Stable Clifford Theory for Divisorially Graded Rings

José Gómez Torrecillas  Blas Torrecillas

Introduction

Dade \([D1, \text{Theorem 7.4}]\) obtained an important result on the equivalence of categories, extending the classical stable Clifford theory. He used the theory of strongly graded rings. Recently, this work has been generalized to arbitrary graded rings, see E. Dade \([D2], [D3]\), J.L. Gómez Pardo and C. Năstăcescu \([GN]\), C. Năstăcescu and F. Van Oystaeyen \([NVO2]\). In the classical case the stable Clifford theory relates isomorphism classes of simple modules on a strongly graded ring \(R\) which are direct sums of a fixed simple \(R_e\)-module, where \(R_e\) is the component of degree \(e\), with the isomorphism classes of simple modules on a crossed product. The aim of this paper is extend the foregoing result to \(\mathcal{C}\)-cocritical modules, where \(\mathcal{C}\) is a localizing subcategory, on divisorially graded rings. We start with a relative version of Clifford theory using the simple objects of the quotient category. We investigated the situation of the so-called divisorially graded rings introduced by F. Van Oystaeyen in the commutative case and then generalized by many other author to more general situations (see the monograph \([LVVO]\) and the references quoted there). We will work in the categories of \(R\text{-Mod}\) and \(R - gr\), thus we prefer use the a general Grothendieck category and the concept of static objects in this kind of category to establish our basic results.

The paper is organized as follows. After a Section of preliminaries, we introduce the notion of static objects in quotient categories in the next section. If we have adjoints functors between two Grothendieck categories \(\mathcal{A}\) and \(\mathcal{B}\) and a localizing subcategory \(\mathcal{C}\) of \(\mathcal{A}\), then we show that under certain conditions it is possible to obtain an equivalence between the category of static objects in \(\mathcal{A}/\mathcal{C}\) and the category of co-static objects of some quotient category of \(\mathcal{B}\). In the last Section, we apply

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this result to study the stable Clifford theory on divisorially graded rings, extending (see Theorem 3.14) the main result of Dade (cf. [D1, Theorem 8.2]).

1 Preliminaries

All the rings considered in this paper are associative with identity element. Let $R$ be a ring, $R$-Mod will denote the category of the unital left $R$-modules.

Let $G$ be a multiplicative group with identity element $e$. A $G$-graded ring $R$ is a ring with identity 1, together with a direct decomposition $R = \bigoplus_{g \in G} R_g$ (as additive subgroups) such that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. Thus $R_e$ is a subring of $R$, $1 \in R_e$, and for every $g \in G$, $R_g$ is an $R_e$-bimodule. A $G$-graded left $R$-module is a left $R$-module $M$ endowed with an internal direct sum decomposition $M = \bigoplus_{g \in G} M_g$ where each $M_g$ is a subgroup of the additive group of $M$ such that $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Let $M$ and $N$ be graded left modules over the graded ring $R$. For every $g \in G$ we set

$$\text{HOM}_R(M, N)_g = \{ f : M \to N \mid f \text{ is } R \text{-linear} \text{ and } f(M_h) \subseteq M_{hg} \}$$

$\text{HOM}_R(M, N)_g$ is an additive subgroup of the group $\text{Hom}_R(M, N)$ of all $R$-linear maps from $M$ to $N$. Observe that

$$\text{HOM}_R(M, N) = \bigoplus_{g \in G} \text{HOM}_R(M, N)_g$$

is a subgroup of $\text{Hom}_R(M, N)$ and it is a graded abelian group of type $G$. Clearly $\text{HOM}_R(M, N)_e$ is just $\text{Hom}_{R-gr}(M, N)$, i.e. the group of all morphisms from $M$ to $N$ in the category $R-gr$ of all graded left $R$-modules. Define for $g \in G$ the $g$-suspension $M(g)$ of a graded left $R$-module $M$ as follows: $M(g)$ is the left $R$-module $M$ graded by $G$ by putting $M(g)_{ih} = M_{ih}$ for all $h \in G$. Observe that

$$\text{HOM}_R(M, N)_g = \text{Hom}_{R-gr}(M, N(g)) = \text{Hom}_{R-gr}(M(g^{-1}), N)$$

It is well known that $R-gr$ is a Grothendieck category (See [NVO1]).

We recall some ideas from torsion theories on Grothendieck categories. Let $\mathcal{A}$ be a Grothendieck category. A non empty subclass $T$ of $\mathcal{A}$ is a torsion class if $T$ is closed under quotient objects, coproducts and extensions. In this case for any $M \in \mathcal{A}$ one can consider the greatest subobject $t_T(M)$ of $M$ belonging to $T$. A torsion class is said to be hereditary if it is closed under subobjects.

Let us now recall the concept of quotient category. A Serre class (or Serre subcategory) of an abelian category $\mathcal{A}$ is a non-empty class $\mathcal{S}$ which is closed under subobjects, quotient objects and extensions. The quotient category $\mathcal{A}/\mathcal{S}$ of $\mathcal{A}$ by $\mathcal{S}$ is the category defined as follows: the objects of $\mathcal{A}/\mathcal{S}$ are those of $\mathcal{A}$ and the morphism are defined by

$$\text{Hom}_{\mathcal{A}/\mathcal{S}}(A, B) = \lim_{\longrightarrow} \text{Hom}_{\mathcal{A}}(A', B/B')$$

where $A'$ runs over the subobjects of $A$ such that $A/A' \in \mathcal{S}$ and $B'$ runs over the subobjects of $B$ such that $B' \in \mathcal{S}$. $\mathcal{A}/\mathcal{S}$ is an abelian category and the canonical
functor $T_S : \mathcal{A} \to \mathcal{A}/\mathcal{S}$, which is the identity on objects and maps morphisms in $\text{Hom}_{\mathcal{A}}(A, B)$ onto their canonical image in the direct limit $\text{Hom}_{\mathcal{A}/\mathcal{S}}(A, B)$, is an exact functor. The Serre class $\mathcal{S}$ is called a localizing subcategory of $\mathcal{A}$ if the canonical functor $T$ has a right adjoint. If $\mathcal{A}$ is a Grothendieck category then the concept of localizing subcategory coincides with that of hereditary torsion class. If $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$, then for any $X \in \mathcal{A}$ we consider the greatest subobject $t_{\mathcal{C}}(X)$ of $X$ belonging to $\mathcal{C}$. If $t_{\mathcal{C}}(X) = 0$ then $X$ is called a $\mathcal{C}$-torsionfree object, if $t_{\mathcal{C}}(X) = X$, then $M$ is said to be a $\mathcal{C}$-torsion object. Following Gabriel [G], if $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$, we can define the quotient category $\mathcal{A}/\mathcal{C}$ which is also a Grothendieck category. We denote by $T_{\mathcal{C}} : \mathcal{A} \to \mathcal{A}/\mathcal{C}, S_{\mathcal{C}} : \mathcal{A}/\mathcal{C} \to \mathcal{A}$, the canonical functors. It is well known [G] that $T_{\mathcal{C}}$ is an exact functor, and $S_{\mathcal{C}}$ is right adjoint of $T_{\mathcal{C}}$. Moreover, $S_{\mathcal{C}}$ is a left exact functor.

2 Static objects in quotient categories.

Let $\mathcal{A}$ and $\mathcal{B}$ be two Grothendieck categories, and consider two additive functors $F$ and $G$, such that $F$ is a left adjoint of $G$.

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{F} & \mathcal{A} \\
\uparrow \quad & & \downarrow \\
\mathcal{A} & \xrightarrow{G} & \mathcal{B}
\end{array}
$$

For an $R - S$-bimodule $M$ over associative rings $R$ and $S$, the adjunction $M \otimes_S \dashv \text{Hom}_R(M, -)$ was used in [Na] to define full subcategories of $R\text{-Mod}$ and $S\text{-Mod}$ in order to have that the restrictions of the functors $M \otimes_S \dashv \text{Hom}_R(M, -)$ to such subcategories establish an equivalence of categories between them. The aim of this section is to carry this construction to a quotient category of $\mathcal{A}$.

