Weak Annihilator over Extension Rings

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Abstract. Let $R$ be a ring and $\text{nil}(R)$ the set of all nilpotent elements of $R$. For a subset $X$ of a ring $R$, we define $N_R(X) = \{a \in R \mid xa \in \text{nil}(R) \text{ for all } x \in X\}$, which is called the weak annihilator of $X$ in $R$. In this paper we mainly investigate the properties of the weak annihilator over extension rings.

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1. Introduction

Throughout this paper $R$ denotes an associative ring with unity, $\alpha : R \rightarrow R$ is an endomorphism, and $\delta$ an $\alpha$-derivation of $R$, that is, $\delta$ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for $a, b \in R$. We denote by $R[x; \alpha, \delta]$ the Ore extension whose elements are the polynomials over $R$, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$. We use $P(R)$ and $\text{nil}(R)$ to represent the prime radical and the set of all nilpotent elements of $R$ respectively. Due to Birkenmeier et al. [3], a ring $R$ is called 2-primal if $P(R) = \text{nil}(R)$. Every reduced ring (i.e. nil$(R) = 0$) is obviously a 2-primal ring. Other examples and properties of 2-primal rings can be founded in [4, 5, 6]. Let $\alpha$ be an endomorphism and $\delta$ an $\alpha$-derivation of a ring $R$. Following E. Hashemi and A. Moussavi [11], a ring $R$ is said to be $\alpha$-compatible if for each $a, b \in R, ab = 0 \iff a\alpha(b) = 0$. Moreover, $R$ is called to be $\delta$-compatible if for each $a, b \in R, ab = 0 \Rightarrow a\delta(b) = 0$. If $R$ is both $\alpha$-compatible and $\delta$-compatible, then $R$ is said to be $(\alpha, \delta)$-compatible.

For a subset $X$ of a ring $R$, $r_R(X) = \{a \in R \mid Xa = 0\}$ and $l_R(X) = \{a \in R \mid aX = 0\}$ will stand for the right and left annihilator of $X$ in $R$, respectively. Properties of the right (left) annihilator of a subset in a ring $R$ are studied by many authors (see [2, 8, 9, 14, 15]). As a generalization of the right (left) annihilator, in this paper we introduce the notion of a weak
annihilator of a subset in a ring, and investigate the weak annihilator properties over the Ore extension ring \( R[x; \alpha, \delta] \).

In this paper all subsets are nonempty. Let \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x; \alpha, \delta] \). We say that \( f(x) \in \text{nil}(R[x; \alpha, \delta]) \) if and only if \( a_i \in \text{nil}(R) \) for all \( 0 \leq i \leq n \). Let \( I \) be a subset of \( R \), \( I[x; \alpha, \delta] \) means \( \{ u_0 + u_1 x + \cdots + u_n x^n \in R[x; \alpha, \delta] \mid u_i \in I \} \), that is, for any skew polynomial \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x; \alpha, \delta] \), \( f(x) \in I[x; \alpha, \delta] \) if and only if \( a_i \in I \) for all \( 0 \leq i \leq n \). If \( f(x) \in R[x; \alpha, \delta] \) is a nilpotent element of \( R[x; \alpha, \delta] \), then we say that \( f(x) \in \text{nil}(R[x; \alpha, \delta]) \). For \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x; \alpha, \delta] \), we denote by \( \{ a_0, a_1, \ldots, a_n \} \) or \( C_f \) the set comprised of the coefficients of \( f(x) \), and for a subset \( U \subseteq R[x; \alpha, \delta] \), \( C_U \) = \( \bigcup_{f \in U} C_f \).

2. Weak annihilator

**Definition 2.1.** Let \( R \) be a ring. For a subset \( X \) of a ring \( R \), we define \( N_R(X) = \{ a \in R \mid xa \in \text{nil}(R) \text{ for all } x \in X \} \), which is called the weak annihilator of \( X \) in \( R \). If \( X \) is singleton, say \( X = \{ r \} \), we use \( N_R(r) \) in place of \( N_R(\{ r \}) \).

Obviously, for any subset \( X \) of a ring \( R \), \( N_R(X) = \{ a \in R \mid xa \in \text{nil}(R) \text{ for all } x \in X \} = \{ b \in R \mid bx \in \text{nil}(R) \text{ for all } x \in X \} \), and \( r_R(X) \subseteq N_R(X) \) and \( I_R(X) \subseteq N_R(X) \). If \( R \) is reduced, then \( r_R(X) = N_R(X) = I_R(X) \) for any subset \( X \) of \( R \). It is easy to see that for any subset \( X \subseteq R, N_R(X) \) is an ideal of \( R \) in case \( \text{nil}(R) \) is an ideal.

**Example 2.1.** Let \( Z \) be the ring of integers and \( T_2(Z) \) the \( 2 \times 2 \) upper triangular matrix ring over \( Z \). We consider the subset \( X = \{ \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \} \). Clearly, \( r_{T_2(Z)}(X) = 0 \), and \( N_{T_2(Z)}(X) = \{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \mid m \in Z \} \). Thus \( r_{T_2(Z)}(X) \neq N_{T_2(Z)}(X) \). Hence a weak annihilator is not a trivial generalization of an annihilator.

**Proposition 2.1.** Let \( X, Y \) be subsets of \( R \). Then we have the following:

1. \( X \subseteq Y \) implies \( N_R(X) \supseteq N_R(Y) \).
2. \( X \subseteq N_R(N_R(X)) \).
3. \( N_R(X) = N_R(N_R(N_R(X))) \).

**Proof.** (1) and (2) are really easy.

(3) Applying (2) to \( N_R(X) \), we obtain \( N_R(X) \subseteq N_R(N_R(N_R(X))) \). Since \( X \subseteq N_R(N_R(X)) \), we have \( N_R(X) \supseteq N_R(N_R(N_R(X))) \) by (1). Therefore we have \( N_R(X) = N_R(N_R(N_R(X))) \).

Let \( \delta \) be an \( \alpha \)-derivation of \( R \). For integers \( i, j \) with \( 0 \leq i \leq j, f^i_j \in \text{End}(R, +) \) will denote the map which is the sum of all possible words in \( \alpha, \delta \) built with \( i \) letters \( \alpha \) and \( j - i \) letters \( \delta \). For instance, \( f^0_1 = 1, f^1_1 = \alpha, f^2_1 = \delta^j \) and \( f^3_1 = \alpha^{i-1} \delta + \alpha^{i-2} \delta \alpha + \cdots + \delta \alpha^{j-1} \). The next Lemma appears in [12, Lemma 4.1].

**Lemma 2.1.** For any positive integer \( n \) and \( r \in R \), we have \( x^n r = \sum_{i=0}^{n} f^i_n(r) x^i \) in the ring \( R[x; \alpha, \delta] \).

For the proof of the next lemma, see [11].

