ON THE HILBERT INEQUALITY

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Abstract. In this paper it is shown that the Hilbert inequality for double series can be improved by introducing a weight function of the form \( \frac{\sqrt{n}}{n+1} \left( \frac{\sqrt{n}-1}{\sqrt{n+1}} - \frac{\ln n}{\pi} \right) \), where \( n \in \mathbb{N} \). A similar result for the Hilbert integral inequality is also given. As applications, some sharp results of Hardy-Littlewood’s theorem and Widder’s theorem are obtained.

1. Introduction

Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of complex numbers. It is all-round known that the inequality

\[
\left| \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m \overline{b_n}}{m+n} \right|^2 \leq \pi^2 \sum_{n=1}^{\infty} |a_n|^2 \sum_{n=1}^{\infty} |b_n|^2
\]

(1.1)

is called the Hilbert inequality for double series, where \( \sum_{n=1}^{\infty} |a_n|^2 < +\infty \) and \( \sum_{n=1}^{\infty} |b_n|^2 < +\infty \), and that the constant factor \( \pi^2 \) in (1.1) is the best possible. The equality in (1.1) holds if and only if

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\{a_n\}, or \{b_n\} is a zero-sequence (see [?]). The corresponding integral form of (1.1) is that
\[
\left| \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy \right|^2 \leq \pi^2 \left( \int_0^\infty |f(x)|^2 \, dx \right) \left( \int_0^\infty |g(x)|^2 \, dx \right)
\]
where \( \int_0^\infty |f(x)|^2 \, dx < +\infty \) and \( \int_0^\infty |g(x)|^2 \, dx < +\infty \), and that the constant factor \( \pi^2 \) in (1.2) is also the best possible. The equality in (1.2) holds if and only if \( f(x) = 0 \), or \( g(x) = 0 \). Recently, various improvements and extensions of (1.1) and (1.2) appeared in a great deal of papers (see [?]). The purpose of the present paper is to build the Hilbert inequality with the weights by means of a monotonic function of the form \( \sqrt{x} \), thereby new refinements of (1.1) and (1.2) are established, and then to give some of their important applications.

For convenience, we need the following lemmas.

**Lemma 1.1.** Let \( n \in \mathbb{N} \). Then
\[
\int_0^\infty \frac{dx}{(n+x^2)(1+x)} = \frac{1}{n+1} \left( \frac{\pi}{2\sqrt{n}} + \frac{1}{2} \ln n \right)
\]

**Proof.** Let \( a, e \) and \( f \) be real numbers. Then
\[
\int \frac{dx}{(a^2+x^2)(e+fx)} = \frac{1}{e^2 + a^2 f^2} \left\{ f \ln |e+fx| - \frac{1}{2} \ln(a^2+x^2) + \frac{e}{a} \arctan \frac{x}{a} \right\} + C
\]
where \( C \) is an arbitrary constant. This result has been given in the papers (see [3]–[4]). Based on this indefinite integral it is easy to deduce that the equality (1.3) is true.
Lemma 1.2. Let $n \in \mathbb{N}$, $x \in (0, +\infty)$. Define two functions by

$$f(x) = \left(\frac{1}{x + n} \left(\frac{n}{x}\right)^{\frac{1}{2}}\right) \left(1 - \left(\frac{\sqrt{x}}{1 + \sqrt{x}} - \frac{\sqrt{n}}{1 + \sqrt{n}}\right)\right)$$

$$g(x) = \left(\frac{1}{x + n} \left(\frac{n}{x}\right)^{\frac{1}{2}}\right) \left(1 + \left(\frac{\sqrt{x}}{1 + \sqrt{x}} - \frac{\sqrt{n}}{1 + \sqrt{n}}\right)\right),$$

then $f(x)$ and $g(x)$ are monotonously decreasing in $(0, +\infty)$, and

$$\int_{0}^{\infty} f(x) \, dx = \pi - \pi \omega(n) \quad \text{(1.4)}$$

$$\int_{0}^{\infty} g(x) \, dx = \pi + \pi \omega(n) \quad \text{(1.5)}$$

where the weight function $\omega$ is defined by

$$\omega(n) = \frac{\sqrt{n}}{n + 1} \left(\frac{\sqrt{n} - 1}{\sqrt{n} + 1} - \frac{\ln n}{\pi}\right) \quad \text{(1.6)}$$