If $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$ we can induce a subcategory $\mathcal{D}$ of $\mathcal{B}$ setting

$$
\mathcal{D} = \{ Y \in \mathcal{B} \mid F(Y) \text{ is } \mathcal{C} - \text{torsion} \}.
$$

It is evident that $\mathcal{D}$ is stable under homomorphic images and direct sums. To check that $\mathcal{D}$ is closed by extensions, consider

$$
0 \to X \to Y \to Z \to 0
$$

an exact sequence in $\mathcal{B}$ with $X, Z \in \mathcal{D}$. Applying $F$, we obtain an exact sequence in $\mathcal{A}$,

$$
F(X) \xrightarrow{f} F(Y) \xrightarrow{g} F(Z) \to 0
$$

with $F(X)$ and $F(Z)$ $\mathcal{C}$-torsion. Construct the exact sequence

$$
0 \to \text{Ker } g \to F(Y) \to F(Z) \to 0.
$$
Since $\text{Ker}g = \text{Im}f$ and $F(X)$ is $C$-torsion it follows that $\text{Ker}g$ is $C$-torsion and therefore $F(Y)$ is $C$-torsion. This gives $Y \in \mathcal{D}$. In order to have that $\mathcal{D}$ is a localizing subcategory of $\mathcal{B}$, we require that $F$ satisfies certain property of exactness, as reflects the following result.

**Proposition 2.1.** Let $\mathcal{A}, \mathcal{B}$ be two Grothendieck categories. Consider the following situation of adjoints functors.

\[
\begin{tikzcd}
\mathcal{B} & & \\
& F & \mathcal{G} \\
\mathcal{A} & & \\
\end{tikzcd}
\]

Let $\mathcal{C}$ be a localizing subcategory of $\mathcal{A}$ and $\mathcal{D} = \{Y \in \mathcal{B} \mid F(Y) \text{ is } C\text{-torsion}\}$ the induced subcategory of $\mathcal{B}$. The class $\mathcal{D}$ is a localizing subcategory of $\mathcal{B}$ whenever for every monomorphism $f : X \to Y$ in $\mathcal{B}$, $\text{Ker}(F(f))$ is $C$-torsion.

**Proof.** For every monomorphism in $\mathcal{B}, f : X \to Y$ we have the exact sequence

\[0 \to \text{Ker}(F(f)) \to F(X) \to \text{Im}(F(f)) \to 0.\]

If $Y$ is $C$-torsion then $\text{Im}(F(f))$ is $C$-torsion. Thus $F(X)$ is $C$-torsion if and only if $\text{Ker}(F(f))$ is $C$-torsion. The proposition follows from this fact.

**Definition 2.2.** A functor $F$ satisfying Proposition 2.1 is said to be $C$-exact.

Throughout this section the functor $F$ will be assumed to be $C$-exact. Following Gabriel [G] we can define the quotient categories $\mathcal{A}/\mathcal{C}$ and $\mathcal{B}/\mathcal{D}$. We will denote by $T_C : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ and $S_C : \mathcal{A}/\mathcal{C} \to \mathcal{A}$ (resp. $T_D : \mathcal{B} \to \mathcal{B}/\mathcal{D}$ and $S_D : \mathcal{B}/\mathcal{D} \to \mathcal{B}$) the canonical functors (see [G, ch. III]). We have also natural transformations $\Phi_C : T_C S_C \to \text{id}$ and $\Psi_C : \text{id} \to S_C T_C$ such that $\Phi_C$ is a natural isomorphism and for each object $X$ in $\mathcal{A}$ the morphism $(\Psi_C)_X : X \to S_C T_C(X)$ has kernel and cokernel $C$-torsion. Analogous notations will be used for $\mathcal{D}$. The adjunction

\[
\begin{tikzcd}
\mathcal{B} & & \\
& F & \mathcal{G} \\
\mathcal{A} & & \\
\end{tikzcd}
\]

induces functors
defined as \( K = T_D G S_c \) and \( H = T_c F S_D \). There exist natural transformations \( \chi : HK \rightarrow \text{id} \) and \( \nu : \text{id} \rightarrow KH \) described as follows: For each object \( Y \) in \( B/D \) there is a natural \( B \)-morphism

\[
S_D Y \rightarrow GFS_D Y
\]

Applying \( T_D \) we obtain a natural morphism in \( B/D \),

\[
Y \cong T_D S_D Y \rightarrow T_D GFS_D Y.
\]

The natural \( B \)-morphism

\[
FS_D Y \rightarrow S_c T_c FS_D Y
\]

induces a natural \( A \)-morphism

\[
GFS_D Y \rightarrow GS_c T_c FS_D Y.
\]

Applying again \( T_D \) and composing we obtain the natural homomorphism in \( B/D \)

\[
u_Y : Y \cong T_D S_D Y \rightarrow T_D GFS_D Y \rightarrow T_D GS_c T_c FS_D Y = KH Y
\]

On the other hand, given \( X \) in \( A/C \), we can use the natural morphism in \( B \),

\[
GS_c X \rightarrow S_D T_D GS_c X
\]

with kernel and cokernel \( D \)-torsion to define the canonical morphism

\[
FGS_c X \rightarrow FS_D T_D GS_c X.
\]

By a standard argument and the definition of \( D \) it is not difficult to check that this morphism permits us to obtain a natural isomorphism in \( A/C \)

\[
T_c FGS_c X \cong T_c FS_D T_D GS_c X.
\]

Now, the canonical morphism \( FGS_c X \rightarrow S_c X \) gives a morphism in \( A/C \)

\[
T_c FGS_c X \rightarrow T_c S_c X.
\]

Therefore we achieve the definition of a natural homomorphism in \( A/C \)

\[
\chi_X : HK X = T_c F S_D T_D GS_c X \cong T_c FGS_c X \rightarrow T_c S_c X \rightarrow X
\]

We are now ready to define full subcategories of \( A/C \) and \( B/D \) equivalent by restriction of \( K \) and \( H \).
Definition 2.3. An object \( X \) of \( \mathcal{A}/\mathcal{C} \) is said to be \( F \)-static whenever \( \chi_X : HKX \rightarrow X \) is an isomorphism. The category of all \( F \)-static objects in \( \mathcal{A}/\mathcal{C} \) will be denoted by \( \mathcal{A}/\mathcal{C}_F \) which is a full additive subcategory of \( \mathcal{A}/\mathcal{C} \). An object \( Y \) of \( \mathcal{B}/\mathcal{D} \) is said to be \( F \)-co-static if \( \nu_Y : Y \rightarrow KHY \) is an isomorphism. The category of all \( F \)-co-static objects will be denoted by \( \mathcal{B}/\mathcal{D}^F \) which is a full additive subcategory of \( \mathcal{B}/\mathcal{D} \). When the original adjunction is \( \mathcal{M} \otimes_S - \dashv \text{Hom}_R(M, -) \) for some left \( R \)-bimodule with \( S = \text{End}_R(M) \), we will speak of \( M \)-static and \( M \)-co-static objects of \( R \)-Mod/\( \mathcal{C} \) and \( S \)-Mod/\( \mathcal{D} \).

We have immediately the following theorem, that extends [Na, Theorem 2.5].

Theorem 2.4. The restrictions of the additive functors

\[
K = T_DGS_C : \mathcal{A}/\mathcal{C}_F \rightarrow \mathcal{B}/\mathcal{D}^F
\]

and

\[
H = T_CFS_D : \mathcal{B}/\mathcal{D}^F \rightarrow \mathcal{A}/\mathcal{C}_F
\]

form an equivalence between the categories \( \mathcal{A}/\mathcal{C}_F \) and \( \mathcal{B}/\mathcal{D}^F \).

As usual, we will say that an object \( X \) in \( \mathcal{A} \) divides an object \( U \) if there is an object \( X' \) in \( \mathcal{A} \) and an isomorphism \( U \cong X \oplus X' \). When \( X \) divides a finite direct sum of copies of \( U \), we say that \( X \) weakly divides \( U \). We will say that two objects of \( \mathcal{A} \) are weakly isomorphic if each weakly divides the other. It is clear that both functors \( K \) as \( H \) preserve finite direct sums. Therefore, the following result has an easy straightforward proof.

Proposition 2.5. The subcategories \( \mathcal{A}/\mathcal{C}_F \) and \( \mathcal{B}/\mathcal{D}^F \) are closed under finite direct sums and direct summands.

