# Local and Covariant Deformations of Wedge Algebras in Quantum Field Theory

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- Motivation
- Twisted convolution
- Deformation of wedge algebras
- Conclusions

# I. Motivation

Work of Grosse and Lechner:

Input: free QFT on non–commutative Minkowski space

$\Longrightarrow$ "interacting" theory on Minkowski space $\mathbb{R}^d$

$\Longrightarrow$ novel approach to deformation of QFT's

$\Longrightarrow$ novel approach to "rigorous model building"

GL formula:

$Q$ antisymmetric $d \times d$–matrix

$a^*(p)$ creation operator of particle with momentum $p$

$\Longrightarrow$ deformed creation operators

$$a^*(Q, p) \doteq e^{iPQp} \, a^*(p)$$

Observation: Correspondence between wedge regions $W \subset \mathbb{R}^d$ and specific matrices $Q$ such that deformed field operators are wedge local and covariant; two–particle $S$–matrix non–trivial.

BS question:

Is it meaningful to apply this deformation procedure to arbitrary theories and what are the properties of the deformed theories?

Reinterpretation of GL formula:

$$
\begin{aligned}
&(2\pi)^{-d/2} \int dp \, e^{iPQp} \, a^*(p) e^{-ipx} \\
&= (2\pi)^{-d/2} \int dp \left( \int dE(q) e^{iqQp} \right) a^*(p) e^{-ipx} \\
&= \int dE(q) \, \widetilde{a^*}(x + Qq)
\end{aligned}
$$

Resembles "Q–twisted convolution". Latter expression better accessible to analysis.

## II. Twisted convolution

$$U(x) = \int e^{ipx}\, dE(p)$$

$$\alpha_x(\cdot) \doteq U(x) \cdot U(x)^{-1}$$

Task: Definition of right/left integrals

$$\int dE(p)\alpha_{Qp}(A)\,, \quad \int \alpha_{Qp}(A) dE(p)$$

where $Q$ real $d \times d$–matrix.

Not definable on all of $\mathcal{B}(\mathcal{H})$! Restriction to

$$\mathcal{C}(\mathcal{H}) \doteq \{A \in \mathcal{B}(\mathcal{H}) : x \mapsto \alpha_x(A) \text{ smooth}\}\,.$$

Then, $F$ finite dimensional projections,

$$\int \alpha_{Qp}(A) dE(p) \doteq \lim_{F \nearrow 1} \int \alpha_{Qp}(A) F dE(p)$$

$$\int dE(p)\alpha_{Qp}(A) \doteq \lim_{f \to 1} \lim_{F \nearrow 1} \int dE(p) F f(p) \alpha_{Qp}(A)\,.$$

Strong limits exist on

$$\mathcal{D} \doteq \{\Psi \in \mathcal{H} : x \mapsto U(x)\Psi \text{ smooth .}\}$$

Remark: $\mathcal{D}$ stable under action of left/right integrals.

(i) Adjoints

Relation between left/right integrals?

Proofs require proper treatment of expressions such as $dE(p)AdE(q)$ (no product measure!). Formal computation using $dE(p)f(P) = dE(p)f(p)$:

$$\begin{aligned}
&\int \alpha_{Qp}(A)dE(p)^* \supset \int dE(p)\alpha_{Qp}(A^*) \\
&= \int dE(p)U(Qp)A^*U(-Qp)\int dE(q) \\
&= \iint dE(p)e^{ipQp}A^*dE(q)e^{-iqQp} \\
&= e^{iPQP}\iint dE(p)\, e^{-iqQp}A^*e^{iqQq}dE(q)\, e^{-iPQP} \\
&= e^{iPQP}\int dE(p)\int U(-Q^Tq)A^*U(Q^Tq)dE(q)\, e^{-iPQP}.
\end{aligned}$$

Proposition:

$$\int \alpha_{Qp}(A)dE(p)^* \supset e^{iPQP}\int \alpha_{-Q^Tp}(A^*)dE(p)\, e^{-iPQP}$$

Note: If $Q^T = -Q$ one has

$$\int dE(p)\alpha_{Qp}(A) = \int \alpha_{Qp}(A)dE(p)$$

Left and right integrals coincide!

