Indag. Mathem., N.S., 7 (2), 165–183 June 17, 1996

Some Weyl–Heisenberg frame bound calculations

by A.J.E.M. Janssen

Philips Research Laboratories Eindhoven, 5656 AA Eindhoven, the Netherlands

Communicated by Prof. M.S. Keane at the meeting of May 29, 1995

ABSTRACT

We calculate for several $g \in L^2(\mathbb{R})$ and for integer values of $1/ab$ the frame bounds and, when possible, the minimum energy dual functions ${}^\circ\gamma$ for the set of time-frequency translates of g corresponding to the lattice parameters a, b. For this we use a recently derived expression for frame bounds and biorthogonal functions in terms of translates of g corresponding to the complementary lattice with parameters $1/b, 1/a$.

0. INTRODUCTION

Efficient signal representations for the transmission and storage of signals, as they occur, for instance, in audio and video applications, are of increasing importance. These signal representations often consist of the concatenation of an analysis and a synthesis operator. That is, there is a countable collection $g_k \in L^2(\mathbb{R})$, $k \in \mathcal{I}$, such that all occurring data signals $f \in L^2(\mathbb{R})$ can be represented in the form

$$(0.1) \qquad f = S\boldsymbol{c} = \sum_{k \in \mathcal{I}} c_k g_k,$$

where the coefficients $\boldsymbol{c} = (c_k)_{k \in \mathcal{I}}$ depend on f and have the form of inner products,

$$(0.2) \qquad \boldsymbol{c} = Af = ((f, \gamma_k))_{k \in \mathcal{I}}$$

with $\gamma_k \in L^2(\mathbb{R})$, $k \in \mathcal{I}$. Here one calls S the synthesis operator and A the analysis operator of the representation, and (0.1–2) can be written succinctly as

$SA = I$ with I the identity operator of $L^2(\mathbb{R})$. The crucial point in the representations (0.1–2) is to choose the g_k and γ_k such that, on the average over all occurring f, there are only few coefficients c_k that really matter from a perceptive point of view, so that data reduction can be obtained by coarse quantization of the c_k. Further requirements on the representation are that, with a hardware structure of modest complexity, the coefficients can be computed and the signals can be reconstructed fast and reliably, and that the coefficients reflect in some way or another the energy distribution of the signals in the time-frequency plane. For all these reasons the modulated filter banks, in which one has

$$(0.3)\qquad g_k(t) = e^{2\pi imbt} g(t-na), \qquad \gamma_k(t) = e^{2\pi imbt}\gamma(t-na), \quad t \in \mathbb{R},$$

for $k = (n,m) \in \mathcal{I} = \mathbb{Z}\times\mathbb{Z}$ with $g, \gamma \in L^2(\mathbb{R})$ the synthesis and analysis window of the filter bank and $a > 0, b > 0$, are quite popular. Systems of functions as in (0.3) are also known as Weyl–Heisenberg systems or Gabor systems, and the corresponding representations (0.1–2) are called lattice expansions or Gabor representations.

The last decade has witnessed a major development of the theory of the signal representations (0.1–3). In particular, the questions when a synthesis operator S as in (0.1) has a bounded right-inverse A as in (0.2) and whether one can compute an analysis window γ from S have been addressed, see [1–4]. These questions have been put in the mathematical setting of frame operator theory; for the present case the relevant frame operator is $T = SS^*$, and the crucial point for an affirmative answer to these questions is that T is bounded and positive definite as an operator of $L^2(\mathbb{R})$, also see Section 1.

In the present paper we establish for several choices of $g \in L^2(\mathbb{R})$ and integer values of $1/ab$ that the corresponding T is bounded and positive definite by explicitly calculating $\inf \sigma(T)$, $\sup \sigma(T)$, with $\sigma(T)$ the spectrum of T, and we compute, whenever possible, minimum energy analysis windows ${}^\circ\gamma$.

1. DEFINITIONS AND RESULTS

Let $a > 0, b > 0$, and denote for $x, y \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$

$$(1.1)\qquad f_{x,y}(t) = e^{2\pi iyt} f(t-x), \quad t \in \mathbb{R}.$$

When $g \in L^2(\mathbb{R})$ we say that g generates a (Weyl–Heisenberg) frame (for the parameters a, b) when there are $A > 0, B < \infty$ such that

$$(1.2)\qquad A\|f\|^2 \le \sum_{n,m=-\infty}^{\infty} |(f, g_{na,mb})|^2 \le B\|f\|^2, \quad f \in L^2(\mathbb{R}),$$

where $(\,,)$ and $\|\;\|$ denote ordinary $L^2(\mathbb{R})$ inner product and norm. The numbers A, B in (1.2) are called lower, upper frame bounds for g. In case $g \in L^2(\mathbb{R})$ generates a frame, one has that the frame operator T, defined as

$$(1.3)\qquad Tf = \sum_{n,m=-\infty}^{\infty} (f, g_{na,mb}) g_{na,mb}, \quad f \in L^2(\mathbb{R}),$$

maps $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ and that $AI \le T \le BI$ with I the identity operator of $L^2(\mathbb{R})$. For such a g one can define ${}^\circ\gamma = T^{-1}g$; this ${}^\circ\gamma$ is dual in the sense that for any $f \in L^2(\mathbb{R})$ there is the $L^2(\mathbb{R})$-convergent expansion

$$(1.4)\qquad f = \sum_{n,m=-\infty}^{\infty} \alpha_{nm} g_{na,mb}$$

with $\alpha_{nm} = (f, {}^\circ\gamma_{na,mb})$. It is an interesting property of ${}^\circ\gamma$ that for any $f \in L^2(\mathbb{R})$ and any $\boldsymbol{\alpha} \in l^2(\mathbb{Z}^2)$ such that (1.4) holds we have

$$(1.5)\qquad \sum_{n,m=-\infty}^{\infty} |(f, {}^\circ\gamma_{na,mb})|^2 \le \sum_{n,m} |\alpha_{nm}|^2,$$

whence the name minimum energy dual function. It can furthermore be shown that for any $\gamma \in L^2(\mathbb{R})$ such that $((f, \gamma_{na,mb})) \in l^2(\mathbb{Z})$ and

$$(1.6)\qquad f = \sum_{n,m=-\infty}^{\infty} (f, \gamma_{na,mb}) g_{na,mb}$$

for all $f \in L^2(\mathbb{R})$, we have that $\|{}^\circ\gamma\| \le \|\gamma\|$. We refer to [1], [2] for more details on the theory of Weyl–Heisenberg frames, in particular with respect to ranges of values of ab for which $g_{na,mb}$ indeed constitutes a frame.

The determination of the largest lower frame bound and smallest upper frame bound is an important problem in the theory of Weyl–Heisenberg frames. This is so since it is customary to compute ${}^\circ\gamma$ according to the formula

$$(1.7)\qquad {}^\circ\gamma = T^{-1}g = \frac{2}{B+A}(I-V)^{-1}g = \frac{2}{B+A}\sum_{r=0}^{\infty} V^r g,$$

where

$$(1.8)\qquad V = I - \frac{2}{B+A}T; \qquad \|V\| \le \frac{B-A}{B+A}.$$

Hence the convergence speed of the series at the right-hand side of (1.7) is critically determined by A, B. In this note we present for several choices of $g \in L^2(\mathbb{R})$ and for $(ab)^{-1} = q \in \mathbb{N}$ analytic formulas for A, B and, when possible, for ${}^\circ\gamma$. The restriction to values of $ab \le 1$ is a logical one since it is well known that no g can generate a frame when $ab > 1$. Our methods are, however, only of limited value for the cases of non-integral values of $(ab)^{-1}$.