**Lemma 2.2.** Let \( R \) be an \( (\alpha, \delta) \)-compatible ring. Then we have the following:

1. If \( ab = 0 \), then \( \alpha^m(b) = \alpha^n(a)b = 0 \) for all positive integers \( n \).
2. If \( \alpha^k(a)b = 0 \) for some positive integer \( k \), then \( ab = 0 \).
3. If \( ab = 0 \), then \( \alpha^n(a) \delta^m(b) = 0 = \delta^m(a) \alpha^n(b) \) for all positive integers \( m, n \).
Lemma 2.3. Let $\delta$ be an $\alpha$-derivation of $R$. If $R$ is $(\alpha, \delta)$-compatible, then $abc = 0$ implies $abf^j_i(c) = 0$ and $af^j_i(b)c = 0$ for all $0 \leq i < j$ and $a, b, c \in R$.

Proof. Let $abc = 0$ for $a, b, c \in R$. Then $ab\alpha(c) = ab\delta(c) = 0$ since $R$ is $(\alpha, \delta)$-compatible. Thus $abf^j_i(c) = 0$ is clear. To see $af^j_i(b)c = 0$, it suffices to show that if $abc = 0$, then $a\alpha(b)c = 0$ and $a\delta(b)c = 0$. Take $a, b, c \in R$ such that $abc = 0$. Then because $R$ is $(\alpha, \delta)$-compatible,

$$abc = 0 \Rightarrow a\alpha(bc) = a\alpha(b)\alpha(c) = 0 \Rightarrow a\alpha(b)c = 0,$$

and

$$a\alpha(b)c = 0 \Rightarrow a\alpha(b)\delta(c) = 0.$$

Moreover,

$$abc = 0 \Rightarrow a\delta(bc) = a\alpha(b)\delta(c) + a\delta(b)c = 0 \Rightarrow a\delta(b)c = 0.$$

Therefore we obtain $af^j_i(b)c = 0$.

Corollary 2.1. Let $R$ be an $(\alpha, \delta)$-compatible ring. Then $a_1a_2\cdots a_n = 0$ implies

$$f_{i_1}^1(a_1)f_{i_2}^2(a_2)\cdots f_{i_n}^n(a_n) = 0$$

for all $i_j \geq s_j \geq 0$ and $a_i \in R$, $i = 1, 2, \ldots, n$.

Proof. It follows from Lemma 2.3.

Lemma 2.4. Let $\delta$ be an $\alpha$-derivation of $R$. If $R$ is $(\alpha, \delta)$-compatible, then $ab \in \text{nil}(R)$ implies $af^j_i(b) \in \text{nil}(R)$ for all $j \geq i \geq 0$ and $a, b \in R$.

Proof. Since $ab \in \text{nil}(R)$, there exists some positive integer $k$ such that $(ab)^k = abab\cdots ab = 0$. Then by Corollary 2.1, it is easy to see that $af^j_i(b) \in \text{nil}(R)$.

Lemma 2.5. Let $R$ be an $(\alpha, \delta)$-compatible ring. If $a\alpha^n(b) \in \text{nil}(R)$ for $a, b \in R$, and $m$ is a positive integer, then $ab \in \text{nil}(R)$.

Proof. Since $a\alpha^n(b) \in \text{nil}(R)$, there exists some positive integer $n$ such that $(a\alpha^n(b))^n = 0$. In the following computations, we use freely the condition that $R$ is $(\alpha, \delta)$-compatible:

$$(a\alpha^n(b))^n = a\alpha^n(b)a\alpha^n(b)\cdots a\alpha^n(b) = 0$$

$$\Rightarrow a\alpha^n(b)a\alpha^n(b)\cdots a\alpha^n(b)ab = 0$$

$$\Rightarrow a\alpha^n(b)a\alpha^n(b)\cdots a\alpha^n(b)a\alpha^m(ab) = 0$$

$$\Rightarrow a\alpha^n(b)a\alpha^n(b)\cdots a\alpha^n(b)a\alpha^m(bab) = 0$$

$$\Rightarrow a\alpha^n(b)a\alpha^n(b)\cdots a\alpha^n(b)abab = 0$$

$$\Rightarrow \cdots \Rightarrow ab \in \text{nil}(R).$$

Lemma 2.6. Let $R$ be an $(\alpha, \delta)$-compatible 2-primal ring and $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x, \alpha, \delta]$. Then $f(x) \in \text{nil}(R[x, \alpha, \delta])$ if and only if $a_i \in \text{nil}(R)$ for all $0 \leq i \leq n$.

Proof. $(\Rightarrow)$ Suppose $f(x) \in \text{nil}(R[x, \alpha, \delta])$. There exists some positive integer $k$ such that $f(x)^k = (a_0 + a_1x + \cdots + a_nx^n)^k = 0$. Then

$$0 = f(x)^k = \text{“lower terms”} + a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots \alpha^{(k-1)n}(a_n)x^{nk}.$$
Hence \( a_n \alpha^n(a_n) \alpha^{2n}(a_n) \cdots \alpha^{(k-1)n}(a_n) = 0 \), and \( \alpha \)-compatibility of \( R \) gives \( a_n \in \text{nil}(R) \). So by Lemma 2.4, \( a_n \in 1 \cdot a_n \in \text{nil}(R) \) implies \( 1 \cdot f_i^j(a_n) = f_i^j(a_n) \in \text{nil}(R) \) for all \( 0 \leq i \leq j \). Let \( Q = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \). Then we have

\[
0 = (Q + a_n x^n)^k \\
= (Q + a_n x^n)(Q + a_n x^n) \cdots (Q + a_n x^n) \\
= (Q^2 + Q \cdot a_n x^n + a_n x^n \cdot Q + a_n x^n \cdot a_n x^n)(Q + a_n x^n) \cdots (Q + a_n x^n) \\
= \cdots = Q^k + \Delta,
\]

where \( \Delta \in R[x; \alpha, \delta] \). Note that the coefficients of \( \Delta \) can be written as sums of monomials in \( a_i \) and \( f_u^v(a_j) \) where \( a_i, a_j \in \{a_0, a_1, \cdots, a_n\} \) and \( v \geq u \geq 0 \) are positive integers, and each monomial has \( a_i \) or \( f_u^v(a_j) \). Since \( \text{nil}(R) \) of a \( 2\)-\( \text{primal} \) ring \( R \) is an ideal, we obtain that each monomial is in \( \text{nil}(R) \), and so \( \Delta \in \text{nil}(R)[x; \alpha, \delta] \). Thus we obtain

\[
(a_0 + a_1 x + \cdots + a_{n-1} x^{n-1})^k = "\text{lower terms}" + a_{n-1} \alpha^{n-1}(a_{n-1}) \cdots \alpha^{(n-1)(k-1)}(a_{n-1}) x^{(n-1)k} \in \text{nil}(R)[x; \alpha, \delta]
\]

since \( \text{nil}(R) \) is an ideal of \( R \). Hence

\[
a_{n-1} \alpha^{n-1}(a_{n-1}) \cdots \alpha^{(k-1)(n-1)}(a_{n-1}) \in \text{nil}(R)
\]

and so \( a_{n-1} \in \text{nil}(R) \) by Lemma 2.5. Using induction on \( n \) we obtain \( a_i \in \text{nil}(R) \) for all \( 0 \leq i \leq n \).

