EXISTENCE OF SOLUTIONS FOR NONCONVEX FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. We prove the existence of solutions for the functional differential inclusion
\[ x' \in F(T(t)x), \]
where \( F \) is upper semi-continuous, compact-valued multifunction such that
\[ F(T(t)x) \subseteq \partial V(x(t)) \]
on \([0, T]\), \( V \) is a proper convex and lower semicontinuous function, and \( (T(t)x)(s) = x(t+s) \).

1. Introduction

Let \( \mathbb{R}^m \) be the \( m \)-dimensional Euclidean space with the norm \( \| \cdot \| \) and the scalar product \( \langle \cdot, \cdot \rangle \). When \( I \) is a segment in \( \mathbb{R} \), we denote by \( C(I, \mathbb{R}^m) \) the Banach space of continuous functions from \( I \) to \( \mathbb{R}^m \) with the norm \( \| x(\cdot) \|_\infty := \sup \{ \| x(t) \| : t \in I \} \).

When \( \sigma \) is a positive number, we put \( C_\sigma := C([-\sigma, 0], \mathbb{R}^m) \) and for any \( t \in [0, T] \), \( T > 0 \), we define the operator \( T(t) \) from \( C([-\sigma, T], \mathbb{R}^m) \) to \( C_\sigma \) as \( (T(t)x)(s) := x(t+s), s \in [-\sigma, 0] \).

Let \( \Omega \) be a nonempty subset in \( C_\sigma \). For a given multifunction \( F : \Omega \to 2^{\mathbb{R}^m} \), we consider the following functional differential inclusion:
\[ x' \in F(T(t)x), \quad (1.1) \]

We recall that a continuous function \( x(\cdot) : [-\sigma, T] \to \mathbb{R}^m \) is said to be a solution of (1.1) if \( x(\cdot) \) is absolutely continuous on \([0, T]\), \( T(t)x \in \Omega \) for all \( t \in [0, T] \) and \( x'(t) \in F(T(t)x) \) for almost all \( t \in [0, T] \); see [8].

The functional differential equation (1.1) with \( F \) single-valued, has been studied by many authors; for results, references, and applications, see for example [9, 10].

The existence of solutions for the functional differential inclusion (1.1) was proved by Haddad [8] when \( F \) is upper semicontinuous with convex compact values. The nonconvex case in Banach space has been studied by Benchohra and Ntouyas [2]. The case when \( F \) is lower semicontinuous with compact value has been studied by Fryszkowski [7].

In this paper we prove the existence of solutions for functional differential inclusion (1.1) when \( F \) is upper semicontinuous, compact valued multifunction such that \( F(\psi) \subseteq \partial V(\psi(0)) \) for every \( \psi \in \Omega \) and \( V \) is a proper convex and lower semicontinuous function. Our existence result contains Peano's existence theorem as a
particular case. On the other hand, our result may be considered as an extension of the previous result of Bressan, Cellina and Colombo \[3\].

2. Preliminaries and statement of the main result

For \(x \in \mathbb{R}^m\) and \(r > 0\) let \(B(x,r) := \{y \in \mathbb{R}^m; \|y - x\| < r\}\) be the open ball centered at \(x\) with radius \(r\), and let \(\overline{B}(x,r)\) be its closure. For \(\varphi \in \mathcal{C}_\sigma\) let \(B_\varphi(\varphi,r) := \{\psi \in \mathcal{C}_\sigma; \|\psi - \varphi\|_\infty < r\}\) and \(\overline{B}_\varphi(\varphi,r) := \{\psi \in \mathcal{C}_\sigma; \|\psi - \varphi\|_\infty \leq r\}\). For \(x \in \mathbb{R}^m\) and for a closed subset \(A \subset \mathbb{R}^m\) we denote by \(d(x,A)\) the distance from \(x\) to \(A\) given by \(d(x,A) := \inf \{\|y - x\|; y \in A\}\). Given a function \(V : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}\) let

\[
D(V) := \{x \in \mathbb{R}^m; V(x) < +\infty\}
\]

be its effective domain. We say that \(V\) is proper function if \(D(V)\) is nonempty.

Let \(V : \mathbb{R}^m \to \mathbb{R}\) be a proper convex and lower semicontinuous function. The multifunction \(\partial V : \mathbb{R}^m \to 2^{\mathbb{R}^m}\), defined by

\[
\partial V(x) := \{\xi \in \mathbb{R}^m; V(y) - V(x) \geq \langle \xi, y - x \rangle, \quad \forall y \in \mathbb{R}^m\},
\]

is called subdifferential (in the sense of convex analysis) of the function \(V\).

