Weighted Sharp Inequality for Vector-Valued Multilinear Integral Operator

LIU LANZHE

College of Mathematics, Changsha University of Science and Technology, Changsha 410077, P. R. of China
lanzhe163@gmail.com

Abstract. In this paper, we prove the sharp inequality for some vector-valued multilinear integral operators. The operators include Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator. By using this inequality, we obtain the weighted $L^p$-norm inequality and $L \log L$-type inequality for the vector-valued multilinear operators.

2010 Mathematics Subject Classification: 42B20, 42B25

Keywords and phrases: Vector-valued multilinear operator, Littlewood-Paley operator, Marcinkiewicz operator, Bochner-Riesz operator, sharp inequality, BMO, $A_p$-weight.

1. Introduction and theorems

In this paper, we shall study some vector-valued multilinear integral operators which are defined as following.

Suppose $m_j$ are the positive integers $(j = 1, \cdots, l)$, $m_1 + \cdots + m_l = m$ and $A_j$ are the functions on $\mathbb{R}^n(j = 1, \cdots, l)$. Let $F_t(x, y)$ be the function defined on $\mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{\mathbb{R}^n} F_t(x, y)f(y)dy$$

and

$$F_t^A(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^l R_{m_j+1}(A_j; x, y) \frac{F_t(x, y)f(y)dy}{|x-y|^m}$$

for every bounded and compactly supported function $f$, where

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$
Let $H$ be the Banach space $H = \{ h : ||h|| < \infty \}$ such that, for each fixed $x \in \mathbb{R}^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as mappings from $[0, +\infty)$ to $H$. For $1 < s < \infty$, the vector-valued multilinear operator related to $F_t$ is defined by

$$|T^A(f)(x)|_s = \left( \sum_{i=1}^{\infty} (T^A(f_i)(x))^s \right)^{1/s},$$

where

$$T^A(f_i)(x) = ||F_t^A(f_i)(x)||,$$

and $F_t$ satisfies: for fixed $\varepsilon > 0$,

$$||F_t(x, y)|| \leq C|x - y|^{-n}$$

and

$$||F_t(y, x) - F_t(z, x)|| \leq C|y - z|^{\varepsilon}|x - z|^{-n-\varepsilon}$$

if $2|y - z| \leq |x - z|$. We also denote

$$|T(f)(x)|_s = \left( \sum_{i=1}^{\infty} |T(f_i)(x)|^s \right)^{1/s} \quad \text{and} \quad |f(x)|_s = \left( \sum_{i=1}^{\infty} |f_i(x)|^s \right)^{1/s},$$

where

$$T(g)(x) = ||F_t(g)(x)||,$$

and suppose that $|T|_s$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and and weak $(L^1, L^1)$-bounded.

Note that when $m = 0$, $T^A$ is just the vector-valued multilinear commutator of $T$ and $A$ (see [9, 10, 12]). While when $m > 0$, $T^A$ is non-trivial generalizations of the commutator. It is well-known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1–5]). Hu and Yang (see [7]) proved a variant sharp estimate for the multilinear singular integral operators. In [16] and [17], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator. The main purpose of this paper is to prove a sharp inequality for the vector-valued multilinear integral operators. As the application, we obtain the weighted $L^p$-norm inequality and $L\log L$-type inequality for the vector-valued multilinear operators. In Section 4, we shall give some applications.

First, let us introduce some notations. Throughout this paper, $Q$ will denote a cubes of $\mathbb{R}^n$ with sides parallel to the axes. For any locally integrable function $f$, the sharp function of $f$ is defined by

$$f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [6])

$$f^\#(x) \approx \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that $f$ belongs to $BMO(\mathbb{R}^n)$ if $f^\#$ belongs to $L^\infty(\mathbb{R}^n)$ and define $||f||_{BMO} = ||f^\#||_{L^\infty}$. For $0 < r < \infty$, we denote $f_r^\#$ by

$$f_r^\#(x) = [(|f|^r)^\#(x)]^{1/r}. $$
Let $M$ be the Hardy-Littlewood maximal operator defined by
\[ M(f)(x) = \sup_{x \in Q} |Q|^{-1} \int_{Q} |f(y)|dy. \]
For $k \in \mathbb{N}$, we denote by $M^k$ the operator $M$ iterated $k$ times, i.e., $M^1(f)(x) = M(f)(x)$ and
\[ M^k(f)(x) = M(M^{k-1}(f))(x) \text{ when } k \geq 2. \]