(ii) Commutators

Proposition: Let $Q^T = -Q$ and $A, B \in \mathcal{C}(\mathcal{H})$ such that $[\alpha_{Qp}(A), \alpha_{-Qq}(B)] = 0$ for $p, q \in$ supp$E(\cdot)$. Then

$$\Big[ \int \alpha_{Qp}(A) dE(p), \int \alpha_{-Qq}(B) dE(q) \Big] = 0\,.$$

Sketch of proof:

$$\begin{aligned}
&\int \alpha_{Qp}(A) dE(p) \int \alpha_{-Qq}(B) dE(q) \\
&= \int dE(p) \alpha_{Qp}(A) \int \alpha_{-Qq}(B) dE(q) \\
&= \iint dE(p) \alpha_{-Qq}(B) \alpha_{Qp}(A) dE(q) \\
&= \iint dE(p) e^{-ipQq} BU(Qq + Qp) A e^{-iqQp} dE(q) \\
&= \iint dE(p) U(-Qp) BU(Qp + Qq) AU(-Qq) dE(q) \\
&= \iint dE(p) \alpha_{-Qp}(B) \alpha_{Qq}(A) dE(q) \\
&= \int \alpha_{-Qp}(B) dE(p) \int \alpha_{Qq}(A) dE(q) \qquad \text{QED}
\end{aligned}$$

## (iii) Actions

Isometry group $\mathcal{P}_+^\uparrow$ of $\mathbb{R}^d$: $x \mapsto \lambda x \doteq \Lambda x + a$
Continuous unitary representation $U$:

$$U(\lambda)U(x) = U(\lambda x)U(\lambda)$$
$$U(\lambda)dE(p) = dE(\Lambda p)U(\lambda)$$
$$\alpha_\lambda(\cdot) \doteq U(\lambda) \cdot U(\lambda)^{-1}\,.$$

Proposition: Let $Q^T = -Q$ and $A \in \mathcal{C}(\mathcal{H})$. Then

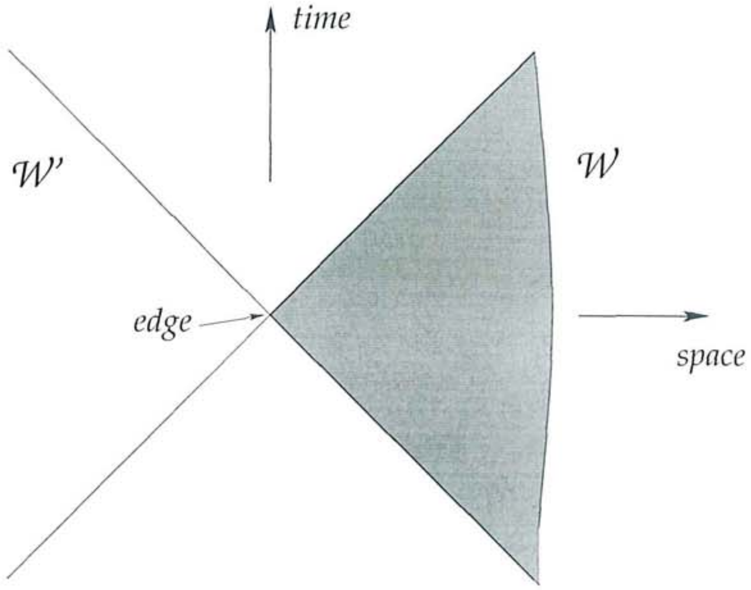
$$U(\lambda)\Big(\int \alpha_{Qp}(A)dE(p)\Big)U(\lambda)^{-1}$$
$$= \int \alpha_{\Lambda Q \Lambda^{-1}p}(\alpha_\lambda(A))dE(p)\,.$$

Twisted convolution has intriguing properties!