(\( \iff \)) Consider the finite subset \( S = \{a_0, a_1, \cdots, a_n\} \subseteq \text{nil}(R) \). Since \( R \) is a \( 2\)-\( \text{primal} \) ring, there exists an integer \( k \) such that any product of \( k \) elements \( a_i a_j \cdots a_k \) from \( \{a_0, a_1, \cdots, a_n\} \) is zero. Then by Corollary 2.1, we obtain

\[
a_{i_1} f_{j_2}^{i_2}(a_{j_2}) f_{j_3}^{i_3}(a_{j_3}) \cdots f_{j_k}^{i_k}(a_{j_k}) = 0.
\]

Now we claim that

\[
f(x)^k = (a_0 + a_1 x + \cdots + a_n x^n)^k = 0.
\]

From

\[
\left( \sum_{i=0}^n a_i x^i \right)^2 = \sum_{k=0}^{2n} \left( \sum_{s+t=k} \left( \sum_{i=s}^n a_i f_s^i(a_i) \right) \right) x^k,
\]

it is easy to check that the coefficients of \( \left( \sum_{i=0}^n a_i x^i \right)^k \) can be written as sums of monomials of length \( k \) in \( a_i \) and \( f_u^v(a_j) \), where \( a_i, a_j \in \{a_0, a_1, \cdots, a_n\} \) and \( v \geq u \geq 0 \) are positive integers. Since each monomial \( a_{i_1} f_{j_2}^{i_2}(a_{j_2}) \cdots f_{j_k}^{i_k}(a_{j_k}) = 0 \), where \( a_{i_1}, a_{j_2}, \cdots, a_{j_k} \in \{a_0, a_1, \cdots, a_n\} \) and \( s_{ip}, t_{ip} \) are nonnegative integers for all \( 2 \leq p \leq k \). We obtain \( f(x)^k = 0 \). Hence \( f(x) \) is a nilpotent element of \( R[x; \alpha, \delta] \).

**Corollary 2.2.** Let \( R \) be an \( (\alpha, \delta) \)-compatible \( 2\)-\( \text{primal} \). Then we have the following:

1. \( \text{nil}(R[x; \alpha, \delta]) \) is an ideal.
2. \( \text{nil}(R[x; \alpha, \delta]) = \text{nil}(R)[x; \alpha, \delta] \).

In particular, if \( R \) is an \( \alpha \)-compatible ring, then \( \text{nil}(R[x; \alpha]) \) is an ideal and \( \text{nil}(R[x; \alpha]) = \text{nil}(R)[x; \alpha] \).

**Theorem 2.1.** Let \( R \) be an \( (\alpha, \delta) \)-compatible \( 2\)-\( \text{primal} \). If for each subset \( X \not\subseteq \text{nil}(R) \), \( N_R(X) \) is generated as an ideal by a nilpotent element, then for each subset \( U \not\subseteq \text{nil}(R[x; \alpha, \delta]) \), \( N_{R[x; \alpha, \delta]}(U) \) is generated as an ideal by a nilpotent element.
Proof. Let $U$ be a subset of $R[x; \alpha, \delta]$ with $U \not\subseteq \text{nil}(R[x; \alpha, \delta])$. Then by Corollary 2.2, we have $C_U \not\subseteq \text{nil}(R)$. So there exists $c \in \text{nil}(R)$ such that $N_R(C_U) = c \cdot R$. Now we show that $N_{R[x; \alpha, \delta]}(U) = c \cdot R[x; \alpha, \delta]$. For any $d(x) = d_0 + d_1x + \cdots + d_{u}x^u \in U$ and $h(x) = h_0 + h_1x + \cdots + h_vx^v \in R[x; \alpha, \delta]$, we have

$$d(x) \cdot ch(x) = \sum_{k=0}^{u+v} \left( \sum_{s+t=k}^{u} \left( \sum_{i=s}^{u} d_i f_s^i(ch_i) \right) \right) x^k.$$

Since $c \in \text{nil}(R)$ and $\text{nil}(R)$ of a 2-primal ring is an ideal, we obtain $d_i ch_i \in \text{nil}(R)$, and so $d_i f_s^i(ch_i) \in \text{nil}(R)$ by Lemma 2.4. Hence $\sum_{i+s=t}^{u} \left( \sum_{i=s}^{u} d_i f_s^i(ch_i) \right) \in \text{nil}(R)$, and so $d(x) \cdot ch(x) \in \text{nil}(R[x; \alpha, \delta])$ by Lemma 2.6, and so $N_{R[x; \alpha, \delta]}(U) \supseteq c \cdot R[x; \alpha, \delta]$. Let $g(x) = b_0 + b_1x + \cdots + b_{n}x^{n} \in N_{R[x; \alpha, \delta]}(U)$, then $f(x)g(x) \in \text{nil}(R[x; \alpha, \delta])$ for any $f(x) = a_0 + a_1x + \cdots + a_{m}x^{m} \in U$. Then

$$f(x)g(x) = \sum_{k=0}^{m+n} \left( \sum_{s+t=k}^{m} \left( \sum_{i=s}^{m} a_i f_s^i(b_i) \right) \right) x^k = \sum_{k=0}^{m+n} \Delta_k x^k \in \text{nil}(R[x; \alpha, \delta]).$$

Then we have the following equations by Lemma 2.6:

\begin{align*}
(2.1) & \quad \Delta_{m+n} = a_m \alpha^m(b_n), \\
(2.2) & \quad \Delta_{m+n-1} = a_m \alpha^m(b_{n-1}) + a_{m-1} \alpha^{m-1}(b_n) + a_m f_{m-1}^m(b_n), \\
(2.3) & \quad \Delta_{m+n-2} = a_m \alpha^m(b_{n-2}) + \sum_{i=m-1}^{m} a_i f_{m-1}^i(b_{n-1}) + \sum_{i=m-2}^{m} a_i f_{m-2}^i(b_{n}), \\
& \quad \vdots \\
(2.4) & \quad \Delta_k = \sum_{s+t=k}^{m} \left( \sum_{i=s}^{m} a_i f_s^i(b_i) \right),
\end{align*}

