quite a job, since it amounts to solving a linear integral equation for all $\Omega$, $0 \leqslant \Omega \leqslant 1$. We consider the important special case that $h^\Delta$ is band-limited for all $\Delta$ to an interval $[-N/2,N/2]$ with integer $N \geqslant 1$. We look for a solution $v$ which is band-limited to $[-N/2,N/2]$ as well. It can be shown that in this case $M$ maps the linear space $\{Zv\,|\,v \text{ band-limited to } [-N/2,N/2]\}$ into itself, and that $M$ is invertible when $\Psi(v) \geqslant m > 0$, $-N/2 \leqslant v \leqslant N/2$ ($\Psi$ is the noise spectrum). Moreover, as we know from subsec. 2.2.7, to reconstruct $V$ from $Zv$ for band-limited v's we only need to know $(Zv)\,(n/N,\Omega)$, $n = 0,1,\ldots,N-1$. Hence, it is sufficient to solve for every $\Omega$ the linear system

$$\sum_{n=0}^{N-1} M\Big(\frac{n}{N},\frac{m}{M};\Omega\Big)\,(Zv)\Big(\frac{m}{N},\Omega\Big) = (Zk)\Big(\frac{n}{N},\Omega\Big), \quad m = 0,1,\ldots,N-1. \quad (8.46)$$

Moreover, the calculation of $(M^{-1}Zk, Zk)_\tau = (Zv,Zk)_\tau$ as required by (8.34) can be done according to (2.46) as

$$(Zv,Zk)_\tau\,(\Omega) = \frac{1}{N}\sum_{n=0}^{N-1} (Zv)\Big(\frac{n}{N},\Omega\Big)\,(Zk)^*\Big(\frac{n}{N},\Omega\Big), \quad 0 \leqslant \Omega \leqslant 1. \quad (8.47)$$

This reduces the computational load considerably and gives at the same time a nice connection with the linear systems obtained in ref. 34. In ref. 23 this point is considered from a fractional tap spacing point-of-view.

d. The computation of the spectral factor $\Gamma(\Omega) = \sum_{k=0}^{\infty} \gamma_k \exp(-2\pi i k \Omega)$ as required by (8.39) can be done efficiently by using the recursion

$$\gamma_0 = \exp\Big(-\tfrac{1}{2}\int_0^1 \log\chi(\Omega)\,\mathrm{d}\Omega\Big), \quad (8.48)$$





$$\gamma_k = \frac{1}{k}\sum_{i=0}^{k-1} (k-i)\,a_{k-i}\,\gamma_i, \quad k = 1,2,\ldots\,, \quad (8.49)$$

where

$$a_k = -\int_0^1 \exp(2\pi i k \Omega)\log\chi(\Omega)\,\mathrm{d}\Omega, \quad k = 1,2,\ldots\,. \quad (8.50)$$

See ref. 35.

We end this section with some conclusions. It has been shown in this section that the Zak transform is the proper substitute for the Fourier transform for solving linear integral equation with a cyclostationary (rather than stationary) kernel. This observation also holds for problems of the Wiener-Hopf type with cyclostationary kernels. This has been demonstrated by solving equations of this kind arising from problems associated with optimal retrieving binary data $a_k$ from a received time-continuous signal $r(t)$ given by

$$r(t) = \sum_{k=-\infty}^{\infty} a_k h^\Delta(t-k) + n(t), \quad -\infty < t < \infty, \quad (8.51)$$

where $h^\Delta$ is a time-continuous, stochastic channel and $n$ is additive stationary noise. As a mathematically slight but technically significant extension of the results of this section, we note that we can handle cases where the data transmission system suffers from jitter, so that $r$ has the form

$$r(t) = \sum_{k=-\infty}^{\infty} a_k h(t-k-\theta_k) + n(t), \quad -\infty < t < \infty, \quad (8.52)$$

with $\theta_k$ as in (4.17), in exactly the same way.

In the equalization problems dealt with in the existing literature (see e.g. refs 33 and 34), using the Zak transform is often an overkill in the sense that it is not always necessary to set up all the machinery as was done in this section. As a useful exercise one could rederive results obtained in the literature from the general ones we have obtained here, and one would find that this can be an easy but laborious job in certain cases. Nevertheless, the solution to the harder problems of the type considered here (e.g. the case of a stochastic channel with feedback, or feedback problems associated with (8.52)) cannot be easily derived by known techniques. The real strong point of the Zak transform, however, is that it provides one framework within which all problems of this type that have been considered can be solved and that the solutions can be compared.

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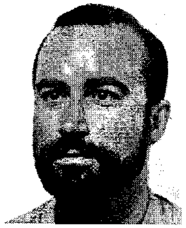

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