2017 ESI Summer School
Between Geometry and Relativity
Problems from Einstein Constraint Equations Mini-course

Basic facts and conventions

- We use the Einstein summation convention: sum over a pair of upper and lower repeated indices.
- A comma in a subscript denotes partial differentiation, whereas a semicolon in a subscript denotes covariant differentiation. For example, if h is a $(1,2)$-tensor with components in a coordinate chart $h^i{}_{jk}$, then the covariant derivative ∇h is a $(1,3)$-tensor with components

$$h^i{}_{jk;\ell} := \nabla h\Big(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\Big) = (\nabla_{\frac{\partial}{\partial x^\ell}} h)\Big(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big) = h^i_{jk,\ell} + \Gamma^i_{m\ell} h^m_{jk} - \Gamma^m_{j\ell} h^i_{mk} - \Gamma^m_{k\ell} h^i{}_{jm}.$$

- The Christoffel symbols for the connection compatible with a metric g (Riemannian or pseudo-Riemannian) are given by the rule (where $g^{i\ell} g_{\ell j} = \delta^i{}_j$)

$$\Gamma^k_{ij} = \frac{1}{2} g^{km}(g_{mj,i} + g_{im,j} - g_{ij,m}).$$

- In normal coordinates (x^i) about a point $p \in M$, we have $g_{ij,k}(p) = 0$ for all i, j and k, so that at p, $\Gamma^k_{ij}(p) = 0$ and $g^{ij}_{\ ,k}(p) = (-g^{i\ell} g_{\ell m,k} g^{mj})(p) = 0$.
- Our convention for the Riemann curvature tensor is as follows:

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$R\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^k}\right) = R^\ell_{ijk} \tfrac{\partial}{\partial x^\ell}$$
$$R_{ijk\ell} = g_{m\ell} R^m_{ijk}.$$

- In our convention the Ricci tensor is then given by

$$\mathrm{Ric}(X,Y) = dx^i\Big(R\big(\frac{\partial}{\partial x^i}, X, Y\big)\Big) = g^{k\ell} g\left(R\big(\frac{\partial}{\partial x^k}, X, Y\big), \frac{\partial}{\partial x^\ell}\right)$$
$$R_{ij} = \mathrm{Ric}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = R^\ell_{\ell ij}.$$

- The scalar curvature is $R(g) = g^{ij} R_{ij}$. The Einstein tensor is $G = G(g) = \mathrm{Ric}(g) - \frac{1}{2} R(g) g$.
- Let $g_{\mathbb{E}}$ be the Euclidean metric on $\mathbb{R}^n$, with Cartesian coordinates $x = (x^1, \ldots, x^n)$ so that $g_{\mathbb{E}} = \delta_{ij} dx^i dx^j$, and let $|x| = \sqrt{\sum_{i=1}^n (x^i)^2}$. If $r = |x|$, $g_{\mathbb{E}} = dr^2 + r^2\, \mathring{g}_{\mathbb{S}^{n-1}}$, where $\mathring{g}_{\mathbb{S}^{n-1}}$ is the standard round unit metric on the sphere $\mathbb{S}^{n-1}$. We recall the notation $\mathring{g}_{\mathbb{S}^2} = d\Omega^2 = d\phi^2 + \sin^2(\phi)\, d\theta^2$. Note that the angle convention is the mathematics convention, as opposed to the physics convention,

Problems. Except for the first problem, take $\Lambda = 0$.