with $\Delta_i \in \text{nil}(R)$ for all $0 \leq i \leq m + n$. From Lemma 2.5 and Equation (2.1), we obtain $a_m b_n \in \text{nil}(R)$, and so $b_n a_m \in \text{nil}(R)$. Now we show that $a_i b_n \in \text{nil}(R)$ for all $0 \leq i \leq m$. If we multiply Equation (2.2) on the left side by $b_n$, then $b_n a_{m-1} \alpha^{m-1}(b_n) = b_n \Delta_{m+n-1} = b_n a_m \alpha^m(b_{n-1}) + b_n a_m f_{m-1}^m(b_n) \in \text{nil}(R)$ since the nil(R) of a 2-primal ring is an ideal. Thus by Lemma 2.5, we obtain $b_n a_{m-1} b_n \in \text{nil}(R)$, and so $b_n a_{m-1} b_n \in \text{nil}(R)$. If we multiply Equation (2.3) on the left side by $b_n$, then we obtain $b_n a_{m-2} f_{m-2}^{m-2}(b_n) = b_n \Delta_{m+n-2} = b_n a_m \alpha^m(b_{n-2}) - b_n a_{m-1} f_{m-1}^{m-1}(b_{n-1}) - b_n a_m f_{m-1}^m(b_{n-1}) - b_n a_{m-2} f_{m-2}^{m-2}(b_n) = b_n \Delta_{m+n-2} = b_n a_m \alpha^m(b_{n-2}) - b_n a_{m-1} f_{m-1}^{m-1}(b_{n-1}) - b_n a_{m-2} f_{m-2}^{m-2}(b_n) \in \text{nil}(R) \text{ since nil}(R) \text{ is an ideal of } R. \text{ Thus we obtain } a_{m-2} b_n \in \text{nil}(R) \text{ and } b_n a_{m-2} \in \text{nil}(R). \text{ Continuing this procedure yields that } a_i b_n \in \text{nil}(R) \text{ for all } 0 \leq i \leq m, \text{ and so } a_i f_s^i(b_n) \in \text{nil}(R) \text{ for any } t \geq s \geq 0 \text{ and } 0 \leq i \leq m \text{ by Lemma 2.4. Thus it is easy to verify that } (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} b_j x^j) \in \text{nil}(R)[x; \alpha, \delta].$

Applying the preceding method repeatedly, we obtain $a_i b_j \in \text{nil}(R)$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. Thus $b_j \in N_R(C_U) = c \cdot R$ for all $0 \leq j \leq n$. Thus there exists $r_j \in R$ such that $b_j = cr_j$. Hence $g(x) = b_0 + b_1x + \cdots + b_{n}x^{n} = c(r_0 + r_1x + \cdots + r_n x^n) \in c \cdot R[x; \alpha, \delta]$. Therefore $N_{R[x; \alpha, \delta]}(U) = c \cdot R[x; \alpha, \delta]$ where $c \in \text{nil}(R[x; \alpha, \delta])$. \hfill \qed

Corollary 2.3. Let $R$ be an $(\alpha, \delta)$-compatible 2-primal ring, and $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]$. Then $f(x)g(x) \in \text{nil}(R[x; \alpha, \delta])$ if and only if $a_i b_j \in \text{nil}(R)$ for all $i, j$. \hfill \qed
Proof. (⇐) Suppose \( a_ib_j \in \text{nil}(R) \) for all \( i, j \). Then \( a_if_s^i(b_j) \in \text{nil}(R) \) for all \( i, j \) and all positive integer \( i \geq s \geq 0 \) by Lemma 2.4. Thus
\[
\sum_{s+i=k} \left( \sum_{i=s}^{m} a_if_s^i(b_i) \right) \in \text{nil}(R), k = 0, 1, 2, \ldots, m+n.
\]
Hence \( f(x)g(x) = \sum_{k=0}^{m+n} \left( \sum_{s+i=k} \left( \sum_{i=s}^{m} a_if_s^i(b_i) \right) x^k \right) \in \text{nil}(R[x; \alpha, \delta]) \) by Lemma 2.6.

(⇒) By analogy with the proof of Theorem 2.1, we complete the proof. □

**Theorem 2.2.** Let \( R \) be an \( \alpha \)-compatibe 2-primal ring. Then the following statements are equivalent:

1. For each subset \( X \not\subseteq \text{nil}(R) \), \( N_R(X) \) is generated as an ideal by a nilpotent element.
2. For each subset \( U \not\subseteq \text{nil}(R[x; \alpha]) \), \( N_{R[x; \alpha]}(U) \) is generated as an ideal by a nilpotent element.

Proof. By Theorem 2.1, it suffices to show (2) ⇒ (1). Let \( X \) be a subset of \( R \) with \( X \not\subseteq \text{nil}(R) \). Then \( X \not\subseteq \text{nil}(R[x; \alpha]) \). So there exists \( f(x) = a_0 + a_1x + \cdots + a_mx^m \in \text{nil}(R[x; \alpha]) \) such that \( N_{R[x; \alpha]}(X) = f(x) \cdot R[x; \alpha] \). Note that \( f(x) = a_0 + a_1x + \cdots + a_mx^m \in \text{nil}(R[x; \alpha]) \), we have \( a_i \in \text{nil}(R) \) for all \( 0 \leq i \leq m \) by Corollary 2.2. Clearly, we may assume that \( a_0 \neq 0 \). Now we show that \( N_{R}(X) = a_0R \). Since \( a_0 \in \text{nil}(R) \) and \( \text{nil}(R) \) is an ideal of \( R \), we obtain \( p \cdot a_0R \subseteq \text{nil}(R) \) for each \( p \in X \). So \( N_R(X) \supseteq a_0R \). If \( m \in N_R(X) \), then \( m \in N_{R[x; \alpha]}(X) \). Thus there exists \( h(x) = h_0 + h_1x + \cdots + h_qx^q \in R[x; \alpha] \) such that
\[
m = f(x)h(x) = \sum_{s=0}^{m+q} \left( \sum_{i+j=s} a_i\alpha^i(h_j) \right) x^s.
\]
Thus we have \( m = a_0h_0 \in a_0R \), and so \( N_R(X) \subseteq a_0R \). Hence \( N_R(X) = a_0R \) where \( a_0 \in \text{nil}(R) \).

For any \( p \in R \), we denote by \( p \cdot R \) the principal right ideal of \( R \) generated by \( p \). Then we have the following results. □

**Theorem 2.3.** Let \( R \) be an \( (\alpha, \delta) \)-compatible 2-primal ring. If for each principal right ideal \( p \cdot R \not\subseteq \text{nil}(R) \), \( N_R(p \cdot R) \) is generated as an ideal by a nilpotent element, then for each principal right ideal \( f(x) \cdot R[x; \alpha, \delta] \not\subseteq \text{nil}(R[x; \alpha, \delta]) \), \( N_{R[x; \alpha, \delta]}(f(x) \cdot R[x; \alpha, \delta]) \) is generated as an ideal by a nilpotent element.