Let $\Phi$ be a Young function and $\tilde{\Phi}$ be the complementary associated to $\Phi$, we denote the $\Phi$-average by, for a function $f$,
\[ \|f\|_{\Phi, Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_{Q} \Phi \left( \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\} \]
and the maximal function associated to $\Phi$ by
\[ M_{\Phi}(f)(x) = \sup_{x \in Q} \|f\|_{\Phi, Q}; \]
The Young functions to be using in this paper are $\Phi(t) = t(1 + \log t)^r$ and $\tilde{\Phi}(t) = \exp(t^{1/r})$, the corresponding average and maximal functions denoted by $\|\cdot\|_{L(\log L)^r, Q}$, $M_{L(\log L)^r}$ and $\|\cdot\|_{\exp L^{1/r}, Q}$, $M_{\exp L^{1/r}}$. Following [13–16], we know the generalized Hölder’s inequality:
\[ \frac{1}{|Q|} \int_{Q} |f(y)g(y)|dy \leq \|f\|_{\Phi, Q} \|g\|_{\tilde{\Phi}, Q} \]
and the following inequality, for $r, r_j \geq 1, j = 1, \ldots, l$ with $1/r = 1/r_1 + \cdots + 1/r_l$ and any $b \in BMO(R^n)$,
\[ \|f\|_{L(\log L)^{1/r_j}, Q} \leq M_{L(\log L)^{1/r_j}}(f) \leq CM^{l+1}(f), \]
\[ \|b - b_Q\|_{\exp L^r, Q} \leq C\|b\|_{BMO}, \]
\[ |b_{2k+1} - b_{2Q}| \leq Ck\|b\|_{BMO}. \]
The Muckenhoupt $A_p$ weight is defined by (see [6], pp. 389–390)
\[ A_p = \left\{ w : \sup_Q \left( \frac{1}{|Q|} \int_{Q} w(x)dx \right) \left( \frac{1}{|Q|} \int_{Q} w(x)^{-1/(p-1)}dx \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty, \]
and
\[ A_1 = \{ w : M(w)(x) \leq Cw(x), a.e. \}. \]

We shall prove the following theorems.

**Theorem 1.1.** Let $1 < s < \infty$, $D^\alpha A_j \in BMO(R^n)$ for all $\alpha$ with $|\alpha| = m_j$ and $j = 1, \ldots, l$. Then there exists a constant $C > 0$ such that for any $f = \{ f_i \} \in C_0^\infty(R^n)$, $0 < r < 1$ and $x \in R^n$,
\[ (|T^A(f)|_s)^r(x) \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M^{l+1}(|f|_s)(x). \]
Theorem 1.2. Let \( 1 < s < \infty, D^\alpha A_j \in BMO(R^n) \) for all \( \alpha \) with \( |\alpha| = m_j \) and \( j = 1, \cdots, l \). Then \( |T^A|_s \) is bounded on \( L^p(w) \) for any \( 1 < p < \infty \) and \( w \in A_p \), that is

\[
|||T^A(f)|||_s \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} |||D^{\alpha_j}A_j|||_{BMO} \right) |||f|||_s \quad \text{on } L^p(w).
\]

Theorem 1.3. Let \( 1 < s < \infty, w \in A_1, D^\alpha A_j \in BMO(R^n) \) for all \( \alpha \) with \( |\alpha| = m_j \) and \( j = 1, \cdots, l \). Then there exists a constant \( C > 0 \) such that for all \( \lambda > 0 \),

\[
w(\{ x \in R^n : |T^A(f)(x)|_s > \lambda \}) \leq C \int_{R^n} \frac{|f(x)|_s}{\lambda} \left[ 1 + \log^+ \left( \frac{|f(x)|_s}{\lambda} \right) \right]^l w(x)dx.
\]

2. Proof of theorem

To prove the theorems, we need the following lemmas.

Lemma 2.1. [3] Let \( A \) be a function on \( R^n \) and \( D^\alpha A \in L^q(R^n) \) for all \( \alpha \) with \( |\alpha| = m \) and some \( q > n \). Then

\[
|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} \left( \frac{1}{Q(x, y)} \int_{\tilde{Q}(x, y)} |D^\alpha A|qdz \right)^{1/q},
\]

where \( \tilde{Q} \) is the cube centered at \( x \) and having side length \( 5\sqrt{n}|x - y| \).

