Strongly nonlinear degenerated elliptic unilateral problems via convergence of truncations

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Abstract

We prove an existence theorem for a strongly nonlinear degenerated elliptic inequalities involving nonlinear operators of the form $Au + g(x, u, \nabla u)$. Here, $A$ is a Leray-Lions operator, $g(x, s, \xi)$ is a lower order term satisfying some natural growth with respect to $|\nabla u|$. There is no growth restrictions with respect to $|u|$, only a sign condition. Under the assumption that the second term belongs to $W^{-1,p'}(\Omega, w^*)$, we obtain the main result via strong convergence of truncations.

1 Introduction

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ and $p$ a real number such that $1 < p < \infty$. Let $w = \{w_i(x), 0 \leq i \leq N\}$ be a vector of weight functions on $\Omega$, i.e. each $w_i(x)$ is a measurable a.e. strictly positive function on $\Omega$, satisfying some integrability conditions (see section 2). The aim of this paper, is to prove an existence theorem for unilateral degenerate problems associated to nonlinear operators of the form $Au + g(x, u, \nabla u)$. Where $A$ is a Leray-Lions operator from $W^{1,p}_0(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$, defined by,

$$Au = -\text{div}(a(x, u, \nabla u))$$

and where $g$ is a nonlinear lower order term having natural growth with respect to $|\nabla u|$. With respect to $|u|$ we do not assume any growth restrictions, but we assume a sign condition. Bensoussan, Boccardo and Murat have proved in the second part of [2] the existence of at least one solution of the unilateral problem

$$\langle Au, v - u \rangle + \int_\Omega g(x, u, \nabla u)(v - u) \, dx \geq \langle f, v - u \rangle \quad \text{for all } v \in K_\psi$$

$$u \in W^{1,p}_0(\Omega) \quad u \geq \psi \ \text{a.e.}$$

$$g(x, u, \nabla u) \in L^1(\Omega) \quad g(x, u, \nabla u)u \in L^1(\Omega)$$

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where \( f \in W^{-1,p'}(\Omega) \) and \( K_\psi = \{ v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), v \geq \psi \ \text{a.e.}\) Here \( \psi \) is a measurable function on \( \Omega \) such that \( \psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \). For that the authors obtain the existence results by proving that the positive part \( u^+ \) (resp. \( u^- \)) of \( u_e \) strongly converges to \( u^+ \) (resp. \( u^- \)) in \( W_0^{1,p}(\Omega) \), where \( u_e \) is a solution of the approximate problem. In the present paper, we study the variational degenerated inequalities. More precisely, we prove the existence of a solution for the problem (3.3) (see section 3), by using another approach based on the strong convergence of the truncations \( T_k(u_e) \) in \( W_0^{1,p}(\Omega) \), where \( u_e \) is a solution of the approximate problem. In the present paper, we study the variational degenerated inequalities. More precisely, we prove the existence of a solution for the problem (3.3) (see section 3), by using another approach based on the strong convergence of the truncations \( T_k(u_e) \) in \( W_0^{1,p}(\Omega) \), where \( u_e \) is a solution of the approximate problem.

This paper is organized as follows: Section 2 contains some preliminaries and basic assumptions. In section 3 we state and prove our main results.

## 2 Preliminaries and basic assumption

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) (\( N \geq 1 \)), let \( 1 < p < \infty \), and let \( w = \{ w_i(x), \ 0 \leq i \leq N \} \) be a vector of weight functions, i.e. every component \( w_i(x) \) is a measurable function which is strictly positive a.e. in \( \Omega \). Further, we suppose in all our considerations that for \( 0 \leq i \leq N \),

\[
\begin{align*}
  w_i & \in L^1_{\text{loc}}(\Omega) \\
  w_i^{-\frac{1}{p-1}} & \in L^1_{\text{loc}}(\Omega)
\end{align*}
\]

We define the weighted space \( L^p(\Omega, \gamma) \) where \( \gamma \) is a weight function on \( \Omega \) by,

\[
L^p(\Omega, \gamma) = \{ u = u(x), \ u \gamma^{1/p} \in L^p(\Omega) \}
\]

with the norm

\[
\| u \|_{p, \gamma} = \left( \int_{\Omega} |u(x)|^p \gamma(x) \, dx \right)^{1/p}.
\]

We denote by \( W^{1,p}(\Omega, w) \) the space of all real-valued functions \( u \in L^p(\Omega, w_0) \) such that the derivatives in the sense of distributions satisfies

\[
\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \quad \text{for all} \quad i = 1, \ldots, N,
\]

which is a Banach space under the norm

\[
\| u \|_{1,p,w} = \left( \int_\Omega |u(x)|^p w_0(x) \, dx + \sum_{i=1}^N \int_\Omega \left| \frac{\partial u(x)}{\partial x_i} \right|^p w_i(x) \, dx \right)^{1/p}.
\]

Since we shall deal with the Dirichlet problem, we shall use the space

\[
X = W_0^{1,p}(\Omega, w)
\]
defined as the closure of \( C_0^\infty(\Omega) \) with respect to the norm (2.3). Note that, \( C_0^\infty(\Omega) \) is dense in \( W_0^{1,p}(\Omega, w) \) and \( (X, \|\cdot\|_{1,p,w}) \) is a reflexive Banach space.

