CONVERGENCE OF BANACH LATTICE VALUED STOCHASTIC PROCESSES
WITHOUT THE RADON-NIKODYM PROPERTY

V. MARRAFFA

Abstract. We obtain almost sure convergence theorems for stochastic processes consisting of Bochner integrable functions taking values in a Banach lattice without assuming the Radon-Nikodym property. It is shown that if the limit exists in a weak sense then the almost sure convergence follows.

1. Introduction

For Banach lattice valued submartingales the Radon-Nikodym property is equivalent to the convergence a.e. (see [4], [11] and [6]). If the Radon-Nikodym property is not assumed it is natural to ask how small can be the class $T$ of functionals $f$ such that the a.s. convergence of $fX_n$ to $fX$ for $f \in T$ implies the convergence of $X_n$ to $X$ in some stronger sense. In case of Banach valued processes it was established that $T$ can be a total set. In particular in [8] it was proved that an amart $(X_n)$ converges scalarly almost surely to a random variable $X$ if $fX_n$ converges to $fX$ a.s for each $f$ in a total subset of the dual. In [3], under the same assumption, the strong a.s. convergence for martingales follows. Analogous results has been obtained also for weak amarts and uniform amarts in [1].

In §3 we obtain similar results for submartingales taking values in a Banach lattice (see Theorem 2).

In §4, under a suitable covering condition (Vitali condition $V$), we generalize the submartingales result to directed sets.

Received December 22, 2003.
2000 Mathematics Subject Classification. Primary 60G48, 28B05.
Key words and phrases. Random variable, stopping time, submartingale, scalar convergence.
Supported by MURST of Italy.
2. Definitions and notations

Throughout this note \((\Omega, \mathcal{F}, P)\) is a probability space and \((\mathcal{F}_n)_{n \in \mathbb{N}}\) a family of sub-\(\sigma\)-algebras of \(\mathcal{F}\) such that \(\mathcal{F}_m \subset \mathcal{F}_n\) if \(m < n\). Moreover, without loss of generality, we will assume that \(\mathcal{F}\) is the completion of \(\sigma(\bigcup_n \mathcal{F}_n)\). From now on \(E\) will denote a Banach lattice with norm \(\| \cdot \|\) and \(E^*\) its dual. A subset \(T\) of \(E^*\) is called a total set over \(E\) if \(f(x) = 0\) for each \(f \in T\) implies \(x = 0\). For an element \(x \in E\) we denote by \(x^+\) the least upper bound between \(x\) and 0. The Banach lattice \(E\) is said to have the order continuous norm or, briefly, to be order continuous, if for every downward directed set \(\{x_\alpha\}_\alpha\) in \(E\) with \(\wedge_\alpha x_\alpha = 0\), then \(\lim_\alpha \|x_\alpha\| = 0\). The norm on \(E\) has the Kadec-Klee property with respect to a set \(D \subset E^*\) if whenever \(\lim_n f(x_n) = f(x)\) for every \(f \in D\) and \(\lim_n \|x_n\| = \|x\|\), then \(\lim_n x_n = x\) strongly. If \(D = E^*\) we say that the norm has the Kadec-Klee property. It was proved in [2] the following renorming theorem for Banach lattices.

**Theorem 1.** A Banach lattice \(E\) is order continuous if and only if there is an equivalent lattice norm on \(E\) with the Kadec-Klee property.

It is obvious that if \(E\) is separable, the equivalent norm has the Kadec-Klee property with respect to a countable set of functionals.

A stopping time is a map \(\tau : \Omega \to \mathbb{N} \cup \{\infty\}\) such that, for each \(n \in \mathbb{N}\), \(\{\tau \leq n\} = \{\omega \in \Omega : \tau(\omega) \leq n\} \in \mathcal{F}_n\). We denote by \(\Gamma\) the collection of all simple stopping times (i.e. taking finitely many values and not taking the value \(\infty\)). Then \(\Gamma\) is a set filtering to the right.