Proof. Let \( f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x; \alpha, \delta] \) with \( f(x) \cdot R[x; \alpha, \delta] \not\subseteq \text{nil}(R[x; \alpha, \delta]) \). We show that \( N_{R[x; \alpha, \delta]}(f(x) \cdot R[x; \alpha, \delta]) \) is generated as an ideal by a nilpotent element. If \( a_iR \subseteq \text{nil}(R) \) for all \( 0 \leq i \leq m \), then by Corollary 2.2, it is easy to see that \( f(x) \cdot R[x; \alpha, \delta] \subseteq \text{nil}(R[x; \alpha, \delta]) \), a contradiction. So there exists \( 0 \leq i \leq m \) such that \( a_iR \not\subseteq \text{nil}(R) \). Thus there exists \( c \in \text{nil}(R) \) such that \( N_R(a_iR) = c \cdot R \). Now we show that \( N_{R[x; \alpha, \delta]}(f(x) \cdot R[x; \alpha, \delta]) = c \cdot R[x; \alpha, \delta]) \). For any \( u(x) = u_0 + u_1x + \cdots + u_fx^f \in R[x; \alpha, \delta] \) and \( v(x) = v_0 + v_1x + \cdots + v_qx^q \in R[x; \alpha, \delta] \), we have \( a_iu_jcv_k \in \text{nil}(R) \) for each \( i, j, k \), since \( c \in \text{nil}(R) \) and \( \text{nil}(R) \) is an ideal of \( R \). Thus \( a_if_s^i(u_j)cv_k \in \text{nil}(R) \) for all \( i, j, k \) and \( s \leq i \) by Lemma 2.4, and so it is easy to see that \( (f(x)u(x)) \cdot cv \in \text{nil}(R[x; \alpha, \delta]) \) for all \( u(x) \in R[x; \alpha, \delta] \) and \( v(x) \in R[x; \alpha, \delta] \) by Corollary 2.3. Hence \( cv(x) \in N_{R[x; \alpha, \delta]}(f(x) \cdot R[x; \alpha, \delta]) \) and so \( N_{R[x; \alpha, \delta]}(f(x) \cdot R[x; \alpha, \delta]) \supseteq c \cdot R[x; \alpha, \delta]) \). On the other hand, assume that \( p(x) = p_0 + p_1x + \cdots + p_xx^x \in N_{R[x; \alpha, \delta]}(f(x) \cdot R[x; \alpha, \delta]) \). Then \( f(x) \cdot R[x; \alpha, \delta] \cdot p(x) \subseteq \text{nil}(R[x; \alpha, \delta]) \) and so \( f(x) \cdot R \cdot
\( p(x) \subseteq \text{nil}(R[x; \alpha, \delta]). \) Thus we obtain \( a R \cdot p_j \subseteq \text{nil}(R) \) for all \( 0 \leq j \leq s. \) So \( p_j \in N_R(a_i R) = c R. \) Thus there exists \( r_j \in R \) such that \( p_j = c r_j \) for all \( 0 \leq j \leq s. \) Hence \( p(x) = p_0 + p_1 x + \cdots + p_s x^s = c(r_0 + r_1 x + \cdots + r_s x^s) \in c \cdot R[x; \alpha, \delta]. \) Hence \( N_{R[x; \alpha, \delta]}(f(x) \cdot R[x; \alpha, \delta]) \subseteq c \cdot R[x; \alpha, \delta]. \) Therefore \( N_{R[x; \alpha, \delta]}(f(x) \cdot R[x; \alpha, \delta]) = c \cdot R[x; \alpha, \delta]. \)

**Theorem 2.4.** Let \( R \) be an \( \alpha \)-compatible 2-primal ring. Then the following statements are equivalent:

1. For each principal right ideal \( p \cdot R \nsubseteq \text{nil}(R), N_R(p \cdot R) \) is generated as an ideal by a nilpotent element.
2. For each principal right ideal \( f(x) \cdot R[x; \alpha] \nsubseteq \text{nil}(R[x; \alpha]), N_{R[x; \alpha]}(f(x) \cdot R[x; \alpha]) \) is generated as an ideal by a nilpotent element.

**Proof.** It follows by the same method of proof as in Theorem 2.2.

Using the same way as above, we also obtain the next two theorems:

**Theorem 2.5.** Let \( R \) be an \((\alpha, \delta)\)-compatible 2-primal ring. If for each \( p \nsubseteq \text{nil}(R), N_R(p) \) is generated as an ideal by a nilpotent element, then for each \( f(x) \nsubseteq \text{nil}(R[x; \alpha, \delta]), N_{R[x; \alpha, \delta]}(f(x)) \) is generated as an ideal by a nilpotent element.

**Theorem 2.6.** Let \( R \) be an \( \alpha \)-compatible 2-primal ring. Then the following statements are equivalent:

1. For each \( p \nsubseteq \text{nil}(R), N_R(p) \) is generated as an ideal by a nilpotent element.
2. For each skew polynomial \( f(x) \nsubseteq \text{nil}(R[x; \alpha]), N_{R[x; \alpha]}(f(x)) \) is generated as an ideal by a nilpotent element.

**Example 2.2.** Let \( R \) be a domain and let

\[
R_3 = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_1 \end{pmatrix} \mid a_i \in R \right\}
\]

be the subring of \( 3 \times 3 \) upper triangular matrix ring. Let \( X \) be any subset of \( R_3 \) with \( X \nsubseteq \text{nil}(R_3). \) We show that \( N_{R_3}(X) \) is generated as a ideal by a nilpotent element. Let

\[
U = \left\{ x \in R \mid \begin{pmatrix} x & y & z \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} \in X \right\}.
\]

If \( U = \{0\}, \) then \( X \nsubseteq \text{nil}(R_3). \) This is contrary to the fact that \( X \nsubseteq \text{nil}(R_3). \) Thus we have \( U \neq \{0\}. \) In this case, we have

\[
N_{R_3}(X) = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & u \\ 0 & 0 & 0 \end{pmatrix} \mid u, v \in R \right\} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot R_3,
\]

where \( \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \text{nil}(R_3) \) by a routine computations. Therefore \( N_{R_3}(X) \) is generated as an ideal by a nilpotent element.

**Example 2.3.** Let \( Z \) be the ring of integers, and \( T(Z, Z) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in Z \right\} \) the trivial extension of \( Z \) by \( Z. \) Let \( p = \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \in T(Z, Z). \) If \( a = 0, \) then we have \( p \cdot T(Z, Z) \subseteq \text{nil}(T(Z, Z)). \) So we assume that \( a \neq 0. \) By a routine computations, we obtain

\[
N_{T(Z,Z)}(p \cdot T(Z, Z)) = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \mid m \in Z \right\} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot T(Z, Z),
\]

\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \text{nil}(T(Z, Z))
\]
where \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) is a nilpotent element.

### 3. Nilpotent associated primes

Given a right \( R \)-module \( N_R \), the right annihilator of \( N_R \) is denoted by \( r_R(N_R) = \{ a \in R \mid Na = 0 \} \). We say that \( N_R \) is prime if \( N_R \neq 0 \), and \( r_R(N_R) = r_R(N'_R) \) for every nonzero submodule \( N'_R \subseteq N_R \) (see [1]). Let \( M_R \) be a right \( R \)-module, an ideal \( \wp \) of \( R \) is called an associated prime of \( M_R \) if there exists a prime submodule \( N_R \subseteq M_R \) such that \( \wp = r_R(N_R) \). The set of associated primes of \( M_R \) is denoted by \( \text{Ass}(M_R) \) (see [1]). Associated primes are well-known in commutative algebra for their important role in the primary decomposition, and has attracted a lot of attention in recent years. In [7], Brewer and Heinzer used localization theory to prove that, over a commutative ring \( R \), the associated primes of the polynomial ring \( R[x] \) (viewed as a module over itself) are all extended: that is, every \( \wp \in \text{Ass}(R[x]) \) may be expressed as \( \wp = \wp_0[x] \), where \( \wp_0 = \wp \cap R \in \text{Ass}(R) \). Using results of Shock in [13] on good polynomials, C. Faith has provided a new proof in [10] of the same result which does not rely on localization or other tools from commutative algebra. In [1], Scott Annin showed that Brewer and Heinzer’s result still holds in the more general setting of a polynomial module \( M[x] \) over a skew polynomial ring \( R[x; \alpha, \delta] \), with possibly noncommutative base \( R \). So the properties of associated primes over a commutative ring can be profitably generalized to noncommutative setting as well.

