Collineations of the Subiaco generalized quadrangles

S. E. Payne

Dedicated to J. A. Thas on his fiftieth birthday

Abstract

Each generalized quadrangle (GQ) of order \((q^2, q)\) derived in the standard way from a conical flock via a \(q\)-clan with \(q = 2^e\) has subquadrangles of order \(q\) associated with a family of \(q + 1\) (not necessarily projectively equivalent) ovals in \(\text{PG}(2, q)\). A new family of these GQ is announced in [1] and named the Subiaco GQ. We begin a study of their collineation groups. When \(e\) is odd, \(e \geq 5\), the group is determined. In the standard notation for the GQ, the collineation group is transitive on the lines through the point \((\infty)\). As a corollary we have that up to the usual notions of equivalence, just one conical flock, one oval in \(\text{PG}(2, q)\), and one subquadrangle of order \((q, q)\) arise.

1 Introduction

The objects studied in this paper are introduced in [1], and we thank its authors for making their work available to us as it was being developed. Moreover, Tim Penttila and Gordon Royle helped us eliminate a serious error in an early version of this work.

Let \(F = \text{GF}(q)\), \(q = 2^e\). For each \(t \in F\), let \(A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}\) be a \(2 \times 2\) matrix over \(F\). Put \(\mathcal{C} = \{A_t : t \in F\}\). Then \(\mathcal{C}\) is a \(q\)-clan provided \(A_t - A_s\) is anisotropic (i.e., \(\alpha(A_t - A_s)\alpha^T = 0\) if and only if \(\alpha = (0, 0)\)) whenever \(t, s \in F\), \(t \neq s\). This holds if and only if \((x_t - x_s)(z_t - z_s)(y_t - y_s)^{-2}\) has trace 1 whenever \(s \neq t\). From

Received by the editors in February 1994

AMS Mathematics Subject Classification : Primary 51E12, Secondary 05B25

Keywords : Generalized quadrangles, \(q\)-clan geometry, flocks

now on we assume that $C$ is a $q$-clan, so the three maps $t \mapsto x_t, t \mapsto y_t, t \mapsto z_t$ are all permutations. Put $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. And for $A_t \in C$, put $K_t = A_t + A_t^T = y_t P$.

There is a standard construction of a generalized quadrangle (GQ) $S = S(C)$ as a coset geometry starting with the group

$$\mathcal{G} = \{ (\alpha, c, \beta) : \alpha, \beta \in F^2, c \in F \}$$

whose binary operation is given by

$$(\alpha, c, \beta) * (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta(\alpha')^T, \beta + \beta'),$$

and a certain 4-gonal family of subgroups. Specifically, put $\mathcal{A}(\infty) = \{ (0, 0, \beta) \in \mathcal{G} : \beta \in F^2 \}$, and for $t \in F$, $\mathcal{A}(t) = \{ (\alpha, \alpha A_t \alpha^T, \alpha K_t) \in \mathcal{G} : \alpha \in F^2 \}$. Put $\mathcal{F} = \{ \mathcal{A}(t) : t \in F \cup \{ \infty \} \}$, and $C = \{ (0, 0, \beta) \in \mathcal{G} : c \in F \}$. For $A \in \mathcal{F}$, put $A^* = AC$. Then $\mathcal{F}$ is a 4-gonal family for $\mathcal{G}$ with associated groups (tangent spaces) $\mathcal{F}^* = \{ A^* : A \in \mathcal{F} \}$. We assume the reader is familiar with W. M. Kantor’s construction of a GQ $S(G, \mathcal{F})$ (cf. [3], [4], [8]). In [8], pp. 213–214, it is shown that for a fixed $t_0 \in F$, a new $q$-clan may be constructed so that each $A_t \in C$ is replaced with $A_t - A_{t_0}$, and the “new” GQ is isomorphic to the original. Then the new matrices may be reindexed so that $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

For two matrices $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}, B = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ over $F$, let $A \equiv B$ mean that $x = r, w = u,$ and $y + z = s + t$. So $A \equiv B$ if and only if $\alpha A_0 A^T = \alpha B 0 A^T$ for all $\alpha \in F^2$.

Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, F)$. For $A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} \in C$, put

$$\mathcal{A}_t = B A_t B^T \equiv \begin{pmatrix} a^2 x_t + ab y_t + b^2 z_t & (ad + bc) y_t \\ 0 & c^2 x_t + cd y_t + d^2 z_t \end{pmatrix}.$$  

Then

$$(\alpha, c, \beta) \mapsto (\alpha B^{-1}, c, \beta B^T)$$  

is an automorphism of $\mathcal{G}$ that replaces $\mathcal{F}$ with a 4-gonal family derived from the $q$-clan $\mathcal{G} = \{ \mathcal{A}_t : t \in F \}$ and that produces a GQ isomorphic to the original.

