Chapter 1

SOME REMARKS ON THE VALUE-AT-RISK AND THE CONDITIONAL VALUE-AT-RISK

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Abstract The value-at-risk (VaR) and the conditional value-at-risk (CVaR) are two commonly used risk measures. We state some of their properties and make a comparison. Moreover, the structure of the portfolio optimization problem using the VaR and CVaR objective is studied.

Keywords: Risk measures, Value-at-Risk, Conditional Value-at-Risk, Portfolio optimization

Introduction

Let Y be a random cost variable and let F_Y be its distribution function, i.e. $F_Y(u) = \mathbb{P}\{Y \le u\}$. Let $F_Y^{-1}(v)$ be its left continuous inverse, i.e. $F_Y^{-1}(v) = \min\{u : F_Y(u) \ge v\}$. When no confusion may occur, we write simply F instead of F_Y.

For a fixed level α, we define (as usual) the *value-at-risk* VaR_α as the α-quantile, i.e.

$$\mathrm{VaR}_\alpha(Y) = F^{-1}(\alpha). \tag{1.1}$$

The *conditional value-at-risk* $CVaR_\alpha$ is defined as the solution of an optimization problem

$$\mathrm{CVaR}_\alpha(Y) := \inf\{a + \frac{1}{1-\alpha}\mathbb{E}[Y-a]^+ : a \in \mathbb{R}\}. \tag{1.2}$$

Here $[z]^+ = \max(z, 0)$. Uryasev and Rockafellar (1999) have shown that for smooth F_Y CVaR equals the conditional expectation of Y, given that $Y >$ VaR$_\alpha$, i.e.

$$\mathrm{CVaR}_\alpha(Y) = \mathbb{E}(Y\,|Y > \ \mathrm{VaR}_\alpha(Y)). \tag{1.3}$$

In fact, (1.3) is the usual definition of CVaR$_\alpha$.

We will prove some properties of CVaR and VaR and study the relation between these two measures of risk. To begin with, we show that (1.2) has always a solution and that a minimizer is VaR$_\alpha$, even if F is not continuous.

Proposition 1. Suppose that $F(b) \geq \alpha$ and $F(b-) \leq \alpha$. Then

$$b + \frac{1}{1-\alpha}\mathbb{E}[Y-b]^+ \leq a + \frac{1}{1-\alpha}\mathbb{E}[Y-a]^+$$

for all a.

Proof. Suppose first that $b \leq a$. Then

$$\begin{aligned} \mathbb{E}[Y\mathbb{1}_{\{b<Y\}}] - \mathbb{E}[Y\mathbb{1}_{\{a<Y\}}] &= \mathbb{E}[Y\mathbb{1}_{\{b<Y\leq a\}}] \\ &\leq a[F(a)-F(b)] \leq a[F(a)-\alpha] - b[F(b)-\alpha]. \end{aligned}$$

Therefore

$$\begin{aligned} b[1-\alpha] - b[1-F(b)] + \mathbb{E}[Y\mathbb{1}_{\{b<Y\}}] &\leq a[1-\alpha] - a[1-F(a)] + \mathbb{E}[Y\mathbb{1}_{\{a<Y\}}] \\ b[1-\alpha] + \mathbb{E}[(Y-b)\mathbb{1}_{\{b<Y\}}] &\leq a[1-\alpha] + \mathbb{E}[(Y-a)\mathbb{1}_{\{a<Y\}}] \\ b + \frac{1}{1-\alpha}\mathbb{E}[(Y-b)\mathbb{1}_{\{b<Y\}}] &\leq a + \frac{1}{1-\alpha}\mathbb{E}[(Y-a)\mathbb{1}_{\{a<Y\}}]. \end{aligned}$$

Let now $a \leq b$. Then

$$\begin{aligned} \mathbb{E}[Y\mathbb{1}_{\{a<Y\}}] - \mathbb{E}[Y\mathbb{1}_{\{b\leq Y\}}] &= \mathbb{E}[Y\mathbb{1}_{\{a<Y<b\}}] \\ &\geq a[F(b-)-F(a)] \geq b[F(b-)-\alpha] - a[F(a)-\alpha]. \end{aligned}$$