# III. Deformation of wedge algebras

(i) Input:

- Net: $\mathcal{W} \mapsto \mathcal{A}(\mathcal{W})$, where $\mathcal{W} \subset \mathbb{R}^d$ are wedges




- Locality: $[\mathcal{A}(\mathcal{W}_1), \mathcal{A}(\mathcal{W}_2)] = 0$ if $\mathcal{W}_1 \subset \mathcal{W}_2{}'$.
- Covariance: $\alpha_\lambda(\mathcal{A}(\mathcal{W})) \subset \mathcal{A}(\lambda\mathcal{W})$ for $\lambda \in \mathcal{P}_+^\uparrow$.
- Stability: $\operatorname{Sp} U(\mathbb{R}^d) \subset \overline{V}_+$

Goal: Construction of new nets with these properties.

(ii) Deformation:

Standard wedge $\mathcal{W}_R = \{x \in \mathbb{R}^d : x_1 > |x_o|\}$;
associated $d \times d$–matrix

$$Q = \begin{pmatrix} 0 & -\kappa & \\ \kappa & 0 & \\ & & 0 \end{pmatrix}, \qquad \kappa > 0\,.$$

- Deformed net:

$$\widehat{\mathcal{A}}(\mathcal{W}_R) \doteq \{\widehat{A} : A \in \mathcal{A}(\mathcal{W}_R) \cap \mathcal{C}(\mathcal{H})\}$$

$$\widehat{A} \doteq \int \alpha_{Qp}(A) dE(p)\,.$$

Arbitrary wedges $\mathcal{W} = \lambda \mathcal{W}_R$

$$\mathcal{W} \to \widehat{\mathcal{A}}(\mathcal{W}) \doteq \alpha_\lambda(\widehat{\mathcal{A}}(\mathcal{W}_R))\,.$$

Definition consistent. Deformed net still covariant.

- Locality:

Note that $Q(p_0, p_1, \dots p_{d-1}) = (-\kappa p_1, \kappa p_0, 0, \dots 0)$. Thus if $A \in \mathcal{A}(\mathcal{W}_R)$ then $\alpha_{Qp}(A) \in \mathcal{A}(\mathcal{W}_R)$ for $p \in \overline{V}_+$; similarly, if $B \in \mathcal{A}(\mathcal{W}_R{}')$ then $\alpha_{-Qq}(B) \in \mathcal{A}(\mathcal{W}_R{}')$ for $q \in \overline{V}_+$. Locality and stability of the original net and the commutation properties of twisted convolutions imply: $[\widehat{A}, \widehat{B}] = 0$.

So the deformed net $\mathcal{W} \mapsto \widehat{\mathcal{A}}(\mathcal{W})$ is still local.

(iii) Scattering:

If original theory describes a massive boson, then two particle scattering states can be constructed in deformed theory (frame–dependent). For $\mathcal{W}_R$ fixed:

$$\widehat{|p,q\rangle_{in}} = e^{-ipQq}\,|p,q\rangle_{in} \quad p_1 > q_1$$
$$\widehat{|p,q\rangle_{out}} = e^{ipQq}\,|p,q\rangle_{out} \quad p_1 > q_1\,.$$

Scattering kernel: $p_1 > q_1, p_1' > q_1'$

$$_{out}\widehat{\langle p,q}|\widehat{p',q'\rangle}_{in} = e^{-ipQq-ip'Qq'}\,{}_{out}\langle p,q|p',q'\rangle_{in}\,.$$

Remarks:

• Deformed S–matrix not Lorentz invariant in spite of $\mathcal{W}$–locality and covariance of deformed net. Resembles "long range forces".

• collision cross sections do not change under deformations. Thus observation of breakdown of Lorentz invariance (e.g. due to non–commutative background) only possible in special types of experiments (e.g. time delay).

## IV Conclusions

- "Twisted convolution" provides a novel deformation procedure for arbitrary wedge–local, stable and covariant theories
- Deformed theories for different $Q$ are different; procedure can be extended
- Theories describe long range effects and/or non–commutative backgrounds; weak localization properties of field operators are quite natural in this context
- New approach to the (algebraic) construction of interacting models