First, by reindexing the members of $C$ we may assume $x_t = t$ for all $t \in F$. So $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 1 & y_t \\ 0 & z_t \end{pmatrix}$. Then using $B = \begin{pmatrix} 1 & 0 \\ 0 & y_t^{-1} \end{pmatrix}$ in equation (2), we may assume $A_1 = \begin{pmatrix} 1 & 1 \\ 0 & \delta \end{pmatrix}$, where $\delta \in F$ is some element with trace 1. We again reindex the members of $C$ to obtain $y_t = t$ (probably destroying $x_t = t$) for all $t \in F$. So without loss of generality we may assume that the $q$-clan $C$ has been normalized to satisfy the following:
Collineations of the Subiaco generalized quadrangles

\[ A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad A_1 = \begin{pmatrix} 1 & 1 \\ 0 & \delta \end{pmatrix} \text{(with } \text{tr}(\delta) = 1); \quad A_t = \begin{pmatrix} x_t & t \\ 0 & z_t \end{pmatrix}, \quad t \in F. \quad (3) \]

From [5] recall the following notation: For \( \alpha \in F^2, [\alpha]_\infty = (\overrightarrow{0}, 0, \alpha) \in \overrightarrow{G}; \) for \( t \in F, [\alpha]_t = (\alpha, \alpha A_t \alpha^T, \alpha K_t); \) for \( c \in F, [c] = (\overrightarrow{0}, c, \overrightarrow{0}) \in \overrightarrow{G}. \) For \( t, u \in F \cup \{ \infty \}, t \neq u, \) put \( ([\alpha]_t, [c], [\beta]_u) := [\alpha]_t * [c] * [\beta]_u. \) A simple computation shows that

\[ ([\alpha]_\infty, [c], [\beta]_0) * ([\alpha']_\infty, [c'], [\beta']_0) = ([\alpha + \alpha']_\infty, [c + c' + \beta(\alpha')^T, [\beta + \beta']_0). \quad (4) \]

And with \( \gamma = \alpha K_t, \)

\[ [\alpha]_t = (\alpha, \alpha A_t \alpha^T, \alpha K_t) = ([\alpha K_t]_\infty, [\alpha A_t \alpha^T], [\alpha]_0) \]

\[ = ([\gamma]_\infty, [\gamma K_t^{-1} A_t K_t^{-1} \gamma^T], [\gamma K_t^{-1}]_0). \quad (5) \]

Since in the original description of \( \overrightarrow{G}, (\alpha, c, \beta) = ([\alpha]_0, [c], [\beta]_\infty), \) it follows that by interchanging the roles of \( \infty \) and 0 in the description of \( \overrightarrow{G} \) the matrix \( A_t \in C \) is replaced with \( K_t^{-1} A_t K_t^{-1} \equiv y_t^{-2} \begin{pmatrix} z_t & y_t \\ 0 & x_t \end{pmatrix}. \) Now by using \( B = P \) in equation (2), we have that \( \hat{C} = \left\{ A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \{ y_t^{-2} A_t : 0 \neq t \in F \} \) is a q-clan associated with a GQ isomorphic to that derived from \( C. \) By combining this operation with the shift \( \overline{A}_t = A_t - A_{t_0} \) mentioned earlier, we obtain

\[ \hat{C} = \{ (y_t - y_{t_0})^{-2} (A_t - A_{t_0}) : t_0 \neq t \in F \} \cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \]

(6)

is a q-clan associated with a GQ isomorphic to (essentially the same as) that derived from \( C, \) but recoordinitized so that the line \([A(t_0)] \) of \( S(C) \) through the point \( (\infty) \) corresponds to the line \([A(\infty)] \) through \( (\infty) \) in \( S(\hat{C}). \) (Also see [7], [9].)

A truly satisfactory geometric interpretation of the construction of a GQ from a q-clan (equivalent to a conical flock by J. A. Thas [10]) must somehow explain the existence of these \( q + 1 \) distinct (and not always equivalent) q-clans. For the purpose of distinguishing flocks, it is important to note that there is a collineation of \( S(C) \) (fixing \( (\infty) \) and \( (\overrightarrow{0}, 0, \overrightarrow{0}) \)) moving the line \([A(t_0)] \) to the line \([A(\infty)] \) if and only if the flock associated with \( C \) is equivalent to that associated with the q-clan \( \hat{C} \) obtained in equation (6).

We now recall the slightly modified description of \( \overrightarrow{G} \) introduced in [6] (cf. also [4]). In characteristic 2, this revised version seems to us to be more natural and useful.