Let \( M \) be a \( \mathcal{C} \)-closed object in \( \mathcal{A} \), i.e., \( M \cong S_C T_C M \) naturally, and \( S = \text{End}_A(M) \). We can take \( G = \text{Hom}_A(M, -) \) and we know by [P, Corollary 7.3] that there exists a left adjoint \( F \) of \( G \), satisfying \( F(S) = M \). Assume that \( F \) is \( \mathcal{C} \)-exact. As in a foregoing argument, we have that

\[
HK(T_C M) = T_C(FS_D T_D GS_C T_C M) \cong T_C(FGS_C T_C M)
\]

\[
\cong T_C(FS) \cong T_C M.
\]

This gives immediately that \( M \) is \( F \)-static. Therefore \( T_D S \) is \( G \)-co-static. If \( S \) is the filter of left ideals of \( S \) associated with the torsion theory \( (\mathcal{D}, \mathcal{G}) \) and \( Y \) is a left \( S \)-module, then \( Y_\mathcal{S} = \lim_{\alpha \in \mathcal{S}} \text{Hom}_S(\alpha, Y/\text{t}_\mathcal{D}(Y)) \) denotes the localized \( S \)-module.

Indeed, \( S_\mathcal{S} \) is an \( S \)-algebra via the canonical map \( S \rightarrow S_\mathcal{S} \) and \( Y_\mathcal{S} \) is a left \( S_\mathcal{S} \)-module. It is well known [S, Ch. X] that there exists a full and faithful functor \( S_\mathcal{S} : S - \text{Mod}/\mathcal{D} \rightarrow S_\mathcal{S} - \text{Mod} \) such that the diagram of functors

\[
\begin{array}{ccc}
S - \text{Mod} & \xleftarrow{U} & S_\mathcal{S} - \text{Mod} \\
\downarrow S_D & & \downarrow S_\mathcal{S} \\
S - \text{Mod}/\mathcal{D} & \xrightarrow{S_\mathcal{S}} & S_\mathcal{S} - \text{Mod}
\end{array}
\]
commutes, where \( U \) is the restriction of scalars. Moreover, \( US_3 \) is isomorphic to \( S_D \). It follows from this that the restriction of \( S_3 \) to the full subcategory of \( S\text{-Mod}/D \) of the objects that weakly divides \( T_D S \) gives an isomorphism between this category and the category of the projective \( S_3 \)-modules of finite type. This last category will be denoted by \( (S_3 \mid weak S_3) \). In what follows, these categories will be identified. Analogously, the subcategory of \( A/C_M \) consisting of the objects that weakly divides \( T_C M \) will be denoted by \( (A/C \mid weak M) \). These are equivalent categories, as shows the following result, that generalizes \([Na, Theorem 3.7]\).

**Corollary 2.6.** The restrictions of the additive functors

\[
H = T_D GS_C : (A/C \mid weak M) \rightarrow (S_3 \mid weak S_3)
\]

and

\[
K = T_C FS_D : (S_3 \mid weak S_3) \rightarrow (A/C \mid weak M)
\]

form an equivalence between the categories \( (A/C \mid weak M) \) and \( (S_3 \mid weak S_3) \).

We specialize to the case of modules. For a left \( R \)-module \( M \) we have the adjoint functors

\[
\begin{array}{ccc}
S - Mod & \overset{\cong}{\leftrightarrow} & Hom_R(M,-) \\
M \otimes_S & \downarrow & \\
R - Mod
\end{array}
\]

where \( S = \text{End}_R(M) \). Let \( C \) denote a localizing subcategory of \( R\text{-Mod} \). In order to assure that the subcategory of \( S\text{-Mod} \)

\[
D = \{X \in S - Mod : M \otimes_S X \in C\}
\]

is a localizing subcategory of \( S\text{-Mod} \) we will assume that the functor \( M \otimes_S - \) is \( C \)-exact (see Proposition 2.1 and Definition 2.2). In this case we will say that \( M \) is \( weakly C\text{-flat} \). In the absolute case \([Na, Theorem 3.7]\), under the condition that \( M \) is finitely generated, it is proved that there is an equivalence between the category of left \( R \)-modules that divide \( M \) and the category of all projective left \( S \)-modules. In our quotient situation we will need some finiteness condition on the torsion theory \((C,F)\) defined by the localizing subcategory \( C \) of \( A = R\text{-Mod} \). Actually, \((C,F)\) is said to be a torsion theory of \textit{finite type} if the Gabriel filter of left ideals \( \mathfrak{R} \) associated with \((C,F)\) has a cofinal subset of finitely generated left ideals. On the other hand, the assumption on \( M \) can be weakened, and we will prove a generalization of \([Na, Theorem 3.7]\) for a \( C \)-closed left \( R \)-module \( M \) which is \( C\text{-finitely generated} \), i.e., there is a finitely generated submodule \( N \) of \( M \) such that \( M/N \) is \( C\)-torsion. When \((C,F)\) is of finite type, the functor \( T_C \) preserves arbitrary direct sums. In such case, we will denote by \((R\text{-Mod}/C \mid M)\) the full subcategory of \( R\text{-Mod}/C \) consisting of the
objects that divide some direct sum of copies of $T_C M$ and by $(S_3 \mid S_3)$ the category of all projective $S_3$-modules.

**Theorem 2.7.** Assume that the torsion theory $(\mathcal{T}, \mathcal{F})$ on $R$-$\text{Mod}$ is of finite type and let $M$ be a $C$-finitely generated, $C$-closed and $C$-flat left $R$-module. The restrictions of the additive functors

$$H = T_D(\text{Hom}_R(M, S_C(-)) : (R - \text{Mod}/C \mid M) \to (S_3 \mid S_3)$$

and

$$K = T_C(M \otimes_S S_D(-)) : (S_3 \mid S_3) \to (R - \text{Mod}/C \mid M)$$

form an equivalence between the categories $(R$-$\text{Mod}/C \mid M)$ and $(S_3 \mid S_3)$.

**Proof.** In view of Corollary 2.6 and the discussions made throughout this section, we need only prove that $K$ and $H$ preserve arbitrary direct sums and that they are well defined. First, we claim that $D$ is of finite type. If this claim is assumed, then $T_D$ preserves direct sums by [S, Proposition XIII.2.1]. Since $M$ is $C$-finitely generated and $C$ is of finite type, $\text{Hom}_R(M, S_C(-))$ preserves direct sums and this shows that $K$ preserves direct sums. Moreover, the fact that $D$ is of finite type forces that every projective left $S_3$-module is $D$-closed as left $S$-module. Since $S_3$ is full and faithful, $(S_3 \mid S_3)$ can be identified with the full subcategory of $B/D$ consisting of the objects that divide some direct sum of copies of $T_D S$. This shows that $H$ is well defined. Analogous arguments give that $K$ preserves direct sums and it is well defined. In this way, the proof will be complete if we prove the claim. For, observe that the filter of left ideals of $S$ associated with $D$ is $\mathcal{Z} = \{I \leq_S S : M/MI \text{ is } C\text{-torsion}\}$. Take $I$ a left ideal in $\mathcal{Z}$ and let $m_1, \ldots, m_n$ in $M$ such that $N = Rm_1 + \ldots + Rm_n$ is $C$-dense in $M$. For each $i = 1, \ldots, n$ there is an finitely generated left ideal $\alpha_i$ of $R$ in the filter $\mathcal{R}$ such that $\alpha_i m_i \subseteq MI$. It is straightforward to check that $\alpha_1 m_1 + \ldots + \alpha_n m_n$ is $C$-dense in $N$. Since $N$ is $C$-dense in $M$ this implies that $\alpha_1 m_1 + \ldots + \alpha_n m_n$ is $C$-dense in $M$. Now, for each $i = 1, \ldots, n$ the left ideal $\alpha_i$ can be expressed as $\alpha_i = \sum_{j \in A_i} R_{aij}$, for some $aij$ in $R$ and $A_i$ a finite index set. Thus $\alpha_1 m_1 + \ldots + \alpha_n m_n = \sum_{i=1}^{n} \sum_{j \in A_i} R_{aij} m_i \subseteq MI$. For every $i = 1, \ldots, n$ and for every $j \in A_i$, there is $f_{ij} \in I$ such that $a_{ij} m_i \in \text{Im}f_{ij}$. This implies that $\sum_{i=1}^{n} \sum_{j \in A_i} \text{Im}f_{ij} = M\{f_{ij}\}$ is $C$-dense in $M$. Hence, the left $S$-ideal $I_0$ generated by the $f_{ij}$ verifies that $MI_0$ is $C$-dense in $M$. Hence, $I_0$ is in $\mathcal{Z}$ and this proves that $D$ is of finite type.