A commonly used method to compute frame bounds and dual functions for the $(ab)^{-1} = q \in \mathbb{N}$-case is to employ the Zak transform, see [1–6]. For $f \in L^2(\mathbb{R})$ we let

$$(1.9)\qquad (Z_b f)(\tau, \nu) = b^{-1/2} \sum_{k=-\infty}^{\infty} f(b^{-1}(\tau - k)) e^{2\pi ik\nu}, \quad \tau, \nu \in \mathbb{R},$$

(convergence in a local $L^2(\mathbb{R}^2)$-sense) be the Zak transform of f, where the scaling by b in (1.9) suits our present purposes best. Then the largest lower frame bound A and smallest upper frame bound B are given respectively by

$$(1.10) \quad A = \operatorname{ess\,inf} \sum_{n=0}^{q-1} \left|(Z_b g)\left(\tau + \frac{n}{q}, \nu\right)\right|^2,$$

$$(1.11) \quad B = \operatorname{ess\,sup} \sum_{n=0}^{q-1} \left|(Z_b g)\left(\tau + \frac{n}{q}, \nu\right)\right|^2,$$

where the ess inf and ess sup can be taken over any unit square in $\mathbb{R}^2$. Also, when $0 < A \le B < \infty$ in (1.10–11), we have that

$$(1.12) \quad (Z_b {}^\circ\gamma)(\tau, \nu) = \frac{(Z_b g)(\tau, \nu)}{\sum_{n=0}^{q-1} |(Z_b g)(\tau + n/q, \nu)|^2}, \quad \tau, \nu \in \mathbb{R},$$

from which we can find ${}^\circ\gamma$ according to the formula

$$(1.13) \quad {}^\circ\gamma(t) = b^{1/2} \int_0^1 (Z_b {}^\circ\gamma)(t, \nu)\, d\nu, \quad t \in \mathbb{R}.$$

In this paper we present an alternative to the procedure just given that is occasionally easier to work out. It is based on some results recently obtained in [7] (for these results it is not necessary to assume that $(ab)^{-1}$ is an integer). Consider the linear operator G mapping $f \in L^2(\mathbb{R})$ onto a double sequence Gf according to

$$(1.14) \quad Gf = ((f, g_{k/b, l/a}))_{k, l \in \mathbb{Z}}.$$

Then the operator GG^* maps double sequences onto double sequences, and GG^* has matrix elements

$$(1.15) \quad (GG^*)_{kl;k'l'} = (g_{k'/b, l'/a}, g_{k/b, l/a}), \quad k, l, k', l' \in \mathbb{Z},$$

when we take the standard basis $(\delta_{kk'}\delta_{ll'})_{k', l' \in \mathbb{Z}}$, $k, l \in \mathbb{Z}$ for $l^2(\mathbb{Z}^2)$. According to the main result in [7] we have for all $A \ge 0$, $B < \infty$ that*

$$(1.16) \quad g \text{ generates a frame with frame bounds } A, B \Leftrightarrow AI \le \frac{1}{ab} GG^* \le BI,$$

where the I at the right-hand side of (1.16) now denotes the identity operator of $l^2(\mathbb{Z}^2)$. Also, when at least one of the two statements in (1.16) holds, we can compute ${}^\circ\gamma$ according to

$$(1.17) \quad {}^\circ\gamma(t) = ab \sum_{k, l = -\infty}^{\infty} ((GG^*)^{-1})_{kl;oo}\, g_{k/b, l/a}(t).$$

For the case that $(ab)^{-1} = q \in \mathbb{N}$, it is easy to show that GG^* has a Toeplitz

* After completion of [7] and the present paper the author was informed that (1.16) has also been discovered independently by I. Daubechies, H. Landau and Z. Landau, and by A. Ron and Z. Shen. Both groups also show that (1.17) holds. This rigourizes a result of Wexler and Raz who used the right-hand side of (1.17) in the expansions (1.6) as γ.

matrix, viz.

$$(1.18) \quad (GG^*)_{kl;k'l'} = (g, g_{(k-k')/b, (l-l')/a}), \quad k, l, k', l' \in \mathbb{Z}.$$

Accordingly, the spectrum of $(1/ab)GG^*$ is contained in the interval $[m, M]$ where (observe that GG^* is positive semi-definite)

$$(1.19) \quad m = \frac{1}{ab} \operatorname{ess\,inf} F \ge 0, \qquad M = \frac{1}{ab} \operatorname{ess\,sup} F,$$

and F is defined as

$$(1.20) \quad F(\theta, \tau) = \sum_{k, l = -\infty}^{\infty} (g, g_{k/b, l/a}) e^{2\pi i k\theta + 2\pi i l\tau}.$$

Also, when $0 < m \le M < \infty$ in (1.19) we have that

$$(1.21) \quad ((GG^*)^{-1})_{kl;oo} = \int_0^1 \int_0^1 \frac{e^{-2\pi i k\theta - 2\pi i l\tau}}{F(\theta, \tau)}\, d\theta\, d\tau$$

equals the $(kl)^{\text{th}}$ Fourier coefficient of $1/F$.

Of course, the two methods are not really different. Indeed, it is easy to show that

$$(1.22) \quad \frac{1}{ab} F(\theta, \tau) = \sum_{n=0}^{q-1} \left|(Z_b g)\left(\frac{\tau + n}{q}, \theta\right)\right|^2,$$

and also formula (1.12) can be easily converted into formula (1.17). The only point this paper wants to make is that the approach using the formulas (1.17), (1.21) is somewhat more direct than the approach based on the Zak transform, and therefore has some computational advantages.

We shall encounter a number of cases where GG^* factorizes according to

$$(1.23) \quad (GG^*)_{kl;k'l'} = M^{\text{time}}_{k-k'} M^{\text{freq}}_{l-l'}, \quad k, l, k', l' \in \mathbb{Z},$$

so that

$$(1.24) \quad F(\theta, \tau) = m_{\text{time}}(\theta) m_{\text{freq}}(\tau)$$

with

$$(1.25) \quad m_{\text{time}}(\theta) = \sum_{k=-\infty}^{\infty} M_k^{\text{time}} e^{2\pi i k\theta}; \qquad m_{\text{freq}}(\tau) = \sum_{l=-\infty}^{\infty} M_l^{\text{freq}} e^{2\pi i l\tau}.$$

In that case the frame bounds A, B are given as

$$(1.26) \quad \begin{cases} A = \dfrac{1}{ab} \operatorname{ess\,inf} m_{\text{time}} \cdot \operatorname{ess\,inf} m_{\text{freq}}; \\ B = \dfrac{1}{ab} \operatorname{ess\,sup} m_{\text{time}} \cdot \operatorname{ess\,sup} m_{\text{freq}}, \end{cases}$$

and ${}^\circ\gamma$ is given by (provided that $0 < A \le B < \infty$)

$$(1.27)\qquad {}^{\circ}\gamma(t) = \frac{ab}{m_{\text{freq}}(t/a)} \sum_{k=-\infty}^{\infty} c_k g(t - k/b),$$

with

$$(1.28)\qquad c_k = ((M^{\text{time}})^{-1})_k = \int_0^1 m_{\text{time}}^{-1}(\theta) e^{-2\pi i k\theta} d\theta,$$

the k^{th} Fourier coefficient of m_{time}^{-1}. Formula (1.27) is a special case of the following more general result: when $1/F$ can be written as

$$(1.29)\qquad \frac{1}{F(\theta,\tau)} = \sum_{k=-\infty}^{\infty} \varphi_k(\tau) e^{2\pi i k\theta},$$

then

$$(1.30)\qquad {}^{\circ}\gamma(t) = ab \sum_k \varphi_k(t/a) g(t - k/b).$$

We consider the following examples to illustrate our point.