Let \( E = F(\zeta) = GF(q^2), \zeta^2 + \zeta + \delta = 0 \) (for some \( \delta \in F \) with \( \text{tr}(\delta) = 1). \) Then \( x \mapsto \overline{x} = x^\delta \) is the unique involutionary automorphism of \( E \) with fixed field \( F. \) Here \( \zeta + \overline{\zeta} = 1 \) and \( \zeta \overline{\zeta} = \delta. \) The element \( \alpha = a + b \zeta \in E (a, b \in F) \) is often (without notice) identified with the pair \( (a, b) \in F^2. \) For example, the inner product

\[ \alpha \circ \beta = \alpha \overline{\beta} + \overline{\alpha} \beta \]

(7)
on \( E \) as a vector space over \( F \) may also appear as \( \alpha \circ \beta = \alpha P \beta^T \). Note that \( \alpha \circ \beta = 0 \) if and only if \( \{ \alpha, \beta \} \) is \( F \)-dependent.

Now put \( G = \{(\alpha, \beta) : \alpha, \beta \in E = F^2, c \in F \} \) with binary operation

\[
(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \sqrt{\beta \circ \alpha'}, \beta + \beta'). \tag{8}
\]

It is straightforward to check that \( \overline{G} \to G : (\alpha, c, \beta) \mapsto (\alpha, \sqrt{c}, \beta P) \) is an isomorphism mapping the 4-gonal family \( \overline{\mathcal{F}} \) for \( \overline{G} \) to a 4-gonal family \( \mathcal{F} \) of \( G \) defined as follows:

\[
A(\infty) = \{ (\overline{0}, 0, \beta) \in G : \beta \in E \}; \tag{9}
\]

\[
A^*(\infty) = \{ (\overline{0}, c, \beta) \in G : c \in F, \beta \in F^2 \}.
\]

And for \( t \in F \),

\[
A(t) = \{ (\alpha, \sqrt{\alpha t \alpha^T}, y_t \alpha) \in G : \alpha \in F \}, \tag{10}
\]

\[
A^*(t) = \{ (\alpha, c, y_t \alpha) \in G : \alpha \in E, c \in F \}.
\]

Clearly \( \mathcal{F} = \{ A(t) : t \in F \cup \{ \infty \} \} \) yields a GQ \( S(G, \mathcal{F}) \) isomorphic to \( S(\overline{G}, \overline{\mathcal{F}}) \).

The revised description \( S(G, \mathcal{F}) \) makes it easy to recognize subquadrangles.

2 Subquadrangles and ovals

Let \( \mathcal{C} \) be a \( q \)-clan (normalized as in equation (3)) with corresponding 4-gonal family \( \mathcal{F} \) for \( G \) (in the revised form just given), etc., and let \( S = S(G, \mathcal{F}) \) be the associated GQ. For \( \overline{0} \neq \alpha \in E \), put

\[
G_\alpha = \{ (a\alpha, c, b\alpha) \in G : a, b, c \in F \}. \tag{11}
\]

Since \( \alpha \circ \beta = 0 \) if \( \beta = c\alpha \), \( c \in F \), in \( G_\alpha \) we have

\[
(a\alpha, c, b\alpha) \cdot (a'\alpha, c', b'\alpha) = ((a + a')\alpha, c + c', (b + b')\alpha). \tag{12}
\]

So \( G_\alpha \) is an elementary abelian group with order \( q^3 \). Define the following subgroups of \( G_\alpha \):

\[
A_\alpha(\infty) = A(\infty) \cap G_\alpha = \{ (\overline{0}, 0, ba) \in G : b \in F \} ; \tag{13}
\]

\[
A^*_\alpha(\infty) = A^*(\infty) \cap G_\alpha = \{ (\overline{0}, c, ba) \in G : c, b \in F \}.
\]

And for \( t \in F \),

\[
A_\alpha(t) = A(t) \cap G_\alpha = \{ (a\alpha, a\sqrt{A_t \alpha^T}, at\alpha) \in G : a \in F \}; \tag{14}
\]

\[
A^*_\alpha(t) = A^*(t) \cap G_\alpha = \{ (a\alpha, c, at\alpha) \in G : a, c \in F \}.
\]

Here \( \mathcal{F}_\alpha = \{ A_\alpha(t) : t \in F \cup \{ \infty \} \} \) is immediately seen to be a 4-gonal family for \( G_\alpha \). Moreover, by [6] we may view \( \mathcal{S}_\alpha = S(G_\alpha, \mathcal{F}_\alpha) \) as a subquadrangle (of order
Corollaries of the Subiaco generalized quadrangles

We say that the map $t \mapsto s$ might avoid $F$ with the 3-dimensional $L$-linear space $G$. Hence the oval $O$ is an oval in the projective plane naturally associated with the 3-dimensional $F$-linear space $G$.