We recall that the dual space of weighted Sobolev spaces \( W_0^{1,p}(\Omega, w) \) is equivalent to \( W^{-1,p'}(\Omega, w^*) \), where \( w^* = \{ w_i^* = w_i^{1-p'}, \forall i = 0, \ldots, N \} \), where \( p' \) is the conjugate of \( p \) i.e. \( p' = \frac{p}{p-1} \) (for more details we refer to [5]).

**Definition 2.1** Let \( Y \) be a separable reflexive Banach space, the operator \( B \) from \( Y \) to its dual \( Y^* \) is called of the calculus of variations type, if \( B \) is bounded and is of the form

\[
B(u) = B(u,u),
\]

where \( (u, v) \to B(u,v) \) is an operator from \( Y \times Y \) into \( Y^* \) satisfying the following properties:

\[
\forall u \in Y, \ v \to B(u,v) \text{ is bounded hemicontinuous from } Y \text{ into } Y^*, \quad (2.6)
\]

\[
\forall v \in Y, \ u \to B(u,v) \text{ is bounded hemicontinuous from } Y \text{ into } Y^*, \quad (2.7)
\]

\[
\text{if } u_n \rightharpoonup u \text{ weakly in } Y \text{ and if } (B(u_n, u_n) - B(u_n, u), u_n - u) \to 0, \text{ then, } B(u_n, v) \to B(u, v) \text{ weakly in } Y^*, \forall v \in Y, \quad (2.8)
\]

\[
\text{if } u_n \rightharpoonup u \text{ weakly in } Y \text{ and if } B(u_n, v) \to \psi \text{ weakly in } Y^*, \text{ then, } (B(u_n, v), u_n) \to (\psi, u). \quad (2.9)
\]

**Definition 2.2** Let \( Y \) be a reflexive Banach space, a bounded mapping \( B \) from \( Y \) to \( Y^* \) is called pseudo-monotone if for any sequence \( u_n \in Y \) with \( u_n \rightharpoonup u \) weakly in \( Y \) and \( \limsup_{n \to \infty} \langle Bu_n, u_n - u \rangle \leq 0 \), one has

\[
\liminf_{n \to \infty} \langle Bu_n, v - u \rangle \geq \langle Bu, v \rangle \quad \text{for all } v \in Y.
\]

We start by stating the following assumptions:

**Assumption (H1)** The expression

\[
\|u\| \leq \left( \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u(x)}{\partial x_i} \right|^p w_i(x) \, dx \right)^{1/p}
\]

is a norm on \( X \) and it is equivalent to the norm (2.3). There exist a weight function \( \sigma \) on \( \Omega \) and a parameter \( q \), such that

\[
1 < q < p + p', \quad (2.10)
\]

\[
\sigma^{1-q'} \in L^1_{\text{loc}}(\Omega), \quad (2.11)
\]

with \( q' = \frac{q}{q-1} \). The Hardy inequality,

\[
\left( \int_{\Omega} |u(x)|^q \sigma \, dx \right)^{1/q} \leq c \left( \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u(x)}{\partial x_i} \right|^p w_i(x) \, dx \right)^{1/p}, \quad (2.12)
\]
holds for every \( u \in X \) with a constant \( c > 0 \) independent of \( u \). Moreover, the imbedding

\[
X \hookrightarrow L^q(\Omega, \sigma),
\]

expressed by the inequality (2.12) is compact.

Note that \((X, \| \cdot \|_X)\) is a uniformly convex (and thus reflexive) Banach space.

**Remark 2.1** If we assume that \( w_0(x) \equiv 1 \) and in addition the integrability condition: There exists \( \nu \in [\frac{N}{p}, \infty] \cap [\frac{1}{p-1}, \infty] \) such that

\[
w_i^{-\nu} \in L^1(\Omega)
\]

for all \( i = 1, \ldots, N \) (which is stronger than (2.2)). Then

\[
\|u\|_X = \left( \sum_{i=1}^{N} \int_{\Omega} |\frac{\partial u(x)}{\partial x_i}|^p w_i(x) \, dx \right)^{1/p}
\]

is a norm defined on \( W_{0}^{1,p}(\Omega, w) \) and is equivalent to (2.3). Moreover

\[
W_{0}^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega),
\]

for all \( 1 < q < p^* \) if \( \nu \leq N(\nu + 1) \) and for all \( q \geq 1 \) if \( \nu \geq N(\nu + 1) \), where \( p_1 = \frac{\nu}{\nu + 1} \) and \( p^* = \frac{Np_1}{N(\nu + 1) - \nu} \) is the Sobolev conjugate of \( p_1 \) (see [5]). Thus the hypotheses (H1) is verified for \( \sigma \equiv 1 \) and for all \( 1 < q < \min \{ p^*, p + p' \} \) if \( \nu < N(\nu + 1) \) and for all \( 1 < q < p + p' \) if \( \nu \geq N(\nu + 1) \).