1. a. Suppose M is a hypersurface in a manifold $(\bar{M}, \bar{g})$, which induced metric g, and second fundamental form K, given by $K(X,Y) = \langle \overline{\nabla}_X Y, n\rangle$, where n is a local smooth unit normal field to M. Prove the Codazzi equation, where i, j, k represent tangential directions: $\bar{R}_{ijkn} = K_{jk;i} - K_{ik;j}$.

b. Suppose $(\bar{M}, \bar{g})$ is Lorentzian, with local coordinates (x^α), $\alpha \geq 0$, where $\frac{\partial}{\partial x^0}$ is timelike. Let $\lambda^\alpha = \Box_{\bar{g}} x^\alpha$. Recall the reduced Ricci R^H defined by: $R^H_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}(\bar{g}_{\alpha\mu} \lambda^\alpha_{\ ,\nu} + \bar{g}_{\alpha\nu} \lambda^\alpha_{\ ,\mu})$. Assuming

$R^H_{\mu\nu} = \Lambda\bar{g}_{\mu\nu}$, prove $(G_\Lambda(\bar{g}))_{\mu\nu} = -\frac{1}{2}\bar{g}_{\alpha\mu}\lambda^\alpha{}_{,\nu} - \frac{1}{2}\bar{g}_{\alpha\nu}\lambda^\alpha{}_{,\mu} + \frac{1}{2}\bar{g}_{\mu\nu}\lambda^\alpha{}_{,\alpha}$.

c. Suppose we have arranged the initial data for $\bar{g}$ so that $\bar{g}_{00} = -1$, $\bar{g}_{i0} = 0$ for $i \geq 1$, $\bar{g}_{ij} = g_{ij}$ for $i, j \geq 1$ (where g is a Riemannian metric on M), and so that $\lambda^\alpha = 0$ along $x^0 = 0$ (i.e. $t = 0$). Show that for all α, $\lambda^\alpha{}_{,0} = 0$ along $x^0 = 0$ as well.

2. Linearization of the scalar curvature map. Let $R(g) = g^{ij}R_{ij}$ be the scalar curvature of a metric (not necessarily Riemannian). Consider a variation $g(\epsilon) = g + \epsilon h$ of g in the direction of a symmetric $(0, 2)$-tensor field h (more generally, note that all you will use is that $g(\epsilon)$ is a metric smooth in t, with $g(0) = g$ and $g'(0) = h$). Assume that for small $|\epsilon|$, $g(\epsilon)$ is a metric, as would be the case for h compactly supported. Define $L_g(h) := DR_g(h) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} R(g(\epsilon))$.

a. Derive the scalar curvature formula

$$R(g) = g^{ij}R_{ij} = g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{k\ell}\Gamma^\ell_{ij} - \Gamma^k_{j\ell}\Gamma^\ell_{ik}\right).$$

b. Verify that the difference $S(X, Y) := \nabla_X Y - \widetilde{\nabla}_X Y$ defines a vector-valued $(0, 2)$-tensor (i.e. a $(1, 2)$ tensor $\widehat{S}(\theta, X, Y) = \theta(S(X, Y))$). Thus $\dot{\Gamma}^k_{ij} := \frac{d}{d\epsilon}\Big|_{\epsilon=0} \Gamma^k_{ij}$ form the components $(\delta\Gamma)^k_{ij}$ of a $(1, 2)$-tensor $(\delta\Gamma)$. Argue that $\dot{\Gamma}^k_{ij} = \frac{1}{2}g^{km}(h_{mj;i} + h_{im;j} - h_{ij;m})$, where the covariant derivative is taken with respect to $g(0)$. (Hint: use $g(0)$-normal coordinates at p.)

c. Use the preceding part to aid in verifying the identities $\frac{d}{d\epsilon}\Big|_{\epsilon=0} R_{ij} = (\delta\Gamma)^k_{ij;k} - (\delta\Gamma)^k_{ik;j}$, and then

$$L_g(h) = -\Delta_g(\mathrm{tr}_g(h)) + \mathrm{div}_g(\mathrm{div}_g(h)) - \langle h, \mathrm{Ric}(g)\rangle_g$$

where the inner product of two $(0, 2)$-tensors S and T is given by $\langle S, T\rangle = S_{ij}T_{k\ell}g^{ik}g^{j\ell}$, for example $\mathrm{tr}_g(S) = \langle g, S\rangle$.