Consider the three projective points $p_1 = (\alpha, 0, \bar{0})$, $p_2 = (\bar{0}, 1, \bar{0})$, $p_3 = (\bar{0}, 0, \alpha)$. The scalar triple $(1, \sqrt{\alpha A_t \alpha^T}, t)$ on the points $p_1$, $p_2$ and $p_3$ results in $(\alpha, 0, \bar{0}) \cdot (\bar{0}, \sqrt{\alpha A_t \alpha^T}, 0) \cdot (0, 0, t \alpha) = (\alpha, \sqrt{\alpha A_t \alpha^T}, t \alpha) \leftrightarrow A_\alpha(t)$ considered as a projective point. Hence the oval $O_\alpha$ is the set of $q + 1$ points represented by the following set of triples of coordinates:

$$O_\alpha = \{(0, 0, 1) \cup (1, \sqrt{\alpha A_t \alpha^T}, t) : t \in F\},$$

with nucleus $(0, 0, 1)$. Note. This language is suggested by the term $O$-polynomial in [2], except that we find it more convenient NOT to require that $\gamma$ be normalized so that $\gamma : 1 \mapsto 1$. (We could also avoid $\gamma : 0 \mapsto 0$, but this holds in all the specific examples we consider.)

For $\alpha = (a_1, a_2) \neq (0, 0)$, $A_t = \begin{pmatrix} x_t & t \\ 0 & z_t \end{pmatrix}$, the above becomes

$$t \mapsto a_1 \sqrt{x_t} + \sqrt{a_1 a_2} \sqrt{1} + a_2 \sqrt{z_t}$$

is an $O$-permutation.
3 Collineations

Suppose without loss of generality that the matrices of the $q$-clan $C$ are given in
the form $A_t = \begin{pmatrix} f(t) & t^{1/2} \\ 0 & g(t) \end{pmatrix}$, $t \in F$, with $f(0) = g(0) = 0$ and $f(1) = 1$. Then
$K_t^{-1}A_tK_t^{-1} = \begin{pmatrix} t^{-1}g(t) & t^{-1/2} \\ 0 & t^{-1}f(t) \end{pmatrix}$. Starting with the paragraph following equation (5), it is straightforward to prove the following.

Theorem 3.1 If $f(t^{-1}) = t^{-1}g(t)$ (equivalently, $g(t^{-1}) = t^{-1}f(t)$), then the automorphism $\theta : G \to G : (\alpha, c, \beta) \mapsto (\beta P, c + \sqrt{\alpha P \beta^T}, \alpha P)$ induces a collineation of $S(G, F)$ that interchanges $A(\infty)$ and $A(0)$, and interchanges $A(t)$ and $A(t^{-1})$ for $0 \neq t \in F$.

We now recall a result from [7], but modified to fit our group $G$ introduced at the end of section 1.

Theorem 3.2 Let $C = \{A_t : t \in F\}$ be a $q$-clan with $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Let $\theta$ be a collineation of the GQ $S = S(G, F)$ derived from $C$ which fixes the point $(\infty)$, the line $[A(\infty)]$ and the point $(0, 0, 0)$. Then the following must exist:

(i) A permutation $t \mapsto t'$ of the elements of $F$;

(ii) $\lambda \in F$, $\lambda \neq 0$;

(iii) $\sigma \in \text{Aut}(F)$;

(iv) $D \in \text{GL}(2, q)$ for which $A_{t'} \equiv \lambda D^T A_t^\sigma D - A_0$ for all $t \in F$.

Conversely, given $\sigma$, $D$, $\lambda$ and a permutation $t \mapsto t'$ satisfying the above conditions, the following automorphism $\theta$ of $G$ induces a collineation of $S(G, F)$ fixing $(\infty)$, $[A(\infty)]$ and $(0, 0, 0)$:

$$\theta = \theta(\sigma, D, \lambda) : G \to G : (\alpha, c, \beta) \mapsto (\lambda^{-1}\alpha^\sigma D^{-T}, \lambda^{-1/2}c^\sigma + \lambda^{-1}\sqrt{\alpha^\sigma D^{-T} A_0 D^{-1}(\alpha^\sigma)^T},$$
$$\beta^\sigma PDP + \lambda^{-1}y_0^\sigma \alpha^\sigma D^{-T}).$$

Theorem 3.3 For $A_t = \begin{pmatrix} f(t) & t^{1/2} \\ 0 & g(t) \end{pmatrix}$, the conditions in theorem 3.2 are equivalent to having a permutation $t \mapsto t'$, $0 \neq \lambda \in F$, $D = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \text{GL}(2, F)$, $\sigma \in \text{Aut}(F)$, for which

(i) $t' = \lambda^2(ad + bc)^2 t^\sigma + 0'$, for all $t \in F$.

(ii) $f(t') = \lambda^2 a^2 f(t)^\sigma + ab \sigma + b^2 g(t)^\sigma + f(0')$, for all $t \in F$.