Proof. At first, notice that $1 - \frac{\sqrt{x}}{1 + \sqrt{x}} = \frac{1}{1 + \sqrt{x}}$, hence we can write $f(x)$ in form $f(x) = f_1(x) + f_2(x)$, where

$$f_1(x) = \left(\frac{1}{(x + n)\sqrt{x}}\right) \left(\frac{n}{1 + \sqrt{n}}\right), \quad f_2(x) = \frac{\sqrt{n}}{(x + n)(1 + \sqrt{x})\sqrt{x}}.$$

It is obvious that $f_1(x)$ and $f_2(x)$ are monotonously decreasing in $(0, +\infty)$. Hence $f(x)$ is monotonously decreasing in $(0, +\infty)$. Next, notice that $1 - \frac{\sqrt{n}}{1 + \sqrt{n}} = \frac{1}{1 + \sqrt{n}}$, we can write $g(x)$ in
form $g(x) = g_1(x) + g_2(x)$, where

$$g_1(x) = \frac{\sqrt{n}}{(1 + \sqrt{n})(x + n)\sqrt{x}}, \quad g_2(x) = \frac{\sqrt{n}}{(x + n)(1 + \sqrt{x})}.$$ 

It is obvious that $g_1(x)$ and $g_2(x)$ are monotonously decreasing in $(0, +\infty)$. Hence $g(x)$ is also monotonously decreasing in $(0, +\infty)$. Further we need only to compute two integrals.

$$\int_0^\infty f(x) \, dx = \int_0^\infty \left( \frac{1}{x + n} \left( \frac{n}{x} \right)^{\frac{1}{2}} \right) \left( 1 + \frac{\sqrt{n}}{1 + \sqrt{n}} - \frac{\sqrt{x}}{1 + \sqrt{x}} \right) \, dx$$

$$= \left( 1 + \frac{\sqrt{n}}{1 + \sqrt{n}} \right) \int_0^\infty \left( \frac{1}{x + n} \left( \frac{n}{x} \right)^{\frac{1}{2}} \right) \, dx - \int_0^\infty \left( \frac{1}{x + n} \left( \frac{n}{x} \right)^{\frac{1}{2}} \right) \left( \frac{\sqrt{x}}{1 + \sqrt{x}} \right) \, dx$$

$$= \left( 1 + \frac{\sqrt{n}}{1 + \sqrt{n}} \right) \pi - \int_0^\infty \left( \frac{1}{x + n} \left( \frac{n}{x} \right)^{\frac{1}{2}} \right) \left( \frac{\sqrt{x}}{1 + \sqrt{x}} \right) \, dx$$

$$= \pi - \left\{ 2\sqrt{n} \left( \int_0^\infty \frac{1}{(n + t^2)} \, dt - \int_0^\infty \frac{1}{(n + t^2)(1 + t)} \, dt \right) - \frac{\sqrt{n} \pi}{1 + \sqrt{n}} \right\}$$

$$= \pi - \left\{ \pi - 2\sqrt{n} \int_0^\infty \frac{1}{(n + t^2)(1 + t)} \, dt - \frac{\sqrt{n} \pi}{1 + \sqrt{n}} \right\}$$
By Lemma 1.1, we obtain

\[ \int_0^\infty f(x) \, dx = \pi - \left\{ \pi - \left( \frac{\pi}{n+1} + \frac{\sqrt{n \ln n}}{n+1} \right) - \frac{\sqrt{n} \pi}{1 + \sqrt{n}} \right\} \]

(1.7)

The equality (1.4) follows from (1.7) at once after some simple computations and simplifications. Similarly, the equality (1.5) can be obtained.

\[
\square
\]

2. Main Results

First, we establish a new refinement of (1.1).