(A) Functions g supported by an interval of length $1/b$. We reestablish the results in [1], 3.4.4.A for this case. By observing that ${}^{\circ}\gamma$ can be obtained as the unique solution of the problem

$$(1.31)\qquad \underset{\gamma\in L^2(\mathbb{R})}{\text{minimize}} \sum_{n,m} |(\gamma, g_{na,mb}) - \delta_{no}\delta_{mo}|^2$$

we reestablish a link, originally observed by Veldhuis in [8], with work of Griffin and Lim in [9] on signal estimation from modified short-time Fourier transforms. We treat the truncated exponential as a special case in detail.

(B) Functions g supported by an interval of length $2/b$. We present a condition that guarantees such a g to generate a frame, and we compute $A, B, {}^{\circ}\gamma$ for the truncated exponential. Also, for truncated exponentials, the frame bounds are more generally computed for the case that g is supported by $[0, r/b)$ with $r = 2, 3, \ldots$.

(C) One-sided exponentials. We show that the one-sided exponential generates a frame for any value of $ab \le 1$, and we reestablish the results of Friedlander and Zeira in [6] on the frame bounds and the form of the dual function ${}^{\circ}\gamma$ for the case that $(ab)^{-1} = q \in \mathbb{N}$.

(D) Two-sided exponentials. We present analytic formulas for the frame bounds in case that $(ab)^{-1} = q \in \mathbb{N}$, and compare our results with the numerical results in [2], pp. 982–983, obtained by means of the Zak transform.

(E) Gaussians. We present analytic formulas for the frame bounds in case that $(ab)^{-1} = q \in \mathbb{N}$, and compute ${}^{\circ}\gamma$ for the case that q is even, thereby extending the results in [10], Section 4.

(F) Hyperbolic secants. We present analytic formulas for the frame bounds for $g(t) = (\cosh \pi dt)^{-1}$ in case that $(ab)^{-1} = q \in \mathbb{N}$ in terms of quantities associated to the Jacobian elliptic functions and we show that g generates a frame when $q = 2, 3, \ldots$.

2. FUNCTIONS SUPPORTED BY AN INTERVAL OF LENGTH $1/b$

In this section it is not necessary to assume that $(ab)^{-1}$ is an integer. Assume that $g \in L^2(\mathbb{R})$ is supported by an interval of length $\le 1/b$. We calculate the matrix elements of GG^* as

$$(2.1)\qquad \begin{cases} (GG^*)_{kl;k'l'} = \int g(t - k'/b) g^*(t - k/b) e^{2\pi i(l'-l)t/a} dt \\ \qquad = \int |g(t)|^2 e^{2\pi i(l'-l)t/a} dt, \quad k = k', \end{cases}$$

and $(GG^*)_{kl;k'l'} = 0$ when $k \ne k'$. Hence GG^* has a Toeplitz matrix, and

$$(2.2)\qquad \begin{cases} F(\theta,\tau) = \sum_{k,l=-\infty}^{\infty} (GG^*)_{kl;oo} e^{2\pi i k\theta + 2\pi i l\tau} \\ \qquad = \sum_{l=-\infty}^{\infty} \int |g(t)|^2 e^{-2\pi i l(t/a - \tau)} dt = a \sum_{n=-\infty}^{\infty} |g(a\tau + an)|^2, \end{cases}$$

where in the last step the Poisson summation has been used. Hence we have the frame bounds

$$(2.3)\qquad A \doteq \frac{1}{b} \operatorname{ess\,inf} \sum_{n=-\infty}^{\infty} |g(a\tau + an)|^2, \qquad B = \frac{1}{b} \operatorname{ess\,sup} \sum_{n=-\infty}^{\infty} |g(a\tau + an)|^2.$$

Also, when $0 < A \le B < \infty$ we have in accordance with (1.23–28)

$$(2.4)\qquad {}^{\circ}\gamma(t) = \frac{bg(t)}{\sum_{n=-\infty}^{\infty} |g(t + an)|^2}.$$

This reestablishes the results in [1], 3.4.4.A. Note that ${}^{\circ}\gamma$ and g have the same support.

It is interesting to make a connection, first noted by Veldhuis in [8], Chapter 8, with work by Griffin and Lim in [9] on signal estimation from modified short-time Fourier transforms. To that end we shall produce ${}^{\circ}\gamma$ in a somewhat different manner than is usually done. Assume that g has an upper frame bound $B < \infty$, and that we are given an $F = (F_{nm})_{nm} \in l^2(\mathbb{Z}^2)$. When this F is considered as a noisy/distorted version of $((f, g_{na,mb}))_{n,m\in\mathbb{Z}}$ with $f \in L^2(\mathbb{R})$, one could try to estimate f from F by minimizing

$$(2.5)\qquad J(f) = \sum_{n,m=-\infty}^{\infty} |(f, g_{na,mb}) - F_{nm}|^2.$$

With $h = \sum_{n,m} F_{nm} g_{na,mb} \in L^2(\mathbb{R})$ we can express J as (see (1.3) for the definition of T)

$$(2.6)\qquad J(f) = (Tf, f) - 2\text{Re}(f, h) + \sum_{n,m} |F_{nm}|^2.$$

When g generates a frame, so that T^{-1} exists, $J(f)$ is uniquely minimized by

$$(2.7)\qquad f = T^{-1}h = \sum_{n,m} F_{nm}\, {}^{\circ}\gamma_{na,mb}.$$

In particular, when $F_{nm} = \delta_{no}\delta_{mo}$, we find $f = {}^{\circ}\gamma$.

In [9] a discrete version of the following problem is considered: Estimate an $f \in L^2(\mathbb{R})$ from a noisy/distorted version $F_{n,\nu}$ of

$$(2.8) \qquad (f, g_{na,\nu}) = \int f(t) g^*(t - na) e^{-2\pi i \nu t} dt, \quad n \in \mathbb{Z},\ \nu \in \mathbb{R},$$

by minimizing

$$(2.9) \qquad J_c(f) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} |(f, g_{na,\nu}) - F_{n,\nu}|^2 d\nu.$$

Here it is assumed that $F_{n,\nu} \in L^2(\mathbb{Z} \times \mathbb{R})$. Observing that by Parseval's theorem

$$(2.10) \qquad \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} |(f, g_{na,\nu})|^2 d\nu = \int_{-\infty}^{\infty} |f(t)|^2 G(t) dt,$$

where

$$(2.11) \qquad G(t) = \sum_{n=-\infty}^{\infty} |g(t - an)|^2,$$

we see that minimizing J_c is feasible when $0 < \operatorname{ess\,inf} G \le \operatorname{ess\,sup} G < \infty$. In that case one finds just as before that J_c is minimal for

$$(2.12) \qquad f(t) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F_{n,\nu} \gamma^c_{na,\nu}(t) d\nu; \qquad \gamma^c(t) = g(t)/G(t).$$

Now in case that g is supported by an interval of length $1/b$, we see that $\gamma^c = {}^{\circ}\gamma$ (apart from a constant).
As a special case we consider the choice

$$(2.13) \qquad g(t) = e^{-\alpha t} \chi_{[0,b^{-1})}(t), \quad t \in \mathbb{R}.$$

We then get

$$(2.14) \qquad (GG^*)_{kl;oo} = \begin{cases} 0, & k \ne 0, \\ \dfrac{a(1 - e^{-\alpha/b})}{2\alpha a + 2\pi i l}, & k = 0, \end{cases}$$

and

$$(2.15) \qquad F(\theta, \tau) = \sum_{k,l=-\infty}^{\infty} (GG^*)_{kl;oo} e^{2\pi i k\theta + 2\pi i l\tau} = \frac{a(1 - e^{-\alpha/b})}{1 - e^{-2\alpha a}} e^{-2\alpha a(\tau - \lfloor \tau \rfloor)}.$$