Motivated by the results in [1], [7], [10], in this section, we continue the study of nilpotent associated primes over Ore extension rings. We first introduce the notion of nilpotent associated primes, which are a generalization of associated primes. We next describe all nilpotent associated primes of the Ore extension ring \( R[x; \alpha, \delta] \) in terms of the nilpotent associated primes of the ring \( R \).

### Definition 3.1

Let \( I \) be a right ideal of a nonzero ring \( R \). We say that \( I \) is a right quasi-prime ideal if \( I \not\subseteq \text{nil}(R) \) and \( N_R(I) = N_R(I') \) for every right ideal \( I' \subseteq I \) and \( I' \not\subseteq \text{nil}(R) \).

Let \( R \) be a domain and Let

\[
R_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}
\]

be the subring of \( n \times n \) upper triangular matrix ring. Then \( \text{nil}(R_n) \) is an ideal of \( R_n \) and

\[
\text{nil}(R_n) = \left\{ \begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ 0 & 0 & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \mid x_{ij} \in R \right\}.
\]

By a routine computations, we know that each right ideal \( I \not\subseteq \text{nil}(R_n) \) is a right quasi-prime ideal.

### Definition 3.2

Let \( \text{nil}(R) \) be an ideal of a ring \( R \). An ideal \( \wp \) of \( R \) is called a nilpotent associated prime of \( R \) if there exists a right quasi-prime ideal \( I \) such that \( \wp = N_R(I) \). The set of nilpotent associated primes of \( R \) is denoted by \( \text{NAss}(R) \).
Recall that an ideal $\mathcal{R}$ in a ring $R$ is said to be a prime ideal if $\mathcal{R} \neq R$, and for $a, b \in R$, $aRb \subseteq \mathcal{R}$ implies that $a \in \mathcal{R}$ or $b \in \mathcal{R}$. Suppose nil$(R)$ is an ideal. Then it is easy to see that if $I$ is a right quasi-prime ideal, then $\mathcal{R} = N_R(I)$ is a prime ideal of $R$.

**Example 3.1.** We now provide the following examples:

(a) Let

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

be the subring of $n \times n$ upper triangular matrix ring. Then it is easy to see that NAss$(R_n) = \{\text{nil}(R_n)\}$.

(b) Let $k$ be any field, and consider the ring $R = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ of $2 \times 2$ lower triangular matrices over $k$. One easily checks that $\begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix} \supseteq \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \supseteq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a composition series for $R_R$. In particular, $R_R$ has finite length.

Next we shall determine the set Ass$(R)$. By an easy ad hoc calculation, we can write down all of the proper nonzero ideals of $R$:

$$\left\{ m_1 = \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix}, m_2 = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}, \alpha = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \right\}.$$

Now since $\alpha^2 = 0$, 0 is not a prime ideal. Moreover, since $m_1Rm_2 \subseteq \alpha$, $\alpha$ is not a prime ideal. So the only candidates for the associated primes of $R$ are the maximal ideals $m_1$ and $m_2$.

We claim that $m_2 \not\subseteq \text{Ass}(R)$. Otherwise, there would exists a right ideal $I \supseteq 0$ of $R$ with $m_2 = r_R(I)$. So $I \cdot m_2 = 0$. Now, given $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in I$, we have $0 = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, so $a = b = 0$. Also, $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ implies that $c = 0$. Thus $I = 0$, a contradiction. Hence $m_2 \not\subseteq \text{Ass}(R)$.

By virtue of $R_R$ being noetherian, we know that $\text{Ass}(R) \neq 0$. Hence $\text{Ass}(R) = \{m_1\}$.

Finally, we should determine the set of NAss$(R)$. Clearly, nil$(R) = \alpha$. Thus nil$(R)$ is an ideal. Now we show that $m_1 = N_R(m_2)$ and $m_2$ is a right quasi-prime ideal. Clearly, $m_1 \subseteq N_R(m_2)$ since $m_2m_1 = 0$. Given $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in N_R(m_2)$, we have $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in \text{nil}(R)$. Then $a = 0$ and so $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in m_1$. Hence $m_1 = N_R(m_2)$. Next we see that $m_2$ is a right quasi-prime ideal. Let $n \not\subseteq \text{nil}(R)$ and $n \subseteq m_2$. Since $N_R(n) \supseteq N_R(m_2)$ is clear, we now assume that $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in N_R(n)$, and find $\begin{pmatrix} h & 0 \\ k & 0 \end{pmatrix} \in n$ with $h \neq 0$. Then we have $\begin{pmatrix} h & 0 \\ k & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} ha & 0 \\ ka & 0 \end{pmatrix} \in \text{nil}(R)$. Thus $a = 0$ and so $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in N_R(m_2)$. Hence we obtain $N_R(n) = N_R(m_2)$ and so $m_2$ is a right quasi-prime ideal. Thus we obtain $m_1 \in \text{NAss}(R)$. Similarly, we have $m_2 \in \text{NAss}(R)$. Therefore NAss$(R) = \{m_1, m_2\} \neq \text{Ass}(R)$.

If $R$ is reduced, then $\mathcal{R}$ is a nilpotent associated prime of $R$ if and only if $\mathcal{R}$ is an associated prime of $R$. So NAss$(R) = \text{Ass}(R)$ in case $R$ is reduced.

Given a polynomial $f(x) \in R[x]$. If the polynomial $f(x)$ has the property that each nonzero coefficient has the same right annihilator in $R$, then we say that such a polynomial is a good polynomial. Shock showed in [13] that, given any nonzero polynomial $f(x) \in R[x]$, one can find $r \in R$ such that $f(x)r$ is good. In order to prove the main result of this section, we will need a generalized version of Shock’s result which applies in our skew polynomial setting.
Let $m(x) = m_0 + m_1 x + \cdots + m_k x^k + \cdots + m_n x^n \not\in \text{nil}(R)[x; \alpha, \delta]$. If $m_k \not\in \text{nil}(R)$, and $m_i \in \text{nil}(R)$ for all $i > k$, then we say that the nilpotent degree of $m(x)$ is $k$. To simplify notations, we write $\text{Ndeg}(m(x))$ for the nilpotent degree of $m(x)$. If $m(x) \in \text{nil}(R)[x; \alpha, \delta]$, then we define $\text{Ndeg}(m(x)) = -1$.

**Definition 3.3.** Let $m(x) = m_0 + m_1 x + \cdots + m_k x^k + \cdots + m_n x^n \not\in \text{nil}(R)[x; \alpha, \delta]$ and the nilpotent degree of $m(x)$ be $k$. If $N_R(m_k) \subseteq N_R(m_i)$ for all $i \leq k$, then we say that $m(x)$ is a nilpotent good polynomial.

**Lemma 3.1.** Let $R$ be an $(\alpha, \delta)$-compatible 2-primal ring. For any $m(x) = m_0 + m_1 x + \cdots + m_k x^k + \cdots + m_n x^n \not\in \text{nil}(R)[x; \alpha, \delta]$, there exists $r \in R$ such that $m(x)r$ is a nilpotent good polynomial.