**Theorem 2.1.** Let \{a_n\} and \{b_n\} be two sequences of complex numbers. If \(\sum_{n=1}^{\infty} |a_n|^2 < +\infty\) and \(\sum_{n=1}^{\infty} |b_n|^2 < +\infty\), then

\[ \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{b}_n}{m+n} \right|^4 \leq \pi^4 \left\{ \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^2 - \left( \sum_{n=1}^{\infty} \omega(n) |a_n|^2 \right)^2 \right\} \times \left\{ \left( \sum_{n=1}^{\infty} |b_n|^2 \right)^2 - \left( \sum_{n=1}^{\infty} \omega(n) |b_n|^2 \right)^2 \right\} \]

(2.1)

where the weight function \(\omega(n)\) is defined by (1.6).
Proof. Let \( c(x) \) be a real function and satisfy the condition \( 1 - c(n) + c(m) \geq 0, \ (n, m \in N) \). Firstly we suppose that \( b_n = a_n \). Applying Cauchy’s inequality we have

\[
\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{a}_n}{m + n} \right|^2 = \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{a}_n}{m + n} (1 - c(n) + c(m)) \right|^2 \\
= \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{a_m (1 - c(n) + c(m))^{1/2}}{(m + n)^{1/2}} \frac{m}{n} \right)^{1/4} \right|^2 \\
\times \left( \frac{\bar{a}_n (1 - c(n) + c(m))^{1/2}}{(m + n)^{1/2}} \frac{n}{m} \right)^{1/4} \right|^2 \\
\leq J_1 J_2
\]

(2.2)

where

\[
J_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_m|^2}{m + n} \left( \frac{m}{n} \right)^{1/2} (1 - c(n) + c(m)) \\
J_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|\bar{a}_n|^2}{m + n} \left( \frac{n}{m} \right)^{1/2} (1 - c(n) + c(m))
\]

We can write the double series \( J_1 \) in the following form:

\[
J_1 = \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{1}{m + n} \left( \frac{n}{m} \right)^{1/2} (1 - c(m) + c(n)) \right) |a_n|^2.
\]
Let \( c(x) = \frac{\sqrt{x}}{1 + \sqrt{x}} \). It is obvious that \( 1 - \frac{\sqrt{x}}{1 + \sqrt{x}} + \frac{\sqrt{n}}{1 + \sqrt{n}} \geq 0 \). It is known from Lemma 1.2 that the function \( f(x) \) is monotonously decreasing. Hence we have

\[
J_1 = \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{1}{m + n} \left( \frac{n}{m} \right)^{\frac{1}{2}} \left( 1 - \frac{\sqrt{m}}{1 + \sqrt{m}} + \frac{\sqrt{n}}{1 + \sqrt{n}} \right) \right) |a_n|^2
\leq \sum_{n=1}^{\infty} \left\{ \int_0^{\infty} \left( \frac{1}{x + n} \left( \frac{n}{x} \right)^{\frac{1}{2}} \right) \left( 1 - \left( \frac{\sqrt{x}}{1 + \sqrt{x}} - \frac{\sqrt{n}}{1 + \sqrt{n}} \right) \right) \, dx \right\} |a_n|^2

= \pi \sum_{n=1}^{\infty} |a_n|^2 - \pi \sum_{n=1}^{\infty} \omega(n) |a_n|^2

\]

where the weight function \( \omega(n) \) is defined by (1.6).

Similarly,

\[
J_2 \leq \sum_{n=1}^{\infty} \left\{ \int_0^{\infty} \frac{1}{x + n} \left( \frac{n}{x} \right)^{\frac{1}{2}} \left( 1 + \left( \frac{\sqrt{x}}{1 + \sqrt{x}} - \frac{\sqrt{n}}{1 + \sqrt{n}} \right) \right) \, dx \right\} |\bar{a}_n|^2

= \pi \sum_{n=1}^{\infty} |a_n|^2 + \pi \sum_{n=1}^{\infty} \omega(n) |a_n|^2.

