Non-Archimedian GP-Spaces

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Abstract

We study non-archimedean locally convex spaces in which every limited set is compactoid. In particular, we are interested in spaces of continuous functions.

1 Preliminaries

Throughout this paper K is a non-archimedean valued field that is complete for the metric induced by the non-trivial valuation | . |. Also, E,F are Hausdorff locally convex spaces over K.

A subset B of E is called compactoid if for every zero-neighbourhood U in E there exists a finite set $S \subseteq E$ such that $B \subseteq \text{co}S + U$, where coS is the absolutely convex hull of S.

Obviously every compactoid set is bounded, and spaces in which all the bounded subsets are compactoid have been studied in [5] and [6].

An other interesting subclass of the class of the bounded subsets of E consists of the limited sets (Definition 2.1). It turns out that every compactoid subset is limited and therefore it is quite natural to study the spaces E in which every limited set is compactoid. We call them Gelfand-Philips spaces (GP-spaces) following Lindström and Schlumprecht who studied such spaces in the complex case (see [10]).

The non-archimedean situation is however completely different from the classical one (Remark 2.5). In fact, in our case there are "much more" GP-spaces (Theorem 2.8). In particular - and this is the main objective of this paper- we show that most of the interesting non-archimedean functions spaces are GP-spaces.

*Research partially supported by Comision Mixta Caja Cantabria-Universidad de Cantabria
Received by the editors November 1993
Communicated by R. Delanghe
AMS Mathematics Subject Classification : 46S10
Keywords : $p$-adic functional analysis, locally convex spaces, GP-spaces.

For unexplained terms, notations and background we refer to [15] (locally convex spaces), [16] (normed spaces) and [4] (tensor products and nuclearity).

## 2 Limited sets and GP-spaces

**Definition 2.1** (Compare [10])

A bounded subset $B$ of $E$ is called **limited** in $E$, if every equicontinuous $\sigma(E',E)$-null sequence in $E'$ converges to zero uniformly on $B$.

Using the natural identification of the $\sigma(E',E)$-null sequences in $E'$ with the continuous linear maps from $E$ to $c_0$ ([2], Lemma 2.2) along with the form of the compactoid subsets of $c_0$ ([11], Proposition 2.1), we obtain:

**Lemma 2.2** A bounded subset $B$ of $E$ is limited in $E$ if and only if for each continuous linear map $T$ from $E$ to $c_0$, $T(B)$ is compactoid in $c_0$.

From this Lemma we easily derive,

**Proposition 2.3**

i) Every compactoid subset of $E$ is limited in $E$.

ii) If $B$ is limited in $E$ and $T \in L(E,F)$, then $T(B)$ is limited in $F$ (where $L(E,F)$ denotes the vector space of all continuous linear maps from $E$ to $F$).

iii) If $B$ is limited in $E$ and $D \subset B$, then $D$ is limited in $E$.

iv) Let $M$ be a subspace of $E$ and $B \subset M$. If $B$ is limited in $M$ then $B$ is limited in $E$. The converse is also true when $M$ is complemented or dense in $E$ (For an example showing that the converse is not true in general, see Remark 2.9).

It follows from Lemma 2.2 that if every continuous linear map from $E$ to $c_0$ is compact, then every bounded subset of $E$ is limited. In particular, if the valuation on $K$ is dense, we have

**Corollary 2.4** If the valuation on $K$ is dense then the unit ball of $l^\infty$ is limited (non-compactoid) in $l^\infty$.

**Remark 2.5** Corollary 2.4 shows that, for densely valued fields, the behaviour of limited sets in non-archimedean analysis is in sharp contrast with the one in locally convex spaces over the real or complex field. For this difference compare e.g. [1], Proposition, property 6, [8], Theorem 1 and [9], Proposition 1) with our results.

We'll see in Theorem 2.8.iii) that this difference is even more striking when the valuation on $K$ is discrete.

**Definition 2.6** Compare [10])

A locally convex space $E$ is called a **Gelfand-Philips** space (GP-space in short) if every limited set in $E$ is compactoid.

The following is easily seen:
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Proposition 2.7.

i) A subspace of a GP-space is a GP-space.

ii) The product of a family of GP-spaces is a GP-space.