Hence the lower and upper frame bounds are given by

$$(2.16) \qquad A = \frac{1}{b} \frac{1 - e^{-\alpha/b}}{1 - e^{-2\alpha a}} e^{-2\alpha a}, \qquad B = \frac{1}{b} \frac{1 - e^{-\alpha/b}}{1 - e^{-2\alpha a}},$$

and ${}^{\circ}\gamma$ equals

$$(2.17) \qquad {}^{\circ}\gamma(t) = b \frac{1 - e^{-2\alpha a}}{1 - e^{-\alpha/b}} e^{\alpha t - 2\alpha t \lfloor t/a \rfloor} \chi_{[0,b^{-1})}(t).$$

3. FUNCTIONS SUPPORTED BY AN INTERVAL OF LENGTH $2/b$

In this section we take $(ab)^{-1} = q \in \mathbb{N}$, and we assume that we have a $g \in L^2(\mathbb{R})$ supported by an interval of length $\le 2/b$. We now find

$$(3.1) \qquad (GG^*)_{kl;oo} = \begin{cases} \int_0^{1/b} g(t) g^*\left(t + \frac{1}{b}\right) e^{-2\pi i l t/a} dt, & k = -1, \\ \int_0^{2/b} |g(t)|^2 e^{-2\pi i l t/a} dt, & k = 0, \\ \int_0^{1/b} g\left(t + \frac{1}{b}\right) g^*(t) e^{-2\pi i l t/a} dt, & k = 1, \end{cases}$$

and $(GG^*)_{kl;oo}$ vanishes for $|k| \ge 2$. Then we get

$$(3.2) \qquad F(\theta, \tau) = \sum_{k,l=-\infty}^{\infty} (GG^*)_{kl;oo} e^{2\pi i k\theta + 2\pi i l\tau} = \operatorname{Re}[c_o(\tau) + 2e^{2\pi i\theta} c_1(\tau)],$$

where

$$(3.3) \qquad c_0(\tau) = \sum_{l=-\infty}^{\infty} (GG^*)_{ol;oo} e^{2\pi i l\tau} = a \sum_n |g(a\tau + an)|^2,$$

$$(3.4) \qquad c_1(\tau) = \sum_{l=-\infty}^{\infty} (GG^*)_{1l;oo} e^{2\pi i l\tau} = a \sum_n g\left(a\tau + an + \frac{1}{b}\right) g^*(a\tau + an).$$

Hence the frame bounds A, B are given by

$$(3.5) \qquad A = \operatorname{ess\,inf}(c_0(\tau) - 2|c_1(\tau)|), \qquad B = \operatorname{ess\,inf}(c_0(\tau) + 2|c_1(\tau)|).$$

Let $\tau \in \mathbb{R}$, and let

$$(3.6) \qquad z_n = g(a\tau + an + ar), \quad n = 0, \ldots, 2q - 1,$$

with r such that $g(a\tau + an + ar) = 0$ for $n < 0, n > 2q - 1$. Since

$$(3.7) \qquad c_0(\tau) = \sum_{n=0}^{2q-1} |z_n|^2, \qquad c_1(\tau) = \sum_{n=0}^{q-1} z_n z^*_{n+q},$$

it is easy to see that $|c_1(\tau)| \le \frac{1}{2} c_0(\tau)$ with equality if and only if there is a $\beta \in \mathbb{C}$, $|\beta| = 1$ such that $z_{n+q} = \beta z_n$, $n = 0, 1, \ldots, q - 1$. Hence, when $\operatorname{ess\,inf} c_0(\tau) > 0$, we have that g generates a frame except in the very special case that the sample values of g on one half of its support are arbitrarily close to a unitary multiple of the sample values on the other half.

Assume for simplicity that $c_1(\tau)$ is real for all τ, and that $0 < A \le B < \infty$. Then we have

$$(3.8) \qquad \frac{1}{F(\theta, \tau)} = \frac{1}{c_0(\tau) + 2c_1(\tau)\cos 2\pi\theta} = \sum_{k=-\infty}^{\infty} \varphi_k(\tau) e^{2\pi i k\theta},$$

with

$$(3.9) \qquad \varphi_k(\tau) = \frac{(-p(\tau))^{|k|}}{\sqrt{c_0^2(\tau) - 4c_1^2(\tau)}}, \qquad p(\tau) = \frac{2c_1(\tau)}{c_0(\tau) + \sqrt{c_0^2(\tau) - 4c_1^2(\tau)}}.$$

It follows therefore from (1.29–30) that

$$ {}^\circ\gamma(t) = \frac{ab}{\sqrt{c_0^2(t/a) - 4c_1^2(t/a)}} \sum_{k=-\infty}^{\infty} (-p(t/a))^{|k|} g(t - k/b). \tag{3.10}$$

Consider the special case that

$$ g(t) = e^{-\alpha t}\chi_{[0,2/b)}(t), \quad t \in \mathbb{R}. \tag{3.11}$$

Then we find

$$ \begin{cases} c_0(\tau) = a\,\dfrac{1-e^{-4\alpha/b}}{1-e^{-2\alpha a}}\, e^{-2\alpha a(\tau - \lfloor\tau\rfloor)}, \\ c_1(\tau) = a\,\dfrac{1-e^{-2\alpha/b}}{1-e^{-2\alpha a}}\, e^{-\alpha/b} e^{-2\alpha a(\tau - \lfloor\tau\rfloor)}, \end{cases} \tag{3.12}$$

so that, see (3.9),

$$ p(\tau) = e^{-\alpha/b} =: p. \tag{3.13}$$

It thus follows that

$$ A = \frac{a(1-p^2)(1-p)^2}{1-e^{-2\alpha a}}\, e^{-2\alpha a}, \qquad B = \frac{a(1-p^2)(1+p)^2}{1-e^{-2\alpha a}} \tag{3.14}$$

are the frame bounds, and the dual function is given by

$$ {}^\circ\gamma(t) = \frac{b(1-e^{-2\alpha a})}{(1-p^2)^2}\, e^{2\alpha t - 2\alpha a\lfloor t/a\rfloor} \sum_{k=-\infty}^{\infty} (-p)^k g(t-k/b). \tag{3.15}$$

With some perseverance this ${}^\circ\gamma(t)$ can be shown to be equal to

$$ {}^\circ\gamma(t) = \frac{b(1-e^{-2\alpha a})}{(1-p^2)^2} (-1)^{\lfloor bt\rfloor} e^{\alpha t - 2\alpha a\lfloor t/a\rfloor + 2\alpha\lfloor bt\rfloor/b} \chi_{(-\infty,b^{-1})}(t). \tag{3.16}$$

Observe that ${}^\circ\gamma$ is supported by the set $(-\infty, b^{-1})$, and that ${}^\circ\gamma(t)$ decays as $\exp(\alpha t)$ when $t \to -\infty$.