Lemma 2.2. [6, p.485] Let \( 0 < p < q < \infty \) and for any function \( f \geq 0 \). We define that, for \( 1/r = 1/p - 1/q \)

\[
||f||_{S^q} = \sup_{\lambda > 0} \lambda |\{ x \in R^n : f(x) > \lambda \}|^{1/q}, \quad N_{p,q}(f) = \sup_{E} ||f\chi_E||_{L^p}/||\chi_E||_{L^r},
\]

where the sup is taken for all measurable sets \( E \) with \( 0 < |E| < \infty \). Then

\[
||f||_{S^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/r}||f||_{S^q}.
\]

Lemma 2.3. [17] Let \( r_j \geq 1 \) for \( j = 1, \cdots, l \), we denote that \( 1/r = 1/r_1 + \cdots + 1/r_l \). Then

\[
\frac{1}{|Q|} \int_Q |f_1(x) \cdots f_l(x)|g(x)|dx \leq ||f||_{\exp L^{r_1},Q} \cdots ||f||_{\exp L^{r_l},Q} \|g\|_{L((log L)^{1/r},Q)}.
\]

Proof of Theorem 1.1. It suffices to prove for \( f = \{f_i\} \in C_0^\infty(R^n) \) and some constant \( C_0 \), the following inequality holds:

\[
\left( \frac{1}{|Q|} \int_Q |T^A(f)(x)|^s_C - C_0 \right)^{1/r} \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} |||D^{\alpha_j}A_j|||_{BMO} \right) M^{l+1}(|f|_s)(x).
\]

Without loss of generality, we may assume \( l = 2 \). Fix a cube \( Q = Q(x_0, d) \) and \( \tilde{x} \in Q \). Let \( \tilde{Q} = 5\sqrt{n}Q \) and \( \tilde{A}_j(x) = A_j(x) - \sum_{|\alpha| = m_j} \frac{1}{\alpha} (D^\alpha A_j)_{\tilde{x}} x^\alpha \), then \( R_m(A_j; x, y) = R_m(\tilde{A}_j; x, y) \) and \( D^\alpha A_j = D^\alpha \tilde{A}_j = D^\alpha \tilde{A}_j - (D^\alpha A_j)_{\tilde{x}} \) for \( |\alpha| = m_j \). We split \( f = g + h = \{g_i\} + \{h_i\} \) for \( g_i = f_i\chi_{\tilde{Q}} \) and \( h_i = f_i\chi_{R^n \setminus \tilde{Q}} \). Write \( F^A_i(f_i)(x) \)
then, by Minkowski’s inequality,

\[
\begin{align*}
\left[ & \frac{1}{|Q|} \int_Q \left| T^A(f(x))^r_s - T^\tilde{A}(h(x_0))^r_s \right|^r dx \right]^{1/r} \\
\leq & \left[ \frac{1}{|Q|} \int_Q \left( \sum_{i=1}^{\infty} \left| T^A(f_i(x)) - T^\tilde{A}(h_i(x_0)) \right|^s dx \right)^{r/s} \right]^{1/r} \\
\leq & \left[ \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^{\infty} \left( \int_{R^n} \left| \prod_{j=1}^{2} R_{m_j}(A_j; x, y) \overline{F_i(x, y)} g_i(y) dy \right|^s \right)^{r/s} dx \right]^{1/r} \\
& + \left[ \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^{\infty} \left( \sum_{|\alpha_1| = m_1} \int_{R^n} \left| R_{m_2}(\tilde{A}_2; x, y)(x - y)^{\alpha_1} \right|^{r/s} \right) \right) \left( \sum_{|\alpha_2| = m_2} \int_{R^n} \left| R_{m_1}(\tilde{A}_1; x, y)(x - y)^{\alpha_2} \right|^{r/s} \right) \right]^{1/r} \\
& \times D^{\alpha_1} \tilde{A}_1(y) F_t(x, y) g_i(y) dy \right]^s \right)^{r/s} dx \right]^{1/r} \\
& + \left[ \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^{\infty} \left( \sum_{|\alpha_2| = m_2} \int_{R^n} \left| R_{m_1}(\tilde{A}_1; x, y)(x - y)^{\alpha_2} \right|^{r/s} \right) \right) \left( \sum_{|\alpha_1| = m_1} \int_{R^n} \left| R_{m_2}(\tilde{A}_2; x, y)(x - y)^{\alpha_1} \right|^{r/s} \right) \right]^{1/r} \\
& \times D^{\alpha_2} \tilde{A}_2(y) F_t(x, y) g_i(y) dy \right]^s \right)^{r/s} dx \right]^{1/r} \\
& + \left[ \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^{\infty} \left| \prod_{j=1}^{2} R_{m_j}(A_j; x, y) \overline{F_i(x, y)} g_i(y) dy \right|^s \right)^{r/s} dx \right]^{1/r} \\
\end{align*}
\]
Thus, by Lemma 2.2 and the weak type $(1,1)$ of $|T|$, we get