d. Show that $L^*_g N = -(\Delta_g N)g + \mathrm{Hess}_g N - N\mathrm{Ric}(g)$, by integrating $\int_M N L_g(h)\, dv_g$ by parts (for h compactly supported away from the boundary of M).

e. Suppose (M, g, K) solves the vacuum constraint equations. Let $\pi^{ij} = K^{ij} - (\mathrm{tr}_g K)g^{ij}$. Let $\Phi(g, \pi) = (R(g) - \|\pi\|^2_g + \frac{1}{2}(\mathrm{tr}_g(\pi))^2, \mathrm{div}_g(\pi))$. Show that $\Phi(g, \pi) = 0$.

f. Suppose $(M, g, \pi = 0)$ solves $\Phi(g, \pi) = 0$. Find $D\Phi_{(g,0)}(h, \sigma)$, where h is a symmetric $(0, 2)$ tensor and σ is a symmetric $(2, 0)$ tensor, and then find $D\Phi^*_{(g,0)}(N, X)$ by integrating $\int_M \langle (N, X), D\Phi_{(g,0)}(h, \sigma)\rangle\, dv_g$ by parts, for (h, σ) of compact support (vanishing near the boundary if ∂M is nonempty). Since $\pi = 0$, the linearization simplifies dramatically from the general case. If you want to try your hand at the general linearization, find a Starbucks and see if they serve a coffee larger than a Venti (sometimes they have a Trenta!).

3. a. Show directly (and in one line) that if h is symmetric with compact support, and if $L_{g_{\mathbb{E}}} h \geq 0$, then $L_{g_{\mathbb{E}}} h = 0$.

b. Show by elementary methods that there exists an infinite-dimensional space of TT tensors on $(\mathbb{R}^3, g_{\mathbb{E}})$ *with compact support*. Such tensors automatically satisfy $L_{g_{\mathbb{E}}} h = 0$.

c. (Open problem): If you can construct a non-trivial symmetric TT tensor h on $\mathbb{R}^3$ with compact support *and* so that $|h|^2_{g_{\mathbb{E}}}$ depends only on $|x|$, please let me know—a nice paper would come out of it.

4. Suppose (M, g) is Riemannian.

a. Suppose that $L_g^* N = 0$, and that γ is a unit-speed geodesic in (M^n, g). Let $h(t) = N(\gamma(t))$. Prove that $h(t)$ satisfies a second-order linear ODE. What does this say about the dimension of the kernel of L_g^*?

b. Suppose that $L_g^* N = 0$, but that N is not identically zero. Show that $\Sigma = N^{-1}(0)$ is a regular hypersurface, which is totally geodesic (zero second fundamental form). Hint: If $p \in \Sigma$ and $dN_p = 0$, what does part a. have to say about things?

c. Suppose that (M^n, g) is a closed manifold with negative scalar curvature. Find the kernel of L_g^*.

d. Consider the metric $g = (n-2)^{-1} g_{\mathbb{S}^1} \oplus g_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. Show that $N(t, \omega) = \sin t$ solves $L_g^* N = 0$.

e. Does every Ricci-flat metric have a nontrivial element N in the kernel of L_g^*? What can you say in case a metric (M, g) on a closed manifold with zero scalar curvature admits a nontrivial N with $L_g^* N = 0$?

f. Let $N : M \to \mathbb{R}$ be a smooth function. Define the Lorentzian metric $\bar{g} = -N^2 dt^2 \oplus g$ on the space $\mathcal{S} = I \times \{p \in M : N(p) \neq 0\}$. Prove that for X, Y tangent to M at p with $N(p) \neq 0$, we have $\mathrm{Ric}(\bar{g})(X, Y) = \mathrm{Ric}(g)(X, Y) - \frac{1}{N(p)} \mathrm{Hess}_g N(p)$, $\mathrm{Ric}(\bar{g})(X, \frac{\partial}{\partial t}) = 0$, and $\mathrm{Ric}(\bar{g})(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = N \Delta_g N$.

g. Conclude from part a. that a function N on M is a nontrivial element of the kernel of L_g^* if and only if the metric $\bar{g}$ as above is an Einstein metric. (Note that in the preceding problem you said something about the set $\{p \in M : N(p) = 0\}$ where the metric $\bar{g}$ may have issues.)