Therefore

$$\begin{aligned} a[1-\alpha] - a[1-F(a)] + \mathbb{E}[Y\mathbb{1}_{\{a<Y\}}] &\geq b[1-\alpha] - b[1-F(b-)] + \mathbb{E}[Y\mathbb{1}_{\{b\leq Y\}}] \\ a[1-\alpha] + \mathbb{E}[(Y-a)\mathbb{1}_{\{a<Y\}}] &\geq b[1-\alpha] + \mathbb{E}[(Y-b)\mathbb{1}_{\{b\leq Y\}}] - b[1-F(b-)] \\ b + \frac{1}{1-\alpha}\mathbb{E}[(Y-b)\mathbb{1}_{\{b<Y\}}] &\leq a + \frac{1}{1-\alpha}\mathbb{E}[(Y-a)\mathbb{1}_{\{a<Y\}}]. \end{aligned}$$

□

This Lemma implies the following fact: If F takes the value α in the interval $[a, b)$, then

$$[a,b] = \ \mathrm{argmin}\ \{a + \frac{1}{1-\alpha}\mathbb{E}[Y-a]^+ : a \in \mathbb{R}\}.$$

In particular, $F^{-1}(\alpha) \in \text{ argmin } \{a + \frac{1}{1-\alpha}\mathbb{E}[Y-a]^+\}$.

We will now derive alternative representations of CVaR. Let b be chosen such that $F(b) = \alpha$. Then $\mathbb{P}\{Y > b\} = 1 - \alpha$ and

$$\begin{aligned} \mathbb{E}(Y|Y>b) &= \frac{\mathbb{E}(Y\mathbb{1}_{\{Y>b\}})}{\mathbb{P}\{Y>b\}} \\ &= \frac{\mathbb{E}(b\mathbb{1}_{\{Y>b\}} + [Y-b]^+)}{\mathbb{P}\{Y>b\}} \\ &= b + \frac{1}{1-\alpha}\mathbb{E}([Y-b]^+) \end{aligned}$$

Consequently, if α is in the range of F, i.e. if $F(F^{-1}(\alpha)) = \alpha$, then

$$\begin{aligned} \text{CVaR}_\alpha(Y) &= \mathbb{E}[Y|Y > F^{-1}(\alpha)] \\ &= \frac{1}{1-\alpha}\int_\alpha^1 F^{-1}(v)\,dv \\ &= \frac{1}{1-\alpha}\int_{(F^{-1}(\alpha),\infty)} u\,dF(u). \end{aligned}$$

1. PROPERTIES OF VAR AND CVAR

Properties of risk measures can be formulated in terms of preference structures induced by dominance relations (see Fishburn (1980)).

Let Y_1 and Y_2 be two random variables.

- *Stochastic dominance of order* 1: We say that the relation

  $$Y_1 \prec_{SD(1)} Y_2$$

  holds iff

  $$\mathbb{E}[\psi(Y_1)] \leq \mathbb{E}[\psi(Y_2)]$$

  for all (integrable) monotonic functions ψ.

- *Stochastic dominance of order* 2: We say that the relation

  $$Y_1 \prec_{SD(2)} Y_2$$

  holds iff

  $$\mathbb{E}[\psi(Y_1)] \leq \mathbb{E}[\psi(Y_2)]$$

  for all (integrable) convex, monotonic functions ψ.

- *Monotonic dominance of order* 2: We say that the relation

  $$Y_1 \prec_{MD(1)} Y_2$$

holds iff

$$\mathbb{E}[\psi(Y_1)] \le \mathbb{E}[\psi(Y_2)]$$

for all (integrable) convex functions ψ .

We have the following trivial consequences

$$Y_1 \prec_{SD(1)} Y_2 \quad \text{implies that} \quad Y_1 \prec_{SD(2)} Y_2$$

$$Y_1 \prec_{SD(2)} Y_2 \quad \text{holds iff} \quad Y_1 \prec_{SD(1)} Y_2 \text{ and } Y_1 \prec_{MD(1)} Y_2.$$

$Y_1 \prec_{SD(1)} Y_2$ is equivalent to $F_{Y_1}(u) \ge F_{Y_2}(u)$ for all u. $Y_1 \prec_{SD(2)} Y_2$ is equivalent to $\int_{-\infty}^x F_{Y_1}(u)\ du \ge \int_{-\infty}^x F_{Y_2}(u)\ du$ for all x. Let us prove the latter statement.

Since $\int_{-\infty}^x F(u)\ du = \int_{-\infty}^{\infty} [x-u]^+\ dF(u)$, one sees that $Y_1 \prec_{SD(2)} Y_2$ is equivalent to $\int_{-\infty}^{\infty} \psi(u)\ dF_{Y_1}(u) \le \int_{-\infty}^{\infty} \psi(u)\ dF_{Y_2}(u)$ for all functions of the form $\psi(u) = \sum_k \alpha_k [x_k - u]^+ + \beta_k$, with $\alpha_k \ge 0$. These functions are dense in the set of all convex, monotonic functions.

