

Characterization And Subordination Properties Associated With A Certain Class Of Functions*

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Abstract

This paper introduces a new class of functions which is defined by means of a Hadamard product (or convolution) of analytic functions, and is based on the concept of spirallikeness. The results investigated in the present paper include, the characterization and subordination properties for this class of functions. Several interesting consequences of our results are also pointed out.

1 Introduction

Let \mathcal{A} denote the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C}, |z| < 1\}.$$

If $f, g \in \mathcal{A}$, where $f(z)$ is given by (1), and $g(z)$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (2)$$

then their Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \quad (3)$$

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For two functions f and g analytic in \mathcal{U} , we say that the function f is subordinate to g in \mathcal{U} (denoted by $f \prec g$), if there exists a Schwarz function $w(z)$, analytic in \mathcal{U} with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathcal{U}$), such that $f(z) = g(w(z))$.

We introduce here a class $\mathcal{R}^\lambda(\phi, \psi; \gamma)$ which is defined as follows: Suppose the functions $\phi(z)$ and $\psi(z)$ are given by

$$\phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n, \tag{4}$$

and

$$\psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n, \tag{5}$$

where $\lambda_n \geq \mu_n \geq 0$ ($\forall n \in \mathbb{N} - \{1\}$). We say that $f \in \mathcal{A}$ is in $\mathcal{R}^\lambda(\phi, \psi; \gamma)$ provided that $(f * \psi)(z) \neq 0$, and

$$\Re \left\{ e^{i\lambda} \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} > \gamma \cos \lambda, \quad |\lambda| < \pi/2; \quad 0 \leq \gamma < 1; \quad z \in \mathcal{U}. \tag{6}$$

Several known subclasses of \mathcal{A} can be represented in terms of $\mathcal{R}^\lambda(\phi, \psi; \gamma)$ by suitably choosing the functions $\phi(z)$ and $\psi(z)$. Some of the familiar cases are stated below. Indeed, we have

$$\mathcal{R}^\lambda \left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \gamma \right) \equiv \mathcal{S}(\lambda, \gamma), \tag{7}$$

where $\mathcal{S}(\lambda, \gamma)$ denotes the λ -spirallike functions of order γ due to Libera [1]. Obviously

$$\mathcal{S}(\lambda, 0) = \mathcal{S}_p(\lambda), \quad |\lambda| < \pi/2, \tag{8}$$

and

$$\mathcal{S}(0, \gamma) = \mathcal{S}^*(\gamma), \quad 0 \leq \gamma < 1, \tag{9}$$

are, respectively, the familiar classes of λ -spirallike functions, and starlike functions (of order γ). Further

$$\mathcal{R}^\lambda \left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \gamma \right) \equiv \mathcal{M}(\lambda, \gamma), \tag{10}$$

where $\mathcal{M}(\lambda, \gamma)$ denotes the class of Robertson functions of order γ , studied in [2]. When $\lambda = 0$ in (10), then $\mathcal{M}(0, \gamma) = \mathcal{K}(\gamma)$ denotes the familiar class of convex function of order γ ($0 \leq \gamma < 1$), and $\mathcal{K}(0) = \mathcal{K}$ (see also [6]). Lastly, we observe that

$$\mathcal{R}^\lambda \left(\frac{z}{(1-z)^{p+2}}, \frac{z}{(1-z)^{p+1}}; \frac{1}{2} \right) = \left\{ f \in \mathcal{A} : \Re \left\{ \frac{D^{p+1}f(z)}{D^p f(z)} \right\} > \frac{1}{2}; p > -1; z \in \mathcal{U} \right\} \tag{11}$$

where the right hand side is the class \mathbb{K}_p which involves the Ruschweyh derivative D^p introduced in [3].

In our present investigation, we require the following definition, and also a related result due to Wilf [7].

