Travelling waves for a neural network *

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Abstract

In this note, we give another proof of existence and uniqueness of travelling waves for a neural network equations and prove that all travelling waves are monotonic.

1 Introduction

The following single-layer neural network over the real line was introduced by Ermentrout and Mcleod [6]:

\[ u(x, t) = \int_{-\infty}^{t} ds \int_{-\infty}^{\infty} dy h(t - s)k(x - y)S(u(y, s)) \tag{1.1} \]

where \( x \in \mathbb{R} \) and \( t \in \mathbb{R} \); \( u(x, t) \) is the mean membrane potential of a patch of tissue at position \( x \) and at time \( t \); \( S(u) \) is a nonlinear function and \( S(u(x, t)) \) is the firing rate; \( h \) and \( k \) are nonnegative functions defined on \([0, \infty)\) and \( \mathbb{R} \) respectively. When \( h(t) = e^{-t} \) for \( t > 0 \), then equation (1.1) is equivalent to the following differential equation:

\[ \frac{\partial u(x, t)}{\partial t} + u(x, t) = k \ast S(u)(x, t), \tag{1.2} \]

where \( k \ast S(u) \) denotes the convolution of \( k \) with \( S(u) \), i.e.,

\[ k \ast S(u)(x, t) = \int_{-\infty}^{\infty} k(x - y)S(u(y, t))dy. \]

The existence and uniqueness of travelling waves of (1.1) of the form \( u(x, t) = \phi(x - ct) \) satisfying \( \phi(-\infty) = 0 \) and \( \phi(\infty) = 1 \) are established in [6], where \( \phi \) is a smooth function, called the wave profile, and \( c \) is a constant, called the wave speed. A homotopy argument is employed to prove the existence, which has fostered other studies in similar topics (see [2, 3, 4, 5, 7, 8], for example). This note serves to supplement the results obtained in [6], by applying results in [7], where a comparison argument, together with constructions of appropriate super- and sub- solutions, is used to study travelling waves for (1.2).

First we state the conditions on \( h, k, \) and \( S \). We assume that

\[ (A1) \quad h \in C^1[0, \infty) \text{ is a positive function on } [0, \infty) \text{ with } \int_{0}^{\infty} h(t)dt = 1 \quad \text{and} \quad \int_{0}^{\infty} th(t)dt < \infty. \]

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(A2) $k$ is a nonnegative, continuous function on $\mathbb{R}$ with $\int_{\mathbb{R}} k(x)\,dx = 1$, $k' \in L^1(\mathbb{R})$ and $\text{supp} \, J \cap [0,\infty) \neq \emptyset \neq \text{supp} \, J \cap (-\infty,0)$.

(A3) $S \in C^1([0,1])$ satisfies that $S'(u) > 0$, for $u \in [0,1]$, and that $f(u) = -u + S(u)$ has precisely three zeros at $u = 0, a, 1$ satisfying $f'(0) < 0$ and $f'(1) < 0$, where $0 < a < 1$.

Under the above assumptions, we can improve the results in [6]:

**Theorem 1.1.** Under the above assumptions on $h$, $k$ and $S$, we have

(a) There exists a travelling wave solution $u = \phi(x - ct)$ to (1.1) satisfying $\phi \in C^1$, $\phi(-\infty) = 0$ and $\phi(\infty) = 1$.

(b) Any travelling wave solution to (1.1) satisfying $\phi(-\infty) = 0$ and $\phi(\infty) = 1$ is strictly increasing.

(c) Traveling wave solution to (1.1) is unique modulo spatial translation.

**Remark 1.2.** (a) The monotonicity of travelling wave solutions to (1.1) is established in [6] for special kernels $h$ and $k$ and is conjectured for general case. Our result gives a positive answer.

(b) For the existence and uniqueness in [6], that $k$ is even and $h$ is monotonically decreasing is assumed. While it is natural, we can relax these restrictions.

**2 Proof of Theorem 1.1**

First we need the following result:

**Lemma 2.1.** [7] For any $k$ and $S$ satisfying (A2) and (A3) respectively, there exists one and only one (modulo spatial translation) travelling wave solution $u(x,t) = \phi(x - ct)$ to (1.2) satisfying $\phi(-\infty) = 0$ and $\phi(\infty) = 1$. Moreover, $\phi' > 0$ for all $x \in \mathbb{R}$.