Theorem 2.8.

i) Every locally convex space \( E \) of countable type is a GP-space.

ii) Every Banach space \( E \) with a base is a GP-space.

iii) If the valuation on \( K \) is discrete then every locally convex space over \( K \) is a GP-space.

PROOF

i) From Lemma 2.2 it follows that \( c_0 \) (and hence every normed space of countable type) is a GP-space. Then use the fact that \( E \) can be considered as a subspace of \( \prod_{p \in \mathcal{P}} E_p \), where \( \mathcal{P} \) is a family of seminorms determining the topology of \( E \) and for each \( p \in \mathcal{P}, E_p \) is the normed space \( E/\ker p \). Now all the \( E_p \) are of countable type. Then apply Proposition 2.7.

ii) Let \( A \subseteq E \) be limited. We can assume that \( A \) is absolutely convex. It suffices to prove that every countable subset \( B \) of \( A \) is compactoid. Let \([B]\) stand for the closed linear hull of \( B \). Then ([16] Corollary 3.18) \([B]\) is complemented in \( E \) and so, by Proposition 2.3.iv) we have that \( B \) is limited in \([B]\). By i) \( B \) is compactoid in \([B]\) and hence in \( E \).

iii) Again use the fact that \( E \subseteq \prod_{p \in \mathcal{P}} E_p \) where now each of the spaces \( E_p \) has a base ([16], Theorem 5.16). Then apply ii) and Proposition 2.7.

Remark 2.9 Property iv) of Proposition 2.3 is not true in general. For example, let the valuation on \( K \) be dense, take \( E = l^\infty, M = c_0 \) and \( B \) the unit ball in \( c_0 \). Then, apply 2.4, 2.3.i) and 2.8.i).

3 Spaces of continuous functions

Let \( X \) be a Hausdorff zero-dimensional topological space. We consider the following \( K \)-valued function spaces:

\( \text{PC}(X) \): The space of all continuous functions \( f : X \to K \) for which \( f(X) \) is precompact, endowed with the topology \( \tau_u \) of uniform convergence.

\( \text{C}(X) \): The space of all continuous functions \( X \to K \) endowed with the compact open topology \( \tau_c \).

\( \text{BC}(X) \): The space of all bounded continuous functions \( X \to K \), endowed with the uniform topology \( \tau_u \) or with the strict topology \( \tau_\beta \). This last one is the topology generated by the seminorms \( p_\phi(f) = \sup_{x \in X} | \phi(x) f(x) |, \) where \( \phi : X \to K \) is a bounded function vanishing at infinity.

Since \( \text{PC}(X) \) is a Banach space with a base ([16] Theorem 3.4) we obtain immediately from 2.8.i),

Theorem 3.1 \( \text{PC}(X) \) is a GP-space.

We now tackle the GP-property for \( \text{C}(X) \) and \( \text{BC}(X) \).
Lemma 3.2 Let $\mathcal{K}$ be a compact subset of $X$. Then, for every clopen set $G$ in $\mathcal{K}$ there exists a clopen set $U_G$ in $X$ such that $G = U_G \cap \mathcal{K}$.

**PROOF**
Let $\tau_X$ be the original topology on $X$ and $\tau_{\mathcal{K}}$ the trace of $\tau_X$ on $\mathcal{K}$.

Let $G \subseteq \mathcal{K}$ be $\tau_{\mathcal{K}}$-clopen. Clearly, there exists $U \subseteq X$, $\tau_X$-open, such that $G = \mathcal{K} \cap U$. Also, for each $a \in G \subseteq U$, there exists a $\tau_X$-clopen set $W_a$ in $X$ with $a \in W_a \subseteq U$. Then use a compactness argument.

**Theorem 3.3** (Compare [12], Theorem 3.3) For a set $\mathcal{F} \subset C(X)$, the following properties are equivalent:

i) $\mathcal{F}$ is compactoid in $C(X)$.

ii) For every compact set $\mathcal{K} \subset X$ the set $\mathcal{F} \mid \mathcal{K}$ is compactoid in $C(\mathcal{K})$ (where $\mathcal{F} \mid \mathcal{K}$ is the set of the restrictions $f \mid \mathcal{K}$ of $f$ to $\mathcal{K}$ with $f \in \mathcal{F}$).