DEFINITION 1. A sequence $\{b_n\}_1^\infty$ of complex numbers is said to be a subordination factor sequence, if whenever $f(z)$ given by (1) is regular, univalent and convex in \mathcal{U} , and

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \text{ in } \mathcal{U}. \quad (12)$$

LEMMA 1. The sequence $\{b_n\}_1^\infty$ is a subordinating factor sequence if and only if

$$\Re \left[1 + 2 \sum_{n=1}^{\infty} b_n z^n \right] > 0, \quad z \in \mathcal{U}. \quad (13)$$

The purpose in this paper is to investigate the characterization and subordination properties for the class of functions $\mathcal{R}^\lambda(\phi, \psi; \gamma)$ (defined above by (6)). Some interesting consequences of the main results are also discussed.

2 Characterization Properties

We first prove a characterization property for the class $\mathcal{R}^\lambda(\phi, \psi; \gamma)$, which is contained in the following.

THEOREM 1. Let $f(z) \in \mathcal{A}$ such that

$$\left| \frac{(f * \phi)(z)}{(f * \psi)(z)} - 1 \right| < 1 - \alpha, \quad 0 \leq \alpha < 1; \quad z \in \mathcal{U}, \quad (14)$$

then $f \in \mathcal{R}^\lambda(\phi, \psi; \gamma)$, provided that

$$|\lambda| \leq \cos^{-1} \left(\frac{1 - \alpha}{1 - \gamma} \right). \quad (15)$$

PROOF. In view of (14), we write

$$\frac{f * \phi}{f * \psi} - 1 = (1 - \alpha)w(z), \quad \text{where } |w(z)| < 1 \text{ for } z \in \mathcal{U}.$$

Now

$$\begin{aligned} \Re \left\{ e^{i\lambda} \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} &= \cos \lambda + (1 - \alpha) \Re \{ e^{i\lambda} w(z) \} \\ &\geq \cos \lambda - (1 - \alpha) |e^{i\lambda} w(z)| \\ &> \cos \lambda - (1 - \alpha) \\ &\geq \gamma \cos \lambda, \end{aligned}$$

provided that $|\lambda| \leq \cos^{-1} \left(\frac{1-\alpha}{1-\gamma} \right)$. This completes the proof.

If we set $\alpha = 1 - (1-\gamma) \cos \lambda$, where $|\lambda| < \pi/2$, $0 \leq \gamma < 1$, in Theorem 1, we obtain the following.

COROLLARY 1. If $\left| \frac{(f*\phi)(z)}{(f*\psi)(z)} - 1 \right| < (1-\gamma) \cos \lambda$, then $f \in \mathcal{R}^\lambda(\phi, \psi; \gamma)$ for $|\lambda| < \pi/2$, $0 \leq \gamma < 1$.

REMARK 1. If we set the functions $\phi(z)$ and $\psi(z)$ as in (7), and make use of (8), then Theorem 1 yields the known result of Silverman [4, p.644].

We establish now a coefficient inequality for the class $\mathcal{R}^\lambda(\phi, \psi; \gamma)$.

THEOREM 2. Let $f \in \mathcal{A}$ satisfy the inequality

$$\sum_{n=2}^{\infty} \left\{ \left(\frac{\lambda_n - \mu_n}{1 - \gamma} \right) \sec \lambda + \mu_n \right\} |a_n| \leq 1, \quad |\lambda| < \pi/2, \tag{16}$$

then $f \in \mathcal{R}^\lambda(\phi, \psi; \gamma)$.

PROOF. Suppose the inequality (16) holds true. Then, on using (1), (4) and (5), we find that

$$\begin{aligned} & |(f * \phi)(z) - (f * \psi)(z)| - (1 - \gamma) \cos \lambda |(f * \psi)(z)| \\ &= \left| \sum_{n=2}^{\infty} (\lambda_n - \mu_n) a_n z^n \right| - (1 - \gamma) \cos \lambda \left| z + \sum_{n=2}^{\infty} a_n \mu_n z^n \right| \\ &\leq \left\{ \sum_{n=2}^{\infty} (\lambda_n - \mu_n) |a_n| - (1 - \gamma) \cos \lambda + (1 - \gamma) \cos \lambda \sum_{n=2}^{\infty} \mu_n |a_n| \right\} |z| \\ &= \left\{ \sum_{n=2}^{\infty} [(\lambda_n - \mu_n) + (1 - \gamma) \cos \lambda \mu_n] |a_n| - (1 - \gamma) \cos \lambda \right\} |z| \\ &\leq 0. \end{aligned}$$

Thus implies that $f \in \mathcal{R}^\lambda(\phi, \psi; \gamma)$.