(iii) $g(t') = \lambda[c^2 f(t)^\sigma + cd t^\sigma + d^2 g(t)^\sigma + g(0')]$, for all $t \in F$. (21)
For completeness, we note that right multiplication by elements of \( G \) induces a group of \( q^5 \) collineations of \( \mathcal{S}(G, F) \) acting regularly on the set of points not collinear with \( \langle \infty \rangle \) and fixing each line through \( \langle \infty \rangle \).

The Subiaco GQ introduced in \cite{1} all have \( q \)-clans of the form used in theorem 3.3 with the following additional specializations:

\begin{align*}
    (i) & \quad f(t) = \frac{f(t)}{k(t)} + Ht^{1/2}, \ t \in F; \\
    (ii) & \quad g(t) = \frac{g(t)}{k(t)} + Kt^{1/2}, \ t \in F; \text{where} \\
    (iii) & \quad k(t) \text{ is the square of an irreducible quadratic polynomial (say } k(t) = t^4 + ct^2 + c_0); \\
    (iv) & \quad f'(t) \text{ and } g'(t) \text{ are polynomials over } F \text{ of degree at most } 4 \\
    & \quad \text{with } f'(0) = g'(0) = 0 \text{ (and } f(1) = 1); \text{ and} \\
    (v) & \quad H \text{ and } K \text{ are nonzero elements of } F.
\end{align*}

Then the conditions of theorem 3.3 can be rewritten.

**Theorem 3.4** Suppose \( A_t = \begin{pmatrix} \frac{f(t)}{k(t)} + Ht^{1/2} & t^{1/2} \\ 0 & \frac{g(t)}{k(t)} + Kt^{1/2} \end{pmatrix}, \ t \in F, \text{ with the conditions of } (22) \text{ satisfied. Then the conditions in equation } (21) \text{ of theorem 3.3 take on the following form:}\

\begin{align*}
    (i) & \quad t' = \lambda^2(ad + bc)^2t^\sigma + 0'. \\
    (ii) & \quad f'(t')k(t')k(0') + \lambda(a^2f(t')^\sigma + b^2g(t')^\sigma)k(t')k(0') + k(t')k(t)g'(0') \\
    & \quad + \lambda(k(t')k(t)k(0'))[a^2H^\sigma + ab + b^2K^\sigma + H(ad + bc)]t^{\sigma/2} = 0 \\
    (iii) & \quad g'(t')k(t')k(0') + \lambda(c^2f(t')^\sigma + d^2g(t')^\sigma)k(t')k(0') + k(t')k(t)g'(0') \\
    & \quad + \lambda(k(t')k(t)k(0'))[c^2H^\sigma + cd + d^2K^\sigma + K(ad + bc)]t^{\sigma/2} = 0.
\end{align*}

In equations (23)(ii) and (iii) replace \( t' \) with \( \lambda^2(ad + bc)^2t^\sigma + 0' \) and write the resulting expressions as polynomials in \( t^\sigma \). Now square both sides. The terms touched by \( t^{\sigma/2} \) (before squaring) have odd positive integer exponents \( \leq 17 \). The other terms have even exponents \( \leq 16 \). Since \( e \geq 5 \), the coefficients on \( t^{\sigma/2} \) in equations (23)(ii) and (iii) must be zero. Hence equation (23) can be replaced with

\begin{align*}
    (i) & \quad (a^2H^\sigma + ab + b^2K^\sigma)/H = ad + bc = (c^2H^\sigma + cd + d^2K^\sigma)/K \not\equiv 0. \\
    (ii) & \quad f'(t')k(t')k(0') + \lambda[a^2f(t')^\sigma + b^2g(t')^\sigma]k(t')k(0') = k(t')k(t)g'(0'). \\
    (iii) & \quad g'(t')k(t')k(0') + \lambda[c^2f(t')^\sigma + d^2g(t')^\sigma]k(t')k(0') = k(t')k(t)g'(0').
\end{align*}

### 4 Subiaco GQ with \( q = 2^e, \ e \text{ odd} \)

From now on we assume \( F = GF(q), \ q = 2^e, \ e \text{ odd} \text{ and } e \geq 5 \text{.} \text{ So } 1 + t^2 + t^4 \not\equiv 0 \text{ for all } t \in F. \text{ } \mathcal{C} = \left\{ A_t = \begin{pmatrix} h(t) + t^{1/2} & t^{1/2} \\ 0 & t^2h(t) + t^{1/2} \end{pmatrix}; \ t \in F \right\}, \text{ where } h(t) = (t + t^2)/(1 + t^2 + t^4). \text{ Let } \mathcal{F} \text{ denote the corresponding } 4 \text{-gonal family for } G, \text{ and let } \mathcal{S} = \mathcal{S}(\mathcal{C}) = \mathcal{S}(G, \mathcal{F}) \text{ be the associated GQ. This example arises as a specialization of the general construction of Subiaco GQ in } [1].
Theorem 4.1 Each \( \sigma \in \text{Aut}(F) \) induces a collineation of \( S \) fixing \([A(\infty)]\) and mapping \([A(t)]\) to \([A(t^\sigma)]\) for \( t \in F \).