$$R_m(A_j; x, y) \leq C|x - y|^m \sum_{|\alpha_j| = m_j} \|D^{\alpha_j} A_j\|_{BMO},$$

thus, by Lemma 2.2 and the weak type $(1,1)$ of $|T|$, we obtain

$$I_1 \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \left( \frac{1}{|Q|} \int_Q |T(g)(x)|_r^r dx \right)^{1/r}$$

$$\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) |Q|^{-1} \left| \|T(g)\|_{s} \chi_Q \right|_{L^r} \left| \frac{Q}{|Q|^{1/r-1}} \right|$$

$$\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) |Q|^{-1} \left| |T|_{s} \right|_{W^{1,1}}$$

$$\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \left| \tilde{Q} \right|^{-1} \left| \|g\|_{s} \right|_{L^1}$$

$$\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M(|f|_s)(\tilde{x}).$$

For $I_2$, we get, by Lemma 2.2 and generalized Hölder’s inequality,

$$I_2 \leq C \sum_{|\alpha_2| = m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1| = m_1} \left( \frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} A_1 g)(x)|_{r}^r dx \right)^{1/r}$$

$$\leq C \sum_{|\alpha_2| = m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1| = m_1} \left| \frac{Q}{|Q|^{1/r-1}} \right| \left| T(D^{\alpha_1} A_1 g) \right|_{s} \chi_Q \left| \frac{Q}{|Q|^{1/r-1}} \right|$$

$$\leq C \sum_{|\alpha_2| = m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1| = m_1} \left| \frac{Q}{|Q|^{1/r-1}} \right| \left| T(D^{\alpha_1} A_1 g) \right|_{s} \chi_Q \left| \frac{Q}{|Q|^{1/r-1}} \right|$$

$$\leq C \sum_{|\alpha_2| = m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1| = m_1} \left| \frac{Q}{|Q|^{1/r-1}} \right| \left| \|f\|_{s} \right|_{L^1}$$

$$\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M^2(|f|_s)(\tilde{x}).$$
For $I_3$, similar to the proof of $I_2$, we get

$$I_3 \leq C \prod_{j=1}^{2} \left( \sum_{\alpha_j = m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M^2(|f|_s)(\hat{x}).$$

Similarly, for $I_4$, taking $r, r_1, r_2 \geq 1$ such that $1/r = 1/r_1 + 1/r_2$, we obtain, by Lemma 2.3,

$$I_4 \leq C \sum_{\alpha_j = m_j} \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} |T(D^{\alpha_j} A_j D^{\alpha_2} \tilde{A}_2 g)(x)|^r dx \right)^{1/r}$$

$$\leq C \sum_{\alpha_j = m_j} \left| \sum_{\alpha_j = m_j} \frac{|Q|^{-1}}{|Q|^{1/r-1}} \left\| |T(D^{\alpha_j} A_j D^{\alpha_2} \tilde{A}_2 g)|_{BMO} \right\|_{L^{r}} \cdot \left\| |s\chi_{Q}|_{L^{r}} \right\|_{L^{r}} \cdot \left\| |s\chi_{Q}|_{L^{r}} \right\|_{L^{r}}$$

$$\leq C \sum_{\alpha_j = m_j} \left( \prod_{j=1}^{2} \left( \sum_{\alpha_j = m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M^2(|f|_s)(\hat{x}) \right).$$