\]

Whence \( J_1J_2 \leq \pi^2 \left\{ \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^2 - \left( \sum_{n=1}^{\infty} \omega(n) |a_n|^2 \right)^2 \right\} \).
Consequently, we have

\[
\left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{a}_n}{m+n} \right)^2 \leq \pi^2 \left\{ \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^2 - \left( \sum_{n=1}^{\infty} \omega(n) |a_n|^2 \right)^2 \right\}
\]

(2.3)

where the weight function \( \omega(n) \) is defined by (1.6).

If \( b_n \neq a_n \), then we can apply Schwarz’s inequality to estimate the right-hand side of (2.1) as follows:

\[
\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{b}_n}{m+n} \right|^4 = \left\{ \int_0^1 \left( \sum_{m=1}^{\infty} a_m t^{m-\frac{1}{2}} \right) \left( \sum_{n=1}^{\infty} \bar{b}_n t^{n-\frac{1}{2}} \right) dt \right\}^2
\]

\[
\leq \int_0^1 \left( \sum_{m=1}^{\infty} |a_m| t^{m-\frac{1}{2}} \right)^2 dt \int_0^1 \left( \sum_{n=1}^{\infty} |b_n| t^{n-\frac{1}{2}} \right)^2 dt
\]

(2.4)

And then by using the relation (2.3), from (2.4) and the inequality (2.1), we obtain at once. □

Similarly, we can establish a new refinement of (1.2).
**Theorem 2.2.** Let $f(x)$ and $g(x)$ be two functions in complex number field. If $\int_0^\infty |f(x)|^2 \, dx < +\infty$, $\int_0^\infty |g(x)|^2 \, dx < +\infty$, then

$$\left| \int_0^\infty \int_0^\infty \frac{f(x)\overline{g(y)}}{x+y} \, dx \, dy \right|^4 \leq \pi^4 \left\{ \left( \int_0^\infty |f(x)|^2 \, dx \right)^2 - \left( \int_0^\infty \omega(x) |f(x)|^2 \, dx \right)^2 \right\}^2$$

(2.5)

where the weight function $\omega$ is defined by

$$\omega(x) = \begin{cases} 0 & x = 0 \\ \frac{\sqrt{x}}{x+1} \left( \frac{\sqrt{x} - 1}{\sqrt{x} + 1} - \frac{\ln x}{\pi} \right) & x > 0 \end{cases}$$

(2.6)

Its proof is similar to that of Theorem 2.1, it is omitted here.

For the convenience of the applications, we list the following result.

**Corollary 2.3.** Let $f(x)$ be a function in complex number field. If $\int_0^\infty |f(x)|^2 \, dx < +\infty$, then

$$\left| \int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y} \, dx \, dy \right|^2 \leq \pi^2 \left\{ \left( \int_0^\infty |f(x)|^2 \, dx \right)^2 - \left( \int_0^\infty \omega(x) |f(x)|^2 \, dx \right)^2 \right\}^2$$

(2.7)

where the weight function $\omega$ is defined by (2.6).
3. Applications

As applications, we shall give some new refinements of Hardy-Littlewood’s theorem and Widder’s theorem.

Let \( f(x) \in L^2(0, 1) \) and \( f(x) \neq 0 \) for all \( x \). Define a sequence \( \{a_n\} \) by \( a_n = \int_0^1 x^n f(x) \, dx \), \( n = 0, 1, 2, \ldots \). Hardy-Littlewood ([1]) proved that

\[
\sum_{n=0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x) \, dx,
\]

where \( \pi \) is the best constant that the inequality (3.1) keeps valid.