**Proof.** Assume the result is false, and let $m(x) = m_0 + m_1 x + \cdots + m_k x^k + \cdots + m_n x^n \not\in \text{nil}(R)[x; \alpha, \delta]$ be a counterexample of minimal nilpotent degree $\text{Ndeg}(m(x)) = k \geq 1$. In particular, $m(x)$ is not a nilpotent good polynomial. Hence there exists $i < k$ such that $N_R(m_k) \not\subseteq N_R(m_i)$. So we can find $b \in R$ with $m_i b \not\in \text{nil}(R)$, and $m_k b \in \text{nil}(R)$. Note that the degree $k$ coefficient of $m(x)b$ is $m_k \alpha^k(b) + \sum_{i=k+1}^{n} m_i f_i^k(b)$ and $m_k \alpha^k(b) \in \text{nil}(R)$ due to the $(\alpha, \delta)$-compatibility of $R$. On the other hand, we have $\text{Ndeg}(m(x)) = k$, so $m_i \in \text{nil}(R)$ for all $i > k$. Since $\text{nil}(R)$ of a 2-primal ring is an ideal, $m_i f_i^k(b) \in \text{nil}(R)$ for all $i > k$. Hence it is easy to see that $m(x)b$ has nilpotent degree at most $k - 1$. Since $m_k b \not\in \text{nil}(R)$, by Corollary 2.3, we have $m(x)b \not\in \text{nil}(R)[x; \alpha, \delta]$. By the minimality of $k$, we know that there exists $c \in R$ with $m(x)bc$ nilpotent good. But this contradicts the fact that $m(x)$ is a counterexample to the statement.

**Theorem 3.1.** Let $R$ be an $(\alpha, \delta)$-compatible 2-primal ring. Then

$$\text{NAss}(R[x; \alpha, \delta]) = \{ \varphi[x; \alpha, \delta] \mid \varphi \in \text{NAss}(R) \}.$$

**Proof.** We first prove $\supseteq$. Let $\varphi \in \text{NAss}(R)$. By definition, there exists a right ideal $I \not\subseteq \text{nil}(R)$ with $I$ a right quasi-prime ideal of $R$ and $\varphi = N_R(I)$. It suffices to prove

$$\varphi[x; \alpha, \delta] = N_{R[x; \alpha, \delta]}(I[x; \alpha, \delta])$$

and

$$I[x; \alpha, \delta] \text{ is quasi-prime.}$$

For Equation (3.1), let $f(x) = a_0 + a_1 x + \cdots + a_l x^l \in \varphi[x; \alpha, \delta]$, and let $i(x) = i_0 + i_1 x + \cdots + i_m x^m \in I[x; \alpha, \delta]$. Since $i_k a_j \in \text{nil}(R)$ for each $k$, $j$, applying Corollary 2.3 yields that $i(x)f(x) \in \text{nil}(R[x; \alpha, \delta])$. Hence $\varphi[x; \alpha, \delta] \subseteq N_{R[x; \alpha, \delta]}(I[x; \alpha, \delta])$.

Conversely, if $f(x) = a_0 + a_1 x + \cdots + a_l x^l \in N_{R[x; \alpha, \delta]}(I[x; \alpha, \delta])$, then $i(x)f(x) \in \text{nil}(R[x; \alpha, \delta])$ for all $i(x) = i_0 + i_1 x + \cdots + i_m x^m \in I[x; \alpha, \delta]$. Using Corollary 2.3 again, we obtain that $i_k a_j \in \text{nil}(R)$ for each $k$, $j$. Thus for all $0 \leq j \leq l$, $a_j \in N_R(I) = \varphi$, and so $f(x) \in \varphi[x; \alpha, \delta]$. Hence $N_{R[x; \alpha, \delta]}(I[x; \alpha, \delta]) \subseteq \varphi[x; \alpha, \delta]$. Therefore $\varphi[x; \alpha, \delta] = N_{R[x; \alpha, \delta]}(I[x; \alpha, \delta])$.

Note that the right ideal $I$ is a right quasi-prime ideal. Then we have $I \not\subseteq \text{nil}(R)$. Thus

$$I[x; \alpha, \delta] \not\subseteq \text{nil}(R)[x; \alpha, \delta] = \text{nil}(R[x; \alpha, \delta]).$$

To see (3.2), we must show that if a right ideal $\mathfrak{q} \not\subseteq \text{nil}(R[x; \alpha, \delta])$ and $\mathfrak{q} \subseteq I[x; \alpha, \delta]$, then

$$N_{R[x; \alpha, \delta]}(\mathfrak{q}) = N_{R[x; \alpha, \delta]}(I[x; \alpha, \delta]).$$
To this end, let $D$ be a subset of $R$ consisting of all coefficients of elements of $\mathcal{U}$. Then let $\mathfrak{a}_0$ denote the right ideal of $R$ generated by $D$. Since $\mathcal{U} \subseteq \text{nil}(R[x; \alpha, \delta]) = \text{nil}(R)[x; \alpha, \delta]$, $D \subseteq \text{nil}(R)$, and hence $\mathfrak{a}_0 \subseteq I$. So we have $N_R(\mathfrak{a}_0) = N_R(I) = \varnothing$ because $I$ is a right quasi-prime ideal. Since $N_{R[x; \alpha, \delta]}(\mathcal{U}) \supseteq N_{R[x; \alpha, \delta]}(I[x; \alpha, \delta])$ is clear, we now assume that

$$h(x) = h_0 + h_1x + \ldots + h_dx^d \in N_{R[x; \alpha, \delta]}(\mathcal{U}),$$

and

$$s(x) = s_0 + s_1x + \ldots + s_ex^e \in \mathcal{U}.$$

Then we have $s(x)h(x) \in \text{nil}(R[x; \alpha, \delta])$. By Corollary 2.3, we obtain $s_jh_j \in \text{nil}(R)$ for all $0 \leq j \leq u$. Since $\text{nil}(R)$ of a 2-primal ring is an ideal, $s_jh_j \in \text{nil}(R)$ implies $h_js_i \in \text{nil}(R)$ and so $s_jRh_jRi = (s_jRh_j)^2 \subseteq \text{nil}(R)$. Hence $s_jRh_j \in \text{nil}(R)$. Thus we obtain

$$h_j \in N_R(\mathfrak{a}_0) = N_R(I) = \varnothing$$

for all $0 \leq j \leq u$. Let $i(x) = x_0 + i_1x + \ldots + i_px^p \in I[x; \alpha, \delta]$, we have $i_mh_j \in \text{nil}(R)$ for all $0 \leq m \leq p$, $0 \leq j \leq u$. Then $i(x)h(x) \in \text{nil}(R[x; \alpha, \delta])$ by Corollary 2.3. Hence $N_{R[x; \alpha, \delta]}(\mathcal{U}) \subseteq N_{R[x; \alpha, \delta]}(I[x; \alpha, \delta])$ is proved, and so is $\subseteq$ in Theorem 3.1.