Let \( A \) be a nonlinear operator from \( W_{0}^{1,p}(\Omega, w) \) into its dual \( W^{-1,p'}(\Omega, w^*) \) defined by,

\[
Au = -\text{div}(a(x, u, \nabla u)),
\]

where \( a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a Carathéodory vector function satisfying the following assumptions:

**Assumption (H2)**

\[
|a_i(x, s, \xi)| \leq \beta w_i^{1/p}(x)[k(x) + \sigma \frac{p}{p'} |s|^\frac{p'}{p} + \sum_{j=1}^{N} w_j^{\frac{1}{p-1}}(x) |\xi_j|^{p-1}] \quad \text{for } i = 1, \ldots, N,
\]

\[
[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0, \quad \text{for all } \xi \neq \eta \in \mathbb{R}^N,
\]

\[
a(x, s, \xi) \xi \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p,
\]

where \( k(x) \) is a positive function in \( L^p(\Omega) \) and \( \alpha, \beta \) are strictly positive constants.
Assumption (H3) Let \( g(x, s, \xi) \) be a Carathéodory function satisfying the following assumptions:

\[
g(x, s, \xi)s \geq 0 \tag{2.17}
\]

\[
|g(x, s, \xi)| \leq b(|s|) \left( \sum_{i=1}^{N} w_i |\xi_i|^p + c(x) \right), \tag{2.18}
\]

where \( b : \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous increasing function and \( c(x) \) is a positive function which lies in \( L^1(\Omega) \). Now we recall some lemmas introduced in [1] which will be used later.

Lemma 2.1 (cf. [1]) Let \( g \in L^r(\Omega, \gamma) \) and let \( g_n \in L^r(\Omega, \gamma) \), with \( \|g_n\|_{r, \gamma} \leq c \) \((1 < r < \infty)\). If \( g_n(x) \to g(x) \) a.e. in \( \Omega \), then \( g_n \to g \) weakly in \( L^r(\Omega, \gamma) \), where \( \gamma \) is a weight function on \( \Omega \).

Lemma 2.2 (cf. [1]) Assume that (H1) holds. Let \( F : \mathbb{R} \to \mathbb{R} \) be uniformly Lipschitzian, with \( F(0) = 0 \). Let \( u \in W^{1,p}_0(\Omega, w) \). Then \( F(u) \in W^{1,p}_0(\Omega, w) \). Moreover, if the set \( D \) of discontinuity points of \( F' \) is finite, then

\[
\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u) \frac{\partial u}{\partial x_i} & \text{a.e. in } \{ x \in \Omega : u(x) \notin D \} \\ 0 & \text{a.e. in } \{ x \in \Omega : u(x) \in D \}. \end{cases}
\]

Lemma 2.3 (cf. [1]) Assume that (H1) holds. Let \( u \in W^{1,p}_0(\Omega, w) \), and let \( T_k(u), k \in \mathbb{R}^+ \), is the usual truncation then \( T_k(u) \in W^{1,p}_0(\Omega, w) \). Moreover, we have

\[
T_k(u) \to u \quad \text{strongly in } W^{1,p}_0(\Omega, w).
\]

Lemma 2.4 Assume that (H1) holds. Let \( (u_n) \) be a sequence of \( W^{1,p}_0(\Omega, w) \) such that \( u_n \rightharpoonup u \) weakly in \( W^{1,p}_0(\Omega, w) \). Then, \( T_k(u_n) \rightharpoonup T_k(u) \) weakly in \( W^{1,p}_0(\Omega, w) \).

Proof. Since \( u_n \rightharpoonup u \) weakly in \( W^{1,p}_0(\Omega, w) \) and by (2.13) we have for a subsequence \( u_{n_k} \rightharpoonup u \) strongly in \( L^q(\Omega, \sigma) \) and a.e. in \( \Omega \). On the other hand,

\[
\|T_k(u_{n_k})\|_X = \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial T_k(u_{n_k})}{\partial x_i} \right|^p w_i = \sum_{i=1}^{N} \int_{\Omega} \left| T_k u_{n_k} \right| \frac{\partial u_{n_k}}{\partial x_i} \right|^p w_i \\
\leq \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_{n_k}}{\partial x_i} \right|^p w_i = \|u_n\|_X^p.
\]

Then \((T_k(u_{n_k}))\) is bounded in \( W^{1,p}_0(\Omega, w) \), hence by using (2.13), \( T_k(u_{n_k}) \rightharpoonup T_k(u) \) weakly in \( W^{1,p}_0(\Omega, w) \).

Lemma 2.5 (cf. [1]) Assume that (H1) and (H2) are satisfied, and let \( (u_n) \) be a sequence of \( W^{1,p}_0(\Omega, w) \) such that \( u_n \rightharpoonup u \) weakly in \( W^{1,p}_0(\Omega, w) \) and

\[
\int_{\Omega} [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)] \nabla(u_n - u) \, dx \to 0.
\]

Then \( u_n \rightharpoonup u \) strongly in \( W^{1,p}_0(\Omega, w) \).
3 Main result

Let $\psi$ be a measurable function with values in $\mathbb{R}$ such that

$$\psi^+ \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega).$$  \hfill (3.1)

Set

$$K_\psi = \{ v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega) \mid v \geq \psi \text{ a.e.} \}. \hfill (3.2)$$

Note that (3.1) implies $K_\psi \neq \emptyset$. Consider the nonlinear problem with Dirichlet boundary conditions,

$$\langle Au, v - u \rangle + \int_\Omega g(x, u, \nabla u)(v - u) \, dx \geq \langle f, v - u \rangle \text{ for all } v \in K_\psi$$

$$u \in W_0^{1,p}(\Omega, w) \quad u \geq \psi \text{ a.e.}$$

$$g(x, u, \nabla u) \in L^1(\Omega), \quad g(x, u, \nabla u) u \in L^1(\Omega).$$ \hfill (3.3)

Then, the following result can be proved for a solution $u$ of this problem.

**Theorem 3.1** Assume that the assumptions (H1)–(H3) and (3.1) hold and let $f \in W^{-1,q'}(\Omega, w^*)$. Then there exists at least one solution of (3.3).