**Theorem 3.1.** Let \( f(x) \in L^2(0, 1) \) and \( f(x) \neq 0 \) for all \( x \). Define a sequence \( \{a_n\} \) by \( a_n = \int_0^1 x^{n-1/2} f(x) \, dx \) \( n = 1, 2, \ldots \). Then

\[
\left( \sum_{n=1}^{\infty} a_n^2 \right)^2 \leq \pi \left\{ \left( \sum_{n=1}^{\infty} a_n^2 \right)^2 - \left( \sum_{n=1}^{\infty} \omega(n) a_n^2 \right)^2 \right\} \frac{1}{2} \int_0^1 f^2(x) \, dx
\]

where \( \omega(n) \) is defined by (1.6).

**Proof.** By our assumptions, we may write \( a_n^2 \) in the form

\[
a_n^2 = \int_0^1 a_n x^{n-1/2} f(x) \, dx.
\]
Applying Cauchy-Schwarz’s inequality we estimate the right hand side of (3.2) as follows

\[
\left( \sum_{n=1}^{\infty} a_n^2 \right)^2 = \left( \sum_{n=1}^{\infty} \int_0^1 a_n x^{n-1/2} f(x) \, dx \right)^2 = \left\{ \int_0^1 \left( \sum_{n=1}^{\infty} a_n x^{n-1/2} \right) f(x) \, dx \right\}^2
\]

\[
\leq \int_0^1 \left( \sum_{n=1}^{\infty} a_n x^{n-1/2} \right)^2 \, dx \int_0^1 f^2(x) \, dx
\]

\[
= \int_0^1 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m a_n x^{m+n-1} \, dx \int_0^1 f^2(x) \, dx
\]

\[
= \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m a_n}{m+n} \right) \int_0^1 f^2(x) \, dx
\]

(3.3)

It is known from (2.3) and (3.3) that the inequality (3.2) is valid. Therefore the theorem is proved. \(\square\)

Let \(a_n \geq 0\) \((n = 0, 1, 2, \ldots)\), \(A(x) = \sum_{n=0}^{\infty} a_n x^n\), \(A^*(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}\). Then

\[
\int_0^1 A^2(x) \, dx \leq \pi \int_0^\infty \left( e^{-x} A^*(x) \right)^2 \, dx
\]

(3.4)

This is Widder’s theorem (see [1]).
Theorem 3.2. With the assumptions as the above-mentioned, it yields

\[
(3.5) \quad \left( \int_0^1 A^2(x) \, dx \right)^2 \leq \pi^2 \left\{ \left( \int_0^\infty (e^{-x} A^*(x))^2 \, dx \right)^2 - \left( \int_0^\infty \omega(x) (e^{-x} A^*(x))^2 \, dx \right)^2 \right\}
\]

where \( \omega(x) \) is defined by (2.6).

Proof. At first we have the following relation:

\[
\int_0^\infty e^{-t} A^*(tx) \, dt = \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{a_n (xt)^n}{n!} \, dt = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \int_0^\infty t^n e^{-t} \, dt = \sum_{n=0}^{\infty} a_n x^n = A(x)
\]

Let \( tx = s \). Then we have

\[
\int_0^1 A^2(x) \, dx = \int_0^1 \left\{ \int_0^\infty e^{-t} A^*(tx) \, dt \right\}^2 \, dx = \int_0^i \left( \int_0^\infty e^{-\frac{u}{x}} A^*(s) \, ds \right)^2 \frac{1}{x^2} \, dx
\]

\[
= \int_0^\infty \left( \int_0^\infty e^{-sy} A^*(s) \, ds \right)^2 \, dy = \int_0^\infty \left( \int_0^\infty e^{-s(u+1)} A^*(s) \, ds \right)^2 \, du
\]

\[
(3.6) \quad \int_0^\infty \left( \int_0^\infty e^{-su} f(s) \, ds \right)^2 \, du = \int_0^\infty \int_0^\infty \frac{f(s) f(t)}{s + t} \, ds \, dt
\]
where \( f(x) = e^{-x} A^*(x) \). By Corollary 2.3, the inequality (3.5) follows from (3.6) at once. □


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