For $I_5$, we write

$$F_2^\Delta(h_i)(x) - F_1^\Delta(h_i)(x_0)$$

$$= \int_{\mathbb{R}^n} \left( \frac{F_1(x, y)}{|x - y|^m} - \frac{F_1(x_0, y)}{|x_0 - y|^m} \right) \prod_{j=1}^{2} R_m(j, \tilde{A}_j; x, y) h_i(y) dy$$

$$+ \int_{\mathbb{R}^n} \left( R_m(\tilde{A}_1; x, y) - R_m(\tilde{A}_1; x_0, y) \right) \frac{R_m(\tilde{A}_2; x, y)}{|x - y|^m} F_1(x, y) h_i(y) dy$$

$$+ \int_{\mathbb{R}^n} \left( R_m(\tilde{A}_2; x, y) - R_m(\tilde{A}_2; x_0, y) \right) \frac{R_m(\tilde{A}_1; x, y)}{|x - y|^m} F_1(x, y) h_i(y) dy$$

$$- \sum_{|\alpha_1| = m_1} \frac{1}{\alpha_1} \int_{\mathbb{R}^n} \left[ \frac{R_m(\tilde{A}_2; x, y)}{|x - y|^m} F_1(x, y) - \frac{R_m(\tilde{A}_2; x_0, y)}{|x_0 - y|^m} F_1(x_0, y) \right]$$

$$\times D^{\alpha_1} \tilde{A}_1(y) h_i(y) dy$$

$$- \sum_{|\alpha_2| = m_2} \frac{1}{\alpha_2} \int_{\mathbb{R}^n} \left[ \frac{R_m(\tilde{A}_1; x, y)}{|x - y|^m} F_1(x, y) - \frac{R_m(\tilde{A}_1; x_0, y)}{|x_0 - y|^m} F_1(x_0, y) \right]$$

$$\times D^{\alpha_2} \tilde{A}_2(y) h_i(y) dy$$

$$+ \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \frac{1}{\alpha_1 \alpha_2} \int_{\mathbb{R}^n} \left[ \frac{(x - y)^{\alpha_1 + \alpha_2}}{|x - y|^m} F_1(x, y) - \frac{(x_0 - y)^{\alpha_1 + \alpha_2}}{|x_0 - y|^m} F_1(x_0, y) \right]$$

$$\times D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y) h_i(y) dy.$$
By Lemma 2.1 and the following inequality (see [18]):

$$|b_{Q_1} - b_{Q_2}| \leq C \log(\|Q_2/|Q_1|\|b\|_{BMO})$$

for $Q_1 \subset Q_2$, we know that, for $x \in Q$ and $y \in 2^{k+1}Q \setminus 2^k\tilde{Q}$,

$$|R_m(\tilde{A}; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} (\|D^\alpha A\|_{BMO} + \|D^\alpha A\|_{BMO}(x, y) - (D^\alpha A)\tilde{Q})$$

$$\leq Ck|x - y|^m \sum_{|\alpha| = m} \|D^\alpha A\|_{BMO}.$$

Note that $|x - y| \sim |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the conditions on $F_t$,

$$\|I^{(1)}_5\| \leq C \int_{R^n} \left( \frac{|x - x_0|}{|x_0 - y|^{m+n+1}} + \frac{|x - x_0|^\epsilon}{|x_0 - y|^{m+n+\epsilon}} \right) \prod_{j=1}^2 R_{m_j} (\tilde{A}; x, y)||h_i(y)|dy$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha| = m_j} \|D^\alpha A_j\|_{BMO} \right) \times \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k^2 \left( \frac{|x - x_0|}{|x_0 - y|^{n+1}} + \frac{|x - x_0|^\epsilon}{|x_0 - y|^{n+\epsilon}} \right) |f_i(y)|dy$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha| = m_j} \|D^\alpha A_j\|_{BMO} \right) \sum_{k=1}^\infty k^2 (2^{-k} + 2^{-\epsilon k}) \frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f_i(y)|dy.$$

thus, by Minkowski’s inequality,

$$\left( \sum_{i=1}^\infty \|I^{(1)}_5\|^s \right)^{1/s} \leq C \prod_{j=1}^2 \left( \sum_{|\alpha| = m_j} \|D^\alpha A_j\|_{BMO} \right) \times \sum_{k=1}^\infty k^2 (2^{-k} + 2^{-\epsilon k}) \frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|s dy$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha| = m_j} \|D^\alpha A_j\|_{BMO} \right) M(|f|)(\tilde{x}).$$