Some of the calculations just given for the truncated exponential can also be made for the case that

$$ g(t) = e^{-\alpha t}\chi_{[0,r/b)}(t), \quad t \in \mathbb{R}, \tag{3.17}$$

where $r = 3, 4, \ldots$. With $p = e^{-\alpha/b}$, it can be shown that

$$ (GG^*)_{kl;oo} = \begin{cases} 0, & |k| \ge r, \\ \dfrac{a(p^{|k|} - p^{2r-|k|})}{2\alpha a + 2\pi i l}, & |k| < r. \end{cases} \tag{3.18}$$

It is furthermore found that

$$ F(\theta,\tau) = \frac{a(1-p^2)}{1-e^{-2\alpha a}} \left| \frac{1-p^r e^{2\pi i r\theta}}{1-pe^{2\pi i\theta}} \right|^2 e^{-2\alpha a(\tau - \lfloor\tau\rfloor)}. \tag{3.19}$$

This yields the frame bounds

$$ \begin{cases} A = \dfrac{(1-p^2)e^{-2\alpha a}}{b(1-e^{-2\alpha a})} \min\limits_{|z|=p} \left|\dfrac{1-z^r}{1-z}\right|^2, \\ B = \dfrac{1-p^2}{b(1-e^{-2\alpha a})} \max\limits_{|z|=p} \left|\dfrac{1-z^r}{1-z}\right|^2. \end{cases} \tag{3.20}$$

One can even give a series expansion for ${}^\circ\gamma$ as in (3.15); this is so since

$$ \left| \frac{1-pe^{2\pi i\theta}}{1-p^r e^{2\pi i r\theta}} \right|^2 = \frac{|1-pe^{2\pi i\theta}|^2}{1-p^{2r}} \sum_{j=-\infty}^{\infty} p^{|j|r} e^{2\pi i j r\theta} \tag{3.21}$$

(also see (1.29–30)). A further elaboration of ${}^\circ\gamma$ as in (3.16), though possible, yields unwieldy expressions.

4. ONE-SIDED EXPONENTIALS

In this section we consider the one-sided exponential

$$ g(t) = \sqrt{2\alpha}\, e^{-\alpha t}\chi_{[0,\infty)}(t), \quad t \in \mathbb{R} \tag{4.1}$$

(as in [6], we have normalized g so as to have unit $L^2(\mathbb{R})$-norm). We shall first show that g generates a frame when $ab \le 1$. According to Proposition A in [11], it is sufficient to show that g has a finite upper frame bound and to display a γ with a finite upper frame bound such that

$$ (\gamma, g_{k/b,l/a}) = ab\,\delta_{ko}\delta_{lo}, \quad k, l \in \mathbb{Z}. \tag{4.2}$$

To show that g has an upper frame bound, we can use [1], Section 3.4.2. We have (with $a = t_0$, $b = \omega_0/2\pi$) for the present case

$$ \beta(s) := \sup \sum_n |g(x-na)g(x-na+s)| = \frac{2\alpha e^{-\alpha|s|}}{1-e^{-2\alpha a}}. \tag{4.3}$$

Hence

$$ \beta(0) + \sum_{k\neq 0} |\beta(k/b)\beta(-k/b)|^{1/2} < \infty, \tag{4.4}$$

so that g has indeed a finite upper frame bound. Next a γ satisfying (4.2) is given by

$$ \gamma(t) = \frac{b}{\sqrt{2\alpha}}\, e^{\alpha t}(\chi_{[0,a)}(t) - \chi_{[-a,0)}(t)). \tag{4.5}$$

This γ has an upper frame bound since the β of [1], Section 3.4.2 corresponding to γ is bounded and has a bounded support.
We next consider the case that $(ab)^{-1} = q \in \mathbb{N}$. We compute

$$ (GG^*)_{kl;oo} = \frac{2\alpha a}{2\alpha a + 2\pi i l}\, e^{-\alpha|k|/b}, \tag{4.6}$$

so that

$$F(\theta,\tau)=m_{\mathrm{time}}(\theta)m_{\mathrm{freq}}(\tau) \tag{4.7}$$

with

$$m_{\mathrm{time}}(\theta)=\frac{1-p^2}{1+p^2-2p\cos 2\pi\theta},\qquad m_{\mathrm{freq}}(\tau)=\frac{2\alpha a}{1-e^{-2\alpha a}}\,e^{-2\alpha a(\tau-\lfloor\tau\rfloor)}. \tag{4.8}$$

In (4.8) we have set $p=e^{-\alpha/b}$, as usual. It thus follows that lower and upper frame bound are given by

$$A=\frac{2\alpha}{b}\,\frac{1-p}{1+p}\,\frac{e^{-2\alpha a}}{1-e^{-2\alpha a}},\qquad B=\frac{2\alpha}{b}\,\frac{1+p}{1-p}\,\frac{1}{1-e^{-2\alpha a}}. \tag{4.9}$$

Apart from the factors 2α, this agrees with (3.20) when we let $r\to\infty$. Finally, ${}^\circ\gamma$ follows easily from (1.23–28) as

$$^\circ\gamma(t)=\frac{ab}{m_{\mathrm{freq}}(t/a)}\,\frac{1}{1-p^2}\,(-pg(t+1/b)+(1+p^2)g(t)-pg(t-1/b)) \tag{4.10}$$

which can be worked out as

$$^\circ\gamma(t)=\frac{b}{\sqrt{2\alpha}}\,\frac{1-e^{-2\alpha a}}{1-p^2}\,e^{\alpha t-2\alpha a\lfloor t/a\rfloor}(\chi_{[0,1/b)}(t)-p^2\chi_{[-1/b,0)}(t)). \tag{4.11}$$

The results (4.9) and (4.11) agree with the expressions found by Friedlander and Zeira in [6] by using the Zak transform. Observe that ${}^\circ\gamma$ is supported by $[-1/b,1/b)$, and that ${}^\circ\gamma$ and the γ of (4.5) are different.

5. TWO-SIDED EXPONENTIALS

In this section we take $(ab)^{-1}=q\in\mathbb{N}$, and we compute the frame bounds for

$$g(t)=\alpha^{1/2}e^{-\alpha|t|},\quad t\in\mathbb{R}. \tag{5.1}$$

We find

$$(GG^*)_{kl;oo}=\left(\frac{4\alpha^2a^2}{4\alpha^2a^2+4\pi^2l^2}+\frac{\alpha}{b}\,|k|\delta_{lo}\right)e^{-\alpha|k|/b}. \tag{5.2}$$

Therefore

$$F(\theta,\tau)=m^I_{\mathrm{time}}(\theta)m^I_{\mathrm{freq}}(\tau)+\frac{\alpha}{b}\,m^{II}_{\mathrm{time}}(\theta). \tag{5.3}$$

Here we have (with $p=e^{-\alpha/b}$ as usual)

$$m^I_{\mathrm{time}}(\theta)=\sum_{k=-\infty}^{\infty}p^{|k|}e^{2\pi ik\theta}=\frac{1-p^2}{1+p^2-2p\cos 2\pi\theta}, \tag{5.4}$$

$$m^I_{\mathrm{freq}}(\tau)=\sum_{l=-\infty}^{\infty}\frac{4\alpha^2a^2}{4\alpha^2a^2+4\pi^2l^2}\,e^{2\pi il\tau}=\frac{\alpha a\cosh\alpha a(\tau-\lfloor\tau\rfloor-\frac12)}{\sinh\alpha a}, \tag{5.5}$$

$$m^{II}_{\mathrm{time}}(\theta)=\sum_{k=-\infty}^{\infty}|k|p^{|k|}e^{2\pi ik\theta}=p\,\frac{\partial}{\partial p}\,[m^I_{\mathrm{time}}(\theta)]=m^I_{\mathrm{time}}(\theta)m^{III}_{\mathrm{time}}(\theta) \tag{5.6}$$

with

$$m^{III}_{\mathrm{time}}(\theta)=\frac{2p}{1-p^2}\,\frac{(1+p^2)\cos 2\pi\theta-2p}{(1+p^2)-2p\cos 2\pi\theta}. \tag{5.7}$$

Hence

$$F(\theta,\tau)=m^I_{\mathrm{time}}(\theta)\left\{m^I_{\mathrm{freq}}(\tau)+\frac{\alpha}{b}\,m^{III}_{\mathrm{time}}(\theta)\right\}. \tag{5.8}$$