5. Conformal changes of metric.

a. Suppose (M^n, g) is a Riemannian metric and $\tilde{g} = e^{\varphi} g$. Show that

$$R(\tilde{g}) = e^{-\varphi} \left(R(g) - (n-1)\Delta_g \varphi - \frac{1}{4}(n-1)(n-2)|\nabla \varphi|_g^2 \right).$$

b. In case $n \geq 3$, if we write $e^{\varphi} = u^{\frac{4}{n-2}}$ for $u > 0$, then

$$R(\tilde{g}) = u^{-\frac{n+2}{n-2}} \left(R(g) u - \frac{4(n-1)}{(n-2)} \Delta_g u \right).$$

c. Suppose M is compact with empty boundary. Let $c(n) = \frac{n-2}{4(n-1)}$. Let $L_g u = \Delta_g u - c(n) R(g) u$, the *conformal Laplacian*. Show that the total scalar curvature of $\tilde{g} = u^{\frac{4}{n-2}} g$ is given by

$$\int_M R(\tilde{g})\, dv_{\tilde{g}} = c(n)^{-1} \int_M \left(|\nabla u|_g^2 + c(n) R(g) u^2 \right) dv_g.$$

Hint: Show that $dv_{\tilde{g}} = u^{\frac{2n}{n-2}}\, dv_g$.

6. Recall the conformal Killing operator L is related to the Lie derivative $\mathcal{L}$ by the relationship $L_g W = \mathcal{L}_W g - \frac{2}{n}\mathrm{div}_g W g$, so that $(L_g W)_{ab} = W_{a;b} + W_{b;a} - \frac{2}{n}(W^c_{\ ;c}) g_{ab}$.

a. Suppose (M, g_0) is three-dimensional. If $\phi > 0$ on M and $g = \phi^4 g_0$, show that for any trace-free symmetric $(2,0)$-tensor Ξ^{ab},

$$(\mathrm{div}_g(\phi^{-10}\Xi))^a = \phi^{-10}(\mathrm{div}_{g_0}\Xi)^a$$

and

$$(L_g W)^{ab} = \phi^{-4}(L_{g_0} W)^{ab}.$$

b. Can you figure out what the analogous statements would be in higher dimensions?

c. Define the operator $(\widetilde{\mathcal{L}}_g(X))_{ij} = X_{i;j} + X_{j;i} - X^k_{\ ;k} g_{ij}$.

If γ is a metric on M^3, let $g = u^4\gamma$ and $\pi_{ij} = u^2(\widetilde{\mathcal{L}}_\gamma(X))_{ij}$ for $u > 0$. Compute the constraints map $\Phi(g, \pi) = (R(g) - |\pi|_g^2 + (\mathrm{tr}_g\pi)^2, \mathrm{div}_g\pi)$, and show that in case $\gamma = g_{\mathbb{E}}$, show that the vacuum constraints can be written, in a Cartesian coordinate system for the background $g_{\mathbb{E}}$, as follows (subscripts for the flat metric omitted):

$$8\Delta u = u\big(- |\widetilde{\mathcal{L}}X|^2 + \tfrac{1}{2}(\mathrm{tr}(\widetilde{\mathcal{L}}X))^2\big)$$
$$\Delta X^i + 4u^{-1}u_{,j}(\widetilde{\mathcal{L}}X)_i^j - 2u^{-1}u_{,i}\mathrm{tr}(\widetilde{\mathcal{L}}X) = 0$$

Note that the system above has principle part the Laplacian in components, which explains the form of $\widetilde{\mathcal{L}}_g(X)$ (as opposed to, say, the conformal Killing operator).