Proof. Clearly \( f(t^\sigma) = f(t') \) and \( g(t^\sigma) = g(t') \). In equation (21) put \( \lambda = a = d = 1, b = c = 0' = 0 \). Then the conditions are all satisfied with \( t' = t^\sigma \). \( \square \)

Theorem 4.2 There is a collineation of \( S \) interchanging \([A(\infty)]\) and \([A(0)]\) and interchanging \([A(t)]\) and \([A(t^{-1})]\) for \( 0 \neq t \in F \).

Proof. Check that \( f(t^{-1}) = t^{-1}g(t) \), with \( f(t) = h(t) + t^{1/2} \) and \( g(t) = t^2h(t) + t^{1/2} \), and use theorem 3.1.

From the form of \( f \) and \( g \) we know that \( S \) is not classical. (Alternatively, we will show that the group of collineations fixing the point \((\infty)\) and the line \([A(\infty)]\) is not transitive on the other lines through \((\infty)\).) Hence the point \((\infty)\) is fixed by the full collineation group of \( S \) (cf. \cite{9}). And because of theorem 4.1, to find all collineations fixing \([A(\infty)]\), it suffices to find all solutions of equation (24) (since the \( q \)-clan of this section has the form given in theorem 3.4 with \( \sigma = id \). And we use \( g'(t) = t^2f'(t), f'(t) = t + t^2, t' = \lambda^2(ad + bc)^2t + 0' \). Now compute \((t')^2\) times equation (24)(ii) added to equation (24)(iii), and divide by \( k(t') \) to obtain:

\[
\begin{align*}
\lambda(t + t^2)k(0')[a^2(\lambda^4(ad + bc)^4t^2 + (0')^2] \\
+ b^2t^2(\lambda^4(ad + bc)^4t^2 + (0')^2) + c^2 + d^2t^2] \\
= \lambda^4(ad + bc)^4t^2(1 + t^2 + t^4)f'(0'). 
\end{align*}
\] (25)

The coefficient on \( t \) in equation (25) is \( \lambda k(0')[a^2(0')^2 + c^2] \), implying

\[ c = a0'. \] (26)

The coefficient on \( t^2 \) is then \( \lambda k(0')\lambda^4(ad + bc)^4f'(0') \), implying \( f'(0') = 0 \). Hence

\[ 0' \in \{0, 1\}. \] (27)

The coefficient on \( t^5 \) is \( \lambda k(0')b^2\lambda^4(ad + bc) \), implying

\[ b = 0 \quad \text{and} \quad ad \neq 0. \] (28)

Then from equation (24)(i), \( a^2 = ad \), so

\[ a = d. \] (29)

Now equation (25) appears as \( \lambda(t + t^2)k(0')[a^2\lambda^4a^8t^2 + a^2t^2] = 0 \), from which we conclude

\[ \lambda a^2 = 1. \] (30)

We now have established the following:

\[
\begin{align*}
\text{(i)} & \quad 0' \in \{0, 1\} \\
\text{(ii)} & \quad c = a0' \\
\text{(iii)} & \quad b = 0 \neq a = d \\
\text{(iv)} & \quad \lambda a^2 = 1. \quad (31)
\end{align*}
\]

A straightforward check shows that if the conditions of equation (31) all hold, then the conditions of equation (24) all hold with \( t' = t+0' \). This establishes the following:
Theorem 4.3 The group of collineations of $S$ fixing $[A(\infty)]$ and $(\emptyset, 0, \emptyset)$ (and of course $(\infty)$) has order $2e(q - 1)$ and has $\{[A(0)], [A(1)]\}$ as an orbit.

At this point, including collineations induced by right multiplication by elements of $G$, we have a group of collineations of $S = S(G, \mathcal{F})$ with order $6e(q - 1)q^5$, and which as a permutation group acting on the set of indices of the lines through the point $(\infty)$ includes the following:

(i) For $\sigma \in \text{Aut}(F)$, $\sigma : \infty \mapsto \infty$ and $\sigma : t \mapsto t^\sigma$ for $t \in F$.
(ii) $\theta : \infty \mapsto 0$, and $\theta : t \mapsto t^{-1}$ for $0 \neq t \in F$.
(iii) $\phi : \infty \mapsto \infty$, and $\phi : t \mapsto t + 1$ for $t \in F$.