Now

$$m^I_{\mathrm{time}}(\tfrac12)=\frac{1-p}{1+p}\le m^I_{\mathrm{time}}(\theta)\le\frac{1+p}{1-p}=m^I_{\mathrm{time}}(0), \tag{5.9}$$

$$m^I_{\mathrm{freq}}(\tfrac12)=\frac{\alpha a}{\sinh\alpha a}\le m^I_{\mathrm{freq}}(\tau)\le\frac{\alpha a\cosh\alpha a}{\sinh\alpha a}=m^{III}_{\mathrm{freq}}(0), \tag{5.10}$$

$$m^{III}_{\mathrm{time}}(\tfrac12)=\frac{-2p}{1-p^2}\le m^{III}_{\mathrm{time}}(\theta)\le\frac{2p}{1-p^2}=m^{III}_{\mathrm{time}}(0). \tag{5.11}$$

Hence (observe that $F(\theta,\tau)\ge 0$, see (1.19))

$$\min_{\theta,\tau}F(\theta,\tau)=F(\tfrac12,\tfrac12),\qquad \max_{\theta,\tau}F(\theta,\tau)=F(0,0), \tag{5.12}$$

and we obtain the frame bounds

$$A=\frac{\tanh(\alpha/2b)}{ab}\left\{\frac{\alpha a}{\sinh\alpha a}-\frac{\alpha/b}{\sinh\alpha/b}\right\}, \tag{5.13}$$

$$B=\frac{\operatorname{cotanh}(\alpha/2b)}{ab}\left\{\frac{\alpha a\cosh\alpha a}{\sinh\alpha a}+\frac{\alpha/b}{\sinh\alpha/b}\right\}. \tag{5.14}$$

Observe that $A\ge 0$ with equality if and only if $a=1/b$.

A numerical inspection of the bounds A,B for the case $\alpha=1$, $(ab)^{-1}=q\in\mathbb{N}$ shows agreement with Table IV in [2], p. 983 although for instance the case $a=1$, $b=\frac14$ (i.e. $q_0=1$, $p_0=\pi/2$ in the table) shows 2.724 for A_{exact}, while we obtain

$$A=4\tanh 2\left\{\frac{1}{\sinh 1}-\frac{4}{\sinh 4}\right\}=2.716027577, \tag{5.15}$$

which is slightly off.

We refer to [12] where an explicit expression for ${}^\circ\gamma$ is presented when $ab=1$.

6. GAUSSIANS

We consider in this section the Gaussian

$$g(t)=2^{1/4}\exp(-\pi t^2),\quad t\in\mathbb{R}. \tag{6.1}$$

It is known that g generates a frame when $ab<1$ (see [11] for an elementary proof). We shall find the frame bounds for the case that $(ab)^{-1}=q\in\mathbb{N}$, and we compute ${}^\circ\gamma$ for the case that q is even.

We compute (when $(ab)^{-1}=q\in\mathbb{N}$)

$$(GG^*)_{kl;oo}=(-1)^{klq}\exp(-\tfrac12\pi(k^2/b^2+l^2/a^2)). \tag{6.2}$$

We shall show that

$$F(\theta,\tau) = \sum_{k,l} (GG^*)_{kl;oo}\, e^{2\pi ik\theta + 2\pi il\tau} \tag{6.3}$$

is minimal for $(\theta,\tau) = (\frac{1}{2},\frac{1}{2})$ and maximal for $(\theta,\tau) = (0,0)$.

We distinguish between the cases that q is even and q is odd. When q is even we have that

$$F(\theta,\tau) = \vartheta_3(\pi\theta; e^{-\pi/2b^2})\vartheta_3(\pi\tau; e^{-\pi/2a^2}), \tag{6.4}$$

where we have used the definition (for $0 < c < 1$)

$$\vartheta_3(z;c) = \sum_{n=-\infty}^{\infty} c^{n^2} e^{2inz} \tag{6.5}$$

of the theta function (see [12], Chapter 21). It is a well-known fact, and an easy consequence of the product expansion

$$\begin{cases} \vartheta_3(z;c) = C \prod_{n=1}^{\infty} (1 + 2c^{2n-1}\cos 2z + c^{4n-2}); \\ C = \prod_{n=1}^{\infty} (1 - c^{2n}) > 0, \end{cases} \tag{6.6}$$

that $\vartheta_3(z;c)$ is minimal at all points $z = (r+\frac{1}{2})\pi$ and maximal at all points $z = r\pi$ with $r \in \mathbb{Z}$. Hence F is minimal at $(\theta,\tau) = (\frac{1}{2},\frac{1}{2})$ and maximal at $(\theta,\tau) = (0,0)$. Accordingly we get for the frame bounds

$$A = \frac{1}{ab}\sum_{k,l} (-1)^{k+l} \exp(-\tfrac{1}{2}\pi(k^2/b^2 + l^2/a^2)), \tag{6.7}$$

$$B = \frac{1}{ab}\sum_{k,l} \exp(-\tfrac{1}{2}\pi(k^2/b^2 + l^2/a^2)). \tag{6.8}$$

Also, we compute $^\circ\gamma(t)$ according to (1.23–28) as

$$^\circ\gamma(t) = \frac{ab}{\vartheta_3(\pi t/a; \exp(-\pi/2a^2))} \sum_{k=-\infty}^{\infty} c_k\, g(t - k/b), \tag{6.9}$$

where c_k are the Fourier coefficients of $\vartheta_3^{-1}(\pi\theta; \exp(-\pi/2b^2))$. We have by [13], Chapter 21, p. 489 (Example 14), § 21.61 and § 21.41 that

$$\frac{1}{\vartheta_3(z;c)} = \frac{1}{\vartheta_1'(0)} \sum_{k=-\infty}^{\infty} (-1)^k a_k e^{2ikz}. \tag{6.10}$$

Here we have (with $c = \exp(-\pi/2b^2)$)

$$\begin{cases} \vartheta_1'(0) = \sum_{n=-\infty}^{\infty} (-1)^n (2n+1) c^{(n+\frac{1}{2})^2}; \\ a_k = 2\sum_{m=0}^{\infty} (-1)^m c^{(m+\frac{1}{2})(2|k|+m+\frac{1}{2})}. \end{cases} \tag{6.11}$$

Hence

$$c_k = \frac{\sum_{m=0}^{\infty} (-1)^{m+k} c^{(m+\frac{1}{2})(2|k|+m+\frac{1}{2})}}{\sum_{n=-\infty}^{\infty} (-1)^n (n+\frac{1}{2}) c^{(n+\frac{1}{2})^2}}, \quad k \in \mathbb{Z}. \tag{6.12}$$

From (6.9) and (6.12) it follows that $^\circ\gamma(t)$ decays like $\exp(-\pi|t|/2b)$ (and not faster than that) and that $^\circ\gamma$ (unlike g) does not extend to an entire function. The case q odd is more complicated. Then we can write

$$F(\theta,\tau) = \sum_{k,l} (-1)^{kl} \exp(-\tfrac{1}{2}\pi(k^2/b^2 + l^2/a^2) + 2\pi ik\theta + 2\pi il\tau) \tag{6.13}$$

in the forms

$$\begin{cases} F(\theta,\tau) = \left(\sum_{k,l} - 2\sum_{k,l\ \text{odd}}\right) \exp\left(-\frac{\pi k^2}{2b^2} - \frac{\pi l^2}{2a^2} + 2\pi ik\theta + 2\pi il\tau\right) \\ \quad = \vartheta_3(\pi\theta; e^{-\pi/2b^2})\vartheta_3(\pi\tau; e^{-\pi/2a^2}) \\ \qquad - 2\vartheta_2(2\pi\theta; e^{-2\pi/b^2})\vartheta_2(2\pi\tau; e^{-2\pi/a^2}) \end{cases} \tag{6.14}$$

and

$$\begin{cases} F(\theta,\tau) = \left(-\sum_{k,l} (-1)^{k+l} + 2\sum_{k,l\ \text{even}}\right) \exp\left(-\frac{\pi k^2}{2b^2} - \frac{\pi l^2}{2a^2} + 2\pi ik\theta + 2\pi il\tau\right) \\ \quad = -\vartheta_4(\pi\theta; e^{-\pi/2b^2})\vartheta_4(\pi\tau; e^{-\pi/2a^2}) \\ \qquad + 2\vartheta_3(2\pi\theta; e^{-2\pi/b^2})\vartheta_3(2\pi\tau; e^{-2\pi/a^2}). \end{cases} \tag{6.15}$$