We are now ready to state the properties of CVaR_α.

Proposition 2. CVaR_α exhibits the following properties:

(i) CVaR_α is *translation-equivariant*, i.e.

$$\text{CVaR}_\alpha(Y + c) = \text{CVaR}_\alpha(Y) + c.$$

(ii) CVaR_α is *positively homogeneous*, i.e.

$$\text{CVaR}_\alpha(cY) = c\ \text{CVaR}_\alpha(Y),$$

if $c > 0$.

(iii) If Y has a density,

$$\mathbb{E}(Y) = (1-\alpha)\ \text{CVaR}_\alpha(Y) - \alpha\ \text{CVaR}_{(1-\alpha)}(-Y).$$

(iv) CVaR_α is *convex* in the following sense: For arbitrary (possibly dependent) random variables Y_1 and Y_2 and $0 < \lambda < 1$,

$$\text{CVaR}_\alpha(\lambda Y_1 + (1-\lambda) Y_2) \le \lambda\ \text{CVaR}_\alpha(Y_1) + (1-\lambda)\ \text{CVaR}_\alpha(Y_2).$$

(v) CVaR_α is monotonic w.r.t. SD(2) (and a fortiori w.r.t. SD(1)), i.e. if

$$Y_1 \prec_{SD(2)} Y_2$$

then

$$\text{CVaR}_\alpha(Y_1) \le \text{CVaR}_\alpha(Y_2).$$

(vi) CVaR_α is monotonic w.r.t. MD(2), i.e. if

$$Y_1 \prec_{MD(2)} Y_2$$

then

$$\text{CVaR}_\alpha(Y_1) \leq \text{CVaR}_\alpha(Y_2).$$

Proof. (i) and (ii) are obvious from the definition of $\text{CVaR}_\alpha(Y)$. Let us prove (iii). Since

$$\begin{aligned} \text{CVaR}_{(1-\alpha)}(-Y) &= \mathbb{E}(-Y \,|\, -Y > \text{VaR}_{(1-\alpha)}(-Y)) \\ &= \mathbb{E}(-Y \,|\, -Y > -\text{VaR}_\alpha(Y)) \\ &= -\mathbb{E}(Y | Y < \text{VaR}_\alpha(Y)) \end{aligned}$$

one sees that

$$\begin{aligned} \mathbb{E}(Y) &= \alpha\mathbb{E}(Y|Y < \text{VaR}_\alpha(Y)) + (1-\alpha)\mathbb{E}(Y|Y > \text{VaR}_\alpha(Y)) \\ &= -\alpha \, \text{CVaR}_{(1-\alpha)}(-Y) + (1-\alpha) \, \text{CVaR}_\alpha(Y). \end{aligned}$$

Now we prove (iv). Let a_i be such that $\text{CVaR}_\alpha(Y_i) = a_i + \frac{1}{1-\alpha}\mathbb{E}[Y_i - a_i]^+$. Since $y \mapsto [y-a]^+$ is convex, we have

$$\begin{aligned} & \text{CVaR}_\alpha(\lambda Y_1 + (1-\lambda)Y_2) \\ \leq\ & \lambda a_1 + (1-\lambda)a_2 + \frac{1}{1-\alpha}\mathbb{E}[\lambda Y_1 + (1-\lambda)Y_2 - \lambda a_1 + (1-\lambda)a_2)]^+ \\ \leq\ & \lambda a_1 + (1-\lambda)a_2 + \frac{\lambda}{1-\alpha}\mathbb{E}[Y_1 - a_1]^+ + \frac{1-\lambda}{1-\alpha}\mathbb{E}[Y_2 - a_2]^+ \\ \leq\ & \lambda \, \text{CVaR}_\alpha(Y_1) + (1-\lambda) \, \text{CVaR}_\alpha(Y_2). \end{aligned}$$

(v) and (vi) follow from the fact, that $y \mapsto [y-a]^+$ is monotone and convex. □

Artzner, Delbaen, Eber and Heath call a risk measure *coherent*, if it is translation-invariant, convex, positively homogeneous and monotonic w.r.t. $\prec_{SD(1)}$. One sees that CVaR_α is coherent in this sense.