By appealing to (7), (10) and (11), when the functions $\phi(z)$, $\psi(z)$ and the parameter γ , in (6) are chosen appropriately, Theorem 2 would then yield the following results.

COROLLARY 2. Let $f \in \mathcal{A}$, satisfy the inequality

$$\sum_{n=2}^{\infty} \left\{ \left(\frac{n-1}{1-\gamma} \right) \sec \lambda + 1 \right\} |a_n| \leq 1, \quad |\lambda| < \pi/2, \tag{17}$$

then $f \in \mathcal{S}(\lambda, \gamma)$.

REMARK 2. If we put $\gamma = 0$ in (17), then in view of (18), we get the result of Silverman [4, p.643].

COROLLARY 3. Let $f \in \mathcal{A}$, satisfy the inequality

$$\sum_{n=2}^{\infty} \left\{ \left(\frac{n-1}{1-\gamma} \right) \sec \lambda + 1 \right\} n |a_n| \leq 1, \quad |\lambda| < \pi/2, \tag{18}$$

then $f \in \mathcal{M}(\lambda, \gamma)$.

COROLLARY 4. Let $f \in \mathcal{A}$, satisfy the inequality

$$\sum_{n=2}^{\infty} \left\{ \frac{(p+2n-1)\Gamma(p+n)}{(n-1)!\Gamma(p+2)} \right\} |a_n| \leq 1, \quad (19)$$

then $f \in \mathbb{K}_p$.

3 Subordination Theorem

THEOREM 3. Let $f(z) \in \mathcal{A}$ satisfy the inequality (16), and the sequences $\langle \lambda_n \rangle$ and $\langle \mu_n \rangle$ are nondecreasing. Then

$$\frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{2 \left[\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1 \right]} (f * g)(z) \prec g(z), \quad \lambda_n \geq \mu_n \geq 0; \quad 0 \leq \gamma < 1; \quad |\lambda| < \pi/2; \quad z \in \mathcal{U} \quad (20)$$

for every function $g \in \mathcal{K}$, and

$$\Re\{f(z)\} > -\frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1}{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}, \quad z \in \mathcal{U}. \quad (21)$$

The following constant factor in the subordination result (20):

$$\frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{2 \left[\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1 \right]}$$

cannot be replaced by a larger one.

PROOF. Let $f(z)$ defined by (1) belong to the class \mathcal{A} satisfying the inequality (16), and $g(z)$ defined by (2) be any function in the class \mathcal{K} . It follows then

$$\frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{2 \left[\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1 \right]} (f * g)(z) = \frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{2 \left[\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1 \right]} \left(z + \sum_{n=2}^{\infty} a_n b_n z^n \right). \quad (22)$$

By invoking Definition 1, the subordination (20) of our theorem will hold true if the sequence

$$\left\{ \frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{2 \left[\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1 \right]} a_n \right\}_{n=1}^{\infty}, \quad (23)$$

is a subordinating factor sequence. By virtue of Lemma 1, this is equivalent to the inequality:

$$\Re \left(1 + 2 \sum_{n=1}^{\infty} \frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{2 \left[\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1\right]} a_n z^n \right) > 0, \quad z \in \mathcal{U}. \tag{24}$$

By noting the fact that

$$\left(\frac{\lambda_n - \mu_n}{1 - \gamma}\right) \sec \lambda + \mu_n, \quad \forall n \in \mathbb{N} - \{1\}; |\lambda| < \pi/2,$$

is a nondecreasing function of n , and, in particular:

$$\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 \leq \left(\frac{\lambda_n - \mu_n}{1 - \gamma}\right) \sec \lambda + \mu_n, \quad \forall n \in \mathbb{N} - \{1\}; |\lambda| < \pi/2,$$

therefore, for $|z| = r < 1$, we obtain

$$\begin{aligned} & \Re \left(1 + \sum_{n=1}^{\infty} \frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{\left[\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1\right]} a_n z^n \right) \\ &= \Re \left(1 + \frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1} z \right. \\ & \quad \left. + \frac{1}{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1} \sum_{n=2}^{\infty} \left[\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2\right] a_n z^n \right) \\ &> 1 - \frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1} r \\ & \quad - \frac{1}{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1} \sum_{n=2}^{\infty} \left[\left(\frac{\lambda_n - \mu_n}{1 - \gamma}\right) \sec \lambda + \mu_n\right] |a_n| r^n \\ &> 1 - \frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1} r - \frac{1}{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1} r \\ &> 0. \end{aligned}$$