Here we have used the definitions (6.5) and

$$\vartheta_2(z;c) = \sum_{n=-\infty}^{\infty} c^{(n+\frac{1}{2})^2} e^{(2n+1)iz}, \tag{6.16}$$

$$\vartheta_4(z;c) = \vartheta_3(z + \tfrac{1}{2}\pi; c) = \sum_{n=-\infty}^{\infty} (-1)^n c^{n^2} e^{2niz} \tag{6.17}$$

for the theta functions. From the product expansions (6.6) and

$$\vartheta_2(z;c) = 2Cc^{1/4}\cos z \prod_{n=1}^{\infty} (1 + 2c^{2n}\cos 2z + c^{4n}), \tag{6.18}$$

$$\vartheta_4(z;c) = C\prod_{n=1}^{\infty} (1 - 2c^{2n-1}\cos 2z + c^{4n-2}) \tag{6.19}$$

with $C > 0$ as in (6.6), we see that

– $\vartheta_2(z;c)$ is maximal at all $z = 2r\pi$, minimal at all $z = (2r+1)\pi$, and that $\vartheta_2(2r\pi;c) = -\vartheta_2((2r+1)\pi;c) > 0$ for $r \in \mathbb{Z}$,

– $\vartheta_3(z;c)$ is maximal at all $z = r\pi$, minimal at all $z = (r+\frac{1}{2})\pi$, and that $\vartheta_3((r+\frac{1}{2})\pi;c) > 0$ for $r \in \mathbb{Z}$,

– $\vartheta_4(z;c)$ is minimal at all $z = r\pi$, maximal at all $z = (r+\frac{1}{2})\pi$, and that $\vartheta_4(r\pi;c) > 0$ for $r \in \mathbb{Z}$.

We easily conclude from (6.14) that $F(\theta,\tau)$ is minimal at all $(\theta,\tau) = (r+\frac{1}{2}, s+\frac{1}{2})$, and from (6.15) that $F(\theta,\tau)$ is maximal at all $(\theta,\tau) = (r,s)$ for $r,s \in \mathbb{Z}$. As a consequence we get for the frame bounds A, B the expressions

$$A = \frac{1}{ab} \sum_{k,l=-\infty}^{\infty} (-1)^{k+l+kl} \exp(-\tfrac{1}{2}\pi(k^2/b^2 + l^2/a^2)), \tag{6.20}$$

$$B = \frac{1}{ab} \sum_{k,l=-\infty}^{\infty} (-1)^{kl} \exp(-\tfrac{1}{2}\pi(k^2/b^2 + l^2/a^2)). \tag{6.21}$$

The results of this section extend the ones given in [10], Section 4, where it was shown that for the case that $(ab)^{-1}$ is even the upper frame bound B is given by (6.8). We also note that the F in (6.13) has been analyzed in [14], Section 3 for the case that $ab = 1$; it has been shown there, among other things, that $F(\theta,\tau) \geq F(\frac{1}{2},\frac{1}{2}) = 0$. We finally note that computation of $^{\circ}\gamma$ is very cumbersome for this case.

7. HYPERBOLIC SECANTS

We consider in this section for $d > 0$ the hyperbolic secant

$$g(t) = \frac{1}{\cosh \pi dt}, \quad t \in \mathbb{R}, \tag{7.1}$$

whose Fourier transform $\mathcal{F}g$ is given by

$$(\mathcal{F}g)(\nu) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i\nu t}}{\cosh \pi dt}\, dt = \frac{1}{d\cosh(\pi\nu/d)}, \quad \nu \in \mathbb{R}. \tag{7.2}$$

Using the formula

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i\nu t}}{\cosh\gamma + \cosh\beta t}\, dt = \frac{2\pi\sin(2\pi\nu\gamma/\beta)}{\beta\sinh\gamma\sinh(2\pi^2\nu/\beta)}, \quad \nu \in \mathbb{R}, \tag{7.3}$$

(with the usual precautions when $\nu = 0$ or $\gamma = 0$) we compute the matrix elements of GG^* as

$$(GG^*)_{kl;k'l'} = \frac{4\pi\exp(-\pi i(l-l')(k+k')/ab)\sin(\pi(l-l')(k-k')/ab)}{\beta\sinh(\pi d(k-k')/b)\sinh(\pi(l-l')/ad)}. \tag{7.4}$$

Consider the case that $(ab)^{-1} = q \in \mathbb{N}$. Then GG^* has a Toeplitz matrix and (7.4) vanishes unless $(k-k')(l-l') = 0$. Moreover,

$$(GG^*)_{ko;oo} = \frac{2k}{b\sinh(\pi dk/b)}, \qquad (GG^*)_{ol;oo} = \frac{2l}{ad^2\sinh(\pi l/ad)} \tag{7.5}$$

for $k \neq 0 \neq l$, and $(GG^*)_{oo;oo} = 2/\pi d$. It follows that

$$\begin{cases} F(\theta,\tau) = \sum_{k=-\infty}^{\infty} (GG^*)_{ko;oo} e^{2\pi ik\theta} + \sum_{l=-\infty}^{\infty} (GG^*)_{ol;oo} e^{2\pi il\tau} - (GG^*)_{oo;oo} \\ \qquad = \frac{1}{d}\left\{\frac{4}{\alpha}\sum_{k=1}^{\infty}\frac{k\cos 2\pi k\theta}{\sinh(\pi k/\alpha)} + \frac{4}{\beta}\sum_{l=1}^{\infty}\frac{l\cos 2\pi l\tau}{\sinh(\pi l/\beta)} + \frac{2}{\pi}\right\}, \end{cases} \tag{7.6}$$

where we have set

$$\alpha = \frac{b}{d}, \qquad \beta = ad. \tag{7.7}$$

By using the definition of $(GG^*)_{ko;oo}$ and $(GG^*)_{ol;oo}$ directly, the Poisson summation formula, and, for the first series in the middle member of (7.6), Parseval's theorem together with (7.2), we can write $F(\theta,\tau)$ also as

$$F(\theta,\tau) = \frac{1}{d}\left\{\sum_{k=-\infty}^{\infty}\frac{\alpha}{\cosh^2(\pi\alpha(\theta+k))} + \sum_{l=-\infty}^{\infty}\frac{\beta}{\cosh^2\pi\beta(\tau+l)} - \frac{2}{\pi}\right\}. \tag{7.8}$$

To obtain the frame bounds

$$A = \frac{1}{ab}\min F(\theta,\tau), \qquad B = \frac{1}{ab}\max F(\theta,\tau), \tag{7.9}$$

we have to use some results from the theory of Jacobian elliptic functions, see [13], Chapter 21–22. We have from [13], p. 520 (Example 5)

$$\sum_{n=1}^{\infty}\frac{n\cos 2nx}{\sinh ny} = 2\sum_{n=1}^{\infty}\frac{nc^n\cos 2nx}{1-c^{2n}} = \frac{1}{\pi^2}\{K^2 - KE - (kK)^2\mathrm{sn}^2 u\}, \tag{7.10}$$

where $c = \exp(-y)$ and K, E, k, u are as in [13]. In particular,

$$k = \frac{\vartheta_2^2}{\vartheta_3^2}, \qquad u = \vartheta_3^2 x, \qquad \mathrm{sn}\, u = k^{-1/2}\frac{\vartheta_1(x;c)}{\vartheta_4(x;c)} \tag{7.11}$$

where $\vartheta_i = \vartheta_i(0;c)$ (the theta functions are as in Section 6, and $\vartheta_1(x;c) = \vartheta_2(x-\frac{1}{2}\pi,c)$). Now by [13], p. 478 we have

$$\frac{d}{dx}\left(\frac{\vartheta_1(x;c)}{\vartheta_4(x;c)}\right) = \frac{\vartheta_4^2\vartheta_2(x;c)\vartheta_3(x;c)}{\vartheta_4^2(x;c)}. \tag{7.12}$$