The set $\{\infty, 0, 1\}$ is invariant under the permutations exhibited in equation (32). $S_3 \cong \langle \theta, \phi \rangle$ has $\{\infty, 0, 1\}$ as one orbit, and for $t \in F \setminus \{0, 1\}$ has $\Omega_t = \{t, t + 1, (t + 1)^{-1}, t/(t + 1), (t + 1)/t, t^{-1}\}$ as an orbit. Note that $|\Omega_t| = 6$ since $q = 2^e$ with $e$ odd. But the automorphisms $\sigma \in \text{Aut}(F)$ act on the $\Omega_t$ differently for different $t$ and different $e$.

For example, when $e = 5$, there are five disjoint $\Omega_t$ on which $\text{Aut}(F)$ acts transitively: $\Omega_t \cap \Omega_{t'} \neq \emptyset$ if and only if $\sigma = \text{id}$. So in this case (i.e., $q = 32$), the collineation group of $S$ either has two orbits on the lines through the point $(\infty)$ (one of which is $\{[A(\infty)], [A(0)], [A(1)]\}$), or it has just one orbit. (We will show in section 5 that the full collineation group is transitive on the set of $q+1$ lines through $(\infty)$ for all odd $e \geq 5$.)

Note. If $3|e$, then $x^3 + x^2 + 1 = 0$ has a root $t_0 \in F$. In this case $\Omega_{t_0}$ is broken into two orbits under $\text{Aut}(F)$, since for each $t \in \Omega_{t_0}$, $t^2 = (t + 1)^{-1}$.

5 Recordinatizing the GQ

Start with the standard form of the $q$-clan given at the beginning of section 4. Fix $w \in F$. The idea is to let $w$ play the role of $\infty$, i.e., $w$ plays the role of $t_0$ in equation (6).

$$C^w = \left\{ A^w_t = \begin{pmatrix} h(t) + h(w) & (t + w)^{-1/2} \\ t + w & (t + w)^{-1/2} \\ 0 & \frac{t^2h(t) + w^2h(w)}{t + w} + (t + w)^{-1/2} \end{pmatrix} : w \neq t \in F \right\}$$

$$\cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$  \hspace{1cm} (33)

Put $x = (t + w)^{-1}$, so $t = w + x^{-1}$, and substitute into equation (33) to obtain

$$C^w = \left\{ A^w_x = \begin{pmatrix} f^w(x) + x^{1/2} \\ 0 \\ g^w(x) + x^{1/2} \end{pmatrix} : x \in F \right\},$$

where

(i) $f^w(x) = f'(x)/k(x)$, $g^w(x) = g'(x)/k(x)$, and
(ii) $f'(t) = t^4(1 + w^2 + w^4) + t^3(1 + w + w^4) + t(w + w^2)$,
(iii) $g'(t) = t^4(w^2 + w^4 + w^6) + t^3(w + w^4 + w^5) + t^2(1 + w^2 + w^3)$,
(iv) $k(t) = t^4(1 + w^2 + w^4)^2 + (t^2 + 1)(1 + w^2 + w^4)$.
If there is a collineation of the GQ $S$ (with $q$-clan in standard form) mapping $[A(\infty)]$ to $[A(w)]$, then there must be a collineation $\theta = \theta(\sigma, D, \lambda)$ (in the new coordinatization) which is an involution fixing $[A(\infty)]$ and interchanging all other lines through $(\infty)$ in pairs. So there is an involution $\theta$ of the form

$$\theta : (\alpha, c, \beta) \mapsto \begin{pmatrix} \alpha + \beta & \sigma D^{-T} & \lambda^{-1/2} & \sqrt{\lambda^{\alpha} D^{-T} A_{w} D^{-1} (\sigma D^{-T} \beta P D P + \lambda^{-1} y_{0} \alpha^{\alpha} D^{-T})} \end{pmatrix},$$

with $D = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GF(2, q)$.

Since theorem 3.4 applies (with the new coordinates), equation (24)(i) holds with $H = K = 1$.

$$ad + bc = a^{2} + ab + b^{2} = c^{2} + cd + d^{2}. \quad (37)$$

Compute the effect of $\theta^{2} = id$ on $((0, 0, \beta)) \in G : ((0, c, 0)) = ((0, 0, \lambda^{-1/2} c^{2}/2, 0))$ for all $c \in F$. This implies $\sigma^{2} = id$, so $\sigma = id$ since $c$ is odd, and also $\lambda = 1$. Similarly, $id = \theta^{2} : ((0, 0, \beta)) \mapsto ((0, \beta (P D P)^{2}))$ for all $\beta \in F^{2}$, implying $D^{2} = I$. This implies $a = d$ and $1 = \det(D) = a^{2} + bc = a^{2} + ab + b^{2} = a^{2} + ab + c^{2}$, forcing $a + b = bc = ab + b^{2} = ac + c^{2}$. Hence $b(a + b + c) = c(a + b + c) = 0$. Suppose first that $a + b + c \neq 0$. Then $D = I$, and from the fact that the coefficient on $t^{5} = t$ in equation (24)(ii) must be 0 it follows that $0' = 0$. This means that $\theta$ is not the involution we seek, hence $a + b + c = 0$.