We say that a multifunction \(F : \Omega \subset \mathcal{C}_\sigma \to 2^{\mathbb{R}^m}\) is upper semicontinuous if for every \(\varphi \in \Omega\) and for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[
F(\psi) \subset F(\varphi) + B(0,\varepsilon), \quad \forall \psi \in \Omega \cap B_\varphi(\varphi,\delta).
\]

The definition of the upper semicontinuous multifunctions is the same as \([6\], Definition 1.2\].

For a multifunction \(F : \Omega \to 2^{\mathbb{R}^m}\) we consider the functional differential inclusion (1.1) under the following assumptions:

(H1) \(\Omega \subset \mathcal{C}_\sigma\) is an open set and \(F\) is upper semicontinuous with compact values;

(H2) There exists a a proper convex and lower semicontinuous function \(V : \mathbb{R}^m \to \mathbb{R}\) such that

\[
F(\psi) \subset \partial V(\psi(0)) \text{ for every } \psi \in \Omega.
\]

Remark. A convex function \(V : \mathbb{R}^m \to \mathbb{R}\) is continuous in the whole space \(\mathbb{R}^m\) \([11\], Corollary 10.1.1\] and almost everywhere differentiable \([11\], Theorem 25.5\]. Therefore, (H2) restricts strongly the multivaluedness of \(F\).

Our main result is the following:

Theorem 2.1. If \(F : \Omega \to 2^{\mathbb{R}^m}\) and \(V : \mathbb{R}^m \to \mathbb{R}\) satisfy assumptions (H1) and (H2) then for every \(\varphi \in \Omega\) there exists \(T > 0\) and \(x(\cdot) : [-\sigma,T] \to \mathbb{R}^m\) a solution of the functional differential inclusion (1.1) such that \(T(0)x = \varphi\) on \([-\sigma,0]\).

3. Proof of the main result

Let \(\varphi \in \Omega\) be arbitrarily fixed. Since the multifunction \(x \to \partial V(x)\) is locally bounded \([4\], Proposition 2.9\], there exists \(r > 0\) and \(M > 0\) such that \(V\) is Lipschitz continuous with constant \(M\) on \(B(\varphi(0),r)\). Since \(\Omega\) is an open set we can choose \(r\) such that \(\overline{B}_\varphi(\varphi,r) \subset \Omega\). Moreover, by \([1\], Proposition 1.1.3\], \(F\) is locally bounded; therefore, we can assume that

\[
\sup\{\|y\| : y \in F(\psi), \psi \in B(\varphi,r)\} \leq M.
\]

Since \(\varphi\) is continuous on \([-\sigma,0]\) we can choose \(\eta > 0\) such that

\[
\|\varphi(t) - \varphi(s)\| < r/4 \text{ for all } t, s \in [-\sigma,0] \text{ with } |t - s| < \eta.
\]
Let $0 < T \leq \min\{\eta, r/4M\}$. We shall prove the existence of a solution of (1.1) defined on the interval $[-\sigma, T]$. For this, we define a family of approximate solutions and we prove that a subsequence converges to a solution of (1.1).

First, for a fixed $n \in \mathbb{N}^*$, we set
\[ x_n(t) = \varphi(t), \quad t \in [-\sigma, 0]. \tag{3.3} \]
Furthermore, we partition $[0, T]$ by points $t_n^j := \frac{jT}{n}$, $j = 0, 1, \ldots, n$, and, for every $t \in [t_n^j, t_n^{j+1}]$, we define
\[ x_n(t) := x_n^j + (t - t_n^j)y_n^j, \tag{3.4} \]
where $x_n^0 = x_n(0) := \varphi(0)$ and
\[
\begin{align*}
x_n^j &= x_n^{j-1} + \frac{T}{n}y_n^{j-1}, \tag{3.5} \\
y_n^j &\in F(T(t_n^j)x_n) \tag{3.6}
\end{align*}
\]
for every $j \in \{1, 2, \ldots, n\}$. It is easy to see that for every $j \in \{1, 2, \ldots, n\}$ we have
\[
x_n^j = \varphi(0) - \frac{T}{n}(y_n^0 + y_n^1 + \cdots + y_n^{j-1}). \tag{3.7} \]
By (3.1) and (3.7) we infer $\|x_n^j - \varphi(0)\| \leq \frac{T}{n}M < r/4$, proving that
\[
x_n(t_n^j) = x_n^j \in B(\varphi(0), r/4) \tag{3.8}
\]
for every $j \in \{1, 2, \ldots, n\}$.