In contrast, VaR_α is not coherent, since it is not convex. On the other hand, it is comonotone additive as is shown below.

Definition. Two random variables Y_1 and Y_2 defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ are said to be *comonotone*, if for all $\omega, \omega' \in \Omega$,

$$[Y_1(\omega) - Y_2(\omega)][Y_1(\omega') - Y_2(\omega')] \geq 0 \qquad a.s.$$

Equivalently, Y_1 and Y_2 are comonotone, if there is a representation $Y_1 = f(U); Y_2 = g(U)$, with f, g monotonically increasing and U uniform in [0,1] (see Shaun Wang).

Proposition 3. VaR_α exhibits the following properties:

(i) VaR_α is *translation-equivariant*, i.e.

$$\mathrm{VaR}_\alpha(Y+c) = \mathrm{VaR}_\alpha(Y) + c.$$

(ii) VaR_α is *positively homogeneous*, i.e.

$$\mathrm{VaR}_\alpha(cY) = c\ \mathrm{VaR}_\alpha(Y),$$

if $c > 0$.

(iii) $\mathrm{VaR}_\alpha(Y) = -\ \mathrm{VaR}_{(1-\alpha)}(-Y)$.

(iv) VaR_α is monotonic w.r.t. SD(1) i.e. if

$$Y_1 \prec_{SD(1)} Y_2$$

then

$$\mathrm{VaR}_\alpha(Y_1) \leq \mathrm{VaR}_\alpha(Y_2).$$

(v) VaR_α is *comonotone additive*, i.e. if Y_1 and Y_2 are comonotone, then

$$\mathrm{VaR}_\alpha(Y_1+Y_2) = \mathrm{VaR}_\alpha(Y_1) + \mathrm{VaR}_\alpha(Y_2).$$

Proof. (i) - (iv) are nearly obvious. Only (v) has to be proved in detail. If $Y_1 = f(U)$ with U uniform[0,1] and f monotonically increasing, then $\mathrm{VaR}_\alpha(Y_1) = f(\alpha)$. Similarly $\mathrm{VaR}_\alpha(Y_2) = g(\alpha)$ and therefore $\mathrm{VaR}_\alpha(Y_1+Y_2) = f(\alpha)+g(\alpha) = \mathrm{VaR}_\alpha(Y_1) + \mathrm{VaR}_\alpha(Y_2)$. □

2. RELATIONS BETWEEN VAR AND CVAR

In principle, VaR and CVaR measure different properties of the distribution. VaR is a quantile and CVaR is a conditional tail expectation. The two values coincide only if the tail is cut off.

Let $[Y]^c$ be the right censored cost variable $[Y]^c = \min(Y,c)$. If we set $c = \mathrm{VaR}_\alpha(Y)$, then $\mathrm{CVaR}_\alpha([Y]^c) = \mathrm{VaR}_\alpha(Y)$.

Proposition 4.

(i) $\mathrm{CVaR}_\alpha(Y) \geq \mathrm{VaR}_\alpha(Y)$

(ii) $\mathrm{VaR}_\alpha(Y) = \sup\{v : \mathrm{CVaR}_\alpha([Y]^v) = v\}$.

(iii) If Y is nonnegative, then as $n \to \infty$,

$$\left[\frac{\mathbb{E}(Y^n) - (1-\alpha)\ \mathrm{CVaR}_\alpha(Y^n)}{\alpha}\right]^{1/n} \to \mathrm{VaR}_\alpha(Y).$$

Proof. (i) is obvious. To prove (ii) notice that from the representation $\text{CVaR}(Y) = \frac{1}{1-\alpha}\int_\alpha^1 F^{-1}(u)\ du$ one sees that $\text{CVaR}([Y]^c) = \frac{1}{1-\alpha}\int_\alpha^1 \min(F^{-1}(u), c)\ du$, which implies (ii).