This evidently establishes the inequality (24), and consequently the subordination relation (20) of our Theorem 3 is proved. The assertion (21) follows readily from (20) when the function $g(z)$ is selected as

$$g(z) = \frac{z}{1 - z} = z + \sum_{n=2}^{\infty} z^n \in \mathcal{K}. \tag{25}$$

The sharpness of the multiplying factor in (20) can be established by considering a function $h(z)$ defined by

$$h(z) = z - \frac{1}{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2} z^2, \quad \lambda_2 \geq \mu_2 \geq 0; \quad |\lambda| < \pi/2; \quad z \in \mathcal{U},$$

which belongs to the class $\mathcal{R}^\lambda(\phi, \psi; \gamma)$. Using (20), we infer that

$$\frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{2 \left[\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1 \right]} h(z) \prec \frac{z}{1 - z}.$$

It can easily be verified that

$$\min_{|z| \leq 1} \Re \left[\frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{2 \left[\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1 \right]} h(z) \right] = -\frac{1}{2}, \quad (26)$$

which shows that the constant $\frac{\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2}{2 \left[\left(\frac{\lambda_2 - \mu_2}{1 - \gamma}\right) \sec \lambda + \mu_2 + 1 \right]}$ is best possible.

Before concluding this paper, we deem it worthwhile to mention some useful consequences of the subordination Theorem 3. On choosing the arbitrary function $\phi(z), \psi(z)$, and the parameters λ and γ , suitably in accordance with the subclasses defined by (7), (10) and (11), we arrive at the following results:

COROLLARY 5. Let $f(z) \in \mathcal{A}$ satisfy the inequality (17), then for every function g in \mathcal{K} , we have

$$\frac{1 - \gamma + \sec \lambda}{2[2 - 2\gamma + \sec \lambda]} (f * g)(z) \prec g(z), \quad z \in \mathcal{U}, \quad (27)$$

and

$$\Re\{f(z)\} > -\frac{2 - 2\gamma + \sec \lambda}{1 - \gamma + \sec \lambda}, \quad z \in \mathcal{U}, \quad (28)$$

where the constant $\frac{1 - \gamma + \sec \lambda}{2[2 - 2\gamma + \sec \lambda]}$ is best possible.

COROLLARY 6. Let $f(z) \in \mathcal{A}$ satisfy the inequality (18), then for every function g in \mathcal{K} , we have

$$\frac{1 - \gamma + \sec \lambda}{3 - 3\gamma + 2 \sec \lambda} (f * g)(z) \prec g(z), \quad z \in \mathcal{U}, \quad (29)$$

and

$$\Re\{f(z)\} > -\frac{3 - 3\gamma + 2 \sec \lambda}{2(1 - \gamma + \sec \lambda)}, \quad z \in \mathcal{U}, \quad (30)$$

where the constant $\frac{1 - \gamma + \sec \lambda}{3 - 3\gamma + 2 \sec \lambda}$ is best possible.

COROLLARY 7. Let $f(z) \in \mathcal{A}$ satisfy the inequality (19), then for every function g in \mathcal{K} , we have

$$\frac{p+3}{2[p+4]}(f * g)(z) \prec g(z), \quad z \in \mathcal{U}, \quad (31)$$

and

$$\Re\{f(z)\} > -\frac{p+4}{p+3}, \quad z \in \mathcal{U}, \quad (32)$$

where the constant $\frac{p+3}{2[p+4]}$ is best possible.

REMARK 3. For $\gamma = 0$, Corollary 5 corresponds to the main result of Singh [5, p.434, Theorem 1].

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