The coefficients on $(t^{5})^{2} = t^{7}$ in equation (24)(ii) and (iii) must both be 0. This leads to a system of two linear equations in $a^{2}$ and $b^{2}$ which is easily solved. And then $c^{2} = a^{2} + b^{2}$ leads to the following:

Put $v = (1 + w + w^{2})^{1/2}$. Then

$$(i) \quad a = d = (1 + w^{5})/v^{5},$$
$$(ii) \quad b = (1 + w + w^{4})/v^{5},$$
$$(iii) \quad c = (w + w^{4} + w^{5})/v^{5}. \quad (38)$$

Finally, considering the coefficient of $t^{5} = t$ in equation (24)(ii), we compute

$$0' = 1/v^{2}. \quad (39)$$

It is now a tedious but uninspired task to show that the unique possible involution $\theta$ determined by equations (38) and (39) does satisfy the conditions of equation (24). Tracing back through the recoordinatization process, we find that the involution $\theta$ induces the following permutation on the indices of the lines through $(\infty)$ in the original coordinatization of section 4:

$$t \mapsto (t(1 + w^{2}) + w^{2})/(t + 1 + w^{2}). \quad (40)$$

In particular, $\infty \leftrightarrow 1 + w^{2}$ and $w \leftrightarrow w$. Here $w$ can be any element of $F$, and $w = 1$ gives the original involution found in theorem 4.2. This proves the following:

**Theorem 5.1** The full collineation group $G$ of the GQ given in section 4 (with $q \geq 32$) has order $2e(q^{2} - 1)q^{5}$ and acts transitively on the lines through the point $(\infty)$. 
6 The Action of $G$ on the subquadrangles $S_\alpha$

The action of the full collineation group $G$ on the subquadrangles $S_\alpha$ is determined by its action on the subgroups $G_\alpha$. Clearly the stabilizer $G_0$ of the point $(0,0,0)$ must leave the dual grid $\Gamma$ invariant, so it must permute the $S_\alpha$ among themselves.

Let $w \in F$. Then the map $\phi_w$ defined by

$$\phi_w : G \to G : (\alpha, c, \beta) \mapsto ((y_w \alpha + \beta)P, c + \sqrt{\alpha A_w \alpha^T + \alpha P \beta^T}, \alpha P)$$

is an automorphism of $G$ that corresponds to the recoordinatization of section 5. It is convenient to have its inverse

$$\phi_w^{-1} : G \to G : (\gamma, d, \delta) \mapsto (\delta P, d + \sqrt{\delta P A_w P \delta^T + \delta P \gamma^T}, (y_w \delta + \gamma)P).$$

Now consider an involution of the recoordinatized GQ of the type $\theta_w = \theta_w(id, D, 1)$ with $D$ as in equation (38). For such a $D$, $PD^{-T}P = D$ and $PDP = D^T = D^{-T}$.

$$\theta_w : G \to G :$$

$$(\alpha, c, \beta) \mapsto (\alpha D^{-T}, c + \sqrt{\alpha D^{-T} A_w D^{-1} \alpha^T}, (\beta + \alpha' \alpha)D^{-T}),$$

with $0' = (1 + w + w^2)^{-1}$.

Then $\overline{\theta}_w = \phi_w \circ \theta_w \circ \phi_w^{-1}$ (doing $\phi_w$ first) as an automorphism of $G$ is an involution of the GQ $S$ expressed in the original coordinates. To consider its effect on the $G_\alpha$ we do not need to compute the middle coordinate.

$$\overline{\theta}_w : (\alpha, c, \beta) \mapsto ((\alpha + \alpha' (y_w \alpha + \beta))D, -, \alpha' y_w^2 \alpha D + (y_w 0' + 1) \beta D).$$

From equation (44) it is clear that $G_\alpha \to G_{\alpha D}$.

From here on we just use the group $G_0$. The stabilizer $G_{0,\infty}$ of $[A(\infty)]$ (see theorem 4.3) has order $2e(q - 1)$, and for $0 \neq a \in F, \sigma \in \text{Aut}(F)$, $0' \in \{0, 1\}$, consists of the following maps:

$$(\alpha, c, \beta) \mapsto (\alpha a^\sigma K, ac^\sigma + a \sqrt{\alpha^\sigma KA_w K^T (\alpha^\sigma)^T}, a (\beta^\sigma + \alpha' \alpha^\sigma) K),$$

with

$$K = \begin{pmatrix} 1 & 0 \\ 0' & 1 \end{pmatrix}.$$
References


S. E. Payne
Department of Mathematics
CU-Denver Camp Box 170
PO Box 173364
Denver CO 80217-3364
USA.