By (3.1) and (3.4) we have that
\[
\|x_n(t) - x_n(t_n^j)\| = \|x_n(t) - x_n^j\| \leq \frac{jT}{n}M < \frac{r}{4}, \tag{3.9}
\]
for every $j \in \{0, 1, \ldots, n\}$. Hence, from (3.8) and (3.9) we deduce that
\[
\|x_n(t) - \varphi(0)\| \leq \|x_n(t) - x_n(t_n^j)\| + \|x_n(t_n^j) - \varphi(0)\| < \frac{r}{2}
\]
and so
\[
x_n(t) \in B(\varphi(0), \frac{r}{2}), \text{ for every } t \in [0, T]. \tag{3.10}
\]
Moreover, by (3.1), (3.4) and (3.6), we have $\|x_n'(t)\| \leq M$ for every $t \in [0, T]$ and so the sequence $(x_n')$ is bounded in $L^2([0, T], \mathbb{R}^m)$.

For $t, s \in [0, T]$, we have
\[
\|x_n(t) - x_n(s)\| \leq \left| \int_s^t \|x_n'(\tau)\|d\tau \right| \leq M|t - s|
\]
so that the sequence $(x_n)$ is equiuniformly continuous. Hence, by Theorem 0.3.4 in [1], there exists a subsequence, still denoted by $(x_n)$, and an absolute continuous function $x : [0, T] \to \mathbb{R}^m$ such that:

(i) $(x_n)$ converges uniformly on $[0, T]$ to $x$;
(ii) $(x_n')$ converges weakly in $L^2([0, T], \mathbb{R}^m)$ to $x'$.

Moreover, since by (3.3) all functions $x_n$ agree with $\varphi$ on $[-\sigma, 0]$, we can obviously say that $x_n \to x$ on $[-\sigma, T]$, if we extend $x$ in such a way that $x \equiv \varphi$ on $[-\sigma, 0]$. Also, it is clearly that $T(0)x = \varphi$ on $[-\sigma, 0]$.

Further on, if we define $\theta_n(t) = t_n^j$ for all $t \in [t_n^j, t_n^{j+1}]$ then, by (3.4) and (3.6), we have
\[
x_n'(t) \in F(T(\theta_n(t))x_n), \text{ a.e. on } [0, T]. \tag{3.11}
\]
and, by (3.8),

\[ x_n(\theta_n(t)) \in B(\varphi(0), \frac{r}{4}), \text{ for every } t \in [0, T]. \]  

(3.12)

Also, since \(|\theta_n(t) - t| \leq \frac{T}{n}\) for every \(t \in [0, T]\), then \(\theta_n(t) \to t\) uniformly on \([0, T]\). Moreover, by the uniformly converges of \((x_n)\) and \((\theta_n)\), we deduce that \(x_n(\theta_n(t)) \to x(t)\) uniformly on \([0, T]\).

Now, we have to estimate \(||(T(\theta_n(t))x_n)(s) - \varphi(s)||\) for each \(s \in [-\sigma, 0]\). If \(-\theta_n(t) \leq s \leq 0\), then \(\theta_n(t) + s \geq 0\) and there exists \(j \in \{0, 1, \ldots, n - 1\}\) such that \(\theta_n(t) + s \in [t_n, t_{n+1}]\). Thus, by (3.2), (3.10) and by the fact that \(|\theta_n(t) - t| \leq T\) and \(|s| \leq T\), we have

\[
||T(\theta_n(t))x_n(s) - \varphi(s)|| = ||x_n(\theta_n(t) + s) - \varphi(0)|| + ||\varphi(s) - \varphi(0)|| \\
\leq 3r < r.
\]

If \(-\sigma \leq s \leq -\theta_n(t)\) then \(s + \theta_n(t) \leq 0\) and by (3.2) we have

\[
||(T(\theta_n(t))x_n)(s) - \varphi(s)|| = ||\varphi(\theta_n(t) + s) - \varphi(s)|| \leq \frac{r}{4} < r.
\]

Therefore,

\[
T(\theta_n(t))x_n \in B(\varphi, r), \text{ for every } t \in [0, T].
\]

(3.13)

Let us denote the modulus continuity of a function \(\psi\) defined on interval \(I\) of \(\mathbb{R}\) by

\[
\omega(\psi, I, \varepsilon) := \sup\{||\psi(t) - \psi(s)||; s, t \in I, |s - t| < \varepsilon\}, \varepsilon > 0.
\]