## 3 Divisorial Clifford theory

In this section the notation will be slightly modified. So, $R = \bigoplus_{g \in G} R_g$ will denote a $G$-graded ring for $G$ an arbitrary group with neutral element $e$ and $C$ will be a localizing subcategory of $R_e$-$\text{Mod}$ that induces an hereditary torsion theory $(C, \mathcal{F})$ with associated filter of left ideals of $R_e$ denoted by $\mathcal{R}$. We will denote by $R - gr$ the Grothendieck category whose objects are the $G$-graded left $R$-modules. The morphisms in $R - gr$ are the $R$-linear graded maps of degree $e$. Every graded left $R$-module decomposes, when it is considered as left $R_e$-module, as a direct sum $M = \bigoplus_{g \in G} M_g$ of $R_e$-modules and every morphism $f : M \to N$ in $R - gr$ is, after
forgetting the $R$-linear structure, a morphism of left $R_e$-modules $f : M \to N$ that maps the $g$-th component $M_g$ of $M$ to the $g$-th component $N_g$ of $N$. This construction defines an exact functor

$$(\_)_e : R - gr \to R_e - \text{Mod}$$

This permits us to induce a localizing subcategory $C^g$ from $C$ by putting

$$C^g = \{ X \in R - gr \mid X \in C \}$$

This localizing subcategory of $R - gr$ define a rigid [NV01] torsion theory $(C^g, \mathcal{F}^g)$ in $R - gr$ with associated filter of graded left ideals $\mathcal{R}^g$.

Now we can consider the following diagram of functors

$$
\begin{array}{ccc}
R - gr & \xleftarrow{\cong} & R_e - \text{Mod} \\
\downarrow T_{C^g} & & \downarrow T_C \\
R - gr/C^g & \cong & R - \text{Mod}/C
\end{array}
$$

where $R - gr/C^g$ (resp. $R_e\text{-Mod}/C$) is the quotient category of $R - gr$ (resp. $R_e\text{-Mod}$) under the localizing subcategory $C$ (resp. $C^g$). $B$ is defined as $T_C \circ (R \otimes_{R_e} -) \circ S_{C^g}$ and analogously $A$ is defined as $T_{C^g} \circ (\_)_e \circ S_C$.

We remember [D1] that $R$ is strongly graded by $G$ (i.e. $R_g R_h = R_{gh}$ for all $g, h \in G$) if and only if $(\_)_e$ and $R \otimes_{R_e} -$ establish an equivalence between the categories $R - gr$ and $R_e\text{-Mod}$. In [AGT, Theorem 1.1] we analyzed when $B$ and $A$ give an equivalence between the categories $R - gr/C^g$ and $R_e\text{-Mod}/C$. The results there obtained were expressed in the equivalent language of torsion theories. Now, we will restate some of these facts in the formalism of quotient categories.

**Definition 3.1.** Let $R$ be a $G$-graded ring and $C$ a localizing subcategory of $R_e\text{-Mod}$. Following [NR] we will say that $C$ is $G$-stable if $R \otimes_{R_e} A$ is $C$-torsion for every $C$-torsion left $R_e$-module $A$.

The ring morphism $R_e \to R$ permits to define canonically the localizing subcategory $C^*$ of $R\text{-Mod}$ consisting of the left $R$-modules that are $C$-torsion considered as left $R_e$-modules. The following Lemma relates $C^*$ and $C^g$ under the hypothesis of $G$-stability for $C$.

**Lemma 3.2.** Let $R$ be a $G$-graded ring and $C$ a $G$-stable localizing subcategory of $R_e\text{-Mod}$. Let us denote by $\mathcal{R}$ the filter of left ideals of $R_e\text{-Mod}$ associated with $C$ and by $\mathcal{R}^*$ the filter associated with $C^*$. The localizing subcategory $C^*$ is the smallest localizing subcategory of $R\text{-Mod}$ that contains the underlying $R$-modules of the objects in $C^g$. Moreover, $(C^*, \mathcal{F}^*)$ is a graded torsion theory on $R\text{-Mod}$ and

$$\mathcal{R}^* = \{ I \leq_R R \mid \text{there is } H \in \mathcal{R}^g \text{ with } H \subseteq I \} = \{ I \leq_R R \mid I \cap R_e \in \mathcal{R} \}$$

**Proof:** If we prove that $C^*$ is the smallest localizing subcategory of $R\text{-Mod}$ containing the underlying $R$-modules of the objects in $C^g$, then, by [NR, Proposition 1.1],
the rest of the assertions follows. Let $\mathcal{E}$ be any localizing subcategory of $R$-$\text{Mod}$ containing $C^g$ and take $A$ any $C^*$-torsion left $R$-module. Then $R_e A$ is $C$-torsion. By $G$-stability, $R \otimes_{R_e} A$ is $C$-torsion. But $R \otimes_{R_e} A$ is a graded left $R$-module and, so, $R \otimes_{R_e} A$ is $C^g$-torsion. This implies that $R \otimes_{R_e} A$ is $\mathcal{E}$-torsion. Observe that there is a canonical epimorphism of left $R$-modules from $R \otimes_{R_e} A$ onto $A$. This shows that $A$ is $\mathcal{E}$-torsion. Therefore $C^* \subseteq \mathcal{E}$ and the Lemma is proved.

**Definition 3.3.** A $G$-graded ring $R$ is said to be $C$-divisorially graded (see [LVVO]) whenever for every $g, h \in G$, $R_g R_h$ is $C$-dense in $R_e$.

In [AGT, Theorem 1.1] it is essentially proved that the categories $R - gr/C^g$ and $R_e$-$\text{Mod}$/$C$ are canonically equivalent if and only if $R$ is $C$-divisorially graded and $C$ is $G$-stable. We recall this result now.

**Theorem 3.4.** Let $R = \bigoplus_{g \in G} R_g$ be a graded ring and consider $C$ a localizing subcategory of $R_e$-$\text{Mod}$. Then the following assertions are equivalent:

(i) $B$ and $A$ establish an equivalence between the categories $R - gr/C^g$ and $R_e$-$\text{Mod}$/$C$.

(ii) A graded left $R$-module $X$ is $C^g$-torsion if and only if $X_e$ is $C$-torsion.

(iii) $R$ is $C$-divisorially graded and $C$ is $G$-stable.

A graded ring $R$ satisfying Theorem 3.4 is said to be $C$-strongly graded. Note that it is possible that a strongly graded ring fails to be $C$-strongly graded. This happens if $C$ is not $G$-stable. This last condition was studied on strongly graded rings in [NR].

In the part (ii) $\iff$ (i) of the proof of the Theorem 3.4 [AGT, Theorem 1.1] was proved a property of relative flatness on the ring extension $R_e \to R$ that we record in the following proposition.

**Proposition 3.5.** Let $R = \bigoplus_{g \in G} R_g$ be a $C$-strongly graded ring. For each exact sequence $0 \to K \to L \to N \to 0$ in $R_e$-$\text{Mod}$ with $K$ $C$-torsion, the kernel of the canonical morphism $R \otimes_{R_e} L \to R \otimes_{R_e} N$ is $C$-torsion.

For a graded left $R$-module $M$, consider the ring $S = \text{END}_R(M)$ consisting of the graded endomorphism of $M$. $S$ is canonically $G$-graded [D, Sections 3 and 4] by putting $S_g = \{ f \in \text{END}_R(M) : f(M_h) \subseteq M_{gh} \text{ for all } h \in G \}$. Therefore $S_e = \text{End}_{R - gr}(M, M)$. Before to prove the main results on equivalence of certain categories constructed from the module $M$ and the localizing subcategory $C$, we need some technical results. These facts will be stated in the following lemmas.

**Lemma 3.6.** Let $R$ be a $C$-strongly graded ring and $M$ a $C^g$-torsionfree graded left $R$-module. The map $\rho: \text{End}_{R - gr}(M) \to \text{End}_{R_e}(M_e)$, given by $\rho(f) = f_e$ for every $f \in \text{End}_{R - gr}(M)$ is a ring isomorphism.

**Proof:** It is clear that $\rho$ is a ring homomorphism. We will prove that it has trivial kernel and it is surjective. Note that $\text{Ker}(\rho) = \{ f \in \text{End}_{R - gr}(M) : f_e = 0 \}$.
Observe that for $f \in \text{End}_{R_{gr}}(M)$, $f_e = 0$ if and only if $(\text{Im } f)_e = 0$. By Theorem 3.4, $\text{Im } f$ is a $C^g$-torsion graded left $R$-module. Since $\text{Im } f \subseteq M$ and $M$ is $C^g$-torsionfree, $\text{Im } f$ must be trivial, i.e., $f = 0$. This proves that $\text{Ker } \rho = 0$.