7. Schwarzschild Geometry Basics. Recall the spatial Schwarzschild metric $g_S = \left(1 + \frac{m}{2|x|}\right)^4 g_{\mathbb{E}}$, defined on the manifold M given by $M = \mathbb{R}^3 \setminus \{0\}$ for $m > 0$, $M = \mathbb{R}^3$ for $m = 0$, and $M = \{x \in \mathbb{R}^3 : |x| > -\frac{m}{2}\}$ for $m < 0$.

a. We saw that $R(g_S) = 0$. Find $\mathrm{Ric}(g_S)$, which doesn't vanish.

b. Show that

$$m = \frac{1}{16\pi}\lim_{r\to+\infty}\int_{|x|=r}\sum_{i,j=1}^{3}\left((g_S)_{ij,i} - (g_S)_{ii,j}\right)\nu_e^j\, d\sigma_e$$

where the computation is done in the coordinates (x^1, x^2, x^3), and where ν_e is the Euclidean outward unit normal, and $d\sigma_e$ is the Euclidean area measure (where (x^i) are Cartesian coordinates for the Euclidean metric).

c. When $m < 0$, $A(r) \to 0$ as $r \to -(\frac{m}{2})^+$. Show that a radial geodesic from $r = r_0 > -\frac{m}{2}$ to $r = -\frac{m}{2}$ has finite length. Can the Schwarzschild metric with $m < 0$ be smoothly completed by adding in a point?

d. Let g_S be a Schwarzschild metric of non-zero mass m. Show that there is a one-dimensional kernel for $L^*_{g_S}$. Do this by showing first that for any function in the kernel, $\mathrm{Hess}_{g_S}(f) = f\mathrm{Ric}(g_S)$. Write this out in coordinates for which $g_S = (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\varphi^2 + \sin^2\varphi\, d\theta^2)$. Show that $\partial_\theta f = 0$ and $\partial_\varphi f = 0$, and then solve the remaining ODE for f.

e. Let $m > 0$. Find an isometric embedding of (M, g_S) into Euclidean space $(\mathbb{R}^4, g_{\mathbb{E}})$, identified in Cartesian coordinates (x, y, z, w) with $(\mathbb{R}^4, dx^2 + dy^2 + dz^2 + dw^2)$. It might be easiest to use the other coordinates we introduced for the Schwarzschild metric: $(1-\frac{2m}{r})^{-1}dr^2 + r^2\, g_{\mathbb{S}^2}$, $r > 2m$. (This corresponds to "half" of (M, g_S). The map you get will then extend by reflection to the other "half.") For $\omega \in \mathbb{S}^2$, look for an embedding of the form $x = r\omega \mapsto (r\omega, \xi(r)) \in \mathbb{R}^4$. Explain how this justifies the picture we've drawn of the Schwarzschild spatial slice.

f. When $m < 0$ the argument breaks in part e. down. Instead, look for an isometric embedding into Minkowski space $\mathbb{M}^4$, which is identified with $\mathbb{R}^4$ with the metric $dx^2 + dy^2 + dx^2 - dw^2$.

8. Schwarzschild Geometry Basics, continued. Let ∇ be the connection on (M, g_S), and for vector fields X and Y tangent to a surface $\Sigma \subset M$, let $\mathbb{II}(X,Y) = (\nabla_X Y)^{\mathrm{Nor}}$, and let $\mathbf{H} = \mathrm{tr}_\Sigma(\mathbb{II})$.

a. For $m > 0$, show that $r \mapsto \frac{m^2}{4r}$ induces an isometry of g_S which fixes $\Sigma_0 = \{r = \frac{m}{2}\}$.

b. For $m > 0$, show that Σ_0 is totally geodesic in M. Express m in terms of the area of Σ_0.