Then we have:

\[
\|T(\theta_n(t))x_n - T(t)x_n\|_\infty = \sup_{-\sigma \leq s \leq 0} \|x_n(\theta_n(t) + s) - x_n(t + s)\| \\
\leq \omega(x_n, [-\sigma, T], \frac{T}{n}) \\
\leq \omega(\varphi, [-\sigma, 0], \frac{T}{n}) + \omega(x_n, [0, T], \frac{T}{n}) \\
\leq \omega(\varphi, [-\sigma, 0], \frac{T}{n}) + \frac{T}{n}M;
\]

hence

\[
\|T(\theta_n(t))x_n - T(t)x_n\|_\infty \leq \delta_n \text{ for every } t \in [0, T],
\]

(3.14)

where \(\delta_n := \omega(\varphi, [-\sigma, 0], \frac{T}{n}) + \frac{T}{n}M.\) Thus, by continuity of \(\varphi\), we have \(\delta_n \to 0\) as \(n \to \infty\) and hence

\[
\|T(\theta_n(t))x_n - T(t)x_n\|_\infty \to 0 \text{ as } n \to \infty.
\]

Therefore, since the uniform convergence of \(x_n\) to \(x\) on \([-\sigma, T]\) implies

\[
T(t)x_n \to T(t)x \text{ uniformly on } [-\sigma, 0],
\]

(3.15)

we deduce that

\[
T(\theta_n(t))x_n \to T(t)x \text{ in } C_\sigma.
\]

(3.16)

Moreover, by (3.13) and (3.16), we have that \(T(t)x \in \overline{B}_\sigma(\varphi, r) \subset \Omega\). Also, by (3.11) and (3.14), we have

\[
d((T(t)x_n, x'(t)), \text{graph}(F)) \leq \delta_n \text{ for every } t \in [0, T].
\]

(3.17)
By (H2), (ii), (3.16) and [1, Theorem 1.4.1], we obtain
\[ x'(t) \in \text{co}F(T(t)x) \subset \partial V(x(t)) \quad \text{a.e. on } [0, T], \] (3.18)
where \( \text{co} \) stands for the closed convex hull.

Since the functions \( t \to x(t) \) and \( t \to V(x(t)) \) are absolutely continuous, we obtain from [4, Lemma 3.3] and (3.18) that
\[ \frac{d}{dt}V(x(t)) = \|x'(t)\|^2 \quad \text{a.e. on } [0, T]; \]
therefore,
\[ V(x(T)) - V(x(0)) = \int_0^T \|x'(t)\|^2 dt. \] (3.19)

On the other hand, since \( x'_n(t) = y^j_n \in F(T(t^j_n)x_n) \subset \partial V(x_n(t^j_n)) \) for every \( t \in [t^j_n, t^{j+1}_n] \) and for every \( j \in \{0, 1, \ldots, n - 1 \} \), it follows that
\[ V(x_n(t^{j+1}_n)) - V(x_n(t^j_n)) \geq \langle x'_n(t), x_n(t^{j+1}_n) - x_n(t^j_n) \rangle \]
\[ = \langle x'_n(t), \int_{t^j_n}^{t^{j+1}_n} x'_n(t) dt \rangle = \int_{t^j_n}^{t^{j+1}_n} \|x'(t)\|^2 dt. \]

By adding the \( n \) inequalities above, we obtain
\[ V(x_n(T)) - V(x(0)) \geq \int_0^T \|x'_n(t)\|^2 dt \]
and passing to the limit as \( n \to \infty \), we obtain
\[ V(x(T)) - V(x(0)) \geq \limsup_{n \to \infty} \int_0^T \|x'_n(t)\|^2 dt. \] (3.20)

Therefore, by b(3.19) and (3.20),
\[ \int_0^T \|x'(t)\|^2 dt \leq \limsup_{n \to \infty} \int_0^T \|x'_n(t)\|^2 dt \]
and, since \( (x'_n) \) converges weakly in \( L^2([0, T], \mathbb{R}^m) \) to \( x' \), by applying [5] Proposition III.30, we obtain that \( (x'_n) \) converges strongly in \( L^2([0, T], \mathbb{R}^m) \). Hence there exists a subsequence, still denote by \( (x'_n) \), which converges pointwise a.e. to \( x' \).

Since, by (H1), the graph of \( F \) is closed [1, Proposition 1.1.2], by (3.17),
\[ \lim_{n \to \infty} d((T(t)x_n, x'_n(t)), \text{graph}(F)) = 0, \]
we obtain
\[ x'(t) \in F(T(t)x) \quad \text{a.e. on } [0, T]. \]

Therefore, the functional differential inclusion (1.1) has solutions.
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