**Remark 3.1**

1) Theorem 3.1 can be generalized in weighted case to an analogous statement in [2].

2) Note that in [1] the authors have assumed that $\sigma^{1-q'} \in L^1(\Omega)$ which is stronger than (2.11).

In the proof of theorem 3.1 we need the following lemma.

**Lemma 3.1** Assume that $f$ lies in $W^{-1,p'}(\Omega, w^*)$. If $u$ is a solution of (P), then, $u$ is also a solution of the variational inequality

$$\langle Au, T_k(v - u) \rangle + \int_\Omega g(x, u, \nabla u) T_k(v - u) \, dx \geq \langle f, T_k(v - u) \rangle \quad \forall k > 0,$$

for all $v \in W_0^{1,p}(\Omega, w) \quad v \geq \psi \text{ a.e.}$

$$u \in W_0^{1,p}(\Omega, w) \quad u \geq \psi \text{ a.e.}$$

$$g(x, u, \nabla u) \in L^1(\Omega).$$ \hfill (3.4)

Conversely, if $u$ is a solution of (3.4) then $u$ is a solution of (3.3).

The proof of this lemma is similar to the proof of [3, Remark 2.2] for the non weighted case.
Proof of theorem 3.1  Step (1) The approximate problem and a priori estimate. Let $\Omega_\varepsilon$ be a sequence of compact subsets of $\Omega$ such that $\Omega_\varepsilon$ increases to $\Omega$ as $\varepsilon \to 0$. We consider the sequence of approximate problems,

$$
\langle Au_\varepsilon, v - u_\varepsilon \rangle + \int_{\Omega} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)(v - u_\varepsilon) \, dx \geq \langle f, v - u_\varepsilon \rangle
$$

where,

$$
g_\varepsilon(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \varepsilon |g(x, s, \xi)|} \chi_{\Omega_\varepsilon}(x),
$$

and where $\chi_{\Omega_\varepsilon}$ is the characteristic function of $\Omega_\varepsilon$. Note that $g_\varepsilon(x, s, \xi)$ satisfies the following conditions,

$$
g_\varepsilon(x, s, \xi) \geq 0, \quad |g_\varepsilon(x, s, \xi)| \leq |g(x, s, \xi)| \quad \text{and} \quad |g_\varepsilon(x, s, \xi)| \leq \frac{1}{\varepsilon}.
$$

We define the operator $G_\varepsilon : X \to X^*$ by,

$$
\langle G_\varepsilon u, v \rangle = \int_{\Omega} g_\varepsilon(x, u, \nabla u)v \, dx.
$$

Thanks to Hölder's inequality we have for all $u \in X$ and $v \in X$,

$$
|\int_{\Omega} g_\varepsilon(x, u, \nabla u)v \, dx| \leq \left( \int_{\Omega} |g_\varepsilon(x, u, \nabla u)|^{q'} \sigma^{-\frac{q'}{q}} \, dx \right)^{1/q} \left( \int_{\Omega} |v|^q \sigma \, dx \right)^{1/q}
$$

$$
\leq \frac{1}{\varepsilon} \left( \int_{\Omega} \sigma^{1-q'} \, dx \right)^{1/q'} \|v\|_{q, \sigma} \leq c_\varepsilon \|v\|.
$$

(3.6)

The last inequality is due to (2.11) and (2.13).

**Lemma 3.2** The operator $B_\varepsilon = A + G_\varepsilon$ from $X$ into its dual $X^*$ is pseudo-monotone. Moreover, $B_\varepsilon$ is coercive, in the sense that: There exists $v_0 \in K_\psi$ such that

$$
\frac{\langle B_\varepsilon v, v - v_0 \rangle}{\|v\|} \to +\infty \quad \text{as} \quad \|v\| \to \infty, \quad v \in K_\psi.
$$

The proof of this lemma will be presented below. In view of lemma 3.2, (3.5) has a solution by the classical result (cf. Theorem 8.1 and Theorem 8.2 chapter 2 [7]).

With $v = \psi^+$ as test function in (3.5), we deduce that $\int_{\Omega} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)(u_\varepsilon - \psi^+) \geq 0$, then, $\langle Au_\varepsilon, u_\varepsilon \rangle \leq \langle f, u_\varepsilon - \psi^+ \rangle + \langle Au_\varepsilon, \psi^+ \rangle$, i.e.,

$$
\int_{\Omega} a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon \, dx \leq \langle f, u_\varepsilon - \psi^+ \rangle + \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_\varepsilon, \nabla u_\varepsilon) \frac{\partial \psi^+}{\partial x_i} \, dx,
$$

\[\text{(3.5)}\]
then,

\[
\alpha \sum_{i=1}^{N} \int_{\Omega} w_{i}^{p} \left| \frac{\partial u_{\varepsilon}}{\partial x_i} \right|^{p} dx \\
= \alpha \| u_{\varepsilon} \|^{p} \\
\leq \| f \|_{X_{0}^{*}} \left( \| u_{\varepsilon} \| + || \psi^{+} || \right) + \\
+ \sum_{i=1}^{N} \left( \int_{\Omega} \left| a_{i}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \right|^{p} w_{i}^{1-p'} dx \right)^{1/p} \left( \int_{\Omega} \left| \frac{\partial \psi^{+}}{\partial x_i} \right|^{p} w_{i}^{1-p'} dx \right)^{1/p} \\
\leq \| f \|_{X_{0}^{*}} \left( \| u_{\varepsilon} \| + || \psi^{+} || \right) + \\
+ c \sum_{i=1}^{N} \left( \int_{\Omega} (k^{p'} + |u_{\varepsilon}|^{q} \sigma + \sum_{j=1}^{N} \left| \frac{\partial u_{\varepsilon}}{\partial x_j} \right|^{p} w_{j}^{1-p'} dx \right)^{1/p} \| \psi^{+} \|.
\]