c. Find the area $A(r)$ of $S_r = \{x : |x| = r\}$ of S_r in the metric g_S. For $m > 0$, show that $A(r)$ has a global minimum at $r = \frac{m}{2}$.

d. Fix r and find the second fundamental form and the mean curvature vector $\mathbf{H}$ of $S_r = \{x : |x| = r\}$ in the metric g_S.

e. Compare $A'(r)$ to $\int_{S_r} \mathbf{H} \cdot \mathbf{X}\, d\sigma$, where $\mathbf{X} = \frac{\partial}{\partial r}$, and $d\sigma_S$ is the area measure induced by g_S.

f. If ν_S is a unit normal to a surface with mean curvature vector $\mathbf{H}$, let $H = \langle \mathbf{H}, \nu_S \rangle_{g_S}$. The *Hawking mass* of a surface Σ is given by

$$m_H(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 \, d\sigma_S \right).$$

Find $m_H(S_r)$.

g. If you consider the higher-dimensional Riemannian Schwarzschild metric $g_S = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{4/(n-2)} g_{\mathbb{E}}$ with $m > 0$, find the area profile $A(r)$ of $S_r = \{x : |x| = r\}$, and find the radius at which $A(r)$ has a minimum. Compute the mass m in terms of this area.

9. Suppose $(\mathbb{R}^3 \setminus \overline{B_{r_0}(0)}, g)$ is harmonically flat: $g = u^4 g_{\mathbb{E}}$, $R(g) = 0$, i.e. $\Delta_{g_{\mathbb{E}}} u = 0$, with $u(x) \to 1$ as $|x| \to +\infty$. We saw the expansion $u(x) = 1 + \frac{A}{|x|} + \frac{\beta_i x^i}{|x|^3} + O(|x|^{-3})$ via spherical harmonics.

a. Let $x = y + c$, for $c \in \mathbb{R}^3$. For $|y + c| > r_0$, find the asymptotic expansion of u as a function of y. Show for $A \neq 0$ that there is a unique choice of $c \in \mathbb{R}^3$ for which $\tilde{u}(y) := u(y+c) = 1 + \frac{A}{|y|} + O(|y|^{-3})$.

b. Compute $\lim_{r\to+\infty} \int_{|x|=r} x^k \sum_{i,j=1}^{3} (g_{ij,i} - g_{ii,j})\, \nu_e^j \, d\sigma_e$ where $\nu_e^j = \frac{x^j}{r}$. (Warning: this gives the center of mass, but the flux integral isn't the right form for more general asymptotically flat metrics.)

10. Assume that h is a (smooth) transverse-traceless tensor at the Euclidean metric on $\mathbb{R}^3$. Let's use Cartesian coordinates x, so that covariant derivative components are computed via partial

derivatives (the Christoffel symbols vanish). So $0 = \mathrm{tr}_{g_{\mathbb{E}}} h = \sum_{i=1}^{3} h_{ii}$, and $0 = (\mathrm{div}_{g_{\mathbb{E}}} h)_j = \sum_{i=1}^{3} h_{ij,j}$. Now, assume that h has compact support. Let $\gamma_\epsilon = g_{\mathbb{E}} + \epsilon h$, and for $|\epsilon|$ sufficiently small, let $u_\epsilon > 0$ be the associated conformal factor so that with $g_\epsilon = u_\epsilon^4 \gamma_\epsilon$, $R(g_\epsilon) = 0$, and u_ϵ tends to 1 at infinity. Near infinity each u_ϵ is harmonic, and as such has an asymptotic expansion $u_\epsilon = 1 + \frac{m(\epsilon)}{2|x|} + O(|x|^{-2})$.

a. Show that $16\pi m(\epsilon) = -\int_{\mathbb{R}^3} R(\gamma_\epsilon) u_\epsilon \, dv_{g_\epsilon}$.

b. Show that $m'(0) = 0$ and that $16\pi m''(0) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_{g_{\mathbb{E}}} h|^2 \, dv_e$.