For $I^{(2)}_5$, by the formula (see [3]):

$$R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y) = \sum_{|\beta| < m} \frac{1}{|\beta|!} R_{m - |\beta|} (D^\beta \tilde{A}; x, x_0)(x - y)^\beta$$

and Lemma 2.1, we have

$$|R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)| \leq C \sum_{|\beta| < m} \sum_{|\alpha| = m} |x - x_0|^{m - |\beta|} |x - y|^{\beta} \|D^\alpha A\|_{BMO},$$
thus

\[
\left( \sum_{i=1}^{\infty} ||I^{(2)}_{5}||^s \right)^{1/s} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right)
\]

\[
\times \sum_{k=0}^{\infty} \int_{2^k \mathcal{Q} \setminus 2^{k+1} \mathcal{Q}} k^{1/2} \frac{|x-x_0|}{|x_0-y|^{n+1}} |f(y)|_s dy
\]

\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M(|f|_s)(\tilde{x}).
\]

Similarly,

\[
\left( \sum_{i=1}^{\infty} ||I^{(3)}_{5}||^r \right)^{1/r} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M(|f|_s)(\tilde{x}).
\]

For \( I^{(4)}_{5} \), recall that \( |b_{2^{k+1}Q} - b_{2^{k}Q}| \leq C k \| b \|_{BMO} \), we get

\[
\left( \sum_{i=1}^{\infty} ||I^{(4)}_{5}||^s \right)^{1/s} \leq C \sum_{|\alpha|=m_2} ||D^\alpha A_2||_{BMO} \sum_{|\alpha|=m_1} \sum_{k=1}^{\infty} k^{2^{-k} + 2^{-\varepsilon k}} \frac{1}{|2^k \mathcal{Q}|} \int_{2^k \mathcal{Q}} |f(y)|_s |D^\alpha A_1(y)| dy
\]

\[
\leq C \sum_{|\alpha|=m_2} ||D^\alpha A_2||_{BMO} \cdot \sum_{|\alpha|=m_1} \sum_{k=1}^{\infty} k^{2^{-k} + 2^{-\varepsilon k}} ||D^\alpha A_1|| - (D^{\alpha_1} A_1)_Q \|_{\exp L, 2^k \mathcal{Q}} \cdot ||f|_s||_{L(\log L), 2^k \mathcal{Q}}
\]

\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) \sum_{k=1}^{\infty} k^{2^{-k} + 2^{-\varepsilon k}} ||f|_s||_{L(\log L), 2^k \mathcal{Q}}
\]

\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M^2(|f|_s)(\tilde{x}).
\]
Similarly,
\[
\left( \sum_{i=1}^{\infty} \|I_{5}^{(5)}\|^s \right)^{1/s} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M^2(|f|_s)(\tilde{x}).
\]

For \(I_{5}^{(6)}\), similarly to the proof of \(I_4\), we get
\[
\left( \sum_{i=1}^{\infty} \|I_{5}^{(6)}\|^s \right)^{1/s} \leq C \sum_{|\alpha|=m_1,|\alpha|=m_2} \int_{R^n \setminus \tilde{Q}} \left\| \frac{(x-y)^{\alpha_1+\alpha_2}F_l(x,y) - (x_0-y)^{\alpha_1+\alpha_2}F_l(x_0,y)}{|x-y|^m} \right\| \times |D^{\alpha_1} \tilde{A}_1(y)| |D^{\alpha_2} \tilde{A}_2(y)||h(y)|_s dy
\]
\[
\leq C \sum_{|\alpha|=m_1,|\alpha|=m_2} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |f(y)|_s |D^{\alpha_1} \tilde{A}_1(y)||D^{\alpha_2} \tilde{A}_2(y)| dy
\]
\[
\leq C \sum_{|\alpha|=m_1,|\alpha|=m_2} \prod_{j=1}^{2} \left\| D^{\alpha_j} A_j - (D^{\alpha_j} \tilde{A}_j) \hat{Q} \right\|_{L^{r_j}} \exp L^{r_j,2^k \tilde{Q}} \cdot |||f|||_{L(\log L)^{1/r},2^k \tilde{Q}}
\]
\[
\leq C \sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} M^3(|f|_s)(\tilde{x}).
\]

Thus
\[
I_5 \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M^3(|f|_s)(\tilde{x}).
\]

This completes the proof of Theorem 1.1.

By Theorem 1.1 and the \(L^p(w)\)-boundedness of \(M^3\), we may obtain the conclusions of Theorem 1.2. By Theorem 1.1 and [8, 10], we may obtain the conclusions of Theorem 1.3.

3. Applications

Now we give some applications of theorems in this paper.