The property (iii) is based on the fact that for every nonnegative random variable Z, $[\mathbb{E}(Z^n)]^{1/n} \to \inf\{u : \mathbb{P}\{Z > u\} = 0\}$ as $n \to \infty$. From Proposition 2 (iii) we have

$$-\ \text{CVaR}_{(1-\alpha)}(-Y) = \frac{1}{\alpha}[\mathbb{E}(Y) - (1-\alpha)\ \text{CVaR}_\alpha(Y)].$$

On the other hand,

$$-\ \text{CVaR}_{(1-\alpha)}(-Y) = \mathbb{E}(Y|Y \leq F_Y^{-1}(\alpha)) = \sup\{a - \frac{1}{\alpha}\mathbb{E}[Y-a]^- : a \in \mathbb{R}\},$$

which may be proved in analogy to the proof of proposition 1. Since $[\mathbb{E}(Y^n|Y^n \leq F_{Y^n}^{-1}(\alpha))]^{1/n} = [\mathbb{E}(Y^n|Y \leq F_Y^{-1}(\alpha))]^{1/n} \to \text{VaR}_\alpha(Y)$, the result follows. □

3. VAR- AND CVAR-OPTIMAL PORTFOLIOS

Let $\xi = (\xi_1, \ldots, \xi_k)$ be a vector of random returns of asset categories $1, \ldots, k$. Let $x = (x_1, \ldots, x_k)$ be the investments in these categories respectively. Without loss of generality, we may assume that the total budget is 1. The return of the total portfolio is $-Y = x^T\xi$ and we aim at minimizing the VaR of Y under the constraint that the expected return exceeds some prespecified level μ.

We consider the following portfolio optimization problems: The VaR-optimization problem

$$\left\| \begin{array}{ll} \text{Minimize (in } x) & \text{VaR}_\alpha(-x^T\xi) \\ \text{subject to} & \\ x^T\mathbb{E}(\xi) \geq \mu & \\ x^T\mathbb{1} = 1 & \\ x \geq 0 & \end{array} \right. \tag{1.4}$$

and the CVaR-optimization problem

$$\left\| \begin{array}{ll} \text{Minimize (in } x) & \text{CVaR}_\alpha(-x^T\xi) \\ \text{subject to} & \\ x^T\mathbb{E}(\xi) \geq \mu & \\ x^T\mathbb{1} = 1 & \\ x \geq 0 & \end{array} \right. \tag{1.5}$$

In view of Propositions 2 (iii) and 3 (iii) one could equivalently maximize $\text{VaR}_{(1-\alpha)}(x^T\xi)$ and $\text{CVaR}_{(1-\alpha)}(x^T\xi)$.

Problem (1.4) is nonconvex in general and may have several local minima. In contrast, (1.5) can be written as the following (infinite dimensional) linear program:

$$\left\| \begin{array}{ll} \text{Minimize (in } x \text{ and } a) & a + \frac{1}{1-\alpha}\mathbb{E}[Z] \\ \text{subject to} & \\ Z \geq -x^T\xi - a & \text{with probability 1} \\ x^T\mathbb{E}(\xi) \geq \mu & \\ x^T\mathbb{1} = 1 & \\ Z \geq 0 & \\ x \geq 0 & \end{array} \right. \tag{1.6}$$

From the structure of this program it is clear that if there is a solution at all, this solution is either a singleton or a convex polyhedron. Every local optimum is global. This is the big advantage of the CVaR risk measure over the VaR risk measure.

For practical portfolio optimization, Y is a discrete variable which takes the values $-x^T\xi^i$ with equal probability. The vectors $\xi^i, i = 1, \ldots N$ are called the *scenarios*.

Introduce the function $M_{[k:N]}(u^1, \ldots, u^N)$ to denote the k-th largest among $u^1, \ldots, u^N$. Thus $M_{[1:N]}$ denotes the minimum and $M_{[N:N]}$ the maximum. Calculating CVaR and VaR for the discrete distribution, we get

$$\text{VaR}_\alpha(-x^T\xi) = M_{[\lfloor \alpha N \rfloor : N]}(-x^T\xi^1, \ldots, -x^T\xi^N),$$
$$\text{CVaR}_\alpha(-x^T\xi) = \frac{1}{N} \sum_{\{-x^T\xi^i \geq \text{ VaR}_\alpha\}} -x^T\xi^i.$$

The discrete portfolio optimization problem (1.4) is a nonlinear, nonconvex program:

$$\left\| \begin{array}{ll} \text{Minimize (in } x) & M_{[\lfloor \alpha N \rfloor : N]}(-x^T\xi^1, \ldots, -x^T\xi^N) \\ \text{subject to} & \\ x^T e \geq \mu & \\ x^T\mathbb{1} = 1 & \\ x \geq 0 & \end{array} \right. \tag{1.7}$$