Using (2.13) the last inequality becomes,

\[
\alpha \| u_{\varepsilon} \|^{p} \leq c_{1} \| u_{\varepsilon} \| + c_{2} \| \psi^{+} \|^{p} + c_{3} \| u_{\varepsilon} \|^{p-1} + c_{4},
\]

where \( c_{i} \) are various positive constants. Then, thanks to (2.10) we can deduce that \( u_{\varepsilon} \) remains bounded in \( W^{1,p}_{0}(\Omega, w) \), i.e.,

\[
\| u_{\varepsilon} \| \leq \beta_{0}, \quad (3.7)
\]

where \( \beta_{0} \) is some positive constant. Extracting a subsequence (still denoted by \( u_{\varepsilon} \)) we get

\[
u_{\varepsilon} \rightharpoonup u \quad \text{weakly in } X \text{ and a.e. in } \Omega.
\]

Note that \( u \geq \psi \text{ a.e.} \)

**Step (2) Strong convergence of** \( T_{k}(u_{\varepsilon}) \). Thanks to (3.7) and (2.13) we can extract a subsequence still denoted by \( u_{\varepsilon} \) such that

\[
u_{\varepsilon} \rightharpoonup u \quad \text{weakly in } W^{1,p}_{0}(\Omega, w) \]

\[
u_{\varepsilon} \rightarrow u \quad \text{a.e. in } \Omega. \quad (3.8)
\]

Let \( k > 0 \) by lemma 2.4 we have

\[
T_{k}(u_{\varepsilon}) \rightarrow T_{k}(u) \quad \text{weakly in } W^{1,p}_{0}(\Omega, w) \text{ as } \varepsilon \rightarrow 0. \quad (3.9)
\]

Our objective is to prove that

\[
T_{k}(u_{\varepsilon}) \rightarrow T_{k}(u) \quad \text{strongly in } W^{1,p}_{0}(\Omega, w) \text{ as } \varepsilon \rightarrow 0. \quad (3.10)
\]

Fix \( k > \| \psi^{+} \|_{\infty} \), and use the notation \( v_{\varepsilon} = T_{k}(u_{\varepsilon}) - T_{k}(u) \). We use, as a test function in (3.5),

\[
v_{\varepsilon} = u_{\varepsilon} - \eta \varphi_{\lambda}(z_{\varepsilon}) \quad (3.11)
\]

where \( \varphi_{\lambda}(s) = se^{\lambda s^{2}} \) and \( \eta = e^{-4\lambda k^{2}} \). Then we can check that \( v_{\varepsilon} \) is admissible test function. So that

\[
-(Au_{\varepsilon}, \eta \varphi_{\lambda}(z_{\varepsilon})) - \int_{\Omega} g_{\varepsilon}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \eta \varphi_{\lambda}(z_{\varepsilon}) dx \geq -\langle f, \eta \varphi_{\lambda}(z_{\varepsilon}) \rangle
\]
which implies that

\[
\langle Au_\varepsilon, \varphi_\lambda(z_\varepsilon) \rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \varphi_\lambda(z_\varepsilon) \, dx \leq \langle f, \varphi_\lambda(z_\varepsilon) \rangle. 
\] (3.12)

Since \( \varphi_\lambda(z_\varepsilon) \) is bounded in \( X \) and converges a.e. in \( \Omega \) to zero and using (2.13), we have \( \varphi_\lambda(z_\varepsilon) \rightharpoonup 0 \) weakly in \( X \) as \( \varepsilon \to 0 \). Then

\[
\eta_1(\varepsilon) = \langle f, \varphi_\lambda(z_\varepsilon) \rangle \to 0, 
\] (3.13)

and since \( g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \varphi_\lambda(z_\varepsilon) \geq 0 \) in the subset \( \{ x \in \Omega : |u_\varepsilon(x)| \geq k \} \) hence (3.12) and (3.13) yield

\[
\langle Au_\varepsilon, \varphi_\lambda(z_\varepsilon) \rangle + \int_{\{|u_\varepsilon| \leq k\}} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \varphi_\lambda(z_\varepsilon) \, dx \leq \eta_1(\varepsilon). 
\] (3.14)

We study each term in the left hand side of (3.14). We have,

\[
\langle Au_\varepsilon, \varphi_\lambda(z_\varepsilon) \rangle = \int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla (T_k(u_\varepsilon) - T_k(u)) \varphi_\lambda'(z_\varepsilon) \, dx \\
= \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla (T_k(u_\varepsilon) - T_k(u)) \varphi_\lambda'(z_\varepsilon) \, dx \\
- \int_{\{|u_\varepsilon| > k\}} a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u) \varphi_\lambda'(z_\varepsilon) \, dx \\
= \int_\Omega \left( a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - a(x, T_k(u_\varepsilon), \nabla T_k(u)) \right) \nabla (T_k(u_\varepsilon) - T_k(u)) \varphi_\lambda'(z_\varepsilon) \, dx \\
- T_k(u)) \varphi_\lambda'(z_\varepsilon) \, dx + \eta_2(\varepsilon), 
\] (3.15)