To prove that $\rho$ is surjective, take $f \in \text{End}_{R_e}(M_e)$. We construct the morphism of graded left $R$-modules

$$R \otimes_{R_e} f : R \otimes_{R_e} M_e \to R \otimes_{R_e} M_e$$

Consider the canonical morphism of graded left $R$-modules

$$\zeta : R \otimes_{R_e} M_e \to M$$

Because $\zeta_e$ is an isomorphism, it follows from Theorem 3.4 that $\zeta$ has kernel and cokernel $C^g$-torsion. This implies that

$$\zeta \circ (R \otimes_{R_e} f) : R \otimes_{R_e} M_e \to M$$

annihilates $\text{Ker } \zeta$, since $M$ is $C^g$-torsionfree. But this implies that there exists a morphism of graded left $R$-modules $\bar{f} : M \to M$ such that

$$\bar{f} \circ \zeta = \zeta \circ (R \otimes_{R_e} f)$$

It is immediate to check that $\bar{f}_e = f$. Therefore, $\rho$ is surjective and the Lemma is proved.

Assume that $M$ is a $C^g$-torsionfree graded left $R$-module over a $C$-strongly graded ring $R$. The localizing subcategory $C$ of $R_e$-Mod induces (see Section 1) a subcategory $\mathcal{D}$ of the category of left modules on $\text{End}_{R_e}(M_e)$. From Lemma 3.6 it is possible to identify $\text{End}_{R_e}(M_e)$ with $S_e$ and, therefore, $M_e$ can be considered as an $R_e - S_e$-bimodule. Therefore, up to this identification, we rewrite

$$\mathcal{D} = \{ B \in S_e - \text{Mod} : M_e \otimes_{S_e} B \text{ is } C - \text{torsion} \}.$$ 

Proposition 2.1. assures that if the functor $M_e \otimes_{S_e} B : S_e - \text{Mod} \to R_e - \text{Mod}$ is $C$-exact then $\mathcal{D}$ is a localizing subcategory of $S_e - \text{Mod}$. We will change for this functor the nomenclature and we will say that $M_e$ is weakly $C$-flat, i.e., for each monomorphism $0 \to A \xrightarrow{f} B$ in $S_e - \text{Mod}$ the canonical morphism in $R_e - \text{Mod}$, $M_e \otimes_{S_e} f : M_e \otimes_{S_e} A \to M_e \otimes_{S_e} B$, has $C$-torsion kernel.

At this point we can induce a localizing subcategory of $S$-$\text{Mod}$ canonically from $C$ in two ways. The first idea is to use the restriction of scalars $S_e \to S$ to define a localizing subcategory $\mathcal{D}'$ of $S$-$\text{Mod}$. The $\mathcal{D}'$-torsion left $S$-modules are the left $S$-modules that are $\mathcal{D}$-torsion considered as left $S_e$-modules. The second possibility is to define a (at this moment, possibly not localizing), subcategory $\mathcal{P}$ of $S$-$\text{Mod}$ by using the tensor product $M \otimes_S -$.

Dade, in [D, Theorem 4.6], characterized in terms of the graded module $M$ when is $S = \text{END}_R(M)$ strongly graded. Concretely, he found that $S$ is strongly graded if and only if $M$ is weakly $G$-invariant, i.e., $M$ is weakly isomorphic in $R - gr$ to all
its suspensions $M(g)$. It is not hard to prove that a weakly $G$-invariant graded left $R$-module $M$ with $C$-torsionfree $M_e$ must be $C^2$-torsionfree.

**Lemma 3.7.** Let $M$ be a weakly $G$-invariant graded left $R$-module such that $M_e$ is weakly $C$-flat and $C$-torsionfree as left $R_e$-module. Assume that $R$ is $C$-strongly graded. The classes of left $S$-modules

$$\mathcal{D}^* = \{ Y \in S - \text{Mod} : s_e Y \text{ is } D - \text{torsion} \}$$

and

$$\mathcal{P} = \{ Y \in S - \text{Mod} : M \otimes S Y \text{ is } C^* - \text{torsion} \}$$

coincide.

**Proof:** We make the following computation: Given $Y$ a left $S$-module, $Y$ is $\mathcal{D}^*$-torsion if and only if $s_e Y$ is $D$-torsion if and only if $M_e \otimes s_e Y$ is $C$-torsion. But, since $S$ is strongly graded, $M_e \otimes s_e Y \cong M_e \otimes s_e S \otimes S Y \cong M \otimes S Y$. Thus, $Y$ is $\mathcal{D}^*$-torsion if and only if $M \otimes S Y$ is $C$-torsion if and only if $M \otimes S Y$ is $C^*$-torsion.

**Lemma 3.8.** Let $M$ be a weakly $G$-invariant graded left $R$-module such that $M_e$ is weakly $C$-flat and $C$-torsionfree as left $R_e$-module. If $R$ is $C$-strongly graded, then the strongly graded ring $S$ is $D$-strongly graded. Moreover, the torsion theory on $S$-Mod determined by $\mathcal{D}^*$ is a graded torsion theory.

**Proof:** Given $B$ in $S_e$-Mod, $S \otimes S_e B$ is $\mathcal{D}$-torsion if and only if $M_e \otimes S_e S \otimes S_e B$ is $C$-torsion if and only if $M \otimes S_e B$ is $C$-torsion. But $M \otimes S_e B$ is a graded left $R$-module by putting $(M \otimes S_e B)_g = M_g \otimes S_e B$ for each $g$ in the group $G$. So, $M \otimes S_e B$ is $C$-torsion if and only if $M \otimes S_e B$ is $C^2$-torsion and, by Theorem 3.4, this occurs if and only if $(M \otimes S_e B)_e = M_e \otimes S_e B$ is $C$-torsion if and only if $B$ is $\mathcal{D}$-torsion. Therefore, $B$ is $\mathcal{D}$-torsion if and only if $S \otimes S_e B$ is $\mathcal{D}$-torsion and we can use Theorem 3.4 to obtain that $S$ is $\mathcal{D}$-strongly graded.

As in Section 1, we have functors

$$\begin{align*}
\begin{array}{ccc}
R - \text{Mod}/C^* & & \\
F^* & \uparrow G^* & \\
S - \text{Mod}/D^* & & 
\end{array}
\end{align*}$$

defined as $F^* = T_{D^*} \text{-Hom}_R(M, S_{D^*}(-))$ and $G^* = T_{C^*}(M \otimes S S_{C^*}(-))$, where $R$-$\text{Mod}/C^*$ is the quotient category of $R$-$\text{Mod}$ constructed from $C^*$, $S$-$\text{Mod}/D^*$ is the quotient category of $S$-$\text{Mod}$ defined by $D^*$ and $T_{D^*}, S_{D^*}, T_{C^*}, S_{C^*}$ denote the canonical functors. For every object $X$ in $R$-$\text{Mod}/C^*$, we can consider $S_{C^*}X$ as a left $R_e$-module and it is natural ask if $R_i S_{C^*}X$ is $C$-closed, i.e., if $S_{C^*}T_{C^*}S_{C^*}X$ is isomorphic to $S_{C^*}X$ as left $R_e$-modules. But this is true if $R$ is a $C$-strongly graded ring, and the proof of this fact can be constructed analogously to that of $[NR$, Proposition 2.1]. In this case, $S$ is also $\mathcal{D}$-strongly graded by Lemma 3.8 and so we have that
every object in $S$-Mod can be regarded in $S_e$-Mod via the functor $T_D S_D$. We are now ready to establish, by restriction of the functors $F^*$ and $G^*$, an equivalence of categories between certain subcategories of $R$-Mod/$C^*$ and $S$-Mod/$D^*$.