We recall that a stochastic process \((X_n, \mathcal{F}_n)\) is called

(i) a submartingale if \(X_n \leq E(X_{n+1} | \mathcal{F}_n)\) a.s. for each \(n \in \mathbb{N}\), or equivalently if

\[
\int_A X_n \leq \int_A X_{n+1},
\]

for each \(A \in \mathcal{F}_n\) and for each \(n \in \mathbb{N}\);
(ii) a subpramart if for each $\varepsilon > 0$ there exists $\tau_0 \in \Gamma$ such that for all $\tau$ and $\sigma$ in $\Gamma$, $\tau > \sigma > \tau_0$ then

$$P(\{\|X_\sigma - E(X_\tau \mid \mathcal{F}_\sigma)\| > \varepsilon\}) \leq \varepsilon.$$ 

We remind that if $(X_n, \mathcal{F}_n)$ is a positive subpramart (i.e. $X_n(\omega) \geq 0$ for each $n \in \mathbb{N}$ and $\omega \in \Omega$), then for each $f \in (E^*)^+$, where $(E^*)^+$ denotes the nonnegative cone in $E^*$, $(fX_n, \mathcal{F}_n)$ and $(\|X_n\|, \mathcal{F}_n)$ are real valued positive subpramarts [5, Lemma viii.1.12].

3. Convergence theorems for processes indexed by $\mathbb{N}$

We will need the following Propositions.

**Proposition 1.** [5, p. 303] Let $E$ be a Banach space and let $(X_n, \mathcal{F}_n)$ be a $L^1$-bounded stochastic process. Then there exists a subsequence $(n_k)_k$ in $\mathbb{N}$ such that for every $k \in \mathbb{N}$

$$X_{n_k} = Y_{n_k} + Z_{n_k}$$

where $Y_{n_k}$ and $Z_{n_k}$ are $\mathcal{F}_{n_k}$-measurable, $(Y_{n_k})_k$ is uniformly integrable and

$$\lim_k Z_{n_k} = 0 \text{ a.s.}.$$ 

**Proposition 2.** [5, p. 298] Let $(X^m_n, \mathcal{F}_n)_n$ be a sequence of real valued positive subpramarts for which for each $\varepsilon > 0$ there exists $\tau_0 \in \Gamma$ such that for all $\tau$ and $\sigma$ in $\Gamma$, $\tau > \sigma > \tau_0$ then

$$P(\{\sup_m (X^m_\sigma - E(X^m_\tau \mid \mathcal{F}_\sigma)) \leq \varepsilon\}) \geq 1 - \varepsilon.$$ 

Suppose, moreover, that there is a subsequence $(n_k)_k$ such that

$$\sup_k \int \sup_m X^m_{n_k} < \infty.$$
Then each submartingale \((X_n^m, \mathcal{F}_n)_n\) converges a.s. to an integrable function \(X^m\) and we have

\[
\lim_n (\sup_m X_n^m) = \sup_m X^m \text{ a.s..}
\]

We are able to prove the following theorem.

**Theorem 2.** [9, Theorem 3.8] Let \(E\) be an order continuous Banach lattice, which is weakly sequentially complete and let \(T\) be a total subset of \(E^*\). Let \((X_n, \mathcal{F}_n)\) be a positive submartingale with an \(L^1\)-bounded subsequence and let \(X\) be a strongly measurable random variable. Assume that, for each \(f \in T\), \(fX_n\) converges to \(fX\) a.s. (the null depends on \(f\)). Then \(X_n\) converges to \(X\) strongly, a.s..