where,

\[
\eta_2(\varepsilon) = \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u)) \nabla (T_k(u_\varepsilon) - T_k(u)) \varphi_\lambda'(z_\varepsilon) \, dx \\
- \int_{\{|u_\varepsilon| > k\}} a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u) \varphi_\lambda'(z_\varepsilon) \, dx, 
\]
which converges to 0 as $\varepsilon \to 0$. On the other hand,

$$
| \int_{\{ |u_\varepsilon| \leq k \}} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \varphi_\lambda(z_\varepsilon) \, dx |
$$

$$
\leq \int_{\{ |u_\varepsilon| \leq k \}} b(k)|c(x)| \varphi_\lambda(z_\varepsilon) \, dx
$$

$$
\leq b(k) \int_{\{ |u_\varepsilon| \leq k \}} c(x)|\varphi_\lambda(z_\varepsilon)| \, dx + b(k) \int_{\{ |u_\varepsilon| \leq k \}} a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon |\varphi_\lambda(z_\varepsilon)| \, dx
$$

$$
= \eta_3(\varepsilon) + b(k) \frac{1}{\alpha} \int_{\Omega} a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon) - T_k(u_\varepsilon)) \nabla(T_k(u_\varepsilon) - T_k(u)) \, dx + \eta_4(\varepsilon)
$$

(3.16)

where

$$
\eta_3(\varepsilon) = b(k) \int_{\{ |u_\varepsilon| \leq k \}} c(x)|\varphi_\lambda(z_\varepsilon)| \, dx \to 0 \text{ as } \varepsilon \to 0
$$

and

$$
\eta_4(\varepsilon) = \eta_3(\varepsilon) + b(k) \frac{1}{\alpha} \int_{\Omega} a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon) - T_k(u)) \nabla(T_k(u_\varepsilon) - T_k(u)) |\varphi_\lambda(z_\varepsilon)| \, dx
$$

$$
+ \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla(T_k(u_\varepsilon)) \nabla(T_k(u_\varepsilon) - T_k(u)) \, dx \to 0 \text{ as } \varepsilon \to 0.
$$

Note that, when $\lambda \geq \left( \frac{b(k)}{2\alpha} \right)^{\frac{1}{2}}$ we have

$$
\varphi_\lambda'(s) - \frac{b(k)}{\alpha} |\varphi(s)| \geq \frac{1}{2}.
$$

Which combining with (3.14),(3.15) and (3.16) one obtains

$$
\int_{\Omega} (a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla(T_k(u_\varepsilon) - T_k(u)) \, dx
$$

$$
\leq \eta_3(\varepsilon) = 2(\eta_1(\varepsilon) - \eta_2(\varepsilon) + \eta_4(\varepsilon)) \to 0 \text{ as } \varepsilon \to 0.
$$

Finally lemma 2.5 implies (3.10) for any fixed $k \geq \|\psi\|_\infty$.

**Step (3) Passage to the limit.** In view of (3.10) we have for a subsequence,

$$
\nabla u_\varepsilon \rightharpoonup \nabla u \text{ a.e. in } \Omega,
$$

(3.17)

which with (3.8) imply,

$$
a(x, u_\varepsilon, \nabla u_\varepsilon) \to a(x, u, \nabla u) \text{ a.e. in } \Omega,
$$

$$
g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \to g(x, u, \nabla u) \text{ a.e. in } \Omega,
$$

$$
g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \to g(x, u, \nabla u) u \text{ a.e. in } \Omega.
$$

(3.18)
On the other hand, thanks to (2.14) and (3.7) we have $a(x, u_\varepsilon, \nabla u_\varepsilon)$ is bounded in $\prod_{i=1}^{N} L^p(\Omega, w_1^*)$ then by lemma 2.1 we obtain

$$a(x, u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup a(x, u, \nabla u) \quad \text{weakly in } \prod_{i=1}^{N} L^p(\Omega, w_1^*).$$

We shall prove that,

$$g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup g(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega).$$

By (3.18), to apply Vitali’s theorem it suffices to prove that $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)$ is uniformly equi-integrable. Indeed, thanks to (2.17), (3.6) and (3.7) we obtain,

$$0 \leq \int_{\Omega} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \, dx \leq c_0,$$

where $c_0$ is some positive constant. For any measurable subset $E$ of $\Omega$ and any $m > 0$ we have,

$$\int_{E} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx = \int_{E \cap X_m^e} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx + \int_{E \cap Y_m^e} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx$$

where,

$$X_m^e = \{ x \in \Omega, \ |u_\varepsilon(x)| \leq m \}, \quad Y_m^e = \{ x \in \Omega, \ |u_\varepsilon(x)| > m \}.$$  

From these expressions, (2.18), and (3.21), we have

$$\int_{E} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx$$

$$= \int_{E \cap X_m^e} |g_\varepsilon(x, u_\varepsilon, \nabla T_m(u_\varepsilon))| \, dx + \int_{E \cap Y_m^e} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \, dx$$

$$\leq \int_{E \cap X_m^e} |g_\varepsilon(x, u_\varepsilon, \nabla T_m(u_\varepsilon))| \, dx + \frac{1}{m} \int_{E} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \, dx$$

$$\leq b(m) \int_{E} \left( \sum_{i=1}^{N} w_i \left| \frac{\partial T_m(u_\varepsilon)}{\partial x_i} \right|^p + c(x) \right) + c_0$$

Since the sequence $(\nabla T_m(u_\varepsilon))$ strongly converges in $\prod_{i=1}^{N} L^p(\Omega, w_1)$, then (3.22) implies the equi-integrability of $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)$.