**Theorem 3.9.** Consider $R$ a $C$-strongly graded ring and $M$ a weakly $G$-invariant graded left $R$-module such that $M$ is a $C$-torsionfree $C$-flat $R_e - S_e$-bimodule. Let

$$R - \text{Mod}/C_{\text{rest}, M_e}^* = \{ X \in R - \text{Mod}/C^* \mid T C S_C \cdot X \text{ is } M_e - \text{static} \}$$

and

$$S - \text{Mod}/D_{\text{rest}, M_e}^* = \{ Y \in S - \text{Mod}/D^* \mid T_D S_D \cdot Y \text{ is } M_e - \text{co-static} \}$$

The restriction of the functors

$$F^* : R - \text{Mod}/C_{\text{rest}, M_e}^* \to S - \text{Mod}/D_{\text{rest}, M_e}^*$$

and

$$G^* : S - \text{Mod}/D_{\text{rest}, M_e}^* \to R - \text{Mod}/C_{\text{rest}, M_e}^*$$

establish an equivalence of categories between the full subcategories

$$R - \text{Mod}/C_{\text{rest}, M_e}^*$$

of $R$-Mod/$C^*$ and

$$S - \text{Mod}/D_{\text{rest}, M_e}^*$$

of $S$-Mod/$D^*$.

**Proof:** First, we need to prove that the restriction of

$$F^* : R - \text{Mod}/C_{\text{rest}, M_e}^* \to S - \text{Mod}/D_{\text{rest}, M_e}^*$$

and

$$G^* : S - \text{Mod}/D_{\text{rest}, M_e}^* \to R - \text{Mod}/C_{\text{rest}, M_e}^*$$

are well defined. For, take $X \in R$-Mod/$C_{\text{rest}, M_e}^*$ and observe that

$$T_D \cdot \text{Hom}_R(M, S_C \cdot X) \cong T_D \cdot \text{Hom}_R(S_C \cdot T_C \cdot M, S_C \cdot X) \cong$$

$$T_D \cdot \text{Hom}_R(S_C \cdot T_C \cdot (R \otimes_{R_e} M_e), S_C \cdot X) \cong T_D \cdot \text{Hom}_R(R \otimes_{R_e} M_e, S_C \cdot X),$$

where the second isomorphism is given by Theorem 3.4. We have an exact sequence in $S$-Mod

$$0 \to T \to \text{Hom}_R(R \otimes_{R_e} M_e, S_C \cdot X)$$

$$\to S_D \cdot T_D \cdot \text{Hom}_R(R \otimes_{R_e} M_e, S_C \cdot X) \to C \to 0$$

where $T$ and $C$ are $D^*$-torsion left $S$-modules. If we consider this sequence in $S_e$-Mod, then $T$ and $C$ are $D$-torsion then

$$S_D T_D \cdot \text{Hom}_R(R \otimes_{R_e} M_e, S_C \cdot X) \cong S_D T_D S_D \cdot T_D \cdot \text{Hom}_R(R \otimes_{R_e} M_e, S_C \cdot X),$$
since $S_D T_D\cdot \text{Hom}_R(R \otimes_{R_e} M_e, S_{C^*} \cdot X)$ is $D$-closed. Therefore,

$$S_D T_D S_D \cdot T_D \cdot \text{Hom}_R(M, S_{C^*} \cdot X) \cong S_D T_D \text{Hom}_R(R \otimes_{R_e} M_e, S_{C^*} \cdot X) \cong S_D T_D \text{Hom}_{R_e}(M_e, S_C T_C S_{C^*} \cdot X).$$

Since

$$X \in R\text{-Mod} / C^*_\text{rest}.M_e,$$

$T_C S_{C^*} \cdot X$ is $M_e$-static by definition. Hence, $T_D \text{Hom}_{R_e}(M_e, S_C T_C S_{C^*} \cdot X)$ is $M_e$-co-static by Theorem 2.4. But, by the foregoing computations,

$$T_D \text{Hom}_{R_e}(M_e, S_C T_C S_{C^*} \cdot X) \cong T_D S_D \cdot T_D \cdot \text{Hom}_R(M, S_{C^*} \cdot X) = T_D S_D \cdot F^*X,$$

and we have that $T_D S_D \cdot F^*X$ is $M_e$-co-static, that is, $F^*X \in R\text{-Mod} / C^*_\text{rest}.M_e.$ Analogously, for each $Y \in S\text{-Mod} / D^*_\text{rest}.M_e,$ $G^*Y \in R\text{-Mod} / C^*_\text{rest}.M_e.$ Concretely, we have an exact sequence in $R\text{-Mod}$

$$0 \to T \to M \otimes_S S_D \cdot Y \to S_C \cdot T_C \cdot (M \otimes_S S_D \cdot Y) \to C \to 0$$

with both $T$ and $C, C^*$-torsion. As in the foregoing argument, we can deduce that

$$S_C T_C (M \otimes_S S_D \cdot Y) \cong S_C T_C S_C \cdot T_C \cdot (M \otimes_S S_D \cdot Y)$$

as $R_e$-modules. The claim follows as in the foregoing argument after observing that $M \cong M_e \otimes_S S$ since $S$ is strongly graded. Now, we will check that $F^*$ and $G^*$ give the equivalence. For $X \in R\text{-Mod} / C^*_\text{rest}.M_e,$

$$S_C T_C S_C \cdot G^*F^*X = S_C T_C S_C \cdot T_C \cdot (M \otimes_S S_D \cdot T_D \cdot \text{Hom}_R(M, S_{C^*} \cdot X)) \cong$$

$$S_C T_C (M_e \otimes_{S_e} S \otimes_S S_D \cdot T_D \cdot \text{Hom}_R(M, S_{C^*} \cdot X)) \cong$$

$$S_C T_C (M_e \otimes_{S_e} S_D \cdot T_D \cdot \text{Hom}_R(M, S_{C^*} \cdot X)) \cong$$

$$S_C T_C (M_e \otimes_{S_e} S D_T D_Hom_{R_e}(M, S_{C^*} \cdot X)) \cong$$

$$S_C T_C (M_e \otimes_{S_e} S D_T D_Hom_{R_e}(M_e, S_C T_C S_{C^*} \cdot X)) \cong$$

As $S_C$ maps objects from $R\text{-Mod} / C^*$ to $C$-closed $R_e$-modules, we can deduce that $S_C \cdot G^*F^*X \cong S_C \cdot X.$ Since $S_C$ is full and faithful, it follows that $G^*F^*X \cong X.$

Given $Y \in S - \text{Mod} / D^*_\text{rest}.M_e,$

$$S_D T_D S_D \cdot F^*G^*Y = S_D T_D S_D \cdot T_D \cdot \text{Hom}_R(M, S_C \cdot T_C \cdot (M \otimes_S S_D \cdot Y)) \cong$$

$$S_D T_D S_D \cdot T_D \cdot \text{Hom}_R(S_C \cdot T_C \cdot (R \otimes_{R_e} M_e), S_C \cdot T_C \cdot (M \otimes_S S_D \cdot Y)) \cong$$

$$S_D T_D S_D \cdot T_D \cdot \text{Hom}_R(R \otimes_{R_e} M_e, S_C \cdot T_C \cdot (M \otimes_S S_D \cdot Y)) \cong$$

$$S_D T_D \text{Hom}_{R_e}(M_e, S_C T_C S_C \cdot T_C \cdot (M \otimes_S S_D \cdot Y)) \cong$$

$$S_D T_D \text{Hom}_{R_e}(M_e, S_C T_C (M_e \otimes_{S_e} S \otimes_S S_D \cdot Y)) \cong$$