Hence $\mathrm{sn}^2 u$ is extremal when

$$\mathrm{sn}\, u = 0 \quad (\text{i.e. } x = n\pi), \qquad \text{or} \qquad \vartheta_2(x;c) = 0 \quad (\text{i.e. } x = (n+\tfrac{1}{2})\pi), \tag{7.13}$$

where $n \in \mathbb{Z}$. In the second case we have $\mathrm{sn}^2 u = 1$ since $\vartheta_1(\frac{1}{2}\pi;c) = \vartheta_2$, $\vartheta_4(\frac{1}{2}\pi;c) = \vartheta_3$.

It thus follows that F is minimal at $(\theta,\tau) = (\frac{1}{2},\frac{1}{2})$ and maximal at $(\theta,\tau) = (0,0)$, so that the frame bounds are given by

$$\begin{cases} A = \frac{1}{abd}\left\{\frac{4}{\alpha\pi^2}(K_\alpha^2 - K_\alpha E_\alpha - K_\alpha^2 k_\alpha^2)\right. \\ \qquad \left. + \frac{4}{\beta\pi^2}(K_\beta^2 - K_\beta E_\beta - K_\beta^2 k_\beta^2) + \frac{2}{\pi}\right\}, \end{cases} \tag{7.14}$$

$$B = \frac{1}{abd}\left\{\frac{4}{\alpha\pi^2}(K_\alpha^2 - K_\alpha E_\alpha) + \frac{4}{\beta\pi^2}(K_\beta^2 - K_\beta E_\beta) + \frac{2}{\pi}\right\}. \tag{7.15}$$

Here, α, β are as in (7.7) and $K_\gamma, E_\gamma, k_\gamma$ correspond to $c = \exp(-\pi/\gamma)$. For more explicit expressions for A, B one can use (7.6) or (7.8) with $\theta = \tau = \frac{1}{2}, \theta = \tau = 0$.

We shall next show that $A = 0$ when $ab = \alpha\beta = 1$, and that $A > 0$ when $ab =$

$\alpha\beta < 1$. For the case $ab = \alpha\beta = 1$ we argue as follows. We have $A \geq 0$ since GG^* is positive semi-definite. Also, by the Balian–Low theorem, see [1], Theorem 4.1.1, we cannot have $A > 0$ since both g and $\mathcal{F}g$, see (7.1–2), decay rapidly. Hence $A = 0$ in this case. As an aside remark we note that the equality $A = (ab)^{-1}F(\frac{1}{2}, \frac{1}{2}) = 0$ yields by (7.8) for the case $a = b = \alpha = \beta = 1 = d$ the identity

$$(7.16) \qquad \sum_{k=0}^{\infty} \frac{1}{\cosh^2 \pi(k+\frac{1}{2})} = \frac{1}{2\pi},$$

which is [15], (43.8.14).
The case $ab = \alpha\beta < 1$ is slightly harder. Denote for $u > 0$, $v > 0$

$$(7.17) \qquad R(u,v) = Q(u) + Q(v) + \tfrac{1}{2}; \qquad Q(u) = \sum_{k=1}^{\infty} \frac{(-1)^k ku}{\sinh ku},$$

so that

$$(7.18) \qquad A = \frac{1}{ab} F(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{4}{\pi\alpha\beta d} R\left(\frac{\pi}{\alpha}, \frac{\pi}{\beta}\right)$$

by (7.6–7). We shall show that Q in (7.17) is strictly increasing in $u \geq \pi$. Since $R(u,v) = 0$ when $uv = \pi^2$ by what we have proved already, and $\alpha\beta < 1$ it then follows that $A > 0$.
To show that Q strictly increases in $u \geq \pi$ we fix $u \geq \pi$, and we note that

$$(7.19) \qquad Q(u) = \sum_{k=1}^{\infty} (-1)^k \varphi(ku), \qquad uQ'(u) = \sum_{k=1}^{\infty} (-1)^k ku\varphi'(ku),$$

where we have set $\varphi(z) = z/\sinh z$. Now

$$(7.20) \qquad \varphi'(z) = \frac{\sinh z - z\cosh z}{\sinh^2 z} < 0, \quad z > 0.$$

Hence for proving that $Q'(u) > 0$ it suffices to show that the sequence $(ku\varphi'(ku))_{k=1,2,\ldots}$ increases to 0 as $k \to \infty$. For this it suffices to show that $(z\varphi'(z))' > 0$ when $z \geq \pi$. We compute

$$(7.21) \qquad (z\varphi'(z))' = \frac{z^2 + \sinh^2 z + z\cosh z(z\cosh z - 3\sinh z)}{\sinh^3 z}.$$

Evidently (7.21) is positive for $z > 3$, and the proof is complete.

REFERENCES

1. Daubechies, I. – Ten Lectures on Wavelets. CBMS-NSF regional conference series in Applied Mathematics **61**, Philadelphia (1992).
2. Daubechies, I. – The wavelet transform, time-frequency localization and signal analysis. IEEE Trans. Information Theory **36**, 961–1005 (1990).
3. Daubechies, I., A. Grossmann and Y. Meyer – Painless non-orthogonal expansions. J. Math. Phys. **27**, 1271–1283 (1986).
4. Daubechies, I. and A. Grossmann – Frames of entire functions in the Bargmann space. Comm. Pure Appl. Math. **41**, 151–164 (1988).

5. Zibulski, M. and Y.Y. Zeevi – Oversampling in the Gabor scheme. IEEE Trans. Signal Processing **41**, 2679–2687 (1993).
6. Friedlander, B. and A. Zeira – Oversampled Gabor representation for transient signals. Submitted for publication, March 1994.
7. Janssen, A.J.E.M. – Duality and biorthogonality for Weyl–Heisenberg frames. To appear in Journal of Fourier Analysis and Applications.
8. Veldhuis, R.N.J. – A vector-filter notation for analysis/synthesis systems and its relation to frames. IPO Rapport no. **909**, available on request: IPO, P.O. Box 513, 5600 MB Eindhoven, the Netherlands.
9. Griffin, D.W. and J.S. Lim – Signal estimation from modified short-time Fourier transform. IEEE Trans. Acoustics, Speech, and Signal Processing **32**, 236–243 (1984).
10. Tolimieri, R. and R.S. Orr – Poisson summation, the ambiguity function and the theory of Weyl–Heisenberg frames. To appear in Journal of Fourier Analysis and Applications.
11. Janssen, A.J.E.M. – Signal analytic proofs of two basic results on lattice expansions. Applied Computational Harmonic Analysis **1**, 350–354 (1994).
12. Einziger, P.D. – Gabor expansion of an aperture field in exponential elementary beams. IEE Electronics Letters **24**, 665–666 (1988).
13. Whittaker, E.T. and G.N. Watson – A Course of Modern Analysis, 4th ed. Cambridge Univ. Press, London/New York (1927).
14. Janssen, A.J.E.M. – Gabor representation of generalized functions. Journal Math. Anal. Appl. **83**, 377–394 (1981).
15. Hansen, E.R. – A Table of Series and Products. Prentice-Hall, Inc., Englewood Cliffs, NJ (1975).