Moreover, since $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \geq 0$ a.e. in $\Omega$, then by (3.18), (3.21) and Fatou’s lemma, we have $g(x, u, \nabla u) u \in L^1(\Omega)$. On the other hand, for $v \in L^\infty(\Omega)$, set $h = k + \|v\|_{\infty}$, then

$$\frac{\partial T_k(v - u_\varepsilon)}{\partial x_i} |w_1|^{1/p} = \chi_{\{|v-u_\varepsilon| \leq k\}} \left| \frac{\partial v}{\partial x_i} - \frac{\partial u_\varepsilon}{\partial x_i} \right| |w_1|^{1/p}$$

$$\leq \chi_{\{|u_\varepsilon| \leq k\}} \left| \frac{\partial v}{\partial x_i} - \frac{\partial u_\varepsilon}{\partial x_i} \right| |w_1|^{1/p}$$

$$\leq \left| \frac{\partial v}{\partial x_i} \right| |w_1|^{1/p} + \left| \frac{\partial T_k(u_\varepsilon)}{\partial x_i} \right| |w_1|^{1/p}$$
which implies, using Vitali’s theorem with (3.10) and (3.17) that
\[ \nabla T_k(v - u_\varepsilon) \rightarrow \nabla T_k(v - u) \quad \text{strongly in} \quad \prod_{i=1}^{N} L^p(\Omega, u_i) \]  
(3.23)
for any \( v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega) \). Thanks to lemma 3.1 and from (3.19), (3.20) and (3.23) we can pass to the limit in
\[ \langle Au, T_k(v - u_\varepsilon) \rangle + \int_{\Omega} g(x, u_\varepsilon, \nabla u_\varepsilon) T_k(v - u_\varepsilon) \geq \langle f, T_k(v - u_\varepsilon) \rangle \]
and we obtain,
\[ \langle Au, T_k(v - u) \rangle + \int_{\Omega} g(x, u, \nabla u) T_k(v - u) \geq \langle f, T_k(v - u) \rangle \]  
(3.24)
for any \( v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega) \) and for all \( k > 0 \).

Taking for any \( v \in W_0^{1,p}(\Omega, w) \) and \( v \geq \psi \) the test function \( T_m(v) \) which belongs to \( W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega) \) for \( m \geq \|\psi\|_{L^\infty(\Omega)} \) and passing to the limit in (3.24) as \( m \rightarrow \infty \), then \( u \) is a solution of (3.4). Using again lemma 3.1 we obtain the desired result, i.e., \( u \) is a solution of (3.3).

**Proof of lemma 3.2** By proposition 2.6 chapter 2 [7], it is sufficient to show that \( B_\varepsilon \) is of the calculus of variations type in the sense of definition 2.1. Indeed put,
\[ b_1(u, v, \tilde{w}) = \sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla v) \nabla \tilde{w} \, dx, \quad b_2(u, \tilde{w}) = \int_{\Omega} g(x, u, \nabla u) \tilde{w} \, dx. \]
Then the mapping \( \tilde{w} \mapsto b_1(u, v, \tilde{w}) + b_2(u, \tilde{w}) \) is continuous in \( X \). Then
\[ b_1(u, v, \tilde{w}) + b_2(u, \tilde{w}) = b(u, v, \tilde{w}) = (B_\varepsilon(u, v), \tilde{w}), \quad B_\varepsilon(u, v) \in W^{-1,p'}(\Omega, w^*) \]
and we have
\[ B_\varepsilon(u, u) = B_\varepsilon u. \]
Using (2.14) and Hölder’s inequality we can show that \( A \) is bounded as in [4], and thanks to (3.6) \( B_\varepsilon \) is bounded. Then, it is sufficient to check (2.6)-(2.9).

Next we show that (2.6) and (2.7) are true. By (2.15) we have,
\[ (B_\varepsilon(u, u) - B_\varepsilon(u, v), u - v) = b_1(u, u, u - v) + b_1(u, v, u - v) \geq 0. \]
The operator \( v \rightarrow B_\varepsilon(u, v) \) is bounded hemi-continuous. Indeed, we have
\[ a_i(x, u, \nabla (v_1 + \lambda v_2)) \rightarrow a_i(x, u, \nabla v_1) \quad \text{strongly in} \quad L^p(\Omega, u_i) \quad \text{as} \quad \lambda \rightarrow 0. \]  
(3.25)
On the other hand, \( g_\varepsilon(x, u_1, u_2, \nabla (u_1 + \lambda u_2)) \) is bounded in \( L^{p'}(\Omega, \sigma^{-q'}) \) and \( g_\varepsilon(x, u_1 + \lambda u_2, u_2, \nabla (u_1 + \lambda u_2)) \rightarrow g_\varepsilon(x, u_1, \nabla u_1) \quad a.e. \quad \text{in} \quad \Omega \), hence lemma 2.1 gives
\[ g_\varepsilon(x, u_1 + \lambda u_2, \nabla (u_1 + \lambda u_2)) \rightarrow g_\varepsilon(x, u_1, \nabla u_1) \quad \text{weakly in} \quad L^{q'}(\Omega, \sigma^{-q'}) \quad \text{as} \quad \lambda \rightarrow 0. \]  
(3.26)
Using (3.25) and (3.26) we can write
\[ b(u, v_1 + \lambda v_2, \bar{w}) \to b(u, v_1, \bar{w}) \] as \( \lambda \to 0 \) \( \forall u, v_i, \bar{w} \in X \).