$$S_D T_D \text{Hom}_{R_e}(M_e, S_C T_C (M_e \otimes_{S_e} S_D) \cdot Y) \cong$$

We can again deduce from this that $F^*G^*Y \cong Y.$
In order to extend [D1, Theorem 7.4], we will denote by \( (\text{R-Mod}/\mathcal{C}^* | \text{weak } M_e) \) the full subcategory of \( \text{R-Mod}/\mathcal{C}^* \) consisting of the objects \( X \) such that \( T_\mathcal{C}S_\mathcal{C}X \) weakly divides \( T_\mathcal{C}M_e \). If the hereditary torsion theory \((\mathcal{C}, \mathcal{F})\) determined by the localizing subcategory \( \mathcal{C} \) is of finite type, then the functor \( T_\mathcal{C} \) preserves direct sums. So, it makes sense to consider \((\text{R-Mod}/\mathcal{C}^* | M_e)\), the category of all the objects \( X \) in \( \text{R-Mod}/\mathcal{C}^* \) such that \( T_\mathcal{C}S_\mathcal{C}X \) divides some direct sum of copies of \( T_\mathcal{C}M_e \). Analogously, \((\text{S-Mod}/\mathcal{D}^* | \text{weak } S_e)\) denotes the full subcategory of \( \text{S-Mod}/\mathcal{D}^* \) whose objects \( Y \) satisfy that \( T_\mathcal{D}S_\mathcal{D}Y \) weakly divides \( T_\mathcal{D}S_e \). From the proof of Theorem 2.7, if \((\mathcal{C}, \mathcal{F})\) is of finite type and \( M_e \) is \( \mathcal{C} \)-finitely generated, then \((\mathcal{D}, \mathcal{G})\) is also of finite type and we can define the category \((\text{S-Mod}/\mathcal{D}^* | S_e)\) whose objects \( Y \) verify that \( T_\mathcal{D}S_\mathcal{D}Y \) divides some direct sum of copies of \( T_\mathcal{D}S_e \). Recall from Section 1 that we denote by \( S_3 : \text{S-Mod}/\mathcal{D}^* \to (S_e)_{\mathcal{D}}\text{-Mod} \) the canonical functor that associates uniquely a left \( S_3 \)-module to each object in \( \text{S-Mod}/\mathcal{D}^* \). Analogously, we have the full and faithful functor \( S_3 : S_e\text{-Mod}/\mathcal{D} \to (S_e)_{\mathcal{D}}\text{-Mod} \). First, we remark the easy result.

**Proposition 3.10.** The subcategories

\[
\text{R-Mod}/\mathcal{C}^*_{\text{rest.}M_e}
\]

and

\[
\text{S-Mod}/\mathcal{D}^*_{\text{rest.}M_e}
\]

are closed under finite direct sums and direct summands.

**Lemma 3.11.** If \( R \) is \( \mathcal{C} \)-strongly graded, \((\mathcal{C}, \mathcal{F})\) is of finite type and \( M \) is a graded left \( R \)-module such that \( M_e \) is \( \mathcal{C} \)-finitely generated, then \((\mathcal{C}^*, \mathcal{F}^*)\) is of finite type and \( M \) is \( \mathcal{C}^* \)-finitely generated.

**Proof.** Let \( I \) be a left ideal in the Gabriel topology \( \mathcal{R}^* \) associated with \( \mathcal{C}^* \). By Lemma 3.2, \( \mathcal{R}e \cap I \) is a left ideal of \( \mathcal{R}e \) that is in \( \mathcal{R} \). As \((\mathcal{C}, \mathcal{F})\) is of finite type, \( \mathcal{R}e \cap I \) contains a finitely generated member \( J \) of \( \mathcal{R} \). But \( RJ \) is finitely generated as left ideal of \( R \) and thus we have proved that \((\mathcal{C}^*, \mathcal{F}^*)\) is of finite type, since \( RJ \) is in \( \mathcal{R}^* \). Now assume that \( M \) is a graded left \( R \)-module such that \( M_e \) is \( \mathcal{C} \)-finitely generated. Thus, \( M_e \) contains a \( \mathcal{C} \)-dense finitely generated \( Re \)-submodule \( A \). Consider the exact sequence of left \( R \)-modules

\[
0 \to T \to R \otimes_{Re} A \to R \otimes_{Re} M_e \to R \otimes_{Re} (M_e/A) \to 0
\]

Since \( R \) is \( \mathcal{C} \)-strongly graded, \( R \) is \( \mathcal{C} \)-flat by Proposition 3.5 and \( \mathcal{C} \) is \( G \)-stable. Thus, the exact sequence

\[
0 \to T \to R \otimes_{Re} A \to R \otimes_{Re} M_e \to R \otimes_{Re} (M_e/A) \to 0
\]

has starting and final points \( \mathcal{C} \)-torsion or, equivalently, \( \mathcal{C}^* \)-torsion. This means that the \( R \)-finitely generated image on \( R \otimes_{Re} A \) in \( R \otimes_{Re} M_e \) is \( \mathcal{C}^* \)-dense. An analogous argument shows that the canonical image of \( R \otimes_{Re} M_e \) in \( M \) is \( \mathcal{C}^* \)-dense. By composing these two canonical morphisms we obtain that the image of \( R \otimes_{Re} A \) in \( M \) is \( \mathcal{C}^* \)-dense and finitely generated.
Theorem 3.12. Consider a localizing subcategory $C$ of $R_e$-$\text{Mod}$ for which $R$ is $C$-strongly graded. Let $M$ be a weakly $G$-invariant graded left module over a $G$-graded ring $R$ such that the $R_e - S_e$-bimodule $M_e$ is weakly $C$-flat and $C$-closed. The following statements hold:

(I) The restriction of the functors

$$F^* : (R - \text{Mod} | C^* \mid \text{weak } M_e) \rightarrow (S - \text{Mod} | D^* \mid \text{weak } S_e)$$

and

$$G^* : (S - \text{Mod} | D^* \mid \text{weak } S_e) \rightarrow (R - \text{Mod} | C^* \mid \text{weak } M_e)$$

establish an equivalence of categories between the full subcategories

$$(R - \text{Mod} | C^* \mid \text{weak } M_e)$$

of $R$-$\text{Mod} | C^*$ and

$$(S - \text{Mod} | D^* \mid \text{weak } T_D S_e)$$

of $S$-$\text{Mod} | D^*$.

(II) If $(C, F)$ is of finite type and $M_e$ is $C$-finitely generated, then the restriction of the functors

$$F^* : (R - \text{Mod} | C^* \mid M_e) \rightarrow (S - \text{Mod} | D^* \mid S_e)$$

and

$$G^* : (S - \text{Mod} | D^* \mid S_e) \rightarrow (R - \text{Mod} | C^* \mid M_e)$$

establish an equivalence of categories between the full subcategories

$$(R - \text{Mod} | C^* \mid M_e)$$

of $R$-$\text{Mod} | C^*$ and

$$(S - \text{Mod} | D^* \mid S_e)$$

of $S$-$\text{Mod} | D^*$.

Proof: (I) We need to check that if $X$ is an object in $(R - \text{Mod} | C^* \mid \text{weak } M_e)$ then $F^* X$ is an object in $(S - \text{Mod} | D^* \mid \text{weak } S_e)$ and that if $Y$ is in $(S - \text{Mod} | D^* \mid \text{weak } S_e)$ then $G^* Y$ is in $(R - \text{Mod} | C^* \mid \text{weak } M_e)$. For, take $X$ in $(R - \text{Mod} | C^* \mid \text{weak } M_e)$. There is a splitting monomorphism in $R$-$\text{Mod} | C$, $T_C S_e X \rightarrow (T_C M_e)^n$. Now compute

$$S_D T_D S_D^* F^* X = S_D T_D S_D^* T_D \ast \text{Hom}_R (M, S_C^* X) \cong$$

$$S_D T_D S_D^* T_D \ast \text{Hom}_R (M_e @ S_e, S_C^* X) \cong S_D T_D \text{Hom}_{R_e} (M_e, S_C^* X) \cong$$

$$S_D T_D \text{Hom}_{R_e} (M_e, S_C T_C S_C^* X)$$

and observe that if we apply the functor $S_D T_D \text{Hom}_{R_e} (M_e, S_C (-))$ to the morphism $T_C S_C^* X \rightarrow (T_C M_e)^n$ then we obtain a splitting monomorphism of left $S_e$-modules. But

$$S_D T_D \text{Hom}_{R_e} (M_e, S_C(T_C M_e)^n) \cong (S_D T_D \text{Hom}_{R_e} (M_e, S_C T_C M_e))^n \cong$$

$$(S_D T_D \text{Hom}_{R_e} (M_e, M_e))^n \cong (S_D T_D S_e)^n$$
Since $S_D$ reflects splitting monomorphisms, we conclude that $T_D S_D, F^* X$ weakly divides $T_D S_e$. Following a similar argument it is possible to prove that $G^* Y \in (R-\text{Mod}/C^* \mid \text{weak } M_e)$ whenever $Y \in (S-\text{Mod}/D^* \mid \text{weak } S_e)$. Part (I) follows now from Proposition 3.10 and Theorem 3.9.

(II) By Lemma 3.11, if $(C, F)$ is of finite type and $M_e$ is $C$-finitely generate, then $(C^*, F^*)$ is of finite type and $M$ is $C^*$-finitely generated. Therefore the categories $R-\text{Mod}/C_{\text{rest}, M_e}$ and $S-\text{Mod}/D_{\text{rest}, M_e}$ have arbitrary direct sums and the functors $F^*, G^*, T_C, T_D, T_C, T_D$ preserve direct sums. By using these facts, one can prove part (II) in a similar way that part (I).