For any $c \in \mathbb{R}$, let $J_c(\cdot) = \int_0^\infty h(s)k(\cdot + cs)\,ds$. Then $J_c$ satisfies (A2). For each $c \in \mathbb{R}$, by Lemma 2.1, there is a travelling wave solution $\phi_c(x - \alpha(c)t)$ to the equation (1.2) with $k$ replaced by $J_c$, where $\phi_c$ is the profile and $\alpha(c)$ is the wave speed, depending on $c$. Let $\xi = x - ct$. Then the pair $(\phi_c, \alpha(c))$ satisfies the following equations:

\begin{align*}
-\alpha(c)\phi'_c(\xi) + \phi_c(\xi) - J_c * S(\phi_c)(\xi) &= 0, \quad (2.1) \\
\phi(-\infty) &= 0, \quad \phi(\infty) = 1. \quad (2.2)
\end{align*}

On the other hand, a travelling wave solution $u = u(x - ct)$ to (1.1) satisfies

\begin{equation}
\begin{aligned}
\phi(\xi) &= J_c * S(u)(\xi).
\end{aligned}
\end{equation}
Therefore, if \((u, c)\) is a travelling wave solution to (1.1), \((u, 0)\) is a travelling wave solution to (1.2) corresponding to \(k(x) = J_{c}(x)\). Similarly, if \((\phi_{c}, 0)\) is a travelling wave solution to (1.2) with \(k(x) = J_{c}(x)\), then \((\phi_{c}, c)\) is a travelling wave solution to (1.1). Therefore to prove the existence of a travelling wave, we only need to prove that there is a \(c \in \mathbb{R}\) such that \(\alpha(c) = 0\). To that end, we need:

**Lemma 2.2.** The wave speed \(\alpha(\cdot)\) is a continuous function on \(\mathbb{R}\).

**Proof.** Let \(c_0 \in \mathbb{R}\) and \((\phi_{c_0}, \alpha(c_0))\) be a travelling wave solution to (1.2) corresponding to \(k = J_{c_0}\). Then, \(\phi_{c}^{'} > 0\) for all \(x \in \mathbb{R}\) and \((\phi_{c}, \alpha(c))\) can be obtained as a solution to (2.1) by the Implicit Function Theorem, applying in the neighborhood of \(c_0\) (see [6], for example). Therefore, \(\phi(c)\) is indeed continuously differentiable.

**Lemma 2.3.** \(\alpha(c) < 0\) for \(c\) positively sufficiently large and \(\alpha(c) > 0\) for \(c\) negatively sufficiently large.

**Proof.** We only prove the lemma when \(c\) is positive. The other case can be proved similarly. We can choose \(z_0 \in (0, 1)\) such that \(\epsilon_0 = S(z_0) - z_0 > 0\). For this \(\epsilon_0\), we can choose two positive constants \(A = A(\epsilon_0)\) and \(B = B(\epsilon_0)\) such that \((\int_{-\infty}^{A} + \int_{B}^{\infty})h(s)ds < \epsilon_0/8\) and \((\int_{-\infty}^{-B} + \int_{B}^{\infty})k(s)ds < \epsilon_0/8\). Since \((\phi_{c}, \alpha(c))\) satisfies (2.1), we have

\[
\begin{align*}
- \alpha(c)\phi_{c}^{'}(x) + \phi_{c}(x) - S(\phi_{c})(x) \\
= \int_{0}^{\infty} h(s) \int_{-\infty}^{\infty} k(x + cs - y)\{S(\phi_{c}(y)) - S(\phi_{c}(x))\}dy\,ds \\
\geq \int_{A}^{B} h(s) \int_{x + cs - B}^{x + cs + B} k(x + cs - y)\{S(\phi_{c}(y)) - S(\phi_{c}(x))\}dy\,ds - \epsilon_0/2
\end{align*}
\]

(2.4)

where we have used the fact that \(S(u(x)) \leq 1\). If \(c \geq A^{-1}B\), then \(y > x\) for \(y\) in the range of the integration on the right of (2.4). Therefore the integral on the right side of (2.4) is positive and

\[
- \alpha(c)\phi_{c}^{'}(x) + \phi_{c}(x) - S(\phi_{c})(x) > -\epsilon_0/2.
\]

(2.5)

Since \(\phi_{c}(-\infty) = 0\), and \(\phi_{c}(\infty) = 1\), we choose \(x_0\) such that \(\phi_{c}(x_0) = z_0\). Then we deduce from (2.5) that \(\alpha(c)\phi_{c}^{'}(x_0) < 0\). Therefore, \(\alpha(c) < 0\) since \(\phi_{c}^{'}(x_0) > 0\).

**Proof of Theorem 1.1** By lemma 2.2 and 2.3, there is constant \(c\) such that \(\alpha(c) = 0\). The pair \((\phi_{c}, c)\) is the travelling wave solution to (1.1). By lemma 2.1, \(\phi_{c}^{'} > 0\) for all \(x\). The uniqueness is established in [6], where uniqueness for monotonic travelling waves is proved.
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References


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