**Proof.** Since \((X_n)\) and \(X\) are strongly measurable it is possible to assume that \(E\) is separable. Using Proposition 1 and the fact that a subsequence of \((X_n)_n\), still denoted by \((X_n)_n\), is \(L^1\)-bounded we can also assume that

\[
X_{n_k} = Y_{n_k} + Z_{n_k}
\]

where \(Y_{n_k}\) and \(Z_{n_k}\) are \(\mathcal{F}_{n_k}\)-measurable, \((Y_{n_k})_k\) is uniformly integrable and

\[
\lim_k Z_{n_k} = 0 \text{ a.s..}
\]

For each \(f \in (E^*)^+\), \((fX_n)_n\) is a real valued submartingale with a \(L^1\)-bounded subsequence, then it converges a.s. to a real random variable \(X_f\). Also \(fY_{n_k}\) converges to \(X_f\) a.s. and in \(L^1\). In particular for each \(f \in T\), \(\lim_k fY_{n_k} = fX\). So for \(A \in \mathcal{F}\)

\[
\lim_k \int_A fY_{n_k}
\]

exists in \(\mathbb{R}\). Hence \((\int_A Y_{n_k})_k\) is weakly Cauchy. Since the Banach lattice \(E\) is weakly sequentially complete, let for every \(A \in \mathcal{F}\)

\[
\mu(A) = w - \lim_k \int_A Y_{n_k}.
\]
Then $\mu$ is a measure of bounded variation and it is absolutely continuous with respect to $P$. For each $f \in T$ we have

$$f(\mu(A)) = \lim_k \int_A fY_{n_k} = \int_A fX.$$ 

Let $A_n = \{\|X\| \leq n\}$, then $XI_{A_n}$ is Bochner integrable and

$$f(\mu(A_n)) = \int_{A_n} fX = f \int_{A_n} X.$$

Since $T$ is a total set it follows that

$$\mu(A_n) = \int_{A_n} X.$$

Moreover the uniform integrability of $(Y_{n_k})_k$ implies that

$$\int_{A_n} \|X\| = \|\mu\|(A_n) \leq \sup_k \int_{\Omega} Y_{n_k},$$

and since $X$ is strongly measurable, $P(\cup_n(\|X\| \leq n)) = 1$. Letting $n \to \infty$ in (1), we get that $X$ is Bochner integrable and for each $A \in \mathcal{F}$

$$\mu(A) = \int_A X.$$

It follows that

$$\int_A fX = f(\mu(A)) = \lim_k \int_A fY_{n_k} = \int_A Xf,$$

for each $f \in (E^*)^+$ and $A \in \bigcup \mathcal{F}$. Hence $fX = Xf$ a.s. and for each $f \in (E^*)^+$, $fX_n$ converges to $fX$ a.s.. Let $\|\cdot\|$ denote the Kadec-Klee norm equivalent to $\|\cdot\|$, as in Theorem 1, and let $D \in (E^*)^+$ be a countable norming subset. Applying Proposition 2 to the sequence $\{(fX_n, F_n), n \in \mathcal{N}, f \in D\}$ it follows that $\lim_n \|X_n\| = \|X\|$, a.s.. Now invoking again Theorem 1 we get the strong convergence of $X_n$ to $X$ and the assertion follows. $\square$
The following corollary holds.

**Corollary 1.** Let $E$ be a Banach lattice not containing $c_0$ as an isomorphic copy and let $T$ be a total subset of $E^*$. Let $(X_n, \mathcal{F}_n)$ be a positive submartingale with a $L^1$-bounded subsequence and let $X$ be a strongly measurable random variable. Assume that, for each $f \in T$, $fX_n$ converges to $fX$ a.s. (the null set depends on $f$). Then $X_n$ converges to $X$ strongly a.s..

*Proof.* If $E$ does not contain $c_0$, $E$ is an order continuous Banach lattice which is weakly sequentially complete [7, p. 34] and the assertion follows from Theorem 2. □

Since a submartingale is a submartingale we get

**Corollary 2.** [3, Proposition 11] Let $E$ be a Banach lattice not containing $c_0$ as an isomorphic copy and let $T$ be a total subset of $E^*$. Let $(X_n, \mathcal{F}_n)$ be a $L^1$-bounded positive submartingale and let $X$ be a strongly measurable random variable. Assume that, for each $f \in T$, $fX_n$ converges to $fX$ a.s. (the null set depends on $f$). Then $X_n$ converges to $X$ strongly a.s..