**Application 1.** Littlewood-Paley operators.

Fixed \(\varepsilon > 0\) and \(\mu > (3n+2)/n\). Let \(\psi\) be a fixed function which satisfies the following properties:

1. \(\int_{R^n} \psi(x)dx = 0\),
2. \(|\psi(x)| \leq C(1+|x|)^{-(n+1)}\),
3. \(|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon)}\) when \(2|y| < |x|\);
We denote $\Gamma(x) = \{(y, t) \in R_{+}^{n+1} : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by

$$g_{\psi}^{A}(f)(x) = \left( \int_{0}^{\infty} |F_{t}^{A}(f)(x)|^{2} \frac{dt}{t} \right)^{1/2},$$

$$S_{\psi}^{A}(f)(x) = \left( \int_{\Gamma(x)} \frac{1}{t^{n+1}} \int |F_{t}^{A}(f)(x, y)|^{2} \frac{dydt}{t^{n+1}} \right)^{1/2}$$

and

$$g_{\mu}^{A}(f)(x) = \left( \int \int_{R_{+}^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{n\mu} \ |F_{t}^{A}(f)(x, y)|^{2} \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where

$$F_{t}^{A}(f)(x) = \int_{R^{n}} \prod_{j=1}^{l} R_{m_{j}+1}(A_{j}; x, y) \psi_{t}(x - y) f(y) dy,$$

$$F_{t}^{A}(f)(x, y) = \int_{R^{n}} \prod_{j=1}^{l} R_{m_{j}+1}(A_{j}; x, z) \psi_{t}(y - z) dz$$

and $\psi_{t}(x) = t^{-n} \psi(x/t)$ for $t > 0$. Set $F_{t}(f)(y) = f * \psi_{t}(y)$. We also define that

$$g_{\psi}(f)(x) = \left( \int_{0}^{\infty} |F_{t}(f)(x)|^{2} \frac{dt}{t} \right)^{1/2},$$

$$S_{\psi}(f)(x) = \left( \int_{\Gamma(x)} \frac{1}{t^{n+1}} \int |F_{t}(f)(y)|^{2} \frac{dydt}{t^{n+1}} \right)^{1/2}$$

and

$$g_{\mu}(f)(x) = \left( \int \int_{R_{+}^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{n\mu} \ |F_{t}(f)(y)|^{2} \frac{dydt}{t^{n+1}} \right)^{1/2},$$

which are the Littlewood-Paley operators (see [19]). Let $H$ be the space

$$H = \left\{ h : ||h|| = \left( \int_{0}^{\infty} |h(t)|^{2} dt/t \right)^{1/2} < \infty \right\}$$

or

$$H = \left\{ h : ||h|| = \left( \int \int_{R_{+}^{n+1}} |h(y, t)|^{2} dydt/t^{n+1} \right)^{1/2} < \infty \right\},$$

then, for each fixed $x \in R^{n}$, $F_{t}^{A}(f)(x)$ and $F_{t}^{A}(f)(x, y)$ may be viewed as the mapping from $[0, +\infty)$ to $H$, and it is clear that

$$g_{\psi}^{A}(f)(x) = ||F_{t}^{A}(f)(x)||, \quad g_{\psi}(f)(x) = ||F_{t}(f)(x)||,$$

$$S_{\psi}^{A}(f)(x) = \left| |\chi_{\Gamma(x)} F_{t}^{A}(f)(x, y)| \right|, \quad S_{\psi}(f)(x) = \left| |\chi_{\Gamma(x)} F_{t}(f)(y)| \right|$$

and

$$g_{\mu}^{A}(f)(x) = \left| \left| \frac{t}{t + |x - y|} \right|^{n\mu/2} F_{t}^{A}(f)(x, y) \right|.$$
\[ g_\mu(f)(x) = \left\| \left( \frac{t}{t + |x-y|} \right)^{n\mu/2} F_t(f)(y) \right\|. \]

It is easily to see that \( g_\psi, S_\psi \) and \( g_\mu \) satisfy the conditions of Theorems 1.1, 1.2 and 1.3 (see [8–10]), thus Theorems 1.1, 1.2 and 1.3 hold for \( g_\psi^A, S_\psi^A \) and \( g_\mu^A. \)