Here $e = \frac{1}{N}\sum_{i=1}^N \xi^i$ denotes the expected return vector. The discrete version of (1.5) is linear and may be solved using any LP-solver (see Uryasev and Rockafellar (1999)):

$$
\left\| \begin{array}{ll}
\text{Minimize (in } x, a \text{ and } z) & a + \frac{1}{(1-\alpha)N} \sum_{i=1}^{N} z^i \\
\text{subject to} & \\
z^i \geq -x^T \xi^i - a & i = 1, \dots, N \\
x^T e \geq \mu & \\
x^T \mathbb{1} = 1 & \\
z^i \geq 0 & i = 1, \dots, N \\
x \geq 0 &
\end{array} \right. \qquad (1.8)
$$

Notice that the optimal a in (1.8) is $\text{VaR}_\alpha(-x^T\xi) = -\text{VaR}_{(1-\alpha)}(x^T\xi)$. For the reader, who does not like to formulate a portfolio optimization problem as a minimization program for negative returns, we can reformulate it as a maximization program:

$$
\left\| \begin{array}{ll}
\text{Maximize (in } x, b \text{ and } z) & b - \frac{1}{(1-\alpha)N} \sum_{i=1}^{N} z^i \\
\text{subject to} & \\
z^i \geq b - x^T \xi^i & i = 1, \dots, N \\
x^T e \geq \mu & \\
x^T \mathbb{1} = 1 & \\
z^i \geq 0 & i = 1, \dots, N \\
x \geq 0 &
\end{array} \right. \qquad (1.9)
$$

The optimal b in in (1.9) is $\text{VaR}_{(1-\alpha)}(x^T\xi)$.

3.1 A FIXPOINT FORMULATION OF THE VAR OPTIMIZATION PROBLEM

Since the CVaR optimization is much simpler in structure, it is desirable to solve the VaR optimization by a sequence of CVaR optimization problems. The idea is to represent the solution of the VaR problem as a fixpoint of CVaR problems using the results of section 2.

Consider the following program, which is parametrized by the cut-off point c and an index set I.

$$
P(c,I)\left\|\begin{array}{ll}
\text{Minimize (in } x, a \text{ and } z) & \{a + \frac{1}{(1-\alpha)N}\left[\sum_{i\notin I} z_i + \sum_{i\in I}(c-a)\right]\} \\
\text{subject to} & \\
z_i \geq -x^T\xi - a & i \notin I \\
-x^T\xi^i \geq c & i \in I \\
c \geq a & \\
x^T e \geq \mu & \\
x^T \mathbb{1} = 1 & \\
z^i \geq 0 & i \notin I \\
x \geq 0 &
\end{array}\right. \tag{1.10}
$$

For a vector x and a value a let $I(x,a) \subseteq \{1,\ldots,N\}$ be the following index set:

$$I(x,a) = \{i : -x^T\xi_i > a\}.$$

Proposition 5. Suppose that x^* is the minimizer and a^* is the minimal value of the VaR optimization problem (1.7). Then x^* and a^* are the solutions of the linear program $P(a^*, I(x^*, a^*))$. Conversely, every fixpoint i.e. a vector x and a value a such that x and a are the solutions of $P(a, I(x,a))$ is a local minimizer of (1.7).

Proof. In a neighborhood of (x^*, a^*) we have that

$$[-x^T\xi^i]^a = \begin{cases} a & \text{if} \quad i \in I(x^*, a^*) \\ -x^T\xi^i & \text{if} \quad i \notin I(x^*, a^*) \end{cases}$$

and therefore VaR$(-x^T\xi)$ = CVaR$([-x^T\xi]^a)$ there. Since one cannot find a better portfolio w.r.t. VaR within this neighborhood, the local and hence global solution of $P(a, I(x,a))$ must coincide with the solution of the VaR optimization problem. Conversely, let $\hat{x}$, $\hat{a}$ be the solution of $P(\hat{a}, I(\hat{x}, \hat{a}))$. As before, there is a neighborhood of $(\hat{x}, \hat{a})$, in which VaR$(-x^T\xi)$ and CVaR$([-x^T\xi]^a)$ coincide, hence $\hat{x}$ is at least a local solution of the VaR minimization problem. □

We are ready to state the fixpoint property of the local minimizers:

The fixpoint property: x^* is a local minimizer of (1.7), if and only if there is a value a^* and an index set I^*, such that x^*, a^* and I^* are fixpoints in the following sense: The solution of $P(a^*, I^*)$ is x^* and a^* and, in addition, $I(x^*, a^*) = I^*$. Thus, the VaR optimization problem can be reformulated as a fixpoint problem of solutions of linear optimization problems. This leads immediately to a solution strategy.

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