**PROOF**

i) $\Rightarrow$ ii): This follows directly from the fact that, for each compact set $\mathcal{K} \subset X$, the restriction map $C(X) \to C(\mathcal{K}): f \mapsto f \mid \mathcal{K}$ is linear and continuous.

ii) $\Rightarrow$ i): Let $U$ be a zero-neighbourhood in $C(X)$. We can assume that $U$ has the form

$$U = \{f \in C(X) : \sup_{x \in X} |f(x)| \leq \epsilon\}, \quad \epsilon > 0, \quad \mathcal{K} \text{ compact subset of } X.$$

We have to find $f_1, \ldots, f_r \in C(X)$ such that

$$\mathcal{F} \subseteq \text{co}\{f_1, \ldots, f_r\} + U.$$  

(1)

Put $U_{\mathcal{K}} = \{g \in C(\mathcal{K}) : \sup_{x \in \mathcal{K}} |g(x)| \leq \epsilon\}$. Then, since $\mathcal{F} \mid \mathcal{K}$ is compactoid in $C(\mathcal{K})$, there exist $g_1, \ldots, g_r \in C(\mathcal{K})$ such that

$$\mathcal{F} \mid \mathcal{K} \subseteq \text{co}(g_1, \ldots, g_r) + U_{\mathcal{K}}.$$  

(2)

Fix $m \in \{1, \ldots, r\}$ and put $V = \{e \in K : |e| \leq \epsilon\}$. Since $g_m(\mathcal{K})$ is compact in $K$, there are $e_m^1, \ldots, e_m^s \in K$ such that the sets $e_m^i + V, \ldots, e_m^i + V$ are disjoint and

$$g_m(\mathcal{K}) \subseteq (e_m^1 + V) \cup \ldots \cup (e_m^s + V).$$

Then $\{\mathcal{K}_m^1, \ldots, \mathcal{K}_m^s\}$, where $\mathcal{K}_m^i = \{x \in \mathcal{K} : g_m(x) \in e_m^i + V\}$ ($i = 1, \ldots, s$), constitutes a partition of $\mathcal{K}$ consisting of $\tau_{\mathcal{K}}$-clopen subsets of $\mathcal{K}$. Hence, for each $m \in \{1, \ldots, r\}$ the locally constant function $g_m' : \mathcal{K} \to K$ defined by $g_m'(x) = e_m^i$ for $x \in \mathcal{K}_m^i$ is continuous and it has the property $\sup_{x \in \mathcal{K}} |g_m(x) - g_m'(x)| \leq \epsilon$. So (2) can be changed into

$$\mathcal{F} \mid \mathcal{K} \subseteq \text{co}(g_1', \ldots, g_r') + U_{\mathcal{K}}.$$  

By lemma 3.2, each of the functions $g_m'$ has a locally constant continuous extension $f_m : X \to K$ ($m = 1, \ldots, r$). Then, $f_1, \ldots, f_r$ satisfy (1) and we are done.
Corollary 3.4 \( C(X) \) is a GP-space.

PROOF

Let \( \mathcal{F} \subseteq C(X) \) be a limited set. Then (Proposition 2.3.ii)) for each compact set \( K \subseteq X, \mathcal{F} \mid K \) is limited in \( C(K) \) and hence compactoid in \( C(K) \) (Theorem 3.1). Now apply Theorem 3.3.

Corollary 3.5 \( BC(X), \tau_\beta \) is a GP-space.

PROOF

Let \( \mathcal{F} \subseteq BC(X) \) be a \( \tau_\beta \)-limited set. Since \( \tau_\beta \) is finer than \( \tau_c \) ( [7], Proposition 2.10) we obtain from Proposition 2.3.i) that \( \mathcal{F} \) is \( \tau_c \)-limited in \( BC(X) \). By Proposition 2.7.i) and Corollary 3.4 we have that \( \mathcal{F} \) is compactoid in \( BC(X), \tau_c \). Now apply Corollary 2.9.a) and Proposition 2.11 of [7].

The picture changes completely when we endow \( BC(X) \) with the uniform topology \( \tau_u \). We have:

Theorem 3.6 If the valuation on \( K \) is dense (Compare 2.8.iii)), then \( BC(X), \tau_u \) is a GP-space if and only if \( X \) is pseudocompact.