**Application 2.** Marcinkiewicz operators.

Fix \( \lambda > \max(1, 2n/(n+2)) \) and \( 0 < \gamma \leq 1. \) Let \( \Omega \) be homogeneous of degree zero on \( R^n \) with \( \int_{S^{n-1}} \Omega(x')d\sigma(x') = 0. \) Assume that \( \Omega \in Lip_\gamma(S^{n-1}). \) The Marcinkiewicz multilinear operators are defined by

\[ \mu^A_\Omega(f)(x) = \left( \int_0^\infty |F^A_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \]

\[ \mu^A_S(f)(x) = \left[ \int \int_{\Gamma(x)} |F^A_t(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2} \]

and

\[ \mu^A_\lambda(f)(x) = \left[ \int \int_{R^{n+1}_+} \left( \frac{t}{t + |x-y|} \right)^{n\lambda} |F^A_t(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2}, \]

where

\[ F^A_t(f)(x) = \int_{|x-y|\leq t} \prod_{j=1}^I R_{m_j+1}(A_j; x, y) \frac{\Omega(x-y)}{|x-y|^m} f(y)dy \]

and

\[ F^A_t(f)(x,y) = \int_{|y-z|\leq t} \prod_{j=1}^I R_{m_j+1}(A_j; y, z) \frac{\Omega(y-z)}{|y-z|^m} f(z)dz. \]

Set

\[ F_t(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy; \]

We also define that

\[ \mu_\Omega(f)(x) = \left( \int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \]

\[ \mu_S(f)(x) = \left( \int \int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2} \]

and

\[ \mu_\lambda(f)(x) = \left( \int \int_{R^{n+1}_+} \left( \frac{t}{t + |x-y|} \right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}, \]

which are the Marcinkiewicz operators (see [20]). Let \( H \) be the space

\[ H = \left\{ h : ||h|| = \left( \int_0^\infty |h(t)|^2 dt/t^3 \right)^{1/2} < \infty \right\} \]
or

\[ H = \left\{ h : ||h|| = \left( \int \int_{R_+^{n+1}} |h(y, t)|^2 dy dt / t^{n+3} \right)^{1/2} < \infty \right\} . \]

Then, it is clear that

\[ \mu^A_1(f)(x) = ||F^A_1(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||, \]
\[ \mu^A_2(f)(x) = ||\chi_{\Gamma(x)}F^A_t(f)(x, y)||, \quad \mu_S(f)(x) = ||\chi_{\Gamma(x)}F_t(f)(y)|| \]

and

\[ \mu^A_\lambda(f)(x) = \left\| \left( \frac{t}{t + |x - y|} \right)^{n\lambda/2} F^A_t(f)(x, y) \right\|, \]
\[ \mu_\lambda(f)(x) = \left\| \left( \frac{t}{t + |x - y|} \right)^{n\lambda/2} F_t(f)(y) \right\|. \]

It is easily to see that \( \mu_\Omega, \mu_S \) and \( \mu_\lambda \) satisfy the conditions of Theorems 1.1, 1.2 and 1.3 (see [11] and [20]), thus Theorems 1.1, 1.2 and 1.3 hold for \( \mu^A_1, \mu^A_2 \) and \( \mu^A_\lambda \).

**Application 3.** Bochner-Riesz operator.

Let \( \delta > (n - 1)/2 \), \( B^\delta_t(f)(\xi) = (1 - t^2|\xi|^2)^{\delta/2} \hat{f}(\xi) \) and \( B^\delta_t(z) = t^{-n}B^\delta(z/t) \) for \( t > 0 \). Set

\[ F^A_{\delta,t}(f)(x) = \int_{R^n} \prod_{j=1}^t R_{m_j+1}(A_j; x, y) \frac{B^\delta_t(x - y)f(y)dy}{|x - y|^m}, \]

The maximal Bochner-Riesz multilinear operator are defined by

\[ B^A_{\delta,*}(f)(x) = \sup_{t>0} |B^A_{\delta,t}(f)(x)|. \]

We also define that

\[ B_{\delta,*}(f)(x) = \sup_{t>0} |B^\delta_t(f)(x)| \]

which is the maximal Bochner-Riesz operator (see [13]). Let \( H \) be the space \( H = \{ h : ||h|| = \sup_{t>0} |h(t)| < \infty \} \), then

\[ B^A_{\delta,*}(f)(x) = ||B^A_{\delta,t}(f)(x)||, \quad B^\delta_t(f)(x) = ||B^\delta_t(f)(x)||. \]

It is easily to see that \( B^A_{\delta,*} \) satisfies the conditions of Theorems 1.1, 1.2 and 1.3 (see [21]), thus Theorems 1.1, 1.2 and 1.3 hold for \( B^A_{\delta,*} \).

**Acknowledgement.** The author would like to express his gratitude to the referees for the comments and suggestions.
References