Recall that if $\mathcal{S}$ and $\mathcal{S}$ denotes respectively the Gabriel topologies of left ideals of $S$ and of $S_e$ associated with $D^*$ and $D$, we have a canonical ring morphism $(S_e)_3 \to S_{3^*}$ since $(S_D, T_D, S_e)$ is isomorphic to $S_D T_D S_e$ whenever $S$ is $D$-strongly graded.

By $\text{Mod}(S_{3^*} \mid \text{weak } (S_e)_3)$ we denote the category of all the left $S_{3^*}$-modules that are projective of finite type considered as left $(S_e)_3$-modules. By $\text{Mod}(S_{3^*} \mid (S_e)_3)$ we denote the category of all the left $S_{3^*}$-modules that are projective as left $(S_e)_3$-modules.

Let $Y$ be a left $S_{3^*}$-module such that there exists an isomorphisms of left $(S_e)_3$-modules $f : Y \oplus C \to Z$, where $Z$ is a $D$-closed left $S_e$-module. Assume that $G$ is finite. Following [NRVO, Theorem 3.1], the forgetful functor $(\lambda) : S - \text{gr} \to S-$\text{Mod} has a left and right adjoint $\iota[G] : S-$\text{Mod} $\to S - \text{gr}$ that construct for the $S$-module $Y$ the $G$-graded left $S$-module $Y[G] = \bigoplus_{g \in G}^g Y$, where $gY$ denotes a copy of the abelian group $G$. If $gY$ denotes the natural image of an element $y$ of $Y$ in the subgroup $gY$ of $Y[G]$ then the structure of left graded $S$-module in $Y[G]$ is given by setting $(s_h^g y) = h^g(s_h^g y)$ for $s_h \in S_h$, and $g, h \in G$. We have [NRVO, Remark 3.2.1] a canonical $S$-homomorphism $\alpha : Y \to Y[G]$ that is actually an injective map. It is evident that $Y[G] \cong Y$ as left $S_e$-modules and we have that $Y$ is $D$-closed since it is a direct summand of the $D$-closed left $S_e$-module $Z$. By [NR, Proposition 2.1] this implies that $Y[G]$ is $D^*$-closed and, therefore, $Y$ is $D^*$-torsionfree. Now we have a canonical monomorphism of left $S$ (or $S_{3^*}$-)modules $Y \to Y_G$, that we extend trivially to a monomorphism of left $S_e$-modules $g : Y \oplus C \to Y_G \oplus C$. Since the cokernel of this monomorphism is $D$-torsion, the isomorphism $f$ extends uniquely to an isomorphism $\tilde{f} : Y_G \oplus C \to Z$. This forces that $g$ is an isomorphism. Hence, $Y \cong Y_G$ and we have obtained that $Y$ is $D^*$-closed. Taking $Z$ a free left $(S_e)_3$-module (of finite rank in the case that $D$ is not of finite type), it is possible to deduce the following result.

**Theorem 3.13.** Let $R$ be a $C$-strongly graded ring by a finite group $G$, where $C$ is a localizing subcategory of $R_e-$\text{Mod} and let $M$ be a $G$-invariant graded left module over a $G$-graded ring $R$ such that the $R_e - S_e$-bimodule $M_e$ is weakly $C$-flat and $C$-closed. The following statements hold:

(I) The restriction of the functors

$$F^* : (R - \text{Mod}/C^* \mid \text{weak } M_e) \to (S_{3^*} \mid \text{weak } (S_e)_3)$$

and

$$G^* : (S_{3^*} \mid \text{weak } (S_e)_3) \to (R - \text{Mod}/C^* \mid \text{weak } M_e)$$

have arbitrary direct sums and the functors $F^*, G^*, T_C, T_D, T_C, T_D$ preserve direct sums. By using these facts, one can prove part (II) in a similar way that part (I).
establish an equivalence of categories between the full subcategories

\[(R\text{-Mod}/C^* \mid \text{weak } M_e)\]

of \(R\text{-Mod}/C^*\) and

\[(S_{\mathcal{A}} \mid \text{weak } (S_e)_{\mathcal{A}})\]

of \(S_{\mathcal{A}}\text{-Mod}\).

\((\text{II})\) If \((\mathcal{C}, \mathcal{F})\) is of finite type and \(M_e\) is \(\mathcal{C}\)-finitely generated, then the restriction of the functors

\[F^* : (R \text{-Mod}/C^* \mid M_e) \rightarrow (S_{\mathcal{A}} \mid (S_e)_{\mathcal{A}})\]

and

\[G^* : (S_{\mathcal{A}} \mid (S_e)_{\mathcal{A}}) \rightarrow (R \text{-Mod}/C^* \mid M_e)\]

establish an equivalence of categories between the full subcategories

\[(R\text{-Mod}/C^* \mid M_e)\]

of \(R\text{-Mod}/C^*\) and

\[(S_{\mathcal{A}} \mid (S_e)_{\mathcal{A}})\]

of \(S_{\mathcal{A}}\text{-Mod}\).

Let \(M\) be a weakly \(G\)-invariant graded left \(R\)-module with \(M_e\) \(\mathcal{C}\)-closed. We will assume that \(M_e\) is \(\mathcal{C}\)-cocritical as left \(R_e\)-module, that is, \(T_{\mathcal{C}}M_e\) is a simple object in \(R_e\text{-Mod}/\mathcal{C}\). It is not hard to see that \(S_e = \text{End}_{R_e}(M_e)\) is a division ring and \(S\) is a crossed product. Hence \(M_e\) is flat as right \(S_e\)-module. Since \(S_e\) is a division ring, the localizing subcategory of \(S_e\text{-Mod}, \mathcal{D} = \{A \in S_e\text{-Mod}: M_e \otimes_{R_e} A = 0\} = \{0\}. Therefore we have that the localizing subcategory \(\mathcal{D}^*\) of \(S\text{-Mod}\) is trivial too. Moreover, every left \(S_e\)-module is free and, therefore, \((S \mid S_e) = S\text{-Mod}\) and \((S \mid \text{weak } S_e)\) is the category of such left \(S\)-modules that are finitely generated as left \(S_e\)-modules. In the last case, for \(G\) a finite group, it is possible to prove that a left \(S\)-module is finitely generated as \(S\)-module if and only if it is finitely generated as left \(S_e\)-module. This happens because \(S\) is strongly graded and, so, \(S\) is a projective of finite type left \(S_e\)-module. As a corollary of Theorem 3.12 and the foregoing observations, we have

**Theorem 3.14.** Let \(R\) be a \(\mathcal{C}\)-strongly \(G\)-graded ring for \(\mathcal{C}\) a localizing subcategory of \(R_e\text{-Mod}\). Let \(M\) be a weakly \(G\)-invariant graded left \(R\)-module such that \(M_e\) is a \(\mathcal{C}\)-closed and \(\mathcal{C}\)-cocritical left \(R_e\)-module. The following assertions hold:

\((\text{I})\) If \(\mathcal{C}\) is of finite type then the restriction of the functors

\[\text{Hom}_R(M, S_{\mathcal{C}}(-)) : (R \text{-Mod}/C^* \mid M_e) \rightarrow S - \text{Mod}\]

and

\[T_{\mathcal{C}}(M \otimes S -) : S - \text{Mod} \rightarrow (R \text{-Mod}/C^* \mid M_e)\]

establish an equivalence of categories between the full subcategory

\[(R\text{-Mod}/C^* \mid M_e)\]
of $R\text{-Mod}/C^*$ and the category $S\text{-Mod}$.

(II) If the group $G$ is finite then the restriction of the functors

$$\text{Hom}_R(M, S_{C^*}(-)) : (R - \text{Mod}/C^* | \text{weak } M_e) \rightarrow S - \text{mod}$$

and

$$T_{C^*}(M \otimes_S -) : S - \text{mod} \rightarrow (R - \text{Mod}/C^* | \text{weak } M_e)$$

establish an equivalence of categories between the full subcategory

$$(R - \text{Mod}/C^* | M_e)$$

of $R\text{-Mod}/C^*$ and the category $S\text{-mod}$ of the finitely generated left $S$-modules.

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References


José Gómez Torrecillas - Blas Torrecillas
Universidad de Granada
Campus de Almería
04071 Almería, Spain