Similarly we can prove (2.7).

Proof of assertion (2.8). Assume that \( u_n \rightharpoonup u \) weakly in \( X \) and \( (B(u_n, u_n) - 0, u_n - u) \) \( \to 0 \). We have,
\[
(B(u_n, u_n) - B(u_n, u), u_n - u) = \sum_{i=1}^{N} \int_{\Omega} (a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u)) \nabla(u_n - u) \, dx \to 0,
\]
then, by lemma 2.5, \( u_n \to u \) strongly in \( X \), which gives
\[ b(u_n, v, \bar{w}) \to b(u, v, \bar{w}) \] \( \forall \bar{w} \in X \),
i.e., \( B(\epsilon u_n, v) \to B(\epsilon u, v) \) weakly in \( X^* \). It remains to prove (2.9). Assume that
\[ u_n \rightharpoonup u \] weakly in \( X \) \hspace{1cm} (3.27)
and that
\[ B(u_n, v) \to \psi \] weakly in \( X^* \). \hspace{1cm} (3.28)
Thanks to (2.13), (2.14) and (3.27) we obtain,
\[ a_i(x, u_n, \nabla v) \to a_i(x, u, \nabla v) \] in \( L^{p'}(\Omega, w_i^*) \) as \( n \to \infty \),
then,
\[ b_1(u_n, v, u_n) \to b_1(u, v, u). \] \hspace{1cm} (3.29)
On the other hand, by H"older’s inequality,
\[
|b_2(u_n, u_n - u)| \leq \left( \int_{\Omega} |g(x, u_n, \nabla u_n)|^{q'} \sigma^{1 / q'} \, dx \right)^{1 / q'} \left( \int_{\Omega} |u_n - u|^q \sigma \, dx \right)^{1 / q}
\leq \frac{1}{\varepsilon} \left( \int_{\Omega} \sigma^{\frac{q'}{q'}} \, dx \right)^{1 / q'} \|u_n - u\|_{L^q(\Omega, \sigma)} \to 0 \] as \( n \to \infty \),
i.e.,
\[ b_2(u_n, u_n - u) \to 0 \] as \( n \to \infty \), \hspace{1cm} (3.30)
but in view of (3.28) and (3.29) we obtain
\[ b_2(u_n, u) = (B(\epsilon u_n, v), u) - b_1(u_n, v, u) \to (\psi, u) - b_1(u, v, u) \]
and from (3.30) we have \( b_2(u_n, u_n) \to (\psi, u) - b_1(u, v, u) \). Then,
\[ (B(\epsilon u_n, v), u_n) = b_1(u_n, v, u_n) + b_2(u_n, u_n) \to (\psi, u). \]
Now show that $B_\varepsilon$ is coercive. Let $v_0 \in K_\psi$. From Hölder’s inequality, the growth condition (2.14) and the compact imbedding (2.13) we have

\[
\langle Av, v_0 \rangle = \sum_{i=1}^{N} \int_{\Omega} a_i(x, v, \nabla v) \frac{\partial v_0}{\partial x_i} \, dx 
\leq \sum_{i=1}^{N} \left( \int_{\Omega} |a_i(x, v, \nabla v)|^{p'} w_i^{\frac{p}{p'}} \, dx \right)^{\frac{1}{p'}} \left( \int_{\Omega} |\frac{\partial v_0}{\partial x_i}|^{p} w_i \, dx \right)^{1/p} 
\leq c_1 \|v_0\| \left( \int_{\Omega} k(x)^{p'} + |v|^{q\sigma} + \sum_{j=1}^{N} |\frac{\partial v}{\partial x_j}|^{p} w_j \, dx \right)^{\frac{1}{p'}} 
\leq c_2 (c_3 + \|v\|^{\frac{q}{p'}} + \|v\|^{p-1}),
\]

where $c_i$ are various constants. Thanks to (2.16), we obtain

\[
\frac{\langle Av, v \rangle}{\|v\|} - \frac{\langle Av, v_0 \rangle}{\|v\|} \geq \alpha \|v\|^{p-1} - \|v\|^{p-2} - \|v\|^{\frac{q}{p'}} - c \|v\|. 
\]

In view of (2.10) we have $p - 1 > \frac{q}{p'} - 1$. Then,

\[
\frac{\langle Av, v - v_0 \rangle}{\|v\|} \to \infty \quad \text{as} \quad \|v\| \to \infty.
\]

Since $\langle G_\varepsilon v, v \rangle \geq 0$ and $\langle G_\varepsilon v, v_0 \rangle$ is bounded, we have

\[
\frac{\langle B_\varepsilon v, v - v_0 \rangle}{\|v\|} \geq \frac{\langle Av, v - v_0 \rangle}{\|v\|} - \frac{\langle G_\varepsilon v, v_0 \rangle}{\|v\|} \to \infty \quad \text{as} \quad \|v\| \to \infty.
\]

**Remark 3.2** Assumption (2.10) appears to be necessary to prove the boundedness of $(u_\varepsilon)_\varepsilon$ in $W_0^{1,p}(\Omega, w)$ and the coercivity of the operator $B_\varepsilon$. While Assumption (2.11) is necessary to prove the boundedness of $G_\varepsilon$ in $W_0^{1,p}(\Omega, w)$. Thus, when $g \equiv 0$, we don’t need to assume (2.11).

**References**


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