PROOF

If \( X \) is pseudocompact one verifies that \( BC(X) = PC(X) \). Then apply Theorem 3.1.

If \( X \) is not pseudocompact, then \( BC(X), \tau_u \) contains a subspace which is linearly homeomorphic to \( l^\infty \) (see [14], proof of Corollary 2.7). Then apply Proposition 2.7.i) and Corollary 2.4.

In [4] (resp. [3]) the nuclearity of the locally convex space \( C(X), \tau_c \) (resp. \( BC(X), \tau_\beta \)) is characterized. Combining those results with Corollaries 3.4 and 3.5 we obtain:

Theorem 3.7 The following are equivalent:

i) \( C(X), \tau_c \) is nuclear.

ii) \( BC(X), \tau_\beta \) is nuclear.

iii) Every \( \tau_c \)-bounded subset of \( C(X) \) is limited.

iv) Every \( \tau_\beta \)-bounded subset of \( BC(X) \) is limited.

We now consider the case where the continuous functions have their values in a polar complete locally convex Hausdorff space \( E \). We define the function spaces \( PC(X,E), \tau_u \); \( C(X,E), \tau_c \); \( BC(X;E), \tau_u \) and \( BC(X,E), \tau_\beta \) in the canonical way and we then have:

Theorem 3.8

i) \( PC(X,E) \) is a GP-space if and only if \( E \) is a GP-space.

ii) \( C(X,E) \) is a GP-space if and only if \( E \) is a GP-space.

iii) \( BC(X,E), \tau_\beta \) is a GP-space if and only if \( E \) is a GP-space.

iv) If the valuation on \( K \) is dense, then \( BC(X,E), \tau_u \) is a GP-space if and only if \( X \) is pseudocompact and \( E \) is a GP-space.
PROOF
The proof of ii), iii), iv) is essentially the same as in the K-valued case.
For the proof of i) one needs [13] Theorem 1.3 and the following result.

**Theorem 3.9** Let $E$ and $F$ be complete, polar, locally convex Hausdorff spaces. Then $E \otimes F$, the completion of the tensor product for its canonical topology, is a GP-space if and only if $E$ and $F$ are GP-spaces.

**PROOF**
The consecutive steps are:

i) If $E$ is quasicomplete, then $(E'_e)' = E$, where $E'_e$ is the dual $E'$ of $E$ endowed with the topology $\tau_{cp}$ of uniform convergence on the compactoid subsets of $E$.

ii) If $E$ is quasicomplete. Let $H \subset L(E'_e, F)$ be such that $H(U^\varepsilon)$ is compactoid in $F$ for all zero-neighbourhoods $U$ in $E$ and $H^\varepsilon(V^\varepsilon)$ is compactoid in $E$ for all zero-neighbourhoods $V$ in $F$. Then, $H$ is compactoid in $L_\varepsilon(E'_e, F)$, where the $\varepsilon$ means that we consider on $L(E'_e, F)$ the topology of uniform convergence on the equicontinuous subsets of $E'$.

iii) If $E$ and $F$ are GP-spaces, $E$ quasicomplete, then $L_\varepsilon(E'_e, F)$ is also a GP-space. (As a consequence, $E \otimes F$ is a GP-space if and only if $E$ and $F$ are GP-spaces).

iv) If $E$ and $F$ are complete, then so is $L_\varepsilon(E'_e, F)$. The Theorem is then a direct consequence of this result.

The proofs of ii) and iii) are similar to the archimedean case (see [10]) and are therefore omitted. The proof of iv) is standard. So let us prove i):

Since $E$ is quasicomplete and by [15], Theorem 5.12, it follows that $\tau_{cp}$ is the topology of uniform convergence on the sets $A \subset E$ which are absolutely convex, compactoid, edged and $\sigma(E, E')$-complete. Since the family of these sets form a special covering of $E$ (see [15], Definition 7.3), the conclusion follows from [15], Proposition 7.4.

Note that i) is not true in general.

Indeed, take $E = c_0$ and let $x_1, x_2 \ldots$ be a non-convergent Cauchy sequence in $E$. Then, the map $T : E' \to K : f \mapsto \lim_n f(x_n)$ is an element of $(E'_e)'$. But $T$ cannot be represented by an element of $E$.

**References**


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