Now we turn our attention to proving $\subseteq$ in Theorem 3.1. Let $I \in \text{NAss}(R[x; \alpha, \delta])$. By definition, we have a right quasi-prime ideal $\mathcal{U}$ of $R[x; \alpha, \delta]$ with $I = N_{R[x; \alpha, \delta]}(\mathcal{U})$. Pick any $m(x) = m_0 + m_1x + \ldots + m_kx^k + \ldots + m_nx^n \notin \text{nil}(R)[x; \alpha, \delta]$ in $\mathcal{U}$. By $\mathcal{U} \subseteq \text{nil}(R[x; \alpha, \delta])$ and Lemma 3.1, we may assume that $m(x)$ is nilpotent good, and $\text{Ndeg}(m(x)) = k$. Set $\mathcal{U}_0 = m(x) \cdot R[x; \alpha, \delta]$. Note that $m(x) \notin \text{nil}(R)[x; \alpha, \delta]$, so we get

$$\mathcal{U}_0 = m(x)R[x; \alpha, \delta] \subseteq \text{nil}(R)[x; \alpha, \delta] = \text{nil}(R[x; \alpha, \delta]).$$

Then we have

$$N_{R[x; \alpha, \delta]}(\mathcal{U}) = N_{R[x; \alpha, \delta]}(\mathcal{U}_0) = N_{R[x; \alpha, \delta]}(m(x) \cdot R[x; \alpha, \delta]) = I$$

because $\mathcal{U}$ is quasi-prime. Consider the right ideal $m_kR$, and assume that $U = N_R(m_kR)$. We wish to claim that $I = U[x; \alpha, \delta]$. Let

$$g(x) = b_0 + b_1x + \ldots + b_dx^d \in U[x; \alpha, \delta].$$

Then

$$m_kRb_j \in \text{nil}(R)$$

for all $0 \leq j \leq l$. Since $m(x)$ is nilpotent good, and $\text{Ndeg}(m(x)) = k$, $m_kRb_j \in \text{nil}(R)$ for all $0 \leq i \leq k$, and $0 \leq j \leq l$. On the other hand, for all $i > k$, $m_i \in \text{nil}(R)$. Thus we have $m_kRb_j \in \text{nil}(R)$ for all $0 \leq i \leq n$, $0 \leq j \leq l$. Choose any

$$h(x) = h_0 + h_1x + \ldots + h_px^p \in R[x; \alpha, \delta].$$

From $m_ih_db_j \in \text{nil}(R)$ for all $0 \leq i \leq n$, $0 \leq d \leq p$ and $0 \leq j \leq l$ and $(\alpha, \delta)$-compatibility of $R$, we obtain $m(x)h(x)g(x) \in \text{nil}(R[x; \alpha, \delta])$ by a routine computations. Hence $g(x) \in N_{R[x; \alpha, \delta]}(m(x)R[x; \alpha, \delta]) = I$, and so $U[x; \alpha, \delta] \subseteq I$. Conversely, let $g(x) = b_0 + b_1x + \ldots + b_dx^d \in I$. Then

$$m(x)Rg(x) \in \text{nil}(R[x; \alpha, \delta]).$$

By Corollary 2.3, we get $m_kRb_j \in \text{nil}(R)$ for all $0 \leq i \leq n$, and $0 \leq j \leq l$. Thus $b_j \in N_R(m_kR)$ for all $0 \leq j \leq l$, and so $g(x) \in U[x; \alpha, \delta]$. Hence $I \subseteq U[x; \alpha, \delta]$. Therefore $I = U[x; \alpha, \delta]$. 
We are now to check that $m_kR$ is quasi-prime. Since $m_k \not\in \text{nil}(R)$, $m_kR \not\subseteq \text{nil}(R)$. Assume that a right ideal $Q \subseteq m_kR$, and $Q \not\subseteq \text{nil}(R)$. Then $N_R(Q) \supseteq N_R(m_kR)$ is clear. Now we show that

$$N_R(Q) \subseteq N_R(m_kR).$$

Set $W = \{m(x)r \mid r \in Q\}$, and let $WR[x; \alpha, \delta]$ be the right ideal of $R[x; \alpha, \delta]$ generated by $W$. It is obvious that $WR[x; \alpha, \delta] \subseteq m(x)R[x; \alpha, \delta]$. Since $Q \not\subseteq \text{nil}(R)$, there exists $a \in R$ such that $m_k a \in Q$ and $m_k a \not\in \text{nil}(R)$. If $m_k a \in \text{nil}(R)$, then we have $m_k a \in \text{nil}(R)$. This contradicts the fact that $m_k a \not\in \text{nil}(R)$. Thus $m_k a \not\in \text{nil}(R)$ and so $m(x) \cdot m_k a \not\subseteq \text{nil}(R[x; \alpha, \delta])$ by Corollary 2.3, and this implies that $WR[x; \alpha, \delta] \not\subseteq \text{nil}(R[x; \alpha, \delta])$. Since $x$ is quasi-prime, we obtain

$$N_R(Q) \subseteq N_R(m_kR).$$

Suppose $q \in N_R(Q)$. Then $rq \in \text{nil}(R)$ for each $r \in Q$. For any $m(x)rf(x) \in WR[x; \alpha, \delta]$ where $f(x) = a_0 + a_1 x + \cdots + a_l x^l \in R[x; \alpha, \delta]$. The typical term of $m(x)rf(x)$ is $m_x^l r_j a_j x^l$.

From $rq \in \text{nil}(R)$ and $\text{nil}(R)$ of a 2-primal ring is an ideal, we have

$$rq \in \text{nil}(R) \Rightarrow qr \in \text{nil}(R) \Rightarrow ra_j qa_j q \in \text{nil}(R) \Rightarrow ra_j q \in \text{nil}(R) \Rightarrow m_i ra_j q \in \text{nil}(R).$$

Thus $m_x^l r_j a_j x^l q \in \text{nil}(R[x; \alpha, \delta])$ due to the $(\alpha, \delta)$—compatibility of $R$, and so

$$m(x)rf(x)q \in \text{nil}(R[x; \alpha, \delta]) = \text{nil}(R[x; \alpha, \delta]).$$

Thus for any

$$\sum m(x)r_j f_i(x) \in WR[x; \alpha, \delta],$$

it is easy to see that

$$\left(\sum m(x)r_j f_i(x)\right) \in \text{nil}(R[x; \alpha, \delta]),$$

Hence $q \in N_R[\alpha, \delta](WR[x; \alpha, \delta]) = U[x; \alpha, \delta]$, and so $q \in U = N_R(m_kR)$. So $N_R(Q) \subseteq N_R(m_kR)$, and this implies that $N_R(Q) = N_R(m_kR)$. Thus $m_kR$ is quasi-prime.

Assembling the above results, we finish the proof of Theorem 3.1.

**Corollary 3.1.** Let $R$ be a 2-primal ring. Then $\text{NAss}(R[x]) = \{\varphi[x] \mid \varphi \in \text{NAss}(R)\}$.

**Proof.** Take $\alpha = id$ and $\delta = 0$ in Theorem 3.1.

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**References**