4. **A Convergence Theorem for Submartingales Indexed by a Directed Set**

In this section we will consider stochastic processes indexed by a directed set. Let $J$ be a directed set filtering to the right. Throughout this section we assume that there is an increasing cofinal sequence $(t_n)$ in $J$. Let $(\mathcal{F}_t)$ be a filtration, that is an increasing family of sub-$\sigma$-algebras of $\mathcal{F}$. A filtration $(\mathcal{F}_t)$ is said to satisfy the Vitali condition $V$ if for every adapted family of sets $(A_t)$ and for every $\varepsilon > 0$ there exists a simple stopping time $\tau \in \Gamma$ such that $P(\limsup_t A_t \setminus A_\tau) < \varepsilon$. Even in the real-valued case the Vitali condition on the filtration is necessary for the convergence of classes of random variables. Under the condition $V$, the analogue of Theorem 2 holds for submartingales indexed by directed sets.
Theorem 3. Let the filtration satisfy the condition V and let $E$ be a separable order continuous Banach lattice, which is weakly sequentially complete. Let $(X_t, F_t)$ be a $L^1$-bounded positive submart and let $X$ be a strongly measurable random variable. Let $T$ be a total subset of $E^*$ and assume that, for each $f \in T$, $fX_t$ converges to $fX$ a.s.. Then $X_t$ converges to $X$ strongly a.s..

Proof. Let $(t_n)$ be an increasing cofinal sequence in $J$. Set $X_{t_n} = Y_n$ and $F_{t_n} = G_n$. We first show that $(Y_n, G_n)$ is a submartingale sequence. Since $(X_t)$ is a submartingale, for every $\varepsilon > 0$ there exists $\tau_0 \in \Gamma$ such that if $\tau > \sigma > \tau_0$ then

$$P(\{(X_{\sigma} - E(X_{\tau} | F_{\sigma}))^+ \| > \varepsilon \}) \leq \varepsilon.$$ 

Now if $\sigma$ is a stopping time for $G$ then $t_\sigma$ is a stopping time for $F_t$. Thus choose $\sigma_0$ such that $t_\sigma \geq \tau_0$. Now for each $\tau > \sigma > \sigma_0$ it follows

$$P(\{(X_{\sigma} - E(X_{\tau} | G_{\sigma} ))^+ \| > \varepsilon \}) = P(\{(X_{t_\sigma} - E(X_{t_\tau} | F_{t_\sigma} ))^+ \| > \varepsilon \}) \leq \varepsilon.$$ 

Then $Y_n$ is a submartingale sequence. For each $f \in T$, $fY_n$ converges to $fX$ a.s.. Therefore by Theorem 2, $Y_n$ converges to $X$ a.s. and also scalarly. As $E$ is a separable Banach lattice there exists a countable norming subset $D$ of $(E^*)^+$ (i.e. $\|x\| = \sup \{|x^*(x)| : x^* \in D \cap B(X^*)\}$). Now, for each $f \in D$, $fX_t$ is a $L^1$-bounded real valued submart and since the filtration satisfies $V$, by [10] Theorem 4.3, $fX_t$ converges to $fX$ a.s.. Since $fX_{t_n}$ converges to $fX$, it follows that $fX = X_f$. As in Theorem 1, we denote by $\|| \cdot \|$ the Kadec-Klee norm equivalent to $\| \cdot \|$. Applying [6] Lemma 2.3 to the sequence $\{(fX_t, F_t), t \in T, f \in D\}$ it follows that $\lim_t \|X_t\| = \|X\|$, a.s.. Now invoking again Theorem 1 we get the strong convergence of $X_t$ to $X$ and the assertion follows. \qed


V. Marraffa, Department of Mathematics, Via Archirafi 34, 90123 Palermo, Italy,
e-mail: